Quantitative Methods in Portfolio Management

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• Markowitz

• $\mu$-$\sigma$ efficient

• Return

• Efficient Frontier

• Alpha

• CAPM

• BARRA

• Sharpe

• Shortfall

• Information Ratio
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Literature

introductory

- *Modern Portfolio Theory and Investment Analysis*; Elton, Gruber, Brown, Goetzmann; Wiley

- *Portfoliomanagement*; Breuer, Guertler, Schumacher; Gabler

quantitative

- *Risk and Asset Allocation*; Meucci; Springer

- *Quantitative Equity Portfolio Management*; Qian, Hua, Sorensen; CRC

- *Robust Portfolio Optimization and Management*; Fabozzi, Kolm, Pachamanova, Focardi; Wiley
1 Introduction

Investor Choice Under Certainty

- Investor will receive EUR 10000 with certainty in each of two periods
- Only investment available is savings account (yield 5%)
- Investor can borrow money at 5%

How much should the investor save or spend in each period?

Separate problem into two steps:

1. Specify options
2. Specify how to choose between options
Opportunity Set:

**A** save nothing, spend all when received, \((10000, 10000)\)

**B** save first period and consume all in the second \((0, 10000 \times (1 + 0.05) + 10000)\)

**C** consume all in the first, i.e. borrow the maximum in from the second in the first period \((10000 + 10000/(1 + 0.05), 0)\)

\(x_i\) income in period i, \(y_i\) consumption in period i

\[y_2 = x_2 + (x_1 - y_1) \times 1.05\]

Indifference Curve:

“Iso-Happiness Curve” (see graph):

assumption: each additional euro of consumptionforgone in periode 1 requires greater consumption in period 2

ordering due to investor prefers more to less
Solution:
Opportunity set is tangent to indifference set
2 Utility Theory

Use utility function to formalise investors preferences to arrive at optimal portfolio

Base Modell

- Investor has initial wealth $W_0$ at $t = 0$ and an investment universe of $n + 1$ assets ($n$ risky assets, one riskless asset)

- Investment horizon $t = 1$, short-selling, uncertain returns $r_i$ (rv) for risky assets $i = 1, \ldots, n$ and deterministic $r_0$ for bank account.

- Describe portfolio through asset weights, i.e. $P = (x_0, x_1, \ldots, x_n)$, $W_0 = W_0 \sum_i x_i$

- Uncertain wealth (i.e. rv) $W_1$ at $t = 1$, $W_1^P = \sum_i x_i W_0 (1 + r_i)$

- Assign preference through utility function to each possible opportunity set, $U(W_1^P)$, and probability to arrive at expected utility $\mathbb{E}[U(W_1^P)]$.

- Optimization problem: $\max_{x_0, \ldots, x_n} \mathbb{E}[U(W_1^P)]$
Properties of Utility Function

• determined only up to positiv linear transformations (ranking)

• investor prefers more to less, \( W^P < W^Q \) then \( U(W^P) < U(W^Q) \), \( U \) strictly increasing, if differentiable then \( U' > 0 \)

• risk appetite: for \( W_0 = \mathbb{E}[W_1] \)
  
  – risk averse: \( \mathbb{E}[U(W_0)] > \mathbb{E}[U(W_1)] \), \( U \) concave (Jensen)
  
  – risk neutral: \( \mathbb{E}[U(W_0)] = \mathbb{E}[U(W_1)] \), \( U \) linear

  – risk seeking: \( \mathbb{E}[U(W_0)] < \mathbb{E}[U(W_1)] \) \( U \) convex
Some Distributions

**Uniform Distribution**

\[ X \sim U(\mathcal{E}_{\mu, \Sigma}), \quad \mathcal{E}_{\mu, \Sigma} \text{ ellipsoid} \]

\[ f(x) = \frac{\Gamma\left(\frac{N}{2} + 1\right)}{\pi^{N/2}|\Sigma|^{1/2}} \mathbb{1}_{\mathcal{E}_{\mu, \Sigma}}(x), \quad \Phi_{\mu, \Sigma}(\omega) = e^{i\omega'\mu}\Psi(\omega'\Sigma\omega) \]

**Normal Distribution**

\[ X \sim N(\mu, \Sigma) \]

\[ f(x) = (2\pi)^{-N/2}|\Sigma|^{-1/2}e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}, \quad \Phi(\omega) = e^{i\omega'\mu-\frac{1}{2}\omega'\Sigma\omega} \]

