Allocation of Systemic Risk

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Abstract

We critically review the translation of the core originating from game theory to the allocation of systemic risk. Whereas the core is commonly accepted for the portfolio allocation we will see that in a systemic context it might result in unfairnesses for certain members of the system. We observe that due to the presence of possible feedback mechanisms between the single entities, apart from the height of the induced losses, also the ability to transfer these losses has to be considered for the risk allocation problem. Since this new source of risk is only fully assessable in the complete system and not in the subsystems, it can not be captured by a core allocation. Thus, instead of considering upper bounds, as it is done for allocations in the core, we reverse the definition to lower bounds.

Keywords: multivariate risk measures, systemic risk measures, risk allocation, core

MSC 2010 classifications: 91B30, 91B32

1 Introduction

In this work our aim is to study the appropriateness of the transfer of a classical game theoretic allocation concept to the allocation problem for financial systems with interacting institutions. This is due to the fact that in the recent financial crisis it became apparent that a risk evaluation of a financial network on the basis of the single institutions is not sufficient in order to capture the systemic risk inherent from the various feedback mechanisms between the institutions.

For this purpose we position ourself in a stylized market clearing framework for interbank liabilities. This framework traces back to the seminal work of Eisenberg and Noe (2001). Briefly said we have a system of financial institutions

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which are connected via bilateral credit agreements. If now a financial institution defaults due to some adverse market event, then it has to be liquidated immediately and the remaining assets are distributed among the creditors of the institution proportionately to their liabilities. As the liquidation value is less then the total liabilities, the creditor banks face additional losses which might result in a default of one or more creditor banks. These potential defaults can trigger further failures of banks and thus the initial failure of banks spreads into the financial system.

In order to allow for a comprehensive risk assessment of financial systems like the above, systemic risk measures have been introduced. As we have already described how losses propagate into the financial system, we can easily calculate the total losses of the system by summing all the losses of the single institutions after all possible contagion has taken place. Now the risk of the system can be easily obtained by using a univariate risk measure. An axiomatic description of this particular type of systemic risk measures which allow for a decomposition into an aggregation function and a univariate risk measure has been studied in Chen et al. (2013), Kromer et al. (2016) and Hoffmann et al. (2016).

However, a major drawback of a systemic risk measurement compared to a single evaluation is that an appropriate breakdown of the risk of the financial system to the contributing institutions is far from obvious. Therefore, we need to identify appropriate allocations in a separate step. In Chen et al. (2013) and Kromer et al. (2016) this is achieved by using a dual representation of the systemic risk measures. These allocations are essentially equal to the Aumann-Shapley value which is known from the game theoretic literature, cf. Aumann and Shapley (1974). The Aumann-Shapley value is also an example for a coherent allocation as defined in Denault (2001) which gained much attention for the portfolio allocation problem. Among the properties of a coherent allocation the no-undercut property is the most crucial. Moreover, it is also the main building block of the core from the game theory literature, cf. Aubin (1979). It says that for all subgroups the amount of the total risk which is allocated to this subgroup should be smaller than their measured risk. The core property is commonly justified by the following consideration, if a subgroup would get a share of the total risk which is higher than its own risk, then this subsystem would split from the system and consequently obtains a lower risk. Whereas for a portfolio of financial assets it can be easily answered how a split of a subportfolio should be executed, this task is much more complex for a financial network. In this work we will concentrate on two possible ways where the underlying network topology remain intact. For examples where also the interbank liabilities in the financial network are modified, we refer to the works of Drehmann and Tarashev (2013) or Staum et al. (2016). However, in their work they do not study the implications on the core.

For obvious reasons, the core is only meaningful if the risk measurement is subadditive with respect to the subgroups, that is merging two disjoint subgroups should reduce the risk. This diversification effect is usually assumed for the classical risk management of a firm or a portfolio of financial assets. Also for financial systems it can be argued that the risk measurement of the aggregated values should have a diversification benefit. However, for the aggregation itself this really depends on the chosen model on the formation of the subsystems, i.e. we cannot say a priori if the merger of two subsystems will decrease or increase the risk. Hence we have two rivaling streams of diversification benefits and costs.

In Chen et al. (2013) and Kromer et al. (2016) the authors overcome this problem by considering an aggregation of the subsystem which corresponds to a worst-case view, when it comes to the spreading of risk within the system. As a result they also have a diversification benefit on this level. The considered subsystem generation is a generalization from the classical portfolio approach, where the risk factors correspond to profits and losses of certain financial instruments. Thus, considering the risk of the accumulated profits and losses of a subsystem suggests itself as the subsystem risk. That is simply summing up all risk factors which are in the subsystem and equate the remainders to zero.

