Consistent factor models for temperature markets

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Abstract

We propose an approach for pricing and hedging weather derivatives based on including forward looking information about the temperature available to the market. This is achieved by modeling temperature forecasts by a finite dimensional factor model. Temperature dynamics are then inferred in the short end. In analogy to interest rate theory, we establish conditions which guarantee consistency of a factor model with the martingale dynamics of temperature forecasts. Finally, we consider a specific two-factor model and examine in more detail pricing and hedging of weather derivatives in this context.

Key words: temperature models, temperature markets, factor models, consistency, temperature derivatives, pricing and hedging

1 Introduction

Pricing and hedging of weather derivatives has attracted a lot of attention, by a variety of different approaches. One possibility, as employed in e.g. Dornier

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& Queuel [9], Alaton, Djehiche & Stillberger [1] and Benth & Šaltytė-Benth [2], is to model the temperature as an Ornstein-Uhlenbeck process plus a seasonality function, and to obtain the risk-neutral dynamics by adding a market price of risk which is obtained by calibration. Long-range effects have been incorporated in Brody, Syroka & Zervos [5]. As the market for weather derivatives in general is incomplete, Davis [7] proposes to consider the 'zero marginal rate of substitution' price, whereas Platen & West [15] use a benchmark approach to derive a fair price. Density weather forecasts via time series for weather derivatives pricing are studied in Campbell & Diebold [6], whereas Jewson & Caballero [14] propose a 'pricing by pruning' approach: the price is obtained as an expectation under a measure change which incorporates probabilistic weather forecasts. A detailed account on various methods is given in Dischel [8], Geman [10], Geman & Leonardi [11] and Jewson & Brix [13].

In this paper, the primary objective is to specify a temperature market model for pricing and hedging of temperature derivatives which also includes forward looking information about future temperature available to the market. Contrary to traded underlyings, temperature does not reveal all forward looking information available to the market only by its past evolution. Thus all models that consider the filtration generated by the temperature only as market information are built on a fundamental information miss-specification. Instead, we start with a model for the complete curve of meteorological temperature forecasts as an unbiased estimator of future temperature. This can be realized as a conditional expectation with respect to the market filtration. We assume the market filtration is generated by a multidimensional Brownian motion and aim for setting up a parsimonious factor model. In a sense we take a reduced form approach to information modeling: rather then including explicitly specific forward looking information we model a phenomenon (meteorological temperature forecasts) that is assumed to integrate all information about temperature available to the market. The current temperature is then read off in the short end of the forecast curve. An important issue hereby is that the forecast curve has to be consistent. This means that it should satisfy a martingale dynamics in view of our interpretation of it as a conditional expectation of future temperature. One of our main contributions is to characterize consistent factor models for the most popular curve families. We then study in detail a consistent two-factor model. This model is rich enough to capture different types of qualitative behavior of forecast curves in the shorter end, including a temporary increase in forecasted temperature prior to reversion to the seasonal function. Moreover, we address pricing and in particular hedging of temperature derivatives by futures contracts in this in general incomplete market context.

For a more detailed formulation, we move now to a more precise mathematical description of our approach. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space fulfilling the usual conditions. We denote by $\tau(t)$ the temperature at time t at a geographical location of interest and, without loss of generality, we assume the risk free interest rate to be zero. Consider a temperature sensitive derivative with maturity T and payoff $g(\tau(.))$ depending on the temperature path. Then the arbitrage free price process $\pi(t)$ of the derivative is given by

$$\pi(t) = \mathbb{E}_{\mathbb{Q}}[g(\tau(.))|\mathcal{F}_t]\,, \tag{1}$$

where $\mathbb{Q}$ is an equivalent local martingale measure (which in our situation can be any measure equivalent to $\mathbb{P}$ since the underlying temperature is not a traded asset) and the filtration $(\mathcal{F}_t)_{t\geq 0}$ represents the information available to the market at time t. As mentioned above, to compute (1), one approach in the literature is to specify a reduced form model for the temperature. For example, in [2] the following mean reverting dynamics is proposed:

$$\begin{aligned} \tau(t) &= \Lambda(t) + X(t) \\ dX(t) &= -\lambda\, X(t)\, dt + \sigma(t)\, dW(t)\,, \end{aligned} \tag{2}$$

where $\Lambda(t)$ is a deterministic seasonality function, $\lambda > 0$ is the mean reversion rate, $\sigma(t)$ is the deterministic and seasonal varying volatility, and $W(t)$ is Brownian motion. The model is then estimated on historical temperature time series, and it appears that already the fairly simple model (2) together with an appropriately chosen volatility function $\sigma(t)$ yields a rather good fit to temperature dynamics.

To compute derivative prices in analogy to model approaches on classical financial markets, one can assume that the information filtration $(\mathcal{F}_t)_{t\geq 0}$ in (1) is generated by the underlying temperature or respectively by the temperature driving noise. Further, in order to identify the pricing measure $\mathbb{Q}$ the market price of risk could be determined by calibration to market prices. See for example [1] or [2].

However, the transfer of assumptions from classical financial markets with stocks as traded underlyings to new illiquid markets where the underlying (like temperature) might even not be traded at all appears to be problematic. In particular, the assumption that all information available to the market is incorporated in the past evolution of the underlying might be acceptable for storable assets, but for non-storable underlyings (like temperature or electricity) this assumption is fundamentally wrong. In contrast to storable assets, one cannot profit from forward looking information about non-storable assets by taking long or short positions today. Thus, forward looking information available to the market is not reflected in the past evolution of the non-storable underlying and is therefore not included in the filtration generated by the underlying. In the case of temperature there is obviously lots of meteorological forward looking information available to the market that is not included in the past evolution of the temperature.

One way to deal with this information misspecification in the model could be to enlarge the filtration $(\mathcal{F}_t)_{t\geq 0}$ by forward looking information. This ansatz is proposed in [4] in the context of electricity markets. It seems, however, rather difficult to explicitly enlarge the filtration by all forward looking information available to the market. Further, from a mathematical point of view, one encounters the theory of enlargement of filtrations which restricts the type of included forward looking information by its analytic tractability.

Instead, the idea of this paper is to set up a reduced form model for an indicator which integrates all forward looking market information in addition to the evolution of the underlying. More precisely, in this paper about temperature markets we suggest to set up a model for *meteorological temperature forecasts*

$$f(t;T) = \mathbb{E}[\tau(T)|\mathcal{F}_t]\,. \tag{3}$$

Here $f(t;T)$ denotes the forecast given by the meteorologists at time t of the temperature at time T, and $(\mathcal{F}_t)_{t\geq 0}$ is the filtration of all available information. Given $\mathcal{F}_t$, the forecast $f(t;T)$ is thus an unbiased estimator of the temperature $\tau(T)$ and can be expressed as conditional expectation in (3) under the real world probability measure $\mathbb{P}$. To compute the derivative price in (1), the temperature $\tau(t) = f(t;t)$ is inferred in the short end of the forecasts and the evolution of the temperature will now also be governed by the forward looking meteorological information integrated in the forecasts. Note that by assumption the filtration $(\mathcal{F}_t)_{t\geq 0}$ now represents all information available to the market.

In this paper we propose to model the complete forecast curve by finite dimensional factor models. In Section 2 the general theoretical foundations are laid down for consistent forecast factor models, in analogy to consistent factor models in interest rate theory (see ([12], Ch.9) and references therein). Consistent factor models of affine, polynomial, and exponential-polynomial type are examined in more detail. In Section 3 we present a specific consistent 2-factor model for temperature forecasts, before we consider pricing and hedging of derivatives written on temperature futures in this model in Section 4. In general, the resulting market is incomplete and we compute mean-variance hedging strategies in terms of solutions of associated partial differential equations.