**Student-t Distribution**

\[ Z \sim N(\mu, \Sigma), \quad W \sim \chi^2(\nu) \]

\[ \Rightarrow X \equiv \sqrt{\frac{\nu}{W}}Z \sim St(\nu, \mu, \Sigma) \]
Cauchy Distribution

\[ X \sim Ca(\mu, \Sigma) \equiv St(1, \mu, \Sigma) \]

Lognormal Distribution

\[ X \sim LogN(\mu, \Sigma) \iff X = e^Y, Y \sim N(\mu, \Sigma) \]
Distribution Classes

Elliptical Distribution

Definition: rv $Y = (Y_1, \ldots, Y_n)$ has a spherical distribution if, for every orthogonal matrix $U$

$$UY \overset{d}{=} Y$$

Properties: $Y$ spherical $\iff$ exists function $g$ (generator) such that $\Phi_Y(t) = \mathbb{E}[e^{it'Y}] = \psi_n(t't)$

Generator $\Psi$ as function of a scalar variable uniquely describes spherical distribution

$\Rightarrow Y \sim S_n(\psi)$

equivalent representation:

$Y = RU$, where $R = \|Y\|$ is norm (i.e. univariate) and $U = Y/\|Y\|$

$Y \sim S_n(\psi) \iff R$ and $U$ are independent rvs and $U$ is uniformly distributed on the surface of the unit ball

Definition: $X$ has an elliptical distribution if

$$X \overset{d}{=} \mu + AY$$

where $Y \sim S_m(\psi)$ and $A \in \mathbb{R}^{n \times m}, \mu \in \mathbb{R}^n$. $X \sim El(\mu, \Sigma, \psi), \Sigma = AA'$. 
\[ \Sigma \text{ positive definite, then isoprobability contours are surfaces of centered ellipsoids} \]

More Properties:
affine transformation: \( B \mathbf{X} + b \sim EL() \)
marginal distributions: \( \sim EL() \)
conditional distribution: \( \sim EL() \)
convolution with same dispersion matrix \( \Sigma \): \( \sim EL() \)

Examples: Uniform, Normal, Student-t, Cauchy highly symmetric and analytically tractable, yet quite flexible

**Stable Distribution**

**Definition:** \( X, X_1, X_2 \) iid rv. \( X \) is called stable if for all non-negative \( c_1, c_2 \) and appropriate numbers \( a = a(c_1, c_2), b = b(c_1, c_2) \) the following holds:

\[
c_1X_1 + c_2X_2 \overset{d}{=} a + bX
\]

i.e. closed under linear combinations
symmetric-$\alpha$-stable (one dimension) iff

$$\Phi_X(t) = \mathbb{E}[e^{itX}] = \exp\{i\mu t - c|t|^\alpha\}$$

$\mu$ location, $c$ scaling, $\alpha$ tail thickness

symmetric-$\alpha$-stable (multivariate) iff

$$\Phi_X(t) = \mathbb{E}[e^{it'X}] = e^{it'\mu} \exp\{-\int_{\mathbb{R}} |t's|^\alpha m_\Sigma(s) ds\}$$

function $m_\Sigma$ is a symmetric measure, $m_\Sigma(s) = m_\Sigma(-s)$ for all $s \in \mathbb{R}^n$ and

$$m_\Sigma(s) \equiv 0 \quad \text{for all} \quad s \quad \text{such that} \quad s'\Sigma s \neq 1$$

$$X \sim SS(\alpha, \mu, m_\Sigma)$$

Examples: Normal, Cauchy

Counterexamples: Lognormal, Student-t
Normal distribution as symmetric-alpha-stable

\[ X \sim N(\mu, \Sigma) \]

spectral decomposition of \( \Sigma \): \( \Sigma = E\Lambda^{1/2}\Lambda^{1/2}E \)

define \( n \) vectors \( \{v^{(1)}, \ldots, v^{(n)}\} = E\Lambda^{1/2} \)

define measure as

\[ m_\Sigma = \frac{1}{4} \sum_{i=1}^{n} (\delta(v^{(i)}) + \delta(-v^{(i)})) \]

Remark: stability \( \Rightarrow \) additivity, but reverse is not true in general, e.g. Wishart dist.

