

An analytical pricing framework for financial assets with trading suspensions

Lorenzo Torricelli*

Christian Fries†

September 7, 2018

Abstract

In this paper we propose a derivative valuation framework based on Lévy processes which takes into account the possibility that the underlying asset is subject to information-related trading halts/suspensions. For such assets are not traded at all times, we argue that the natural underlying for derivative risk-neutral valuation is not the asset itself, but a forward-type contract that when the asset is suspended at maturity cash-settles the last quoted price plus the interests accrued since the last quote update. elements of potential theory, we devise martingale dynamics and no-arbitrage relations for such a forward process, provide Fourier transform-based pricing formulae for derivatives, and study the asymptotic behavior of the obtained formulae as a function of the halt parameters. The volatility surface analysis reveals that the short term skew of models with suspensions is typically steeper than that of the underlying Lévy models, indicating that the presence of a trade suspension risk is consistent with the well-documented stylized fact of volatility skew persistence/explosion.

Keywords: Market halts and suspensions, time changes, Lévy subordinators, derivative pricing, Lévy processes.

MSC 2010 classification: 91G20, 60H99.

JEL classification: C65, G13.

*Department of Mathematics, Ludwig Maximilians Universität. Email: torricel@math.lmu.de.

†Department of Mathematics, Ludwig Maximilians Universität. Email: email@christian-fries.de.

1 Introduction

Suspending or halting¹ of a stock from trading is a temporary emergency measure taking place in event of abnormal market situations.

Broadly speaking this action is generally triggered by two distinct types of circumstances. The first is the manifestation of severe market anomalies that may prevent the formation of a reliable price (e.g. crashes, order imbalance, excessive bid ask spread/illiquidity holding back buyers). The second is the arrival of news that could have potential high impact on the individual companies quotes. We can thus distinguish between endogenous suspensions, generated by the market activity itself, and exogenous, news-related ones, typically independent from day-to-day trading. Trade generated halts tend to be of fixed time and short-lived, on the order of magnitude of minutes, whereas news-related suspension might last up to hours or days; their duration is typically discretionary. In case of impactful business news arrival, the firm might file in for a trading suspension voluntarily, e.g. motivated by internal management decision, or the action might be directly enforced by the market authority, when there are growing concerns on the ability of the firm of meeting the markets standards. In any case the purpose of a stock suspension, is to give to all of the investors the opportunity to re-assess their positions, facilitate the issuance of a better equilibrium price and reduce market information asymmetries.

Trade suspensions can, and do, occur quite often. Engelen and Kabir [8] observe that in the years between 1992 and 2000 in the EuroNext stock market there were 210 pure information related suspensions, 30% of which lasted more than one trading day, and involved a total of 112 companies whose 49% was halted more than once. Christie et al. [4] study a collective sample of 714 halts in the years 1997-1998 on the NASDAQ. Trading suspensions then appear to be a market-wise repeatable process, of possibly inter-daily duration.

The financial literature surrounding market halts, mainly focuses on whether the market suspension do have the stabilizing effect on trade they are expected to deliver. The evidence is mixed to some extent. Greenwald and Stein [10] suggest that halts facilitate formation of an equilibrium price by reducing transactional risk, whereas statistical analyses in the NYSE and other US stock markets Corwin and Lipson [6], Lee et al. [16] point to an increase in both post-halt trade volume and volatility, at odds with what suspensions are meant to achieve.

However, typically these analyses include suspensions caused by order imbalances, or triggering of the so-called circuit breakers due to some financial variable (especially

¹Depending on the stock markets, halts and suspensions might have slightly different meanings. In this article the two expressions are synonyms.

volatility) breaching a safety threshold, which are market-generated events. Indeed, once halts from order imbalances are removed from the sample, or only inter-daily suspensions are considered, the general findings Christie et al. [4], Engelen and Kabir [8] is that when the suspension last for more than one day the volatility of a stock is not sensibly impacted by the halt.

To our knowledge, to date no research has been put forward to explore the impact of suspensions/halts of stocks on prices of the possible derivatives written on such stocks. In this paper we aim at providing a no-arbitrage pricing frameworks in markets with news related trading halts. Since for derivative pricing the minimum horizon is daily, we do not consider intra-daily stoppages due to circuit breakers or transactional frictions, under the assumption that the changes in trading patterns these might determine are transient, and do not extend to inter-daily trading.

Of course, halts might not produce any significant effect on valuations in the case the expected suspensions are very short lived and maturity is long. However, the effect of a suspension lasting for several days to several hours cannot be ignored altogether in certain cases e.g. for pricing weekly options, a product that has recently drawn much attention.

One difficulty in introducing suspensions in no-arbitrage valuation may be that a security that can be halted *cannot* be used as an underlying for martingale pricing. However paradoxical this might sound, by definition, suspendable assets are not traded at all times, and thus the replication/superreplication arguments establishing the equivalence between no-arbitrage and martingale dynamics of the underlying do not in principle apply.

On the other hand referring to the physical suspendable market quote process is equally problematic. Market prices that repeatedly halt cannot be made to drift to a continuous constant rate after a measure change. A price quote that is subject to halts must preserve the property of the paths being constant during suspension intervals, and therefore the process cannot continuously drift at a risk-free rate after an equivalent measure change. To borrow from the popular rule of thumb: “path properties do not change upon equivalent measure changes”.

In the present paper we propose a solution to the conundrum above. We present a continuous-time semimartingale model based on time-changed Lévy processes that accounts for trading halts in the underlying stock. Firstly, we introduce a model for the fundamental price of the stock recognizing that the evolution of the economic value might follow different dynamics when the asset can be traded or it is suspended. Then we devise an observable last market quote price process Q_t by using a locally constant time change. What we argue is that the natural underlying for derivative valuation on a suspendable stock is a secondary forward process F_t that delivers at t the last observed quote Q_t plus

the accrued interest since the last update of Q_t . This contract has all the characteristics we need: it can always be traded and exhibits martingale dynamics after an appropriate equivalent measure change. In the legal stipulation of an over the counter derivative, F_t effectively represents the real underlying asset if we consider the contract as referencing the last market quote plus the interest rate payment.

From a methodological viewpoint, the framework introduces the idea of using a locally constant time change in option prices, obtained an *inverse* Lévy subordinated time change. Time changes in option prices are a well-established technique, (see e.g. Geman et al. [9] but the literature is immense) that normally is used to capture the evolution of the business activity. Our approach is rather different: our random time change is a *continuous*, piece-wise linear time evolution whose paths can be constant at random times. Those time intervals represent the trade halts. However, this is not yet sufficient: in order to consistently model a market quote that undergoes halting we must further introduce a second time change representing the last observable traded price of the stock and generating the trade reopening asset price jump. The time change achieving this is the so-called *last sojourn process* of a subordinator.

We devise no-arbitrage relations for the model by identifying a set of martingale measures under which both the asset S_t and the forward contract F_t are martingales. Adding a suspension process to a pricing Lévy model enriches its class of equivalent martingale measures. In other words, the intrinsic market incompleteness of these new models also accounts for an additional source of unhedgeable risk, the trading suspension risk, whose market price is embedded in the risk neutral parameters of the Lévy subordinator generating the halts.

Remarkably, the whole framework produces closed form formulae for the characteristic function of F_t . This means that the well-established machinery of Fourier pricing (e.g. Lewis [17]) is available, producing efficient option pricing algorithms.

Finally we consider the potential applications to the volatility surface modelling. Our numerical experiments show a volatility skew which is at the same time much steeper on the short term section and declines more slowly than that of the underlying Lévy models, thus generating a volatility term structure better matching the one observed in the markets.

In Section 2 we discuss equity derivatives with suspensions and outline the economic foundations of the framework. In Section 3 we introduce the stochastic model for the fundamental price of the stock. In Section 4 we define the market quote process and the traded underlying for derivative valuation; Section 5 deals with the equivalent martingale relations for the model. Section 6 is dedicated to the identification of a pricing formula

and its convergence to prices from Lévy models. Finally, in Section 7 we perform some numerical tests for the pricing formula and analyze the arising volatility surfaces; comparisons with the pure Lévy models are drawn. Some concluding remarks are expressed in Section 8.

2 Derivatives on suspendable assets

The starting point for a valuation theory for stocks that undergo halts, is recognizing that the classic theory of no-arbitrage pricing cannot be directly applied. Indeed, the Fundamental Theorem (e.g. Delbaen and Schachermayer [7]) requires that the asset can be traded at any time in order to form hedge and superhedge portfolios, which is not the case when halts are present. Therefore, direct mathematical modelling of the market asset seems not to be the correct way of addressing the problem.

We are thus faced from the very beginning with the problem of manufacturing some form of synthetic underlying which can be traded at any time regardless of possible interruptions of the market activity, so that we can proceed in the usual vein within the theory of no-arbitrage pricing.

Let us denote by Q_t the stochastic process giving at time t the last available market quote for the suspendable equity. Let τ_T the last instant prior to T where the equity was last traded² and

$$F^X(t, T) = e^{r(T-t)} X_t \quad (2.1)$$

the forward value relative to the market traded asset X_t . Denote with $r > 0$ the prevailing constant risk-free rate. The value of such a contract at time t , $\tau_t \leq t \leq T$ is

$$F_t := F^Q(\tau_t, t) = e^{r(t-\tau_t)} Q_{\tau_t} = e^{r(t-\tau_t)} Q_t \quad (2.2)$$

because by definition $Q_t = Q_{\tau_t}$. One can thus consider the security F_t defined by (2.2), promising the cash-settlement at time t of the last available asset market quote Q_{τ_t} plus the risk-free interest accrual (if any) from time τ_t until t . Conditional on time τ_t , F_t is the value of a standard forward contract entered at τ_t and expiring at t on a traded underlying, and as such is a well-defined security traded at all times, with a legally binding effective date t and verifiable initiation date τ_t .