Unfortunately, we will see that in financial networks where contagion might take place this allocation procedure creates wrong incentives to the financial institutions. The reason is that, whereas it was sufficient in the classical approach to measure how much each subsystem spreads into the system, we have now also a second origin of risk, namely the ability of a subsystem to transfer the losses. For example consider two financial systems which are connected exclusively via one intermediate institution having no other operations. Obviously, the intermediate institution can be considered systemic, since it is the only possible way that losses of one financial system can be carried over to the other. However, the systemic relevance of the intermediary cannot be expressed by a core allocation, since each core allocation must be bounded from above by its standalone risk and the intermediary has no other sources of risk apart from the losses from the financial systems.

In order to tackle this problem we invert the definition of the core, i.e. the allocated risk for each subsystem should be at least as much as the risk of this subsystem. We call this allocation principle the reverse core. Clearly, by reversing the core definition there is now also a need for changing the underlying subsystem risk management in such a way that instead of a diversification benefit we have that there is a consolidation cost. In our analysis this is provided by supposing that all institutions outside of a subsystem are equipped with such a high amount of capital that a default is excluded. Contrarily to the classical subsystem generation discussed earlier this supports a best-case view. For our interaction model we will see that this new definition resolves the unfairnesses from before. Moreover, we identify under which assumptions there exist allocations in the intersection of both approaches.

The rest of the paper is structured as follows: In section 2 we state our notion and review the (fuzzy) core concept from the game theoretic literature adapted to more general aggregation functions. In section 3 we apply the core concept to our financial system with contagion. Based on the deficiencies of this allocation we alter the underlying risk measurement for the subsystems from a worst-case to a best-case perspective. Due to this change we introduce in section 4 the notion of a reverse core and how it is related to the core concept from before. Finally, in section 5 we determine a reverse core element for our financial system and show that in most cases it does not coincide with the elements from section 3. In the appendix A we discuss how the non-emptiness of the cores for random risk factors can be inferred from deterministic risk factors.

2 Standard game theoretic approach to systemic risk

Throughout this work we consider a financial system $\mathcal{I} := \{1, ..., d\}$ which consists of $d \in \mathbb{N}$ different financial institutions. In the analysis of the financial system $\mathcal{I}$ subsystems will play a decisive role. We denote the set of all subsystems, that is the powerset of $\mathcal{I}$, by $\mathcal{P} := \mathcal{P}(\mathcal{I})$. Since $\mathcal{I}$ represents the largest system we will denote by J^C the complementary set of $J \in \mathcal{P}$ with respect to $\mathcal{I}$, i.e. $J^C := \mathcal{I} \backslash J$. We will denote the i-th unit vector of $\mathbb{R}^d$ by e_i, i.e. all components are equal to zero except the i-th component which is equal to one. $\mathbf{0}_d$ and $\mathbf{1}_d$ denote d-dimensional vectors where all components are equal to zero or one, resp. As usual $\mathbb{R}^d_+$ is the space of d-dimensional non-negative real valued vectors. By $\mathbf{I}_d$ we denote the $d \times d$ dimensional identity matrix and by $\mathbf{A}^\top$ the transpose of the matrix $\mathbf{A}$. Apart from the usual matrix multiplication, we will sometimes also need the Hadamard product (componentwise multiplication) which we denote by $*$, i.e. $x * y = (x_1 y_1, ..., x_d y_d)^\top$ for $x, y \in \mathbb{R}^d$.

Let $\mathcal{X}^d$ be a space of $\mathbb{R}^d$-valued functions on some measurable space $(\Omega, \mathcal{F})$ representing d-dimensional risk factors of the financial system $\mathcal{I}$. We evaluate the systemic risk of these risk factors via a systemic risk measure.

Definition 2.1. *A function $\rho : \mathcal{X}^d \to \mathbb{R}$ is called systemic risk measure if it is antitone, that is for $X, Y \in \mathcal{X}^d$ with $X \leq Y$ we have that $\rho(X) \geq \rho(Y)$.*

In addition to the measurement of the risk of the whole financial system, we also suppose that we have excess to information on the risk of each subgroup of financial institutions. For this measurement we introduce the notion of a subsystem risk measure.