Infinitely Divisible Distributions

\( X \) is infinitely divisible if, for any integer \( M \) we can decompose it in law

\[ X \overset{d}{=} Y_1 + \cdots + Y_M \]

where \( (Y_i)_{i=1,\ldots,M} \) are iid rvs with possibly different common distributions for different \( M \). Examples: Normal, Lognormal, Chi2 Counterexamples: Wishart
3 Modeling the Market

Market for an investor is represented by an N-dimensional price vector of traded securities, $P_t$:

- Investment decision (allocation) at $T$
- Investment horizon $\tau$
- $P_{T+\tau}$ N-dimensional random variable

Modeling the market means modeling $P_{T+\tau}$:

1. modeling market invariants
2. determining the distribution of market invariants
3. projecting invariants into the future $T + \tau$
4. mapping of invariants to market prices

dimension of randomness $\ll$ numbers of securities $\Rightarrow$ dimension reduction
Market Invariants

\[ D_{t_0, \tilde{\tau}} = \{ t_0, t_0 + \tilde{\tau}, t_0 + 2\tilde{\tau}, \ldots \} \] set of equally spaced observation dates

random variables \( X_t, t \in D_{t, \tilde{\tau}} \) are called invariant if rv are iid and time homogeneous

simple tests: check histograms of two non-overlapping subsets of observations, scatter-plot of values vs. lagged values

Equities

\( P_t, t \in D_{t_0, \tilde{\tau}} \) equally-spaced stock price observations

equity prices are not market invariants (exponential growth)

Total return

\[ H_{t, \tilde{\tau}} = \frac{P_t}{P_{t-\tilde{\tau}}} \]

is a market invariant

g any function, then if \( X_t \) is invariant \( \Rightarrow g(X_t) \) is also an invariant
hence

\[ L_{t, \tilde{\tau}} = H_{t, \tilde{\tau}} - 1 \quad \text{and} \quad C_{t, \tilde{\tau}} = \ln \left( H_{t, \tilde{\tau}} \right) \]

are also market invariants.

**equity invariants: compound returns \( C \)**

Compound returns can be easily projected to any horizon, distribution approximately symmetric.

**Example:**

Continuous-time finance, Black/Scholes, Merton

\[ C_{t, \tilde{\tau}} = \ln \left( \frac{P_t}{P_{t-\tilde{\tau}}} \right) \sim N(\mu, \sigma^2). \]

⇒ total return \( H_{t, \tilde{\tau}} \sim \text{LogN}(\mu, \sigma^2). \)

Other Choices:

Multivariate case

\[ C_{t, \tilde{\tau}} \sim \text{EL}(\mu, \Sigma, g) \quad \text{or} \quad C_{t, \tilde{\tau}} \sim \text{SS}(\alpha, \mu, m_\Sigma) \]
Fixed-Income Market

zero-coupon bonds as building blocks, \( Z_t^{(E)} \), \( E \) maturity

normalization \( Z_E^{(E)} \equiv 1 \)

consider set of bond prices

\( Z_t^{(E)} \), \( t \in D_{t_0,\tilde{\tau}} \)

pull-to-par effect \( \Rightarrow \) bond prices are not market invariants

consider set of non-overlapping total returns

\[
H_{t,\tilde{\tau}}^{(E)} \equiv \frac{Z_t^{(E)}}{Z_{t-\tilde{\tau}}^{(E)}}, \quad t \in D_{t_0,\tilde{\tau}}
\]

pull-to-par also breaks time homogeneity of total return \( \Rightarrow \) total returns are not market invariants

consider total return of bonds with same time to maturity \( \nu \)

\[
R_{t,\tilde{\tau}}^{(\nu)} \equiv \frac{Z_t^{(t+\nu)}}{Z_{t-\tilde{\tau}}^{(t+\nu-\tilde{\tau})}}, \quad t \in D_{t_0,\tilde{\tau}}
\]

ratio of prices of two different securities
test shows that $R_{t, \tilde{\tau}}^{(\nu)}$ is acceptable as market invariant, hence also every function $g$ of $R$.