In this paper we propose using F_t as an underlying asset for derivative valuation on stocks whose trade can be interrupted. This mathematical modelling idea reconnects

²Clearly most of the times $\tau_T = T$ but it is precisely when this does not happen that the discussion is significant.

with the financial practice precisely because of equation (2.2). Indeed, a satisfactory legal definition of an OTC derivative on a suspendable market asset requires an explicit contractual specification of the actions to be undertaken when the market is closed at maturity, because in such a case a current reference market value will not be available. The most natural choice, and the one put in place for exchange-traded options, is using as a reference value the last quoted price Q_t of the underlying. The value to be used for calculation of the payoff would thus be the last market quote Q_t recorded prior to the expiration time T . However, this seems to be unsatisfactory for at least two reasons. Firstly, it completely ignores the time value of money, i.e. the growth of the fundamental value of the asset during suspensions due to the interest rate component. Secondly using Q_t is not fully compliant of the risk-neutral valuation principles, as Q_t is not a traded instrument. Effectively, in view of (2.2), considering the risk-free interest rate accrual when determining the payoff corrects at the same time both of these issues, and as we shall show in this paper, allows for a full analytical derivative valuation framework.

Clearly, when interest rates are zero, $Q_t = F_t$ and in this special case the quote process can indeed be used as a derivative underlying. However what we will show in this paper is that for positive rates, and under some economic assumptions, it is F_t , and not Q_t , to possess martingale dynamics under some pricing measure. This result, together with the previous remarks, seems to implicate that the market practice of using the last available quote Q_t for payoff calculation might be questionable from the theoretical perspective, at least whenever rates are high or suspensions are long-lived, that is, when the difference in valuation between calculating and not calculating interest accrual during the final suspension is significant.

The starting element of our framework is an observable process S_t modelling the fundamental (or intrinsic, or economic) stock value, which is distinct from its market quote Q_t . We emphasize again that neither of these two processes are traded assets. The main idea is that the two must coincide when the asset is tradable and may (will) differ otherwise. When the asset is not traded, Q_t is constant but S_t still evolves to keep track of the economic activity surrounding the real asset. This assumption is naturally rooted in the Efficiency Principle: when an asset can be traded all the available information are reflected in its market quote. The process S_t is modelled by a two factors process; one factor representing the price component purely due to trade, and another one the impact on price of business and markets news and experts valuations. Only the first component is halted during the market suspensions. Our models thus captures the existence of a background noise of business-related information whose contribution to price formation is distinct to that generated purely by the trading activity, and which persists also during

trading halts.

The following two sections are devoted to the identification of a rigorous mathematical model for Q_t and S_t . Once this is done, the considerations expressed in this section will pave the way for a valuation theory for derivatives on suspendable stocks.

3 Fundamental value dynamics

In this section we begin structuring the fundamental value of the market stock S_t . We consider a market filtration $(\Omega, \mathcal{F}_t, \mathcal{F}_\infty, \mathbb{P})$ satisfying the usual conditions and supporting Lévy processes and a money market account process paying a constant rate $r > 0$.

For a càdlàg one-dimensional Lévy process Y_t with Lévy triplet $(\mu_Y, \sigma_Y, \nu_Y(dx))$ and $z \in U \subseteq \mathbb{C}$, for the characteristic function of Y_t we use the notation:

$$\mathbb{E}[e^{-izY_t}] = e^{-t\psi_Y(z)} \quad (3.1)$$

where

$$\psi_Y(z) = iz\mu_Y + \frac{z^2\sigma_Y^2}{2} - \int_{\mathbb{R}} (e^{-izx} - 1 + izx\mathbb{1}_{|x|<1})\nu_Y(dx) \quad (3.2)$$

is the Fourier characteristic exponent of Y_t . We denote the process of the left limits (the “predictable projection”) of Y_t with Y_{t-} . By stochastic continuity, for all fixed $t > 0$ we have $Y_t = Y_{t-}$ almost surely. We write $\Delta Y_t := Y_t - Y_{t-}$ for the process of the jumps of Y_t .

As basic building blocks of our model we consider two one-dimensional independent Lévy processes X_t and R_t , with corresponding Lévy triplets $(\mu_X, \sigma_X, \nu_X(dx))$ and $(\mu_R, \sigma_R, \nu_R(dx))$ and characteristic exponents ψ_X and ψ_R . We also hasten to add the standard conditions:

$$\int_{|x|>1} e^{2x}\nu_X(dx) < \infty, \quad \int_{|x|>1} e^{2x}\nu_R(dx) < \infty \quad (3.3)$$

which are necessary for exponential Lévy models to be square-integrable.

The process X_t and R_t retain the following financial interpretations. The evolution of X_t represents the component of the log-asset price coming purely from the execution of trades. The process R_t (the “*rumor*” process) instead models all of the other external factors that may impact the price, mostly the dissemination of external news, both financial and non-financial. Normally X_t is expected to dominate R_t , but it does not have to be so. When trading for the asset is allowed, the stock returns are defined to be the independent sum of these two factors. However, as explained in the previous section, we

shall require that as a trade halt occurs, X_t does not evolve, while R_t still contributes to the fundamental price formation.

Let us introduce the generator of the market suspensions as a compound Poisson process G_t independent of (X_t, R_t) , of the following form:

$$G_t = t + \sum_{i=0}^{N_t} \xi_i \quad (3.4)$$

with the variables ξ_i being independent identically exponentially distributed of common rate parameter β , and N_t is a Poisson process of intensity λ independent of the ξ_i s and all the remaining processes.

For $s \geq 0$ the Laplace characteristic exponent $\phi_G(s)$ of G_t satisfies

$$\mathbb{E}[e^{-sG_t}] = e^{-t\phi_G(s)} \quad (3.5)$$

and is given by

$$\phi_G(s) = s + \int_{\mathbb{R}^+} (1 - e^{-su}) \nu_G(du) = \frac{\lambda s}{s + \beta} + s. \quad (3.6)$$

Finally we introduce the market suspensions process H_t as the “inverse” of G_t . More precisely, for all t we define H_t as the first exit time of the level t of G_t , that is:

$$H_t = \inf \{s > 0 \mid G_s > t\}. \quad (3.7)$$

When G_t jumps, H_t has a flat spot, and a market suspension occurs. Furthermore the duration of the suspension is exactly given by the size of the jump. In the instants between the jumps of G_t , H_t is just the linear calendar time. In other words, we have the following definition:

Definition 3.1. Let \mathcal{R} be the image of G_t . We say that S_t is *suspended, halted or non-tradable* at $t > 0$ if $t \in \mathcal{R}^c$. If $s > 0$ is such that $G_{s-} \neq G_s$ then ΔG_s is the duration of the halt.

It is important to notice that $H_t \leq G_t$ and the equivalence $\{H_t \leq s\} = \{G_s \geq t\}$, which in particular yields $\{H_t \leq t\} = \Omega$. Since G_t is strictly increasing, H_t is continuous, and it is a stopping time for all fixed t . Also, H_t is almost surely increasing, bounded almost surely, and $\lim_{t \rightarrow \infty} H_t = \infty$ almost surely, and thus it is a valid time change (Jacod [11], Chapter 10).

We are now in the position of describing the fundamental economic value of our suspendable asset S_t . Upon suspension the evolution of the asset value should instead be fully determined by the news arrival. The *fundamental price* S_t is thus defined as follows:

$$S_t := S_0 \exp(X_{H_t} + R_t), \quad S_0 > 0. \quad (3.8)$$

The writing X_{H_t} indicates the time-changed semimartingale process in the sense of Jacod [11]. Since H_t is continuous, X_t is H_t -continuous³. This means that X_{H_t} retains many of the good properties of X_t (again, Jacod [11], Chapter 10); in particular it is an \mathcal{F}_{H_t} -adapted semimartingale. Finally, recalling that $A \in \mathcal{F}_{H_t}$ if and only if $A \cap \{H_t \leq s\} \in \mathcal{F}_s$ for all s , choosing $s = t$ and observing $\{H_t \leq t\} = \Omega$ shows $\mathcal{F}_{H_t} \subset \mathcal{F}_t$, so that S_t is also an \mathcal{F}_t -adapted semimartingale.

The fundamental price evolution S_t has the property we were striving for. Conditionally on the asset being tradable, i.e. G_t not jumping, we have that $X_{H_t} + R_t = X_t + R_t$ and the price process is jointly determined by the economic reaction to trade and an external news flow. When G_t jumps, the stochastic time H_t and thus the price component X_{H_t} are constant, and the fundamental price is driven only by the news dissemination process R_t .

Finally, we associate to S_t the corresponding Lévy exponential model S_t^0 without halts, whose dynamics are given by

$$S_t^0 := S_0 \exp(X_t + R_t). \quad (3.9)$$

Further on, we shall be interested in comparing financial valuations relying on S_t with the analogous on its pure Lévy counterpart S_t^0 , in order to assess the impact of the introduction of trading halt periods in derivative pricing.

4 The asset quote process and the traded underlying

We must at this point rigorously define the quote process Q_t recording the last available market quote of S_t at time t . Recall that \mathcal{R} is the range of G_t .