2 Consistent factor models for temperature forecasts

Let now $W(t)$ be a d-dimensional Brownian motion generating the filtration $(\mathcal{F}_t)_{t\geq 0}$ on our filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We propose to model the complete temperature forecast curve in (3) by an m-dimensional factor model adapting the approach to factor models in interest rate theory as presented in [12]. More precisely, we set

$$f(t, t+x) = H(x, Z(t)), \tag{4}$$

where the m-dimensional state process $(Z(t))_{t\geq 0}$ is given by the diffusion dynamics

$$dZ(t) = b(Z(t))\,dt + \rho(Z(t))\,dW(t), \quad Z(0) = z_0, \tag{5}$$

and $H : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$ is a deterministic function. Note that we have parameterized the model in terms of *time to forecast* $x := T - t$ (which corresponds to the Musiela parametrization in interest rate theory) rather than in terms of forecast time T. This is a convenient parametrization when modeling the evolution of a complete curve. The actual temperature is then given for $x = 0$ by the short end of the curve: $\tau(t) = f(t, t)$.

We call a factor model *admissible* if it fulfills the following three assumptions:

- **A1** $H \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R}^m)$;
- **A2** $b : \mathbb{R}^m \to \mathbb{R}^m$ is continuous and $\rho : \mathbb{R}^m \to \mathbb{R}^{m\times d}$ is measurable and such that the diffusion matrix

$$a(z) := \rho(z)\rho(z)^T$$

is continuous in $z \in \mathbb{R}^m$;

- **A3** The stochastic differential equation (5) has a unique $\mathbb{R}^m$-valued solution $Z(t)$, for every $z_0 \in \mathbb{R}^m$;

Further, an admissible factor model will be called *consistent* if in addition the following assumption is fulfilled

- **A4** For fixed $T > 0$, the process $f(t;T) = H(T-t, Z(t))$ is a $\mathbb{P}$-martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$.

While the three conditions defining an admissible factor model are of a technical nature, the consistency condition A4 stems from (3). Indeed, by assumption A1 we can apply Itô's formula on H and obtain

$$
\begin{aligned}
df(t,T) &= -\partial_x H(T-t, Z(t))dt + \sum_{i=1}^m \partial_{z_i} H(T-t, Z(t))dZ_i(t) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^m \partial_{z_i}\partial_{z_j} H(T-t, Z(t))d\left\langle Z_i, Z_j\right\rangle_t \\
&= \Bigg(-\partial_x H(T-t, Z(t)) + \sum_{i=1}^m b_i(Z(t))\partial_{z_i} H(T-t, Z(t)) \\
&\quad + \frac{1}{2}\sum_{i,j=1}^m a_{ij}(Z(t))\partial_{z_i}\partial_{z_j} H(T-t, Z(t))\Bigg) dt \\
&\quad + \sum_{i=1}^m \sum_{j=1}^d \partial_{z_i} H(T-t, Z(t))\rho_{ij}(Z(t))dW_j(t), \qquad (6)
\end{aligned}
$$

where $a(z) = \rho(z)\rho^T(z)$ as defined above. For $f(t;T)$ to be a martingale, the drift in (6) has to vanish. Letting $t \to 0$ in (6) and replacing $T-t$ by x this is the case if and only if

$$
\partial_x H(x,z) = \sum_{i=1}^m b_i(z)\partial_{z_i} H(x,z) + \frac{1}{2}\sum_{i,j=1}^m a_{ij}(z)\partial_{z_i}\partial_{z_j} H(x,z) \qquad (7)
$$

for all $(x,z) \in \mathbb{R}_+ \times \mathbb{R}^m$. Integrating both sides with respect to x thus leads to the following result.

Theorem 1. *An admissible factor model is consistent if and only if the following two conditions are fulfilled:*
1)

$$\partial_x G(x,z) = H(0,z) + \sum_{i=1}^{m} b_i(z)\partial_{z_i} G(x,z) + \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}(z)\partial_{z_i}\partial_{z_j} G(x,z), \quad (8)$$

for all $(x,z) \in \mathbb{R}_+ \times \mathbb{R}^m$, *where* $G(x,z) = \int_0^x H(u,z)du$ *and* $a(z) = \rho(z)\rho^T(z)$ *as above.*
2) The process

$$Y(s) := \sum_{i=1}^{m}\sum_{j=1}^{d}\int_0^s \partial_{z_i} H(T-t, Z(t))\rho_{ij}(Z(t))dW_j(t) \quad (9)$$

is a $\mathbb{P}$*-martingale.*

Condition (8) is the analogy to the consistency condition of factor models for forward interest rates. Compared to the forward rate case, our consistency condition has one term less and is therefore simpler: there is no term with products of the derivatives of $G(\cdot, z)$ (see ([12], Ch.9) for further details). But contrary to the forward rate case we require $f(t;T)$ to be a martingale and not only a local martingale and thus have to check the additional condition 2) in Theorem 1 for a consistent factor model.

We make the following

Definition 2. *The pair of diffusion characteristics* $\{b, a\} = \{b(z), a(z)\}$ *and the forecast curve parametrization* $H = H(x,z)$ *are said to be* consistent *if the corresponding factor model is consistent.*

Suppose now that we are given a family of forecast curves $H = H(x,z)$ parameterized by the vector $z \in \mathbb{R}^m$. A criterion to choose such a family could for example be obtained by a principle component analysis performed on historically observed forecast curves. The essential question then is if at all there exists a factor process $Z(t)$ which together with H yields an admissible factor model. Or, in terms of Definition 2, if at all there exists a pair of diffusion characteristics $\{b, a\}$ that is consistent with the forecast curve parametrization H. In the rest of this section we want to examine this question in more detail where in particular we focus on the verification of condition (8). To this end, we make the following

Assumption 3. *For the remaining part of this section we assume the factor models to be admissible and such that condition 2) in Theorem 1 is fulfilled.*

By Theorem 1 and the above assumption it is thus sufficient to check condition (8) to verify consistency. A criterion for uniqueness and existence of consistent factor models is given by the following result.

Theorem 4. *Under Assumption 3, suppose that the functions*

$$\partial_{z_i} G(\cdot, z) \quad \text{and} \quad \frac{1}{2}\partial_{z_i}\partial_{z_j} G(\cdot, z) \tag{10}$$

for $1 \leq i \leq j \leq m$*, are linearly independent for all* z *in some dense subset* $\mathcal{D} \subset \mathbb{R}^m$*. Then there exists one and only one consistent pair* $\{b, a\}$*.*

Proof. Set $M = m + m(m+1)/2$, the number of unknown functions b_k and $a_{kl} = a_{lk}$. Let $z \in \mathcal{D}$. Then there exists a sequence $0 \leq x_1 < ... < x_M$ such that the $M \times M$-matrix with k-th row vector built by

$$\partial_{z_i} G(x_k, z) \quad \text{and} \quad \frac{1}{2}\partial_{z_i}\partial_{z_j} G(x_k, z),$$

for $1 \leq i \leq j \leq m$, is invertible. Thus, $b(z)$ and $a(z)$ are uniquely determined by (8). This holds for each $z \in \mathcal{D}$, and by the continuity of b and a hence for all $z \in \mathbb{R}^m$. □

This Theorem tells us the following: if we use a parameterized curve family $\{H(\cdot, z) | z \in \mathbb{R}^m\}$ which fulfills condition (10) for daily estimation of the temperature forecast curve, then any consistent $\mathbb{P}$-diffusion model Z for z is fully determined by H. However, in many situations condition (10) is not fulfilled and one has greater flexibility to choose a consistent factor process. In the remaining part of this section we will investigate this for the most popular curve families: affine, polynomial, and exponential-polynomial (in particular Nelson-Siegel and Svensson) curve families. Also we shortly indicate how the consistency restrictions for forecast models compare to the analogous consistency restrictions for forward rate models.