define yield to maturity $\nu$ as (annualized return of bond)

$$Y_t^{(\nu)} \equiv \frac{1}{\nu} \ln \left( Z_t^{(t+\nu)} \right)$$

consider now changes in yield to maturity

$$X_{t, \tilde{\tau}}^{(\nu)} \equiv Y_t^{(\nu)} - Y_{t-	ilde{\tau}}^{(\nu)} = \frac{1}{\nu} \ln \left( R_{t, \tilde{\tau}}^{(\nu)} \right)$$

fixed-income invariants: changes in yield to maturity $X$

invariant is specific to a given sector $\nu$ of the yield curve

Examples:

$$X_{t, \tilde{\tau}}^{(\nu)} \sim N(\mu, \sigma), \quad X_{t, \tilde{\tau}}^{(\nu)} \sim El(\mu, \Sigma, g), \quad X_{t, \tilde{\tau}}^{(\nu)} \sim SS(\alpha, \mu, m_\Sigma)$$
Derivatives

vanilla european call option

\[ C_t^{(K,E)} = C^{BS} \left( E - t, K, U_t, Z_t^{(E)}, \sigma_t^{(K,E)} \right) \]

boundary condition \( C_E^{(K,E)} = \max (U_E - K, 0) \)

\( E \) expiry date, \( K \) strike

Market variables:
\( U_t \) underlying, \( Z_t^{(E)} \) zero bond, \( \sigma_t^{(K,E)} \) implied percentage volatility (vol surface)

for zero bond \( Z \) and underlying \( U \) market invariants are know

what about implied vol?

consider \textit{at-the-money-forward} (ATMF) implied vol, i.e. implied vol at strike equals forward price

\[ \sigma_t^{(K_t,E)} , \quad t \in D_{t_0,\tilde{\tau}} \quad \text{where} \quad K_t \equiv \frac{U_t}{Z_t^{(E)}} \]

implied vol is not a market invariant as expiry convergence breack time-homogeneity
eliminate expiry date dependency through considering set of implied vols (rolling ATMF vols)

\[
\sigma_t^{(K_t,t+\nu)}, \quad t \in D_{t_0, \tilde{\tau}}
\]

\[
\sigma_t^{(K_t,t+\nu)} \approx \sqrt{\frac{2\pi}{\nu} \frac{C_t^{(K_t,t+\nu)}}{U_t}}
\]

sill has time dependence

consider changes in ATMF implied vols

\[
X_t^{(\nu)} = \sigma_t^{(K_t,t+\nu)} - \sigma_{t-\tilde{\tau}}^{(K_t-t-\tilde{\tau}+\nu)}, \quad t \in D_{t_0, \tilde{\tau}}
\]

derivatives invariants: changes in roll. ATMF implied vol
distribution of changes in roll.ATMF implied vol is symmetrical, hence modeling as

\[
X_t^{(\nu)} \sim N(\mu, \sigma), \quad X_t^{(\nu)} \sim El(\mu, \Sigma, g), \quad X_t^{(\nu)} \sim SS(\alpha, \mu, m_\Sigma)
\]
Projection of the Invariants to the Investment Horizon

invariants $X_{t, \tilde{\tau}}$ relative to estimation interval $\tilde{\tau}$
representation of distribution in form of probability density $f_{X_{t, \tilde{\tau}}}$ or characteristic function $\Phi_{X_{t, \tilde{\tau}}}$
in general, investment horizon $\tau$ is different than estimation interval $\tilde{\tau}$

Figure 1: asset swap.
Projection to investment horizon:
determine distribution of \( X_{T+\tau,\tau} \), i.e. \( f_{X_{T+\tau,\tau}} \) or \( \Phi_{X_{T+\tau,\tau}} \), from estimated distribution
assume \( \tau \) is an integer multiple of \( \tilde{\tau} \)
invariants are \textit{additive}, hence
\[
X_{T+\tau,\tau} = X_{T+\tau,\tilde{\tau}} + X_{T+\tau-\tilde{\tau},\tilde{\tau}} + \cdots + X_{T+\tilde{\tau},\tilde{\tau}}
\]
since invariants are in the form if differences:
- equity return: \( X_{t,\tau} = \ln(P_t) - \ln(P_{t-\tau}) \)
- yield to maturity: \( X_{t,\tau} = Y_t - Y_{t-\tau} \)
- ATFM impl. vol: \( X_{t,\tau} = \sigma_t - \sigma_{t-\tau} \)