The *last sojourn process* of G_t of the level $[0, t]$ is defined as:

$$\tau_t = \sup\{s < t, s \in \mathcal{R}\} \leq t \quad (4.1)$$

This process keeps track of the last position of G_t in all the level sets, and for all fixed t we will interpret as a random time. Indeed, as t varies, this process can be regarded as a time change. We have the following result (see also Bertoin [1], Section 1.4):

Lemma 4.1. *The process τ_t satisfies*

$$\tau_t = G_{H_{t-}} \quad (4.2)$$

and there exists a right-continuous modification of τ_t which is an \mathcal{F}_t -time change .

³A process Y_t is said to be continuous with respect to a time-change T_t if it is almost surely constant on the sets $[T_{t-}, T_t]$.

Proof. As G_{t-} is adapted to \mathcal{F}_t the process $G_{H_{t-}}$ is adapted to \mathcal{F}_{H_t} but since $\mathcal{F}_{H_t} \subset \mathcal{F}_t$ then it is also \mathcal{F}_t -adapted and thus $G_{H_{t-}}$ is an \mathcal{F}_t -stopping times for all t .

To show (4.2) observe that for all s we have:

$$\{G_{H_{t-}} > G_s\} = \{H_t > s\} = \{G_s < t\}, \quad (4.3)$$

so that $G_{H_{t-}} \geq \tau_t$ almost surely. But since $G_{H_{t-}} \leq t$ surely, then the equality holds. Therefore, being τ_t increasing and almost surely finite it is a time changes when looked at as a process if a right-continuous modification exists. This is attained by replacing τ_t with τ_{t+} in the almost-surely null Lebesgue measure set where τ_t is discontinuous, and that the new process is a modification is granted by stochastic continuity. \square

From now on we will make use of the right-continuous version of τ_t . It is crucial to observe from the definitions above that $\tau_t < t$ if and only if $t \in \mathcal{R}^c$, i.e. according to Definition 3.1 if and only if the asset is suspended at t ; conversely, $\tau_t = t$ if and only if the asset is traded at t . We therefore denominate τ_t the *last market quote time* process. Observe that τ_t has a jump discontinuity exactly at the market reopening times given by $t = G_s$, $\Delta G_s \neq 0$, i.e. the points in \mathcal{R} isolated on their left.

To see the importance of τ_t we begin by showing that using this process we can calculate the probability of the asset being tradable at any given time t .

Proposition 4.2. *We have that*

$$\mathbb{P}(\tau_t = t) = \frac{\beta + \lambda e^{-(\lambda+\beta)t}}{\beta + \lambda} \quad (4.4)$$

for all $t > 0$.

Proof. We recall that the q -th potential measure $U^q(dx)$ of a Lévy process L_t is defined as the occupation measure

$$U^q(dx) = \int_0^\infty e^{-qt} \mathbb{P}(L_t \in dx) dt. \quad (4.5)$$

If $U^q(dx)$ is absolutely continuous, its Radon derivative $u^q(x)$ is called the potential density. When $q = 0$ and L_t is a subordinator $U^0(dx)$ and $u^0(x)$ also go under the name of renewal measure (resp. density). In this case we drop the superscript and write $U(dx)$ and $u(x)$.

By Theorem 5 in (Bertoin [2], Chapter 3) we have that since G_t has drift $d = 1$, its renewal density exists, can be chosen continuous, and satisfies $\mathbb{P}(\mathcal{T}_t = t) = u(t)$ where

$\mathcal{T}_t = G_{H_t}$ is the first sojourn of G_t of $[t, \infty)$. Now observe that the well-known relationship (e.g. Bertoin [1], Section 1.3)

$$\mathcal{L}(U(dt), s) = \int_0^\infty e^{-st} u(t) dt = \frac{1}{\phi_G(s)} \quad (4.6)$$

yields

$$\mathcal{L}(U(dt), s) = \frac{1}{s} \frac{\beta + s}{\beta + \lambda + s}. \quad (4.7)$$

On the other and, by a direct calculation

$$\int_0^\infty e^{-st} \frac{\beta + \lambda e^{-(\lambda+\beta)t}}{\lambda + \beta} dt = \frac{1}{s} \frac{\beta}{\beta + \lambda} + \frac{\lambda}{\beta + \lambda} \frac{1}{\lambda + \beta + s} = \frac{1}{s} \frac{\beta + s}{\beta + \lambda + s}. \quad (4.8)$$

The uniqueness of the Laplace transform for continuous functions then yields

$$\mathbb{P}(\mathcal{T}_t = t) = \frac{\beta + \lambda e^{-(\lambda+\beta)t}}{\lambda + \beta} \quad (4.9)$$

To conclude, observe that by (Bertoin [2], Chapter 3, Proposition 2.ii) we know that $\mathbb{P}(\{\tau_t < t, \mathcal{T}_t = t\}) = 0$ for all fixed t entailing $\mathbb{P}(\tau_t = t | \mathcal{T}_t = t) = 1$. Also, $\mathbb{P}(\tau_t = t, \mathcal{T}_t > t) = \mathbb{P}(\Delta G_t \neq 0) = 0$ because a Lévy process is stochastically continuous, so that also $\mathbb{P}(\mathcal{T}_t = t | \tau_t = t) = 1$. By conditioning the event $\{\tau_t = t, \mathcal{T}_t = t\}$ one then sees that $\mathbb{P}(\mathcal{T}_t = t) = \mathbb{P}(\tau_t = t)$ so that (4.4) follows in view of (4.9). □

The last available market quote Q_t of S_t at time t is then nothing else than the value of S_t time-changed with τ_t :

$$Q_t := S_{\tau_t} = S_0 \exp(X_{H_t} + R_{\tau_t}). \quad (4.10)$$

The quote process Q_t is therefore an \mathcal{F}_{τ_t} -semimartingale. The second equality follows from the obvious identity $H_{\tau_t} = H_t$. Moreover, since $\{\tau_t \leq t\} = \Omega$ then $\mathcal{F}_{\tau_t} \subset \mathcal{F}_t$, but since we know $\mathcal{F}_{H_t} \subset \mathcal{F}_t$ and we conclude that Q_t is \mathcal{F}_t -adapted.

The process Q_t acts as expected. Whenever the time runs in a suspension interval, τ_t does not affect the trade component X_{H_t} , which is already halted during such intervals. However such a time change *does* halt the evolution of R_t at the last value R_{τ_t} before the suspension, thus achieving the desired interpretation of S_{τ_t} as the last available market quote. Although R_t is stopped during the suspensions, its background evolution at those time spans still plays an important role in the dynamics of Q_t , as it combines with

the discontinuities of τ_t to determine the “jump” in the reopening price⁴. Outside the suspension intervals we have the plain relation $\tau_t = t$.

Now, according to Section 2, the traded underlying to be used for derivative valuation is the process F_t in (2.2). Observe that this process is \mathcal{F}_t -adapted. Now let f be a square-integrable contingent claim maturing at T . The value V_0 of f on F_t is given by

$$V_0 = \mathbb{E}^{\mathbb{Q}}[e^{-rT} f(F_T)] \quad (4.11)$$

where \mathbb{Q} is a \mathbb{P} -equivalent (local) martingale measure under which the discounted process $e^{-rt}F_t$ is a martingale. Now, observe that:

$$e^{-rt}F_t = e^{-r\tau_t}Q_t. \quad (4.12)$$

Therefore we have the derived following no-arbitrage principle for suspendable stocks.

No-arbitrage principle for securities with market suspensions. *In the model illustrated, the martingale property of the discounted forward value F_t is equivalent to the martingale property of the quote process Q_t discounted with the stochastic discount factor $e^{-r\tau_t}$.*

In the next section we explore the implications of this principle for the determination of equivalent martingale measure/no-arbitrage relations for option pricing on financial securities with market halts. Before moving on, let us briefly summarize the framework.

Construction of an underlying traded security when securities can be suspended:

1. Select processes X_t and R_t for the market trade and rumor price components, as well as a specification of the market halt generating process G_t ;
2. introduce the fundamental value S_t of an asset with market halts as in (3.8);
3. determine the quote process Q_t by time changing S_t with the last price observation time τ_t prior to t ;
4. define the traded forward contract F_t delivering the cash amount Q_{τ_t} in t , whose value in t is $F_t = e^{r(t-\tau_t)}Q_t$.

⁴Strictly speaking also X_t determines the new opening price, but R_t cumulates variation during the suspension intervals, whereas X_t just affects the price through X_{H_t} which in turn - by construction - only releases instantaneous variability at reopening. Thus its contribution to the variance is much inferior.

Using a common drawing from X_t, R_t and G_t we visualize the processes $G_t, H_t, S_t, \tau_t, Q_t$ and F_t in Figures 1 to 4. We have used $S_0 = 100, \mu_X = 0.3, \sigma_X = 0.5, \mu_R = -0.2, \sigma_R = 0.2$ and $\nu_X = \nu_R = 0$, so that conditional on being traded the asset follows the Black-Scholes-Samuelson model with drift $\mu = 0.1$ and volatility coefficient $\sigma = 0.5385$. The asset halt parameters are $\lambda = 1$ and $\beta = 7$.

5 Risk neutral dynamics and price of trade suspension risk

We now proceed to investigate the martingale dynamics of the discounted asset $e^{-rt}F_t$. In view of the no-arbitrage principle, this is equivalent to determine the no-arbitrage dynamics of the stochastically-discounted quote price $e^{-r\tau_t}Q_t$. As we shall see, because of the boundedness of τ_t , to achieve this is sufficient to determine martingale relations on the fundamental price S_t .

To make the discussion more transparent, we introduce the following general proposition, stating that under certain conditions time change and measure change commute. For general background see Jacod and Shiryaev [12], Jacod [11] and Kallsen and Shiryaev [13].