2.1 Affine curve families

From a mathematical point of view, the most tractable curve families are functions H which are affine in z:

$$H(x, z) = g_0(x) + g_1(x) z_1 + ... + g_m(x) z_m. \tag{11}$$

In this case the second-order z-derivatives vanish and the consistency condition (8) reads

$$g_0(x) - g_0(0) + \sum_{i=1}^{m} z_i \left(g_i(x) - g_i(0)\right) = \sum_{i=1}^{m} b_i(z) G_i(x), \tag{12}$$

where we define

$$G_i(x) = \int_0^x g_i(u) du. \tag{13}$$

Since all second-order z-derivatives of $G(\cdot, z)$ are equal to zero, they are not linearly independent, and therefore condition (10) is not fulfilled: there exists no unique solution for the pair $\{a, b\}$. In fact, we will see that b still is uniquely determined but the matrix a is not subject to any consistency restriction.

If the m functions $G_i(x)$ are linearly independent, we can invert and solve the linear equation (12) for a and b, as we did in the proof of Theorem 4. Since the left-hand side of (12) is affine in z and if we assume the $G_i(x)$ to be independent, we obtain that b is also affine of the form

$$b_i(z) = c_i + \sum_{j=1}^{m} \gamma_{ij} z_j, \tag{14}$$

for some constants c_i and γ_{ij}. Inserting this into (12) and matching constant terms and terms with z_k's, we obtain the following system of differential equations:

$$\begin{cases} \partial_x G_0(x) &= g_0(0) + \sum_{i=1}^{m} c_i G_i(x) \\ \partial_x G_k(x) &= g_k(0) + \sum_{i=1}^{m} \gamma_{ik} G_i(x), \quad k = 1, ..., m. \end{cases} \tag{15}$$

We have thus proved:

Theorem 5. *Suppose the functions G_i, $1 \leq i \leq m$, are linearly independent. If the pair $\{b, a\}$ is consistent with the z-affine function H in (11) then b is necessarily affine of the form (14). Moreover, the functions G_i solve the system of equations (15) with initial conditions $G_i(0) = 0$.*
Conversely, suppose Assumption 3 and that b is affine of the form (14), and let $g_i(0)$, $1 \leq i \leq m$, be some given constants. If the functions G_i solve the system of equations (15) with initial conditions $G_i(0) = 0$, then the z-affine function H in (11) is consistent with $\{a, b\}$.

As announced above, there are no additional consistency restrictions on the diffusion matrix $a(z)$. This is in contrast to affine forward rate curve models where also $a(z)$ has to be affine and thus gives more freedom to choose a consistent affine forecast model. As we have seen before, the reason for this greater flexibility compared to forward rate models is the less restrictive consistency condition (8).

2.2 Polynomial curve families

Polynomial curve families are given by a function H of the form

$$H(x,z) = \sum_{|\mathbf{i}|=0}^{n} g_{\mathbf{i}}(x) z^{\mathbf{i}}, \tag{16}$$

where we use the multi-index notation $\mathbf{i} = (i_1, ..., i_m)$, $|\mathbf{i}| = i_1 + ... + i_m$ and $z^{\mathbf{i}} = z_1^{i_1} \cdots z_m^{i_m}$. Here $n > 1$ denotes the degree of the z-polynomial function, which means that there exists an index $\mathbf{i}$ with $|\mathbf{i}| = n$ and $g_{\mathbf{i}} \neq 0$. Note that the case $n = 1$ is the one of affine curve families treated above. In interest rate theory it has been shown that consistent z-polynomial forward curve families can have only degree $n \in \{1, 2\}$ (see [12], Sec. 9.4). For polynomial factor models for temperature forecasts this is not true anymore: the term which induces restrictions on n in the forward rate case lacks in the consistency condition for temperature forecasts. However, there will be restrictions on the diffusion characteristics $\{b, a\}$ (note that contrary to the case $n = 1$, also the diffusion matrix $a(z)$ is restricted for $n > 1$). Indeed, we will see that consistency implies a factor process whose coefficients can be freely chosen only up to degree one.

For notational simplicity we only focus on the special case $m = 1$, where we simply identify $i \equiv |\mathbf{i}| = i_1 \in \{0, ...n\}$ and can write equation (16) as

$$H(x,z) = \sum_{i=0}^{n} g_i(x) z^i.$$

Furthermore, we define G_i as in (13). We remark that the general case goes through analogously.

Theorem 6. *Suppose that G_i, $1 \leq i \leq n$, are linearly independent functions. Then consistency implies that the drift $b(z)$ and the diffusion characteristics*

$a(z)$ are of the form

$$b(z) = b_1(z) + f(z)\,; \quad a(z) = a_1(z) - \frac{2zf(z)}{n-1}\,, \tag{17}$$

where $b_1(z)$ is a polynomial of maximal order one, and $a_1(z)$ and $f(z)$ are a polynomials of maximal order two.

Proof. Equation (8) can be rewritten in the z-polynomial case as

$$\sum_{i=0}^{n} \left(g_i(x) - g_i(0)\right) z^i = \sum_{i=0}^{n} G_i(x) B_i(z), \tag{18}$$

where we define

$$B_i(z) := b(z) i z^{i-1} + \frac{1}{2} a(z) i(i-1) z^{i-2}.$$

By assumption we can solve the linear equation (18) for B, and thus $B_i(z)$ are polynomials in z of order less than or equal n for all $0 \leq i \leq n$. Thus, considering the case $i = n$, it follows that $b(z)$ and $a(z)$ are of the form

$$b(z) = b_1(z) + f(z)\,; \quad a(z) = a_1(z) - \frac{2zf(z)}{n-1}\,, \tag{19}$$

where $b_1(z)$ is a polynomial of maximal order one, $a_1(z)$ is a polynomial of maximal order two, and $f(z)$ is some function. Considering the case $i = n-1$, it then follows that $f(z)$ has to be a polynomial of maximal order two. □

2.3 Exponential-Polynomial families

General exponential-polynomial curve families are of the form

$$H(x) = p_1(x) e^{-\alpha_1 x} + \ldots + p_n(x) e^{-\alpha_n x},$$

where p_i denote polynomials of degree n_i. In the following we will concentrate on the Nelson-Siegel and Svensson families which are very popular special cases of exponential-polynomial curve families.