Definition 2.2. *We say that $\widetilde{\rho} : \mathcal{X}^d \times \mathcal{P} \to \mathbb{R}$ is subsystem risk measure for the systemic risk measure $\rho : \mathcal{X}^d \to \mathbb{R}$ if the function $\mathcal{X}^d \ni X \mapsto \widetilde{\rho}(X, J)$ is a systemic risk measure for all $J \in \mathcal{P}$ and $\widetilde{\rho}(X, \mathcal{I}) = \rho(X)$.*
Moreover, if we just consider deterministic risk factors we will call it a subsystem construction scheme and denote it by $\widetilde{\Lambda} : \mathbb{R}^d \times \mathcal{P} \to \mathbb{R}$.

A fairness criterion known as the core from the game theory literature are individually and coalitionally stable allocations, i.e. allocations where no entity or group of entities has an incentive to deviate from the allocation by "splitting" from the system. Next we give the formal definition of the core as know from the game theoretic literature, cf. Aubin (1979).

Definition 2.3 (Allocation and core)**.** *For a given $X \in \mathcal{X}^d$, we say that $k \in \mathbb{R}^d$ is an allocation of the systemic risk $\rho(X)$ if*

$$\sum_{i=1}^{d} k_i = \rho(X).$$

Moreover, let $\widetilde{\rho} : \mathcal{X}^d \times \mathcal{P} \to \mathbb{R}$ be a subsystem risk measure of ρ. We say that k is in the core $C^-_{\widetilde{\rho}}(X)$ if k is an allocation which additionally fulfills that for all subsystems $J \in \mathcal{P}$

$$\widetilde{\rho}(X, J) \geq \sum_{j \in J} k_j. \tag{2.1}$$

The core has been prominently used for the allocation of the risk of a portfolio consisting of financial assets. In the following example we will review this framework and the motivation for the core.

Example 2.4. The core C^- is a superset of the coherent allocations as postulated by Denault (2001). In his work the property (2.1) also appeared under the name of the no undercut property. In Denault (2001) he measures the risk of a portfolio of financial assets. In our context this translates to a financial system, where there are not feedback mechanisms between the single institutions, i.e. the well-being of a single institution is unaffected by the state of the other banks. However, note that in the absence of feedback mechanisms, the single risk factors, here the profits and losses of the banks, might still be dependent in a probabilistic sense. In the portfolio framework the risk of a multivariate risk factor $X \in \mathcal{X}^d$ is measured by

$$\eta\left(\sum_{j=1}^{d} X_j\right),$$

where $\eta : \mathcal{X} \to \mathbb{R}$ is some coherent risk measure. Note that a coherent risk measure is a functional which is antitone, cash-additive, convex and positive homogeneous. For more details on coherent risk measures we refer to Föllmer and Schied (2011). Since we do not observe any other effects by adding or removing a financial asset apart from the additional or missing profits and losses generated by this asset, the sum is an appropriate aggregation function in this setup. Thus, the risk of a subsystem $J \in \mathcal{P}$ should also be measured via

$$\widetilde{\rho}(X, J) := \eta\left(\sum_{j \in J} X_j\right). \tag{2.2}$$

Recall that every coherent risk measure η is subadditive and thus we have that for all disjoint $J_1, J_2 \in \mathcal{P}$

$$\widetilde{\rho}(X, J_1 \cup J_2) \leq \widetilde{\rho}(X, J_1) + \widetilde{\rho}(X, J_2),$$

which reflects a diversification effect. This implies that it is always profitable to merge subportfolios. In order to have a fair allocation $k \in \mathbb{R}^d$ this diversification benefit should be shared among the different financial assets, i.e. $k_j \leq \widetilde{\rho}(X, \{j\})$ for $j = 1, ..., d$. Otherwise the investor would demerge this asset from the portfolio and would hold it separately. Therefore the allocated risk of every single financial asset should be less than its standalone risk. Similarly we can argue for subportfolios, which then results in (2.1) that is the main property of the core C^-.

The following lemma relates the cores of two subsystem risk measures where one is always more conservative than the other. It is a direct consequence of the definition of the core.

Lemma 2.5. *Let $\widetilde{\rho}_1$ be a subsystem risk measure. If $k \in C^-_{\widetilde{\rho}_1}(X)$, then $k \in C^-_{\widetilde{\rho}_2}(X)$ for all $\widetilde{\rho}_2$ with $\widetilde{\rho}_1(X, J) \leq \widetilde{\rho}_2(X, J)$ for all $J \in \mathcal{P}$ and $\rho_1(