Lemma 5.1. *Let X_t be a semimartingale on a filtered space $(\Omega, \mathbb{P}, \mathcal{F}, \mathcal{F}_t)$ which is continuous with respect to a time change T_t and Z_t a martingale density process having the stochastic exponential representation:*

$$Z_t = \mathcal{E} \left(\int_0^t H_u dX_u^c + \int_0^t (W(u, x) - 1)(\mu^X - \nu^X)(dx \times du) \right) \quad (5.1)$$

where X_t^c is the continuous martingale part of X_t , μ^X and ν^X respectively its jump measure and jump compensator, H_t some square-integrable process integrable with respect to X_t^c , and $W(t, x)$ a random function such that the second integral in (5.1) exists. The symbol $\mathcal{E}(\cdot)$ stands for the stochastic exponential.

Assume further that Z_{T_t} is a true martingale, and denote by \mathbb{Q} and \mathbb{Q}^T the \mathbb{P} -equivalent martingale measures associated respectively with Z_t and Z_{T_t} . We have:

$$(X_{T_t})^{\mathbb{Q}^T} = X_{T_t}^{\mathbb{Q}}. \quad (5.2)$$

Proof. Let $(\mu_t, \sigma_t, \nu(dt \times dx))$ be the \mathbb{P} -characteristics of X_t . By the Girsanov Theorem for semimartingales (Jacod and Shiryaev [12], Chapter III, Theorem 3.24) their \mathbb{Q} counterparts, in the “disintegrated” (Jacod and Shiryaev [12], Chapter II, Proposition 2.9)

form are

$$\mu_t^{\mathbb{Q}} = \mu_t + \int_0^t H_u \sigma_u dA_u + \int_{|x|<1} (W(t, x) - 1) K_t(dx) \quad (5.3)$$

$$\sigma_t^{\mathbb{Q}} = \sigma_t \quad (5.4)$$

$$\nu^{\mathbb{Q}}(dt \times dx) = dA_t W(t, x) K_t(dx) \quad (5.5)$$

for some predictable process A_t and random measure $K_t(dx)$. Furthermore, according to (Jacod and Shiryaev [12], Lemma 2.7), by the adaptedness of X_t to T_t the characteristics of X_{T_t} under \mathbb{Q} are $(\mu_{T_t}, \sigma_{T_t}, \nu(dT_t \times dx))$. Now by (Jacod 11, Theorems 10.19, 10.27) we have that:

$$Z_{T_t} = \mathcal{E} \left(\int_0^t H_{T_u} dX_{T_u}^c + \int_0^t W(T_u, x) (\mu^X - \nu^X)(dx \times dT_u) \right). \quad (5.6)$$

Therefore by applying the Girsanov's Theorem to X_{T_t} with respect to the density Z_{T_t} we obtain, taking into account $\langle \int_0^{T_t} H_{T_u} dX_{T_u}^c \rangle_t = \int_0^{T_t} H_u \sigma_u dA_u$ (because of Jacod 11, Theorem 10.17), the following characteristics:

$$\mu_t^{\mathbb{Q}^T} = \mu_{T_t} + \int_0^{T_t} H_u \sigma_u dA_u + \int_{|x|<1} (W(T_t, x) - 1) K_{T_t}(dx) \quad (5.7)$$

$$\sigma_t^{\mathbb{Q}^T} = \sigma_{T_t} \quad (5.8)$$

$$\nu^{\mathbb{Q}^T}(dt \times dx) = dA_{T_t} W(T_t, x) K_{T_t}(dx) \quad (5.9)$$

which match $(\mu_{T_t}^{\mathbb{Q}}, \sigma_{T_t}^{\mathbb{Q}}, \nu^{\mathbb{Q}}(dT_t \times dx))$. \square

We can then directly state the main result of this section, on the martingale relations for the fundamental asset price process.

Theorem 5.2. *Let S_t be given by (3.8). Under mild assumptions on X_t and R_t , there exists a set of equivalent martingale measures $\mathbb{Q}^{X,R,H}$ with Radon-Nikodym of the form*

$$\frac{d\mathbb{Q}^{X,R,H}}{d\mathbb{P}} = \mathcal{X}_t \mathcal{R}_t \mathcal{H}_t \quad (5.10)$$

for some density processes \mathcal{X}_t and \mathcal{R}_t , and

$$\mathcal{H}_t = \exp \left((\lambda - \lambda^*)t + \sum_{s \leq t} (\beta^* - \beta) \Delta G_s \right) \quad (5.11)$$

such that under $\mathbb{Q}^{X,R,H}$ the discounted fundamental price process $e^{-rt} S_t$ is an \mathcal{F}_{H_t} -martingale of the form $\exp(X_{H_t}^* + R_t^* - rt)$ for some Lévy processes X_t^* , R_t^* and H_t^* a compound Poisson process of drift one, intensity λ^* and exponentially i.i.d. jumps of rate β^* .

Proof. Under the equivalent martingale measure induced by \mathcal{H}_t , we have that the dynamics of G_t are those of a compound Poisson process with intensity λ^* and exponential jump distribution of parameter β^* (e.g. Sato 19, Theorem 33.1). We denote by H_t^* the dynamics of H_t under the \mathbb{P} -equivalent measure induced by \mathcal{H}_t .

The first step is to isolate a change of measure under which $\exp(X_t)$ and $\exp(R_t)$ are individually martingales. By standard arguments (the Esscher transformation, e.g. Cont and Tankov 5, Proposition 9.9), so long as R_t and X_t are not themselves subordinators, it is possible to find martingale density processes \mathcal{R}_t and \mathcal{X}_t^0 under which R_t and X_t are Lévy processes of triplets respectively $(\mu_X^0, \sigma_X, \nu_X^0(dx))$ and $(\mu_R^*, \sigma_R^*, \nu_R^*(dx))$ where:

$$\mu_X^0 = -(\sigma_X^0)^2/2 - \int_{\mathbb{R}} (e^x - 1 - x\mathbb{1}_{|x|<1})\nu_X^0(dx) \quad (5.12)$$

$$\sigma_X^0 = \sigma_X \quad (5.13)$$

$$\frac{d\nu_X^0}{d\nu_X} = \exp(\Phi^0) \quad (5.14)$$

and

$$\mu_R^* = -(\sigma_R^*)^2/2 - \int_{\mathbb{R}} (e^x - 1 - x\mathbb{1}_{|x|<1})\nu_R^*(dx) + r \quad (5.15)$$

$$\sigma_R^* = \sigma_R \quad (5.16)$$

$$\frac{d\nu_R^*}{d\nu_R} = \exp(\Phi^*) \quad (5.17)$$

for functions Φ^* and Φ^0 satisfying certain integrability conditions.

After having operated the \mathbb{P} -equivalent change of measure corresponding to \mathcal{H}_t , we set $\mathcal{X}_t = \mathcal{X}_{H_t^*}^0$. Since \mathcal{X}_t^0 is a martingale and H_t^* a bounded stopping time, by Doob's Optional Sampling Theorem, \mathcal{X}_t is also a martingale. Furthermore \mathcal{X}_t^0 has the exponential representation (5.1), and so by Lemma 5.1, under the equivalent measure induced by $\mathcal{X}_t\mathcal{R}_t$ we calculate the characteristics $(\mu_X^*(t), \sigma_X^*(t), \nu_X^*(dx \times dt))$ of the semimartingale $X_{H_t^*}$ in the measure induced by $\mathcal{X}_t\mathcal{H}_t$ by simply time changing (5.12)-(5.14), yielding:

$$\mu_X^*(t) = -H_t^*(\sigma_X^0)^2/2 - H_t^* \int_{\mathbb{R}} (e^x - 1 - x\mathbb{1}_{|x|<1})\nu_X^0(dx) \quad (5.18)$$

$$\sigma_X^*(t) = \sigma_X H_t^* \quad (5.19)$$

$$\nu_X^*(dx \times dt) = dH_t^* \nu_X^0(dx). \quad (5.20)$$

Recall that the (Fourier) cumulant process $K_t^X(\theta)$ of a quasi-left continuous semimartingale X_t with finite first exponential moment is the almost-surely uniquely determined process $K_t^X(\theta)$ such that in the appropriate domains of definition $\exp(i\theta X_t - K_t^X(\theta))$

is a local martingale. In the case of a Lévy process, in our notation $K_t^X(\theta) = -t\psi_X(-\theta)$. By Kallsen and Shiryaev [13], Lemma 2.7, one has that if T_t is a time change and X_t is T_t -continuous $K_t^{X_T}(\theta) = K_{T_t}^X(\theta)$.

But then observe that X_{H_t} and R_t under the measure $\mathbb{Q}^{X,R,H}$ induced by $\mathcal{X}_t\mathcal{R}_t\mathcal{H}_t$ have dynamics respectively $X_{H_t}^* = \tilde{X}_{H_t^*} - K_{H_t^*}^{\tilde{X}}(-i)$ and $R_t^* = \tilde{R}_t^* - K_t^{\tilde{R}}(-i) + rt$, with $\tilde{X}_{H_t^*}$ and \tilde{R}_t^* being the driftless processes with characteristics given respectively by $(0, \sigma_X^*(t), \nu_X^*(dx \times dt))$ and $(0, t\sigma_R^*, \nu_R^*(dx)dt)$. Therefore, by independence $\exp(X_{H_t}^* + R_t^* - rt)$ is a local martingale under $\mathbb{Q}^{X,R,H}$, with $\exp(R_t^* - rt)$ being a true martingale since \tilde{R}_t^* is a Lévy process. Finally conditioning and using the independence of H_t^* and X_t^* yields that for all t , $\mathbb{E}[\exp(X_{H_t}^*)] = 1$ so that $\exp(X_{H_t}^* + R_t^* - rt)$ is indeed a martingale. This terminates the proof. \square

Comparing to the corresponding results for Lévy processes, the added complexity here is operating the “measure change of a time change”, which may potentially affect the martingale densities for the “spatial” components themselves. However, because of the independence and time-change continuity relationships, the densities factor and this is indeed not the case.