The Nelson-Siegel family. The four-dimensional Nelson-Siegel curves are given by

$$H_{NS}(x, z) = z_1 + (z_2 + z_3 x) e^{-z_4 x}.$$

Theorem 7. *Given Assumption 3, the diffusion characteristics* $\{b, a\}$ *are consistent with* H_{NS} *if and only if*

$$\begin{aligned} &b_1(z) = b_4(z) = 0, \quad b_2(z) = z_3 - z_2 z_4, \quad b_3(z) = -z_3 z_4, \\ &\text{and} \quad a_{4k}(z) = a_{k4}(z) = 0, \quad \forall k = 1, ..., 4, \end{aligned}$$

whereas there are no further consistency restrictions on $a_{ik}(z)$ *for* $i = 1, 2, 3$ *and* $k = 1, ...4$*. The corresponding consistent state process dynamics are thus given by*

$$\begin{aligned} dZ_1(t) &= \sum_{j=1}^{d} \rho_{1j}(Z(t))dW_j(t), \\ dZ_2(t) &= (Z_3(t) - z_4 Z_2(t))dt + \sum_{j=1}^{d} \rho_{2j}(Z(t))dW_j(t), \\ dZ_3(t) &= -z_4 Z_3(t)dt + \sum_{j=1}^{d} \rho_{3j}(Z(t))dW_j(t), \\ dZ_4(t) &\equiv 0, \end{aligned} \tag{20}$$

with initial point $Z(0) = (z_1, ..., z_4)$*.*

Proof. The partial derivative of H_{NS} with respect to x is

$$\partial_x H_S(x, z) = (z_3 - z_2 z_4 - z_3 z_4 x)e^{-z_4 x},$$

the gradient with respect to z reads

$$\nabla_z H_{NS} = \begin{pmatrix} 1 \\ e^{-z_4 x} \\ x e^{-z_4 x} \\ (-z_2 x - z_3 x^2)e^{-z_4 x} \end{pmatrix},$$

and the Hessian matrix is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -x e^{-z_4 x} \\ 0 & 0 & 0 & -x^2 e^{-z_4 x} \\ 0 & -x e^{-z_4 x} & -x^2 e^{-z_4 x} & (z_2 x^2 + z_3 x^3)e^{-z_4 x} \end{pmatrix}.$$

Obviously, condition (10) is not fulfilled, and consequently a and b are not uniquely defined for the Nelson-Siegel family. Nevertheless, the drift coefficients b_i and some of the a_{ij} can still be uniquely determined.
The consistency condition in the form (7) here reads

$$q_1(x) + q_2(x)e^{-z_4 x} = 0, \tag{21}$$

where we assume for the moment that

$$z_i \neq 0 \quad \forall i = 1, ..., 4, \tag{22}$$

and the two polynomials are given by

$$\begin{aligned} q_1(x) &= b_1(z), && (23)\\ q_2(x) &= \left(\frac{1}{2}a_{44}(z)z_3\right)x^3 + \left(-b_4(z)z_3 - a_{34}(z) + \frac{1}{2}a_{44}(z)z_2\right)x^2 \\ &\quad +\left(b_3(z) - a_{24}(z) + z_3 z_4\right)x + \left(b_2(z) - z_3 + z_2 z_4\right), && (24) \end{aligned}$$

where we have used the fact that the diffusion matrix $a(z) = \rho(z)\rho(z)^T$ is symmetric. By (22) we get that for consistency $q_1(x) = q_2(x) = 0$. Hence,

$$\begin{aligned} b_1(z) &= 0,\\ a_{44}(z) &= 0, \end{aligned}$$

and since $a(z)$ is a positive semi-definite symmetric matrix we conclude that

$$a_{4k} = a_{k4} = 0 \quad \forall k = 1, ...4.$$

Therefore, the polynomial $q_2(x)$ reduces to

$$q_2(x) = \Big(- b_4(z)z_3\Big)x^2 + \Big(b_3(z) + z_3 z_4\Big)x + \Big(b_2(z) - z_3 + z_2 z_4\Big),$$

and using this we can solve for the remaining b_i's:

$$\begin{aligned} b_2(z) &= z_3 - z_2 z_4,\\ b_3(z) &= -z_3 z_4,\\ b_4(z) &= 0. \end{aligned}$$

We derived all above results under the assumption (22). But the set of z where this holds is dense in $\mathbb{R}^m$. By the continuity of $a(z)$ and $b(z)$, the above results thus extend for all $z \in \mathbb{R}^m$. □

The Nelson-Siegel family is a popular parametrization for forward interest rates since it is able to reproduce the principle changes in interest rate curve evolution: parallel shift, tilting, and bending. However, it can be shown that there does not exist any non-trivial factor process Z yielding a consistent Nelson-Siegel model for forward rates. Again, in the case of forecast models the situation is thus much less restrictive.

The Svensson family. As an extension of the Nelson-Siegel family, the Svensson family has been introduced to improve the curve flexibility. Two parameters were added, and consequently the six-dimensional Svensson curves are given by:

$$H_S(x,z) = z_1 + (z_2 + z_3 x)e^{-z_5 x} + z_4 x e^{-z_6 x}. \tag{25}$$

With this extension we are able to create a dent additional to the hump given by the Nelson-Siegel family. Surprisingly, consistent forecast factor models do not allow for this additional flexibility as the following Theorem shows.

Theorem 8. *Suppose Assumption 3. To fulfill the consistency condition (8) for H_S we have to set z_4=0, and thus are back to the case of the Nelson-Siegel family.*

Proof. The partial derivative of $H_S(x,z)$ with respect to x is in this case

$$\partial_x H_S(x,z) = (-z_2 z_5 + z_3 - z_3 z_5 x)e^{-z_5 x} + (z_4 - z_4 z_6 x)e^{-z_6 x}.$$

The gradient with respect to z is

$$\nabla_z H_S = \begin{pmatrix} 1 \\ e^{-z_5 x} \\ x e^{-z_5 x} \\ x e^{-z_6 x} \\ (-z_2 x - z_3 x^2)e^{-z_5 x} \\ -z_4 x^2 e^{-z_6 x} \end{pmatrix}$$

and the Hessian matrix with respect to z is given by:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -xe^{-z_5 x} & 0 \\ 0 & 0 & 0 & 0 & -x^2 e^{-z_5 x} & 0 \\ 0 & 0 & 0 & 0 & 0 & -x^2 e^{-z_6 x} \\ 0 & -xe^{-z_5 x} & -x^2 e^{-z_5 x} & 0 & (z_2 x^2 + z_3 x^3)e^{-z_5 x} & 0 \\ 0 & 0 & 0 & -x^2 e^{-z_6 x} & 0 & z_4 x^3 e^{-z_6 x} \end{pmatrix}.$$

As is the case for the Nelson-Siegel family, condition (10) is not fulfilled for the Svensson family as well.
The consistency condition in the form (7) here reads

$$q_1(x) + q_2(x)e^{-z_5 x} + q_3(x)e^{-z_6 x} = 0. \tag{26}$$

Here we assume for the moment that

$$z_5 \neq z_6, \quad \text{and} \quad z_i \neq 0 \text{ for all } i = 1, ..., 6. \tag{27}$$

Then the three polynomials are given by:

$$\begin{aligned} q_1(x) &= b_1(z), & (28)\\ q_2(x) &= \left(\frac{1}{2}a_{55}(z)z_3\right)x^3 + \left(\frac{1}{2}a_{55}(z)z_2 - a_{35}(z) - b_5(z)z_3\right)x^2 \\ &\quad + \Big(z_3 z_5 + b_3(z) - b_5(z)z_2 - a_{25}(z)\Big)x \\ &\quad + \Big(z_2 z_5 - z_3 + b_2(z)\Big), & (29)\\ q_3(x) &= \left(\frac{1}{2}a_{66}(z)z_4\right)x^3 + \Big(-b_6(z)z_4 - a_{46}(z)\Big)x^2 \\ &\quad + \Big(b_4(z) + z_4 z_6\Big)x - z_4, & (30) \end{aligned}$$

where we have used the fact that the diffusion matrix $a(z) = \rho(z)\rho(z)^T$ is symmetric.

