Exercise 1

Let $L_t = L(t, T_1, T_2)$ be the stochastic process representing the Libor rate payed between T_1 and T_2 with dynamics:

$$dL_t = \mu dt + \sigma dW_t,$$

for $t \leq T_1$, some Brownian motion W_t and $\mu, \sigma > 0$. Set the initial forward rate $L_0 = 0.8\%$, a notional $N = 10.000$, $T_1 = 0.5$, $T_2 = 1$ and $\sigma = 0.35L_0$. Assume further that the current yield of the zero coupon bond maturing at T_2 is 0.85%.

Calculate and plot the Black-Scholes implied volatility smile extracted from a caplet having the specified parameters, and using strikes K_i ranging from -20% to $+20\%$ at-the-money, with a 10% step increase.

Exercise 2

Consider the following market instruments: Libor rates $L(0, 0, 0.5) = 0.3\%$, $L(0, 0, 1) = 0.65\%$, a one-to-two years forward rate $L(0, 1, 2) = 0.9\%$, a swap rate $S_3 = 1.2\%$ for a swap with semi annual payments entered at time 0, and a swap rate $S_5 = 1.55\%$ for a swap with annual payments and entered at time 1.

Bootstrap the yield curve and use it to price a cap entered at year 2 and maturing at time 5, with annual reset dates, a notional N and a strike K of your choice. Assume the following physical dynamics for the Libor rates $L_t = L(t, T_{i-1}, T_i)$, $i = 3, 5$ observed in the whole period:

$$dL_t = \mu_{2,3} L_t + \sigma_{2,3} L_t dW_t, \quad t \leq T_2$$

$$dL_t = \mu_{3,4} L_t + \sigma_{3,4} L_t dW_t, \quad t \leq T_3$$

$$dL_t = \mu_{4,5} L_t + \sigma_{4,5} L_t dW_t, \quad t \leq T_4$$

with $\sigma_{23} = 0.15\%$, $\sigma_{34} = 0.25\%$, $\sigma_{45} = 0.4\%$.

Exercise 3

Let S_t be a financial asset of density function $f_t(x)$ and $C(S_0, K, T)$ the set of market-observed prices for call option of corresponding parameters. Assume no interest rate are paid. Prove analytically the *Breeden and Litzenberger formula:*

$$\frac{\partial C}{\partial^2 K}\Big|_{K=k} = f_T(k)$$

(Hint: express the value of the call option as an integral with respect to $f_T(x)$.)