It is now a simple consequence of the structure of the discussion in Section 3 and Theorem 5.2 that F_t is also a martingale under the measures $\mathbb{Q}^{X,R,H}$.

Corollary 5.3. *Let $\mathcal{F}_t^H = \mathcal{F}_{H_t}$. Under the measures $\mathbb{Q}^{X,R,H}$ in Proposition 5.2, F_t is an $\mathcal{F}_{\tau_t}^H$ -martingale.*

Proof. By Proposition 5.2 we have that $e^{-rt}S_t$ is an \mathcal{F}_t^H -martingale under $\mathbb{Q}^{X,H,R}$. But since τ_t is a bounded stopping time, we can apply Doob’s Optimal Sampling Theorem from which follows that $e^{-r\tau_t}S_{\tau_t} = e^{-r\tau_t}Q_t = e^{-rt}F_t$ is an $\mathcal{F}_{\tau_t}^H$ -martingale. \square

In the process of isolating the martingale density process corresponding to the change of measure to the equivalent risk-neutral ones, we can appreciate that this model is intrinsically incomplete, and that pricing incorporates two sources of unhedgeable risk.

First, as in any model based on Lévy processes, the presence of jumps in the drivers X_t and R_t bears a source of systematic risk which cannot be completely hedged by trading in a set of fundamental securities. Second, modelling the asset halts by a random time change driven by a Lévy subordinator introduces an additional source of market risk which is equally unhedgeable in terms of replication. This risk correspond to the “totally inaccessible” events of a suspension taking place. A suspension can happen at any time without notice: suspension times are not predictable times. Hence derivatives on assets with halts cannot be perfectly replicated using a predictable trading strategy.

Consequently, the introduction of a “horizontal jumpiness” of the securities brings about the concept of *market price of suspension risk* embedded in the parameters λ^* and β^* . These parameters encode the premium that the investors should demand for holding an investment which is subject to suspensions.

6 Contingent claim valuation

In order to obtain semi-closed pricing formulae we exploit the fact that since X_t and R_t are independent of G_t , so they are of both τ_t and H_t . Combining this property with the fact that we are effectively able to compute the joint Laplace-Laplace transform of (H_t, τ_t) , we can obtain pricing formulae applying the well-known Fourier techniques.

In this section, we use the \cdot^* notation when we want to emphasize the risk neutral dynamics or parameters of a process.

Let us begin from the derivation of the Laplace-Laplace transform of the joint density of an inverse subordinator H_t and its last sojourn process G_{H_t-} .

Proposition 6.1. *Let G_t be any strictly increasing subordinator and H_t is inverse as defined as in (3.7), denote by $P_t(x, y)$ the joint law of (H_t, G_{H_t-}) and let $\hat{P}_t(q, k) = \mathbb{E}[e^{-qH_t - kG_{H_t-}}]$. The Laplace transform in the variable t of $\hat{P}_t(q, k)$ satisfies:*

$$\mathcal{L}(\hat{P}_t(q, k), s) = \int_0^\infty e^{-st} \hat{P}_t(q, k) dt = \frac{1}{s} \frac{\phi_G(s)}{q + \phi_G(k + s)}. \quad (6.1)$$

Proof. Because of the possibility of an atom at t in G_{H_t-} we must divide

$$\mathcal{L}(\hat{P}_t(q, k), s) = \int_0^\infty e^{-st} \mathbb{E}[e^{-qH_t - kG_{H_t-}} \mathbb{I}_{\{G_{H_t-}=t\}}] + \int_0^\infty e^{-st} \mathbb{E}[e^{-qH_t - kG_{H_t-}} \mathbb{I}_{\{G_{H_t-}<t\}}]. \quad (6.2)$$

It is shown in Kyprianou [14], Chapter 5, that if the drift d of G_t is positive

$$\mathbb{E}[e^{-qH_t} \mathbb{I}_{\{G_{H_t-}=t\}}] = du^q(x) \quad (6.3)$$

where $u^q(x)$ is the q -th potential measure of G_t (and such quantity is zero otherwise), whence

$$\begin{aligned} \int_0^\infty e^{-st} \mathbb{E}[e^{-qH_t - kG_{H_t-}} \mathbb{I}_{\{G_{H_t-}=t\}}] dt &= d \int_0^\infty e^{-(s+k)t} u^q(x) dx \\ &= d \int_0^\infty e^{-(q+\phi_G(k+s))t} dx = \frac{d}{q + \phi_G(k+s)}. \end{aligned} \quad (6.4)$$

Let us turn to study the transform of (H_t, G_{H_t-}) on the set $\{G_{H_t-} < t\}$. By conditioning on $\{H_t = x\}$ applying Fubini's Theorem and writing $f(x, t)$ for the density of H_t (which exists by Meerschaert and Scheffler 18, Theorem 3.1):

$$\begin{aligned} \mathbb{E}[e^{-qH_t - kG_{H_t-}} \mathbb{1}_{\{G_{H_t-} < t\}}] &= \int_0^\infty e^{-qx} f(x, t) dx \int_0^t e^{-ky} \mathbb{P}(G_{H_t-} \in dy | H_t = x) \\ &= \int_0^t e^{-ky} dx \int_0^\infty e^{-qx} \mathbb{P}(G_{H_t-} \in dy, H_t = x). \end{aligned} \quad (6.5)$$

The crucial remark is now that:

$$\{G_{H_t-} < y, H_t = x\} = \{G_{x-} < y, G_x \geq t\} = \{G_{x-} < y, \Delta G_x > t - G_{x-}\}. \quad (6.6)$$

Hence, define the point process:

$$\gamma_x^t = \sum_{s \leq x \leq t} \mathbb{1}_{\{\Delta G_s > t - G_{s-}\}} \quad (6.7)$$

and denote the tail density $\bar{\nu}_G(u) = \nu_G(u, \infty)$. Since G_x has Lévy measure ν_G , for a Borel random set A the point process $\sum_{s \leq x} \mathbb{1}_{\{\Delta G_s \in A\}}$ has compensating measure $\nu_G(A) dx$, and thus γ_x^t has compensating measure $\bar{\nu}_G(t - G_{x-}) dx$. Therefore, as $\mathbb{1}_{\{G_{x-} < y\}}$ is predictable, by virtue of (6.6), the compensation formula (e.g. Last and Brandt 15, Proposition 4.1.6) Fubini's Theorem and stochastic continuity of G_x we calculate:

$$\begin{aligned} \int_0^\infty e^{-qx} \mathbb{P}(G_{H_t-} \leq y, H_t = x) dx &= \mathbb{E} \left[\int_0^\infty e^{-qx} \mathbb{1}_{\{G_{x-} < y\}} d\gamma_x^t \right] \\ &= \mathbb{E} \left[\int_0^\infty e^{-qx} \mathbb{1}_{\{G_{x-} < y\}} \bar{\nu}_G(t - G_{x-}) dx \right] \\ &= \int_0^\infty e^{-qx} dx \int_0^y \mathbb{P}(G_x \in dz) \bar{\nu}_G((t - z)-) \\ &= \int_0^y U^q(dz) \bar{\nu}_G(t - z). \end{aligned} \quad (6.8)$$

The last equality follows because U^q has no atoms and the set of discontinuities of a Lévy measure has Lebesgue measure zero. Thus:

$$\mathbb{E}[e^{-qH_t - kG_{H_t-}} \mathbb{1}_{\{G_{H_t-} < t\}}] = \int_0^t e^{-ky} U^q(dy) \bar{\nu}_G(t - y) dy \quad (6.9)$$

so that, again applying Fubini's Theorem:

$$\begin{aligned} \int_0^\infty e^{-st} \mathbb{E}[e^{-qH_t - kG_{H_t-}} \mathbb{1}_{\{G_{H_t-} < t\}}] dt &= \int_0^\infty e^{-st} dt \int_0^t e^{-ky} U^q(dy) \bar{\nu}_G(t - y) dy \\ &= \int_0^\infty \int_0^t e^{-s(t-y)} e^{-(k+s)y} U^q(dy) \bar{\nu}_G(t - y) dt dy \\ &= \mathcal{L}(\bar{\nu}_G(t), s) \mathcal{L}(U^q(dy), k + s). \end{aligned} \quad (6.10)$$

Using the formula (e.g. Bertoin [2], Section III.1)

$$\mathcal{L}(\bar{\nu}_G(t), s) = \frac{\phi_G(s)}{s} - d \quad (6.11)$$

and the second and third equalities in equation (6.4) again (with $d = 1$), we obtain

$$\int_0^\infty e^{-st} \mathbb{E}[e^{-qH_t - kG_{H_t-}} \mathbb{1}_{\{G_{H_t-} < t\}}] = \frac{1}{q + \phi_G(s+k)} \left(\frac{\phi_G(s)}{s} - d \right) \quad (6.12)$$

and by substituting (6.12) and (6.4) in (6.2) the proof is complete. \square

This proposition extends Bertoin [1], Lemma 1.11, and is somewhat reminiscent of the Wiener-Hopf factorisation formulae and analogous identities in Lévy potential theory (see Bertoin [2], Kyprianou [14]).