By (26) and (27) all three polynomials have to be equal to zero, and we conclude from (30) that

$$z_4 = 0,$$

and plugging this into (25) we are back to the case of the Nelson-Siegel family. □

3 A specific consistent two-factor model for temperature forecasts

For the remaining part of the paper we now introduce a specific (time inhomogeneous) affine 2-factor model for temperature forecast curves and consider pricing and hedging of temperature derivatives in the corresponding market. To this end, we let in this and the remaining sections $W(t) = (W_1(t), W_2(t))$ be a 2-dimensional Brownian motion generating the filtration $(\mathcal{F}_t)_{t\geq 0}$ on our filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

We suggest that the principle building block in the temperature forecast model is the following affine curve family

$$H_\tau(x, z) = z_1 e^{-\lambda x} + z_2 \frac{1}{\lambda - \rho} \left(e^{-\rho x} - e^{-\lambda x} \right) , \tag{31}$$

where $z = (z_1, z_2) \in \mathbb{R}^2$, $x > 0$, and $\lambda, \rho > 0$. Before we have a closer look at our forecast model and the qualitative features of the induced forecast curves and temperature dynamics, we employ Theorem 5 to check for 3-dimensional factor processes

$$dZ(t) = \begin{pmatrix} dZ_1(t) \\ dZ_2(t) \\ dZ_3(t) \end{pmatrix} = \begin{pmatrix} b_1(Z(t))\, dt + \sum_{i=1}^2 \rho_{1i}(Z(t))\, dW_i(t) \\ b_2(Z(t))\, dt + \sum_{i=1}^2 \rho_{2i}(Z(t))\, dW_i(t) \\ b_3(Z(t))\, dt + \sum_{i=1}^2 \rho_{3i}(Z(t))\, dW_i(t) \end{pmatrix}$$

that are consistent with H_τ in (31). Note that although the curve family in (31) only depends on the first two components $(z_1, z_2) \in \mathbb{R}^2$, we enlarge the factor state space by a third component to be able to consider time inhomogeneous dynamics for $Z(t)$ (the third component will be set equal to time later on: $Z_3(t) = t$).

Proposition 9. *The 3-dimensional factor process $Z(t)$ is consistent with our affine curve model H_τ in (31) if and only if the drift is given by*

$$\begin{aligned} b_1(z) &= -\lambda z_1 + z_2, \\ b_2(z) &= -\rho z_2, \\ b_3(z) &= c_3 + \gamma_{31} z_1 + \gamma_{32} z_2 + \gamma_{33} z_3, \end{aligned}$$

where the coefficients of $b_3(z)$ can be chosen arbitrarily.

Proof. With the notation in Subsection 2.1 we have

$$\begin{aligned} g_0(x) &= 0, \\ g_1(x) &= e^{-\lambda x}, \\ g_2(x) &= \frac{1}{\lambda - \rho} \left(e^{-\rho x} - e^{-\lambda x} \right), \\ g_3(x) &= 0, \end{aligned}$$

where $g_0(0) = g_2(0) = g_3(0) = 0$ and $g_1(0) = 1$. The integrals of the $g_i(x)$ read

$$\begin{aligned} G_0(x) &= 0, \\ G_1(x) &= -\frac{1}{\lambda} e^{-\lambda x} + \frac{1}{\lambda}, \\ G_2(x) &= \frac{1}{\lambda - \rho} \left(-\frac{1}{\rho} e^{-\rho x} + \frac{1}{\lambda} e^{-\lambda x} + \frac{1}{\rho} - \frac{1}{\lambda} \right), \\ G_3(x) &= 0, \end{aligned}$$

and therefore $G_1(x), G_2(x), G_3(x)$ are linearly independent. Hence, we can apply part one of Theorem 5 and conclude that the $b_i(z)$ are affine of the form (14):

$$\begin{aligned} b_1(z) &= c_1 + \gamma_{11} z_1 + \gamma_{12} z_2 + \gamma_{13} z_3, \\ b_2(z) &= c_2 + \gamma_{21} z_1 + \gamma_{22} z_2 + \gamma_{23} z_3, \\ b_3(z) &= c_3 + \gamma_{31} z_1 + \gamma_{32} z_2 + \gamma_{33} z_3, \end{aligned}$$

for some constants c_i and γ_{ij}, $1 \leq i, j \leq 3$. The system of differential equations (15) now reads:

$$\begin{aligned} \partial_x G_0(x) &= g_0(0) + c_1 G_1(x) + c_2 G_2(x) + c_3 G_3(x), &(32)\\ \partial_x G_1(x) &= g_1(0) + \gamma_{11} G_1(x) + \gamma_{21} G_2(x) + \gamma_{31} G_3(x), &(33)\\ \partial_x G_2(x) &= g_2(0) + \gamma_{12} G_1(x) + \gamma_{22} G_2(x) + \gamma_{32} G_3(x), &(34)\\ \partial_x G_3(x) &= g_3(0) + \gamma_{13} G_1(x) + \gamma_{23} G_2(x) + \gamma_{33} G_3(x), &(35) \end{aligned}$$

We solve these equations for the constants of the $b_i(x)$. We can rewrite (32) as

$$0 = e^{-\lambda x} \left(-c_1 \frac{1}{\lambda} + c_2 \frac{1}{(\lambda - \rho)\lambda} \right) + e^{-\rho x} \left(-c_2 \frac{1}{(\lambda - \rho)\rho} \right) + \left(c_1 \frac{1}{\lambda} + c_2 \frac{1}{\lambda \rho} \right).$$

From the second term we get that $c_2 = 0$. Inserting this into the first term gives $c_1 = 0$.

We can rewrite (33) as

$$\begin{aligned} -1 = e^{-\lambda x} \left(-1 - \gamma_{11} \frac{1}{\lambda} + \gamma_{21} \frac{1}{(\lambda - \rho)\lambda} \right) + e^{-\rho x} \left(-\gamma_{21} \frac{1}{(\lambda - \rho)\rho} \right) \\ + \left(\gamma_{11} \frac{1}{\lambda} + \gamma_{21} \frac{1}{\lambda \rho} \right). \end{aligned}$$

From the second term we see that $\gamma_{21} = 0$, and plugging this into the first term gives $\gamma_{11} = -\lambda$.

We can rewrite (34) as

$$
\begin{aligned}
0 = e^{-\lambda x} & \left(\frac{1}{\lambda - \rho} - \gamma_{12} \frac{1}{\lambda} + \gamma_{22} \frac{1}{(\lambda - \rho)\lambda} \right) \\
& + e^{-\rho x} \left(-\frac{1}{\lambda - \rho} - \gamma_{22} \frac{1}{(\lambda - \rho)\rho} \right) + \left(\gamma_{12} \frac{1}{\lambda} + \gamma_{22} \frac{1}{\lambda \rho} \right).
\end{aligned}
$$

From the second term we conclude $\gamma_{22} = -\rho$. Inserting this into the first term leads to $\gamma_{12} = 1$.