\( X_{T+n\tilde{\tau},\tilde{\tau}} \) are invariants wrt non-overlapping intervalls, i.e. iid.
Projection via convolution:
\[
\Phi_{X_{T+\tau,\tau}} = \mathbb{E} \left[ e^{i\omega X_{T+\tau,\tau}} \right] = \mathbb{E} \left[ e^{i\omega X_{T+\tau,\tilde{\tau}} + X_{T+\tau-\tilde{\tau},\tilde{\tau}} + \cdots + X_{T+\tilde{\tau},\tilde{\tau}}} \right] \overset{iid}{=} \left( \Phi_{X_{T+\tau,\tilde{\tau}}} \right)^{\tau/\tilde{\tau}}
\]
representation involving the density \( f_{X_{T+\tau,\tau}} \) can be obtained via Fourier transformation
\[
\Phi_X = \mathcal{F}[f_X], \quad f_x = \mathcal{F}^{-1}[\Phi_X]
\]
Example:

\[
X_{t,\tilde{\tau}} \equiv \begin{pmatrix} C_{t,\tilde{\tau}} \\ X_{t,\tilde{\tau}}^\nu \end{pmatrix} = \begin{pmatrix} \ln P_t - \ln P_{t-\tilde{\tau}} \\ Y_t^\nu - Y_{t-\tilde{\tau}}^\nu \end{pmatrix} \sim N(\mu, \Sigma)
\]

Characteristic function:

\[
\Phi_{X_{t,\tilde{\tau}}}(\omega) = e^{i\omega' \mu - \frac{1}{2} \omega' \Sigma \omega}
\]

\[
\Phi_{X_{T+\tau,\tau}} = \left( \Phi_{X_{t,\tilde{\tau}}} \right)^{\tau/\tilde{\tau}} = \left( e^{i\omega' \mu - \frac{1}{2} \omega' \Sigma \omega} \right)^{\tau/\tilde{\tau}} = e^{i\omega' \frac{\tau}{\tilde{\tau}} \mu - \frac{1}{2} \omega' \frac{\tau}{\tilde{\tau}} \Sigma \omega}
\]

\[\implies X_{T+\tau,\tau} \sim N\left( \frac{\tau}{\tilde{\tau}} \mu, \frac{\tau}{\tilde{\tau}} \Sigma \right)\]

Note: normal dist is infinitely divisible, hence $\tau/\tilde{\tau}$ need not to be an integer

Furthermore for moments:

\[
E[X_{T+\tau,\tau}] = \frac{\tau}{\tilde{\tau}} E[X_{t,\tilde{\tau}}], \quad \text{Cov}[X_{T+\tau,\tau}] = \frac{\tau}{\tilde{\tau}} \text{Cov}[X_{t,\tilde{\tau}}], \quad \text{Std}[X_{T+\tau,\tau}] = \sqrt{\frac{\tau}{\tilde{\tau}}} \text{Std}[X_{t,\tilde{\tau}}]
\]

Remarks:

- simplicity of projection formula due to specific formulation of market invariants
- projection formula hides estimation risk, distribution at horizon can only be estimated (estimation error)
From Invariants to Market Prices

how to recover prices from invariants?

market prices of securities at horizon \( T + \tau \) are functions of the the investment-horizon invariants

\[
P_{T+\tau} = g(X_{T+\tau,\tau})
\]

**Equities**

\[
P_{T+\tau} = P_T e^X
\]

(see choice of invariants, i.e. compound return)

**Fixed-Income**

\[
Z_{T+\tau}^{(E)} = Z_T^{(E-\tau)} e^{-X(E-T-\tau)(E-T-\tau)}
\]

in general (equities and fixed-income)

\[
P = e^Y, \quad \text{with} \quad Y \equiv \gamma + \text{diag}(\epsilon)X
\]

\[
\gamma_n \equiv \begin{cases} 
\ln(P_T), & \text{if stock} \\
\ln(Z_T^{(E-\tau)}), & \text{if bond.}
\end{cases}, \quad \epsilon_n \equiv \begin{cases} 
1, & \text{if stock} \\
-(E - T - \tau), & \text{if bond.}
\end{cases}
\]
$\mathbf{Y}$ affine transformation of $\mathbf{X}$ $\Rightarrow$ distribution of $\mathbf{Y}$, e.g. as characteristic function

$$\Phi_{\mathbf{Y}}(\omega) = e^{i\omega'\gamma} \Phi_{\mathbf{X}}(\text{diag}(\epsilon)\omega)$$

Example:

two-security market (stock, bond), maturity $E = T + \tau + \nu$

$$\mathbf{P} = \begin{pmatrix} P_{T+\tau} \\ Z_{T+\tau}^{(E)} \end{pmatrix} = e^{\gamma'\text{diag}(\epsilon)\mathbf{X}}, \quad \gamma = \begin{pmatrix} \ln(P_T) \\ \ln(Z_{T}^{(T+\nu)}) \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 1 \\ -\nu \end{pmatrix}.$$ 

characteristic function, $\mathbf{X}$ multi-normal

$$\Phi_{\mathbf{Y}}(\omega) = e^{i\omega'[\gamma + \frac{\tau}{T}\text{diag}(\epsilon)\mu] - \frac{1}{2}\frac{\tau}{T}\omega'\text{diag}(\epsilon)\Sigma\text{diag}(\epsilon)\omega}$$