Now, the independence of the involved processes allows to transition from the Laplace-Laplace transform of the time changes to the Laplace transform of the characteristic function of the log-value for the traded underlying F_t . After a Laplace inversion, the latter can be in turn used to derive an integral option price representation. The full result reads as follows:

Theorem 6.2. *Let $f(x)$ be a contingent claim on $\log F_t$ maturing at time T , assume that $w(x) = f(e^x)$ is Fourier-integrable and let \mathcal{S}_w be the domain of regularity of its Fourier transform $\hat{w}(z)$. Denote with \mathcal{S}_F the domain of regularity of $\mathbb{E}^\mathbb{Q}[e^{-iz \log F_T}]$, and assume $\mathcal{S}_w \cap \mathcal{S}_F \neq \emptyset$. The price V_0 of the derivative paying off $f(F_T)$ at time T is given by:*

$$V_0 = \mathbb{E}^\mathbb{Q}[e^{-rT} w(\log F_T)] = \frac{e^{-rT}}{2\pi} \int_{i\gamma-\infty}^{i\gamma+\infty} S_0^{-iz} e^{-izrT} \hat{w}(z) \Phi_T(\psi_X^*(z), \psi_R^*(z), \lambda^*, \beta^*) dz \quad (6.13)$$

with

$$\begin{aligned} \Phi_t(z_1, z_2, \lambda, \beta) &= (De^{bt})^{-1} \\ &\left(\beta^2 c_t (d_t - 1)(\lambda + z_2) - z_2 (2e^{bt} \lambda a + e(c_t d_t (a - \lambda - z_1) + c_t (\lambda + z_1 + a))) + \beta c ((d_t - 1)\lambda^2 \right. \\ &\left. + \lambda(z_1 - d_t z_1 + a + d_t(a - z_2) + z_2) + z_2(-2(d_t - 1)z_1 + a + d_t(a - z_2) + z_2)) \right) \end{aligned} \quad (6.14)$$

where

$$\begin{aligned} a &= \sqrt{\beta^2 + 2\beta(\lambda - z_1) + (\lambda - z_1)^2} \\ b_t &= \exp\left(\frac{t}{2}(z_1 + 2z_2 + \lambda + a)\right) \\ c_t &= e^{\beta t/2} \\ d_t &= e^{at} \\ e &= z_1 + z_2 \\ D &= 2a(\beta(\lambda + z_2) - z_2(\lambda + e)) \end{aligned} \quad (6.15)$$

and γ is chosen such that the integration contour lies in $\mathcal{S}_w \cap \mathcal{S}_F$.

Proof. Since $\mathcal{S}_w \cap \mathcal{S}_F \neq \emptyset$, by the discussion in Lewis [17], and conditioning under independence, we have that the value V_0 of a derivative can be represented as the Parseval-type convolution:

$$\begin{aligned} V_0 &= e^{-rt} \mathbb{E}^{\mathbb{Q}}[w(\log(F_T))] = \frac{e^{-rt}}{2\pi} \int_{i\gamma-\infty}^{i\gamma+\infty} \mathbb{E}^{\mathbb{Q}}[e^{-iz \log F_T}] \hat{w}(z) dz \\ &= \frac{e^{-rt}}{2\pi} \int_{i\gamma-\infty}^{i\gamma+\infty} S_0^{-iz} e^{-izrT} \mathbb{E}^{\mathbb{Q}}[e^{-\psi_X^*(z)H_T - \psi_R^*(z)G_{H_T^-}}] \hat{w}(z) dz. \end{aligned} \quad (6.16)$$

for some γ chosen in $\mathcal{S}_w \cap \mathcal{S}_F$. But then using Proposition 6.1 and taking the analytic continuation on the convergence domain of the Laplace transform we have for some $z_1, z_2 \in \mathbb{C}$:

$$\begin{aligned} \int_0^\infty e^{-st} \mathbb{E}^{\mathbb{Q}}[e^{-z_1 H_T - z_2 G_{H_T^-}}] dt &= \frac{1}{s} \frac{\phi_G^*(s)}{z_1 + \phi_G^*(z_2 + s)} \\ &= \frac{1}{\beta^* + s} \frac{(\beta^* + \lambda^* + s)(\beta^* + s z_2)}{(z_1 + z_2 + s)(\beta^* + z_2 + s) + \lambda^*(s + z_2)}. \end{aligned} \quad (6.17)$$

By explicitly calculating the inverse Laplace transform of the last line of (6.17) with MATHEMATICA we obtain equation (6.14)-(6.15). \square

We conclude this section by a natural result that guarantees, in line with the intuition, that the prices of claims written on F_t should converge to those from the benchmark Lévy model without halts S_t^0 , as the halt frequency and average duration tend to zero.

Proposition 6.3. *Let f be a bounded claim maturing at T . We have the following asymptotic relations for V_0 :*

(i) *If V_0^0 is the value of the claim f written on S_t^0 then:*

$$\lim_{\lambda^* \rightarrow 0} V_0 = \lim_{\beta^* \rightarrow \infty} V_0 = V_0^0; \quad (6.18)$$

(ii) *Let ξ an exponential independent time of parameter λ^* . For a stochastic process Y_t define*

$$Y_t^\xi = \begin{cases} Y_t & \text{if } t < \xi \\ Y_\xi, & \text{if } t \geq \xi. \end{cases} \quad (6.19)$$

Then

$$\lim_{\beta^* \rightarrow 0} V_0 = V_0^\xi \quad (6.20)$$

where V_0^ξ is the discounted expectation of f taken with respect to the distribution of the terminal random variable $S_T^\xi = S_0 \exp((X^)_T^\xi + (R^*)_T^\xi)$.*

Proof. From Proposition 6.1 and using independence we see that in the risk-neutral measure

$$\begin{aligned} \lim_{\lambda^* \rightarrow 0} \hat{P}(z, s) &= \lim_{\beta^* \rightarrow \infty} \hat{P}(z, s) = \frac{1}{\psi_X^*(z) + \psi_R^*(z) + s} \\ &= \mathcal{L}(e^{-T(\psi_X^*(z) + \psi_R^*(z))}, s) = \mathcal{L}(\mathbb{E}[e^{-iz(X_T^* + R_T^*)}], s) \end{aligned} \quad (6.21)$$

Taking the limits inside the Laplace integral by dominated convergence and inverting the transform we see by the Lévy Continuity Theorem that S_T tends in distribution to S_T^0 for the given parameter asymptotics. This completes the proof of (i).

For ξ as in (ii) define the killed linear drift

$$\lambda_t^\infty = \begin{cases} t & \text{if } t < \xi \\ \infty, & \text{if } t \geq \xi \end{cases} \quad (6.22)$$

whose De Finetti-Lévy-Kincthine exponent is $\phi_\lambda(s) = \lambda^* + s$, and consider its first exit time process

$$\lambda_t = \inf\{s > 0 \mid \lambda_s^\infty > t\} = \begin{cases} t & \text{if } t < \xi \\ \xi, & \text{if } t \geq \xi \end{cases} \quad (6.23)$$

Evidently $\lambda_{\lambda_t^-}^\infty = \lambda_t$, so we can apply Lemma 1.11 in Bertoin [1] for a subordinator with killing directly to the process λ_t ; note also that $S_t^\xi = S_{\lambda_t}$. Taking the limit in Proposition 6.1 and using independence, shows that

$$\begin{aligned} \lim_{\beta^* \rightarrow 0} \hat{P}(z, s) &= \frac{1}{s} \frac{\lambda^* + s}{\psi_X^*(z) + \psi_R^*(z) + \lambda^* + s} \\ &= \mathcal{L}(\mathbb{E}[e^{-(\psi_X^*(z) + \psi_R^*(z))\lambda_T}], s) = \mathcal{L}(\mathbb{E}[e^{-iz((X^*)_T^\xi + (R^*)_T^\xi)}], s) \end{aligned} \quad (6.24)$$

and the result again follows again by interchanging integration and limit, inverting the transform and applying Lévy Continuity Theorem. \square

The first part of this proposition guarantees convergence of put prices on F_t to those of the associated unhalted model S_t^0 . Convergence of call prices then also holds by call-put parity.

The second part has the interpretation that as the expected length of jumps tends to infinity, the implied asset process tends to a price distribution with only one halt that freezes the asset value at the last recorded price until maturity, and that for every possible maturity. The risk-neutral distribution of the waiting time of such an halt is that of a single Poisson event in G_t , that is, an exponential independent time of parameter λ^* .

In combination with Proposition 4.2, this result can be helpful to assess the relative impact of the halts on prices, something we will pursue in the next section.

7 Numerical experiments

We begin by visualizing equation (4.4) to get a better idea of how the probabilities of suspension, and therefore prices, depend on λ, β . In Figure 5 for a given time horizon $t = 0.5$ we plot the probabilities as a function of λ and β . As $\lambda \rightarrow 0$ this probability tends to 1 and the model converges to S_t^0 . In Figure 6 instead we fixed $\lambda = 1$ and $\beta = 10$ and as time increases the probability of t falling in a suspension decreases to its asymptotic value $\beta/(\beta + \lambda)$. This means that, everything else being equal, we expect the absolute differences of prices compared to S_t^0 to be higher for longer maturities.

For price generation we considered an instantiation of our model where X_t is the CGMY model of Carr et al. [3] with one set of calibrated parameters found therein, namely:

$$C = 6.51, \quad G = 18.75, \quad M = 32.95, \quad Y = 0.57. \quad (7.1)$$

We choose R_t as a Brownian motion with $\mu_R = 0, \sigma_R = 0.2$, and set a risk-free rate $r = 2\%$ and $S_0 = 100$. We take this parameter set as the baseline scenario.