We can rewrite (35) as

$$
\begin{aligned}
0 = e^{-\lambda x} & \left(-\gamma_{13} \frac{1}{\lambda} + \gamma_{23} \frac{1}{(\lambda - \gamma)\lambda} \right) + e^{-\rho x} \left(-\gamma_{23} \frac{1}{(\lambda - \gamma)\gamma} \right) \\
& + \left(\gamma_{13} \frac{1}{\lambda} + \gamma_{23} \frac{1}{\lambda \gamma} \right).
\end{aligned}
$$

From the second term we deduce $\gamma_{23} = 0$, and inserting this into the first term gives $\gamma_{13} = 0$.

We see that the coefficients $c_3, \gamma_{31}, \gamma_{32}, \gamma_{33}$ are not further determined and therefore can be chosen arbitrarily. □

We are now ready to introduce our temperature forecast curve model. The forecast at time t of the temperature at time $t + x$ is given by

$$
\begin{aligned}
f(t; t + x) &= \Lambda(t + x) + H_\tau(x, Z(t)) \\
&= \Lambda(t + x) + Z_1(t) e^{-\lambda x} + Z_2(t) \frac{1}{\lambda - \rho} \left(e^{-\rho x} - e^{-\lambda x} \right), \quad (36)
\end{aligned}
$$

where $\Lambda(s)$ is a deterministic seasonality function (average temperature at time s) and the factor dynamics of $Z_1(t)$ and $Z_2(t)$ are given by

$$
\begin{aligned}
dZ_1(t) &= (-\lambda Z_1(t) + Z_2(t))\, dt + \sigma_1(t)\, dW_1(t) \quad (37) \\
dZ_2(t) &= -\rho Z_2(t)\, dt + \sigma_2(t)\, dW_2(t)
\end{aligned}
$$

for $\lambda, \rho > 0$ and deterministic, continuous volatility functions $\sigma_1(t), \sigma_2(t)$. Note that the time dependence of the volatility specification, $\rho_{11}(Z(t)) = \sigma_1(t)$, $\rho_{22}(Z(t)) = \sigma_2(t)$, and $\rho_{ij}(Z(t)) = 0$ else, is obtained by putting

$Z_3(t) = t$, i.e. by choosing $c_3 = 1$ and $\gamma_{31} = \gamma_{32} = \gamma_{33} = 0$. Re-parameterized in terms of forecast time T we get

$$f(t;T) \quad = \quad \Lambda(T) + Z_1(t)e^{-\lambda(T-t)} + Z_2(t)\frac{1}{\lambda-\rho}\left(e^{-\rho(T-t)} - e^{-\lambda(T-t)}\right) \quad (38)$$

With this choice of factor process we know by Proposition 9 that $H_\tau(T-t, Z(t))$ and thus also $\Lambda(T) + H_\tau(T-t, Z(t))$ is a consistent factor model (for a given T, adding the constant $\Lambda(T)$ obviously does not harm consistency). Further, the choice of $\rho(Z(t))$ evidently guarantees condition 2) in Theorem 1, and we see that (38) is a consistent forecast curve model fulfilling Assumptions A1-A4.

The model proposed in (38) implies the following qualitative behavior of forecast curves:

- As the forecast horizon $T - t$ gets large, the volatility of forecasted temperature diminishes and

  $$f(t;T) \to \Lambda(T) \ \text{ for } \ (T-t) \to \infty\,.$$

  This behavior is realistic since forward looking information decreases the larger the forecast horizon becomes, and finally average temperature is the best prediction.

- Before reverting to the seasonal average temperature in the long end, there are basically two different types of qualitative behavior of forecast curves in the shorter end:

  1. Direct exponential reversion in temperature forecasts from the current temperature level $f(t;t)$ to the seasonal function $\Lambda(T)$.

  2. An increase from $f(t;t)$ (or decrease for current temperatures $f(t;t)$ below seasonal average $\Lambda(t)$) in forecasted temperature prior to exponential reversion to the seasonal function $\Lambda(T)$.

While this type of possible forecast outcomes is certainly a simplification, we believe that it still catches a substantial part of qualitative forecast curve formation in reality. A more thorough empirical analysis of temperature forecast curve data will be part of a future research project.

Also note that when setting the parallel shift parameter $z_1 = 0$ in the Nelson-Siegel family presented in Subsection 2.3, $H_{NS}(x, z)$ would produce the same qualitative behavior of forecast curves. Indeed, in this situation we get

$$H_\tau(x, z) \to H_{NS}(x, z)$$

pointwise for $\lambda \to \rho$ in (31), and our model is seen to be a generalization of the Nelson-Siegel model without parallel shift parameter. The reason we consider this generalization is that then the consistent factor components $Z_1(t)$ and $Z_2(t)$ are allowed to have different mean reversion parameters $\lambda \neq \rho$ while in the Nelson-Siegel model these must be identical.

Finally, for the purpose of temperature derivatives pricing, we are now able to write down the model for the temperature dynamics implied by our forecast curve model. The temperature $\tau(t) = f(t;t)$ at time t can be read off in the short end of the forecast curve and is given by

$$\tau(t) = \Lambda(t) + Z_1(t)\,, \tag{39}$$

where $Z_1(t)$ was introduced in (37). If $Z_2(t) = 0$, our resulting temperature model would thus be the same as the one proposed in [2] and formulated in (2) in the introduction. However, compared to the model in (2), the drift in the evolution of the temperature proposed in (39) is additionally governed by the factor $Z_2(t)$ that integrates the forward looking information contained in meteorological temperature forecasts.

4 Pricing and hedging of temperature derivatives in the two-factor model

We now turn our attention to the pricing and hedging of temperature derivatives in a market where the temperature dynamics is given by the two-factor model presented in (39) in the previous section. The most common exchange traded temperature derivatives are futures contracts. Depending on the market place one can find CAT-futures (Cumulative Average Temperature), HDD-futures (Heating Degree Days), or CDD-futures (Cooling Degree Days). See below for more details on these contracts. Further, besides in futures there is organized trade in plain vanilla options written on the respective futures price as underlying.

Given an option written on a futures price, our aim in the following is to compute hedging strategies in a corresponding futures contract. It is shown that in the case of CAT-futures the market is complete, whereas for general futures contracts (including HDD and CDD futures) the market is incomplete. In the latter case we proceed to compute optimal mean-variance hedging strategies.

According to no-arbitrage theory, pricing of financial assets with the temperature as underlying spot price has to be done under some risk-neutral measure which in our setting can be any probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$. Hence there are *a priori* infinitely many choices of pricing measures, and we will have to find the corresponding market price of risk by calibration to market data which will be explored in a further study.

We assume that the temperature dynamics under a risk-neutral measure $\mathbb{Q}$ are given as $\tau(t) = \Lambda(t) + Z_1(t)$, with

$$dZ_1(t) = (\theta(t) - \lambda Z_1(t) + Z_2(t))\, dt + \sigma_1(t)\, dW_1(t)$$

and

$$dZ_2(t) = (\chi(t) - \rho Z_2(t))\ dt + \sigma_2(t)\, dW_2(t),$$

where θ, χ are some bounded deterministic functions; this imposes a certain restriction on the set of possible pricing measures, but simplifies the calculations considerably. Here W_1, W_2 are independent $\mathbb{Q}$-Brownian motions, hence in general different from the Brownian motions in the preceding section although we use the same letters by a slight abuse of notation.

Also we recall that we assume for convenience that the short rate is $r = 0$.