$\Rightarrow \mathbf{Y}$ multi-normal

$$\mathbf{Y} \sim N \left( \gamma + \frac{\tau}{T}\text{diag}(\epsilon)\mu, \frac{\tau}{T}\text{diag}(\epsilon)\Sigma\text{diag}(\epsilon) \right)$$

$\Rightarrow \mathbf{P}$ log-normal

in most cases not possible to get distribution of future prices in closed form

$\Rightarrow$ usually sufficient to work with moments (Taylor)

$$\mathbb{E}[P_n] = e^{\gamma n} \Phi_{\mathbf{X}}(-\epsilon_n)$$
Example: simple stock

\[ \mathbb{E}[P_{T+\tau}] = P_T e^{\tau \mu + \frac{\tau}{2} \sigma^2} \]

**Derivatives**

Derivative price is a nonlinear function of several investment-horizon invariants (e.g. call option)

\[ P = g(X), \quad \text{e.g.} \quad C_{T+\tau}^{(K,E)} = C^{BS} \left( E - T - \tau, K, U_{T+\tau}, Z_{T+\tau}^{(E)}, \sigma_{T+\tau}^{(K,E)} \right) \]

with

\[ U_{T+\tau} = U_T e^{X_1}, \quad Z_{T+\tau}^{(E)} = Z_T^{(E-\tau)} e^{-X_2(E-T-\tau)}, \quad \sigma_{T+\tau}^{(K,E)} = \sigma_T^{(K,E-\tau)} + X_3 \]

distribution assumptions for \( X_1, X_2, X_3 \), but in general no closed form distribution for \( P, C \).

\[ \implies \] Taylor expansion of \( P \):

\[ P = g(m) + (X - m)' \partial_x g \big|_{x=m} + \frac{1}{2} (X - m)' \partial_{xx} g \big|_{x=m} (X - m) + \ldots \]

1\(^{st}\) order: delta-vega, duration

2\(^{nd}\) order: gamma, convexity
Dimension Reduction

\[ \mathbf{P}_{T+\tau} = g(\mathbf{X}_{T+\tau}) \]

market includes a large number of securities, i.e. market invariant-vector \( \mathbf{X}_{t,\tau} \) has a large dimension
\( \Rightarrow \) dimension reduction to a vector \( \mathbf{F} \) of few common factors

\[ \mathbf{X}_{t,\tau} = h(\mathbf{F}_{t,\tau}) + \mathbf{U}_{t,\tau} \]

with \( K = \text{dim}(\mathbf{F}) \ll N = \text{dim}(\mathbf{X}) \)

If \( \mathbf{X} \) represents market invariant \( \mathbf{F} \), \( \mathbf{U} \) must be invariants too.

common factors \( \mathbf{F} \) should be responsible for most of the randomness, \( \mathbf{U} \) should only be a residual,

\[ \tilde{\mathbf{X}} = h(\mathbf{X}) \approx \mathbf{X} \]

measure goodness of approximation with generalized r-squared:

\[ R^2(\tilde{\mathbf{X}}, \mathbf{X}) \equiv 1 - \frac{\mathbb{E} [(\mathbf{X} - \tilde{\mathbf{X}})'(\mathbf{X} - \tilde{\mathbf{X}})]}{\text{tr}(\text{Cov}(\mathbf{X}))} \]
restrict to *linear factor model* 

\[ X \equiv BF + U \]

\( K \times K \)-matrix \( B \) is called factor loadings

Ideally, \( F \) and \( U \) should be independent variables, but too restrictive for practical purposes, hence impose only

\[ \text{Corr}(F, U) = 0 \]

- explicit factor model: common factors are measurable market variable
- hidden factor model: common factors are synthetic variables

### Explicit Factors

factor loadings for linear regression solve

\[ B_r = \arg \max_B R^2(X, BF) \]

from \( M \equiv \mathbb{E}[(X - BF)(X - BF)'] \) and \( \partial M/\partial B_{ij} = (0)_{ij} \) follows

\[ \Rightarrow B_r = \mathbb{E}[XF']\mathbb{E}[FF']^{-1} \]
which yields

\[ \tilde{X}_r \equiv B_r F, \quad U_r \equiv X - \tilde{X}_r \]

In general, residuals \( U \) do not have zero expectation and are correlated with \( F \) (unless \( \mathbb{E}[F] = 0 \)).