We represent then prices corresponding to an at-the-money call option on F_t with same maturity and same parameters λ^* and β^* of Figure 5. We can see that indeed Figure 7 closely mirrors Figure 5. As $\lambda^* \rightarrow 0$ and $\beta^* \rightarrow \infty$, in accordance to Proposition 6.3, the prices converge to the line 11.883 given by the price in the associated Lévy model S_t^0 . Also, note that this convergence is naturally increasing in both λ and β , since halting the asset has the effect of compressing the volatility and thus lowering the price. As the number of expected halts and their average duration go to zero, the variability of S_t^0 is restored and price convergence attained. In Figure 8 we represent the effect of this lowering on theta. As one could expect, also in view of Figure 6, the option prices grow slower as time to maturity increases.

In Figures 9 to 16 we compare some volatility skews extracted from options on S_t^0 and S_t . We want to show how acting on the halt parameters λ^*, β^* and σ_R^* , dictating respectively the (risk-neutral) frequency and average duration of the halts, and the variance of the price quote jump at re-opening, fundamentally alters the skew structure of the benchmark model S_t^0 . We initially set as baseline $\lambda^* = 2$ and $\beta^* = 50$, corresponding to a bi-yearly suspension frequency with an average length of five days.

Figure 9 shows the baseline scenario with monthly maturity. It can be noticed the excess at-the-money steepness of the halted model compared to the Lévy one, while the two skews retain the same structure in and out of the money. As we shorten the maturity to bi-weekly, this difference gets lost, as can be seen in Figure 10: the likelihood of a halt

λ^*t is too small for the given parameters β^* and σ_R^* to generate any noticeable difference of the implied price distributions from those of S_t^0 .

Therefore, in the bi-weekly maturity case we change σ_R^* to $\sigma_R^* = 0.5$ and hold the other parameters constant. We can see in Figure 11 that the resulting increase in the variance of the reopening price shocks is enough to recreate the excess at-the-money skew already observed in Figure 9. Of course, with this modification the one-month skew difference is exacerbated (Figure 12).

Analogously we proceed to alter λ^* and β^* . Fixing the maturity to monthly and all the remaining parameters to the baseline case, we first change $\lambda^* = 12$ (suspensions expected with monthly frequency) and then $\beta^* = 12$ (monthly expected suspension length). The resulting Figures 13 and 14 show similar effects on the skew that the one attained in Figure 11 by changing σ_R^* . Note also that associated to this parameter change is also a minimal lowering of the level of the skew, consistently with the discussed effect that a decrease in β^* and an increase in λ^* determine a global reduction of the option prices.

Last but not least, we find the effect on the skew for λ^* and β^* to be persistent in time. In Figures 15 and 16 the same situation of Figures 13 and 14 is reproduced, but this time with maturity six months. The halted model skew decay is evidently slower than that of S_t^0 . This effect is in line with the real market volatility skew shapes. Of course, the lowering of the implied volatilities in these examples is even stronger, again following the pattern of Figure 8.

In conclusion, the risk-neutral suspension parameters λ^*, β^* and σ_R^* act as further steepening the surface. Like the jump parameters, these are able to create implicit distribution kurtosis and skewness by a combination of price movements delays and reopening jumps. However, unlike the skewness generated by jumps, which dissipates quickly with maturity because the jump asymmetries “even out” after temporal aggregation, halting generates genuine skew persistence since long run returns process driven by inverse-subordinated Lévy processes do not possess normality features, as explained e.g. in Meerschaert and Scheffler [18] and references therein.

8 Conclusions

In this paper we presented a martingale derivative pricing framework for stocks with news-related suspensions. We did so by observing that the natural underlying of a derivative on a suspendable asset is neither the asset itself nor its last market quote price, but a contract of cash delivery of the last stock quote plus interest, which can always be traded and can be made into an asset earning the risk-free rate after some equivalent measure

change.

In order to mathematically formulate one such a framework we resorted to a Lévy processes setup comprising of two independent price factors, one modelling the trading and the other the news effects on price, together with a finite activity subordinator whose jumps generate the market halts. The economic value of the asset is then recovered by halting the price component with the time change obtained by the first exit time of the halts generator. The last available market quote is then attained by further time changing the asset value to the last sojourn process τ_t of G_t .

Martingale relations pose no difficult, and a class of equivalent martingale measures has been identified. In this context, the concept of market price of suspension risk emerges, as the fraction of the option risk premium borne by the risk-neutral parameters λ^* and β^* .

Furthermore, we have been able to produce an option pricing formula through the popular technique of Fourier integral pricing, by deriving the joint Laplace-Laplace transform of the time changes and then invert it in time to obtain the characteristic function of the log-forward value.

Analyses of the volatility surfaces show that the short time skew of a model with suspension is much steeper than that of the corresponding Lévy model without halts. In addition the smile decays slowly over time, a pattern consistent with real markets volatility term structures which is not normally captured by Lévy models.

Figures and tables

References

- [1] Bertoin, J. (1997). Subordinators: examples and applications. In *Lectures on Probability Theory and Statistics*, volume 1717, chapter 1, pages 1–91. Springer.
- [2] Bertoin, J. (1998). *Lévy Processes*. Cambridge University Press.
- [3] Carr, P., Geman, H., Madan, D., and Yor, M. (2001). The fine structure of asset returns: an empirical investigation. *The Journal of Business*, 75:305–332.
- [4] Christie, W. G., Corwin, S. A., and Harris, J. H. (2002). Nasdaq trading halts: the impact of market mechanisms on price trading activity and execution costs. *The Journal of Finance*, 57:1443–1478.

- [5] Cont, R. and Tankov, P. (2004). *Financial Modelling with Jump Processes*. Chapman & Hall.
- [6] Corwin, S. A. and Lipson, M. L. (2000). Order flow and liquidity around NYSE trading halts. *The Journal of Finance*, 55:1771–1801.
- [7] Delbaen, F. and Schachermayer, W. (2006). *The Mathematics of Arbitrage*. Springer.
- [8] Engelen, P. J. and Kabir, R. (2006). Empirical evidence on the role of trading suspensions in disseminating new information to the capital market. *Journal of Business Finance and Accounting*, 33:1142–1167.
- [9] Geman, H., Madan, D., and Yor, M. (2001). Time changes for Lévy processes. *Mathematical Finance*, 11:79–96.
- [10] Greenwald, B. C. and Stein, J. C. (1991). Transactional risk, market crashes, and the role of circuit breakers. *The Journal of Business*, 64:443–462.
- [11] Jacod, J. (1979). *Calcul Stochastique et Problèmes de Martingales*. Lecture notes in Mathematics. Springer.
- [12] Jacod, J. and Shiryaev, A. N. (2003). *Limit Theorems for Stochastic Processes*. Springer.
- [13] Kallsen, J. and Shiryaev, A. N. (2002). Time change representation of stochastic integrals. *Theory of Probability and its Applications*, 46:522–528.
- [14] Kyprianou, A. E. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Springer.
- [15] Last, G. and Brandt, A. (1995). *Marked Point Processes on the Real Line: The Dynamic Approach*. Springer.
- [16] Lee, C., Ready, M. J., and Seguin, P. J. (1994). Volume, volatility and New York Stock Exchange trading halts. *The Journal of Finance*, 49:183–214.
- [17] Lewis, A. (2001). A simple option formula for general jump-diffusion and other exponential Lévy processes. *OptionCity.net Publications*.
- [18] Meerschaert, M. M. and Scheffler, H. (2008). Triangular array limits for continuous random walks. *Stochastic Processes and their Applications*, 118:1606–1633.

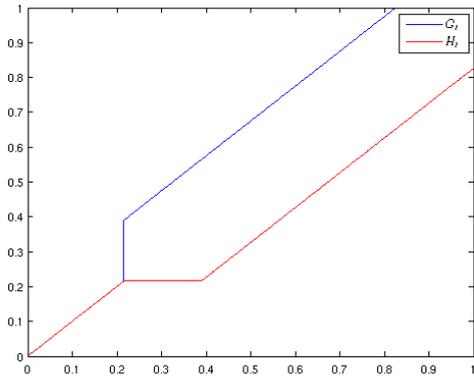


Figure 1: Halts generator G_t and halts process H_t .

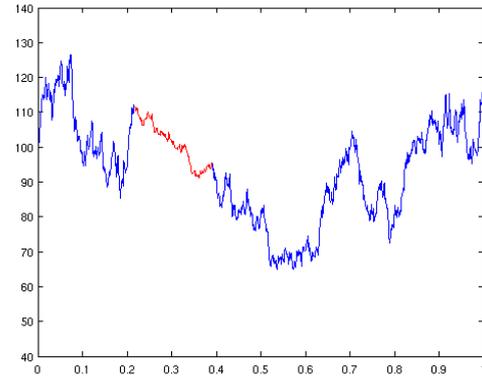


Figure 2: Fundamental asset price value S_t .

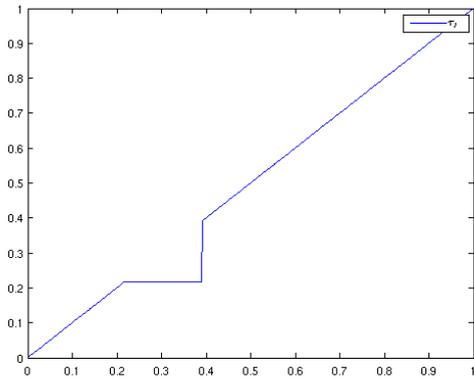


Figure 3: Last quote time processes τ_t . The process is equal to t on the sets where G_t does not jump.