4.1 CAT-futures: a complete Gaussian market

A CAT-futures contract emitted at time t is an instrument whose payoff is the accumulated temperature over a time period $[t_1, t_2]$ in exchange for the futures price $F(t; t_1, t_2)$ agreed on at time t. Given a fixed risk-neutral measure $\mathbb{Q}$, the CAT-futures price $F(t; t_1, t_2)$ is chosen such that the value of the futures contract equals zero at emission time $t \leq t_1$, i.e.

$$E_{\mathbb{Q}} \left[\int_{t_1}^{t_2} \tau(s)\, ds - F(t; t_1, t_2) \middle| \mathcal{F}_t \right] = 0\,,$$

or

$$F(t; t_1, t_2) = E_{\mathbb{Q}} \left[\int_{t_1}^{t_2} \tau(s)\, ds \middle| \mathcal{F}_t \right]\,.$$

By Fubini we can rewrite

$$F(t;t_1,t_2)=\int_{t_1}^{t_2} f_{\mathbb{Q}}(t;s)\,ds,$$

with

$$f_{\mathbb{Q}}(t;s)=E_{\mathbb{Q}}\left[\tau(s)\right|\mathcal{F}_t\left].\right.$$

Now consider an option emitted at time $t=0$ with maturity T written on the futures price $F(T;t_1,t_2)$, $T\le t_1$. Our objective is to determine a hedge of the option in the CAT-futures contract emitted at time $t=0$. Denote by $F(s)$ the value of this futures contract at time s, $0\le s\le t_1$. Then

$$\begin{aligned}F(s)&=E_{\mathbb{Q}}\left[\int_{t_1}^{t_2}\tau\left(r\right)\,dr-F(0;t_1,t_2)\right|\mathcal{F}_s\right]\\&=\int_{t_1}^{t_2} f_{\mathbb{Q}}(s;r)\,dr-F(0;t_1,t_2)\,,\end{aligned}$$

and the dynamics of $F(s)$ and $F(s;t_1,t_2)$ are equal modulo the initial constant $F(0;t_1,t_2)$. To treat the hedging question of options on the futures price it is thus sufficient to consider hedging of options in the market with $F(s)$ as underlying asset price.

As $f_{\mathbb{Q}}\left(\cdot;s\right)$ is a $\mathbb{Q}$-martingale, there is no finite variation part. Therefore, it follows from (38) and the fact that the $\mathbb{Q}$-dynamics of the factors Z_1, Z_2 differ from their $\mathbb{P}$-dynamics only by a deterministic term, that for s fixed

$$df_{\mathbb{Q}}(t,s)=e^{-\lambda(s-t)}\sigma_1(t)\,dW_1(t)+\frac{e^{-\rho(s-t)}-e^{-\lambda(s-t)}}{\lambda-\rho}\sigma_2(t)\,dW_2(t).$$

By stochastic Fubini ([3], Theorem 2.2) we get

$$dF(t)=c\,e^{\lambda t}\,\sigma_1(t)\,dW_1(t)+\left(c_\rho\,e^{\rho t}-c_\lambda\,e^{\lambda t}\right)\,\sigma_2(t)\,dW_2(t),$$

with

$$c=\int_{t_1}^{t_2} e^{-\lambda s}\,ds,\quad c_\rho=\frac{1}{\lambda-\rho}\int_{t_1}^{t_2} e^{-\rho s}\,ds,\quad c_\lambda=\frac{1}{\lambda-\rho}\int_{t_1}^{t_2} e^{-\lambda s}\,ds.$$

Setting

$$Z(t)=\int_0^t\left(c^2e^{2\lambda t}\sigma_1^2(t)+\left(c_\rho\,e^{\rho t}-c_\lambda\,e^{\lambda t}\right)^2\sigma_2^2(t)\right)^{-\frac{1}{2}}\,dF(t),$$

we note that Z is a martingale with $[Z](t) = t$, hence by Lévy's characterization a Brownian motion. We can re-write then the dynamics of $F(t)$ as

$$dF(t) = \left(c^2 e^{2\lambda t}\sigma_1^2(t) + \left(c_\rho\, e^{\rho t} - c_\lambda\, e^{\lambda t}\right)^2 \sigma_2^2(t)\right)^{\frac{1}{2}} \, dZ(t),$$

and conclude that the CAT-futures market is complete, and F even has Gaussian dynamics since the coefficients are deterministic. The delta hedge is then computed by well-known methods.

4.2 Dynamics of general temperature futures contracts

In this section we assume that the value of our futures contract $F(t)$ emitted at time 0 is given as

$$F(t) = \int_{t_1}^{t_2} E_{\mathbb{Q}}\left[h\left(\tau(s)\right) \middle| \mathcal{F}_t\right] \, ds - F(0; t_1, t_2), \quad t \leq t_1,$$

for some suitable function h; popular choices are $h(x) = (x - K)^+$ (Heating Degree Days) or $h(x) = (K - x)^+$ (Cooling Degree Days) for some strike K. Here the futures price $F(t; t_1, t_2)$ at time t is

$$F(t; t_1, t_2) = E_{\mathbb{Q}}\left[\int_{t_1}^{t_2} h(\tau(s)) \, ds \middle| \mathcal{F}_t \right].$$

By the Markov property of the vector stochastic process (Z_1, Z_2) there exist for each $s \in [\tau_1, \tau_2]$ functions $v^{(s)}(t, x, y)$ which we assume to be in $\mathcal{C}^{1,2,2}$ such that

$$v^{(s)}\left(t, Z_1(t), Z_2(t)\right) = E_{\mathbb{Q}}\left[h\left(\tau(s)\right) \middle| \mathcal{F}_t\right].$$

By a Feynman-Kac type argument, the function $v^{(s)}$ satisfies the PDE

$$\begin{aligned} \frac{\partial v^{(s)}}{\partial t} + \left(\theta(t) - \lambda x + y\right) \frac{\partial v^{(s)}}{\partial x} + \left(\chi(t) - \rho y\right) \frac{\partial v^{(s)}}{\partial y} & \\ + \frac{1}{2}\sigma_1^2(t) \frac{\partial^2 v^{(s)}}{\partial x^2} + \frac{1}{2}\sigma_2^2(t) \frac{\partial^2 v^{(s)}}{\partial y^2} &= 0, \end{aligned}$$

with the terminal condition

$$v^{(s)}(s, x, y) = h(\Lambda(s) + x) \qquad \text{for all } (x, y) \in \mathbb{R}^2.$$

Moreover, we have

$$dv^{(s)} = \frac{\partial v^{(s)}}{\partial x}\sigma_1 \, dW_1 + \frac{\partial v^{(s)}}{\partial y}\sigma_2 \, dW_2.$$

Since the two integrands are in general stochastic, the resulting market is in general incomplete. By stochastic Fubini (we assume the corresponding integrability condition to be satisfied), the value $F(t)$ of the futures contract is given as

$$\begin{aligned} F(t) + F(0; t_1, t_2) &= \int_{\tau_1}^{\tau_2} \int_0^t \left\{ \frac{\partial v^{(s)}}{\partial x}\sigma_1(u) \, dW_1(u) + \frac{\partial v^{(s)}}{\partial y}\sigma_2(u) \, dW_2(u) \right\} ds \\ &= \int_0^t \int_{\tau_1}^{\tau_2} \frac{\partial v^{(s)}}{\partial x} \, ds \, \sigma_1(u) \, dW_1(u) + \int_0^t \int_{\tau_1}^{\tau_2} \frac{\partial v^{(s)}}{\partial y} \, ds \, \sigma_2(u) \, dW_2(u). \end{aligned}$$