Enhance linear model with constant factor

\[ X \equiv a + BF + U \]

minimizing \( M = \mathbb{E}[(X - (a + BF))(X - (a + BF))'] \) yields

\[ \tilde{X}_r \equiv \mathbb{E}[X] \text{Cov}[X, F] \text{Cov}[F]^{-1}(F - \mathbb{E}[F]) \]

perturbations \( U \) now have zero expectation and are uncorelated with \( F \).

quality of regression:
- adding factors trivially improves quality but number should be kept at a minimum
- factors should be chosen as diversified as possible (avoid collinearity)
Hidden Factors

factors are not market invariants (not observable)

\[ X \equiv q + BF(X) + U \]

Principal Component Analysis (PCA)

assume that hidden factors are affine transformations of invariants:

\[ F \equiv d + A'X, \]

\(d\) is \(K\)-dim vector, \(A\) is \(K \times N\)-dim matrix

recovered invariants are affine transformation of original invariants

\[ \tilde{X} \equiv m + BA'X, \quad \text{with} \quad m = q + Bd \]

PCA solution from

\[(B, A, m) \equiv \arg \max_{B,A,m} R^2(X, m + BA'X) \]

Impose as additional condition \(\mathbb{E}[F] = 0\)
For this, consider spectral decomposition of covariance matrix

\[
\text{Cov}[\mathbf{X}] = \mathbf{E} \Lambda \mathbf{E}'
\]

\( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \) diag matrix of decreasing positive eigenvalues, eigenvectors \( \mathbf{E} = (\mathbf{e}^1, \ldots, \mathbf{e}^N) \), \( \mathbf{E} \mathbf{E}' = \mathbf{1}_N \)

one hidden factor, \( K = 1 \):
guess

\[
\mathbf{F} = (\mathbf{e}^1)' \mathbf{X},
\]

i.e. orthogonal projection of \( \mathbf{X} \) onto the direction of first eigenvector

recovered invariant is

\[
\tilde{\mathbf{X}} = \mathbf{m} + \mathbf{e}^1 (\mathbf{e}^1)' \mathbf{X}
\]

impose \( \mathbb{E}[\tilde{\mathbf{X}}] = \mathbb{E}[\mathbf{X}] \)

\[
\Rightarrow \mathbf{m} = (\mathbf{1}_N - \mathbf{e}^1 (\mathbf{e}^1)') \mathbb{E}[\mathbf{X}]
\]

satisfying \( \mathbb{E}[\mathbf{F}] = 0 \) yields

\[
\mathbf{F} = (\mathbf{e}^1)' \mathbf{X} - (\mathbf{e}^1)' \mathbb{E}[\mathbf{X}]
\]
$K$ hidden factors:
consider $N \times K$ matrix $E_K = (e^1, \ldots, e^K)$
solution to PCA problem is
\[(B, A, m) = (E_K, E_k, (1_N - E_K E_K') E[X])\]
represent orthogonal projection of original invariants onto hyperplane spanned by the $K$ longest principal axes (i.e. contains the maximum information)
hidden factors:
\[F = E_K'(X - E[X])\]
PCA-invariants:
\[\hat{X} = E[X] + E_K E_K'(X - E[X])\]
residuals $U = \hat{X} - X$ have zero expectation and zero correlation with factors $F$
\[\mathbb{E}[U] = 0, \quad \text{Corr}[U, F] = 0\]
quality of approximation depends on number of hidden factors, with
\[R^2(\hat{X}, X) = \frac{\sum_{n=1}^{K} \lambda_n}{\sum_{n=1}^{N} \lambda_n}\]
$K$th eigenvalue is variance of the $K$th hidden factor

$$\nabla[F_n] = (e^n)'E\Lambda E' e^n = \lambda_n$$

“$n$-th eigenvalue is contribution to the total recovered randomness”

**Explicit vs. Hidden Factors**

explicit factors models are interpretable, hidden factor models tend to have a higher “explanatory” power

PCA: recovered invariants represent projections of the original invariant onto the $K$-th dimensional hyperplane of maximum randomness spanned by the first $K$ principal axes

Explicit Regression: $X = (X_{1,\ldots,K}; X_{K+1,\ldots,N}$; recovered invariants represent projections of the original invariants onto the plane spanned by the $K$ reference invariants
Examples

linear stock returns and index return (explicit)

\[ L^n_{t,\tau} = \frac{P^n_t}{P^n_{t-\tau}} - 1, \quad n = 1, \ldots, N, \quad F_{t,\tau} = \frac{M_t}{M_{t-\tau}} - 1 \]

\[ \tilde{L}^n_{t,\tau} = \mathbb{E}[L^n_{t,\tau}] + \beta^n_{\tau}(F_{t,\tau} - \mathbb{E}(F_{t,\tau})) \quad \text{with} \quad \beta_{\tau} = \frac{\text{Cov}(L^n_{t,\tau}, F_{t,\tau})}{\text{V}(F_{t,\tau})} \]

suppose additional constraint holds:

\[ \mathbb{E}[L^n_{t,\tau}] = \beta_{\tau}\mathbb{E}[F_{t,\tau}] + (1 - \beta_{\tau})R_{t,\tau} \quad \text{with} \quad R_{t,\tau} = \left( \frac{1}{Z^{(t)}_{t-\tau}} - 1 \right) \]

\[ \Rightarrow \tilde{L}^n_{t,\tau} = R_{t,\tau} + \beta_{\tau}(F_{t,\tau} - R_{t,\tau}), \quad \text{CAPM} \]

market-size explicit factors:

(i) broad stock index return
(ii) difference in return between small-cap index and large-cap index (“SmB”)
(iii) difference in return between book-to-market value index and small-book-to-market value index (“HmL”)
Case Study: Modeling the Swap Market

Swap:
plain vanilla interest rate swap (IRS), payer forward start (PFS): commitment initiated at \( t_0 \) to exchange payments between two different legs, starting from a future time at times \( t_1, \ldots, t_m \), ("eight-year swap two years forward").

fixed leg pays

\[
N \delta_i K,
\]

\( N \) nominal, \( K \) fixed rate, \( \delta_i \) day-count fraction

floating leg pays

\[
N \delta_i L(t_{i-1}, t_i),
\]

\( L(t_{i-1}, t_i) \) floating rate between \( t_{i-1} - t_i \) reset at \( t_{i-1} \) paid in arrears.

discounted payoff of PFS at \( t < t_1 \) is

\[
\sum_{i=1}^{m} D(t, t_i) N \delta_i (L(t_{i-1}, t_i) - K)
\]
present value of PFS at $t$:

$$PV(PFS)_t = N \sum_{i=1}^{m} \delta_i Z_t^{(t_i)} (F(t; t_{i-1}, t_i) - K) = NZ_t^{(t_1)} - NZ_t^{(t_m)} - NK \sum_{i=1}^{m} \delta_i Z_t^{(t_i)}$$

$F(t; t_{i-1}, t_i) := \frac{1}{\delta_i} \left( \frac{Z_t^{(t_{i-1})}}{Z_t^{(t_i)}} - 1 \right)$ forward rate

⇒ swap market is completely priced by the set of zero coupon bond prices at all maturities

**market invariants for FI:**

changes in yield-to-maturity

$$X_{t, \tilde{\tau}}^{(\nu)} = Y_t^{(\nu)} - Y_{t-\tilde{\tau}}^{(\nu)} \text{ with yield curve } \nu \mapsto Y_t^{(\nu)} := -\frac{1}{\nu} \ln \left( Z_t^{(t+\nu)} \right)$$

data set of zero-coupon bond prices $Z_t^{(E)}, \tilde{\tau} =$ one week
dimension reduction:

convariance matrix

\[ C(v, p) = \text{Cov} \left[ X^{(v)}, X^{(v+p)} \right] \]

properties

\[ C(v, p + dt) \approx C(v + dt, p) \quad \text{smooth} \]
\[ C(v, 0) \approx C(v + \tau, 0) \quad \text{diagonal elements are similar} \]
\[ C(v, p) \approx C(v + \tau, p) \]
\[ C(v, p) \approx h(p) \quad \text{approximate structure} \]

with \( h(p) = h(-p) \)
4 Estimating the Distribution of Market Invariants
5 Evaluating Allocations
6  Optimizing Allocations
7 Estimating the Distribution of Market Invariants with Estimation Risk
8 Evaluating Allocations under Estimation Risk
9 Optimizing Allocations under Estimation Risk