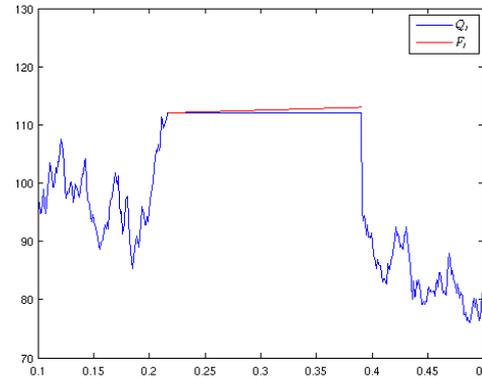


Figure 4: Quote process Q_t and forward process F_t , close up of Figure 2. The processes coincide when S_t is tradable.

[19] Sato, K. I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.

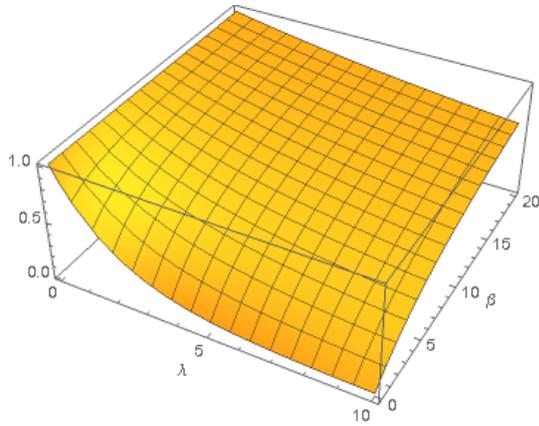


Figure 5: Probability of S_t being tradable at time $t = 0.5$.

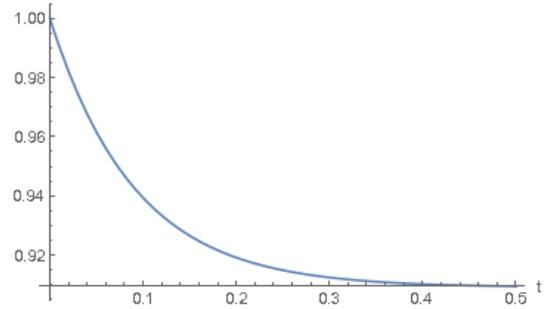


Figure 6: Probability of S_t being tradable as a function of t . $\lambda = 1, \beta = 10$.

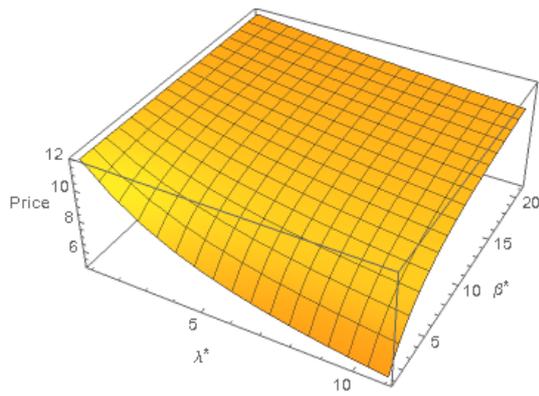


Figure 7: ATM call option prices at $t = 0.5$.

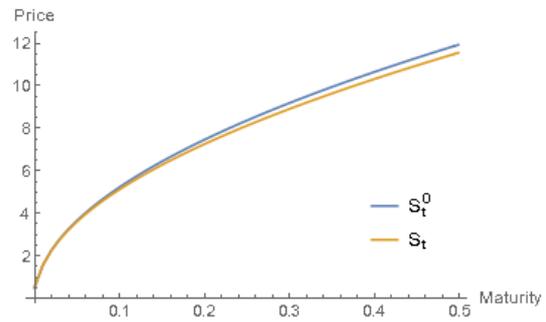


Figure 8: Effect of halts on option prices time growth; ATM option $\lambda^* = 1, \beta^* = 10$.

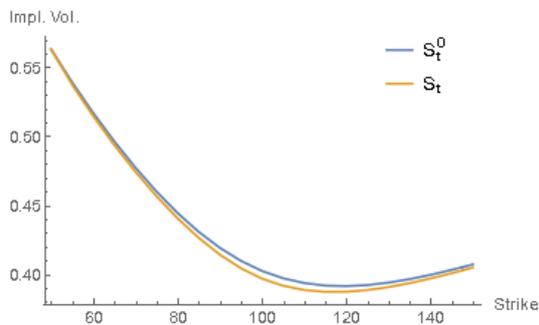


Figure 9: Baseline, $T = 1/12$. Excess skew observed.

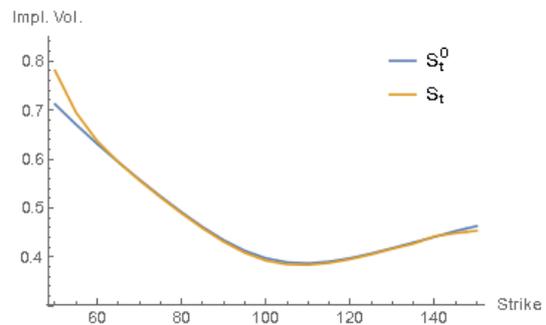


Figure 10: Baseline, $T = 1/24$. At closer maturity the impact of halts is immaterial.

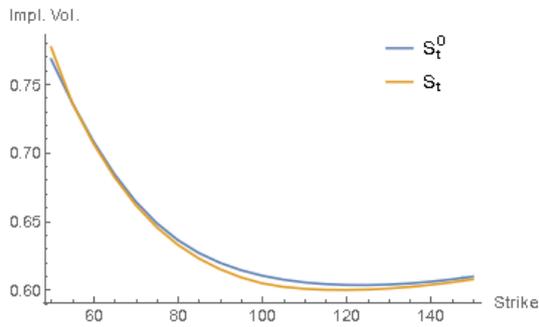


Figure 11: $\sigma_R^* = 0.5$, $T = 1/24$. Increasing σ_R^* recreates skew.

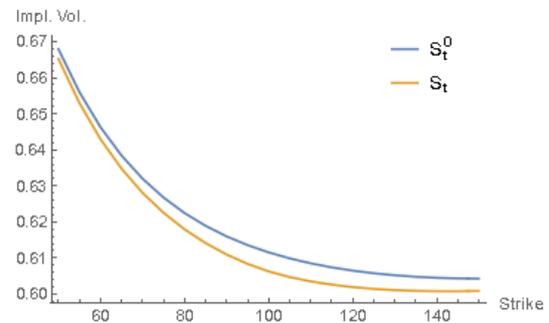


Figure 12: Even more so at monthly level

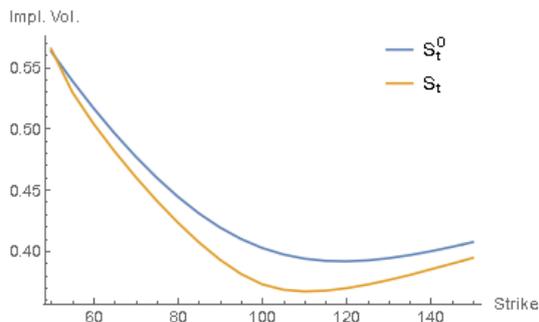


Figure 13: $\lambda^* = 12$, $T = 1/12$. Increasing λ^* steepens and lowers the skew.

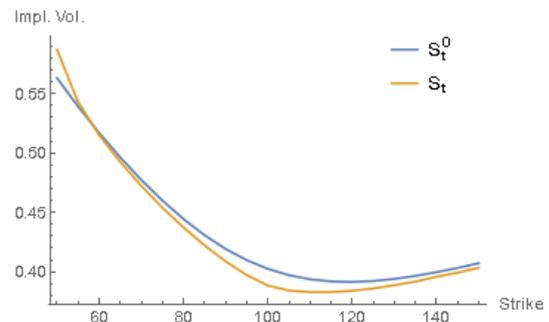


Figure 14: $\beta^* = 12$, $T = 1/12$. Reducing β^* also decreases the level and increases convexity.

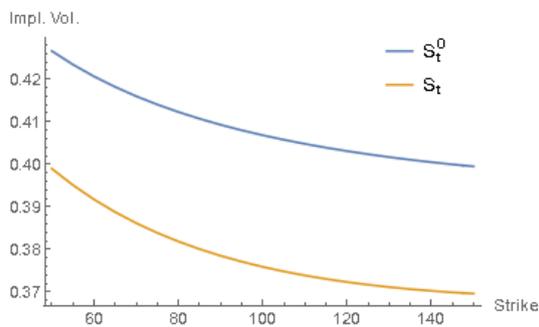


Figure 15: $\lambda^* = 12$, $T = 0.5$. Skew increase and level lowering still visible at longer maturity.

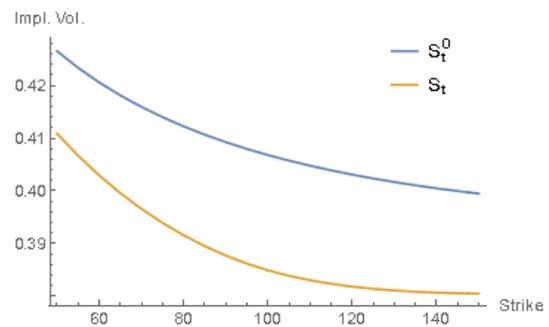


Figure 16: $\beta^* = 12$, $T = 0.5$. Effect even more pronounced for β^* .