4.3 Optimal mean-variance hedging strategy

We consider the stochastic value process F of a temperature futures contract as hedging instrument, and a European option with payoff $g\,(F(T))$ for a suitable function g and maturity $T < \tau_1$. For convenience, we assume the short rate is $r = 0$. Assuming that $g\,(F(T))$ is square-integrable, we associate to such an option its value process

$$V(t) = E_{\mathbb{Q}}\left[g(F(T)) \, | \mathcal{F}_t \right],$$

which is a square-integrable $\mathbb{Q}$-martingale, with final value $V(T) = g\,(F(T))$. As we have seen, the resulting futures market is in general incomplete, hence it is not always possible to replicate the option perfectly with a self-financing hedging strategy (options written on CAT-futures are an exception). We follow here the idea of *mean-variance hedging*: under a fixed martingale measure $\mathbb{Q}$ for $F = (F(t))$, we want to minimize the difference between option payoff and the result from trading in the future via initial capital c and a self-financing hedging strategy ϑ by a quadratic criterion. Formally, we minimize

$$E_{\mathbb{Q}}\left[\left(V(T) - c - \int_0^T \vartheta(t) \, dF(t) \right)^2 \right] \tag{40}$$

over all constants c and all predictable ϑ such that $\int \vartheta \, dF$ is a square-integrable martingale.

Optimizing the quadratic functional (40) can be done via Hilbert space methods, see [16]. It results that the fair price of the option is the initial capital $V(0)$ needed to drive the optimal strategy, and can be calculated as

$$V(0) = E_{\mathbb{Q}}\left[V(T)\right].$$

The optimal hedging strategy is the integrand ξ in the stochastic integral with respect to F in the *Kunita-Watanabe decomposition*

$$V = V(0) + \int \xi \, dF + L. \tag{41}$$

Here L is a square-integrable martingale strongly orthogonal to F, which means that the quadratic co-variation $[F, L]$ equals zero (since all martingales adapted to $(\mathcal{F}_t)$ are continuous). It can be interpreted as the *residual risk process*, that is the part of the risk which is non-attainable through hedging via trading in the underlying future.

As seen in the preceding section, the value of the futures contract can in general be written as

$$dF(t) = \phi(t)\, dW_1(t) + \psi(t)\, dW_2(t)$$

with (for $t \leq \tau_1$)

$$\begin{aligned} \phi(t) &= \int_{t_1}^{t_2} \frac{\partial v^{(s)}}{\partial x}\,(t, Z_1(t), Z_2(t))\; ds\, \sigma_1(t), \qquad (42) \\ \psi(t) &= \int_{t_1}^{t_2} \frac{\partial v^{(s)}}{\partial y}\,(t, Z_1(t), Z_2(t))\; ds\, \sigma_2(t). \end{aligned}$$

Assuming that F is a square-integrable martingale, and since $[W_1, W_2] = 0$, we have that ϕ, ψ are elements of a space Θ defined as

$$\Theta := \left\{ \vartheta \text{ predictable} \middle| E_{\mathbb{Q}}\left[\int_0^T \vartheta^2(t)\, dt\right] < \infty \, . \right\}$$

Here it is understood that we identify ϑ, $\widetilde{\vartheta} \in \Theta$ if

$$\int_0^T \left(\vartheta(t) - \widetilde{\vartheta}(t)\right)^2 dt = 0.$$

By the Itô-isometry, $\int \phi \, dW_1$ and $\int \psi \, dW_2$ are then square-integrable martingales as well. Similarly, fix some square-integrable Q-martingale G strongly orthogonal to F, and write its decomposition with respect to the basis (W_1, W_2) as

$$dG(t) = \phi^G(t) \, dW_1(t) + \psi^G(t) \, dW_2(t),$$

with ϕ^G, $\psi^G \in \Theta$. Such a G always exists, in a complete market one may choose the zero-martingale. However, in an incomplete setting we will choose a non-constant G from the orthogonal complement of F within the Hilbert space of square-integrable martingales. The orthogonality requirement $[F, G] = 0$ yields the condition

$$\phi\phi^G + \psi\psi^G = 0. \tag{43}$$

The choice of G is not unique, but once we have picked a G we shall fix it once and for all. Our goal is now to find the Kunita-Watanabe decomposition

$$V = V_0 + \int \xi \, dF + \int \zeta \, dG,$$

from which we get the optimal mean-variance hedging strategy ξ and the residual risk process $L = \int \zeta \, dG$. This can be done by first finding the martingale representation of the value process associated with the actual option payoff,

$$dV(t) = v^B(t) \, dW_1(t) + v^G(t) \, dW_2(t), \qquad \nu^B, v^G \in \Theta, \tag{44}$$

and then calculating the optimal hedging strategy as

$$\xi = \frac{d\,[V, F]}{d\,[F, F]} = \frac{v^B\phi + v^G\psi}{\phi^2 + \psi^2}.$$

Similarly, we get ζ as

$$\zeta = \frac{d\,[V, G]}{d\,[G, G]} = \frac{v^B\phi^G + v^G\psi^G}{(\phi^G)^2 + (\psi^G)^2}$$

which allows us to calculate the conditional variance of the non-attainable risk process,

$$R_t(\xi) = E_{\mathbb{Q}}\left[\left(\int_t^T \zeta(s) \, dG(s)\right)^2 \middle| \mathcal{F}_t\right].$$

To find the martingale representation (44) of the value process corresponding to an European option on the temperature futures contract with payoff function g, maturing at time $T \leq \tau_1$, we note that by the Markov property of the vector process (F, Z_1, Z_2) there exists a function $u\left(t, x, y, z\right)$ such that for $0 \leq t \leq T$ we have

$$u\left(t, Z_1(t), Z_2(t), F(t)\right) = E_{\mathbb{Q}}\left[\, g\left(F(T)\right) \middle|\, \mathcal{F}_t\right].$$

Furthermore, note that by (42), $\phi(t)$ and $\psi(t)$ are measurable functions of $(Z_1(t), Z_2(t))$, hence we will write $\phi(t) = \phi\left(t, x, y\right)$ and $\psi(t) = \psi\left(t, x, y\right)$.

Assuming that the function u is smooth enough, we get, by applying Itô's formula and setting the drift equal to zero, the linear PDE

$$\mathcal{L}u = 0,$$

with terminal condition

$$u\left(T, x, y, z\right) = g\left(z\right) \qquad \text{for all } x, y, z \in \mathbb{R}^3,$$

where $\mathcal{L}$ denotes the linear second order differential operator

$$\begin{aligned}
&\frac{\partial}{\partial t} + \left(\theta(t) - \lambda x + y\right)\frac{\partial}{\partial x} + \left(\chi(t) - \rho y\right)\frac{\partial}{\partial y} + \\
&\frac{1}{2}\left(\phi^2(t) + \psi^2(t)\right)\frac{\partial^2}{\partial z^2} + \frac{1}{2}\sigma_1^2(t)\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\sigma_2^2(t)\frac{\partial^2 v}{\partial y^2} + \\
&\psi(t)\sigma_1(t)\frac{\partial^2}{\partial x \partial z} + \phi(t)\sigma_2(t)\frac{\partial^2}{\partial y \partial z}.
\end{aligned}$$

Moreover, the martingale dynamics of the value process are

$$dV(t) = \left(\phi(t)\frac{\partial u}{\partial z} + \sigma_1(t)\frac{\partial u}{\partial x}\right) dW_1(t) + \left(\psi(t)\frac{\partial u}{\partial z} + \sigma_2(t)\frac{\partial u}{\partial y}\right) dW_2(t).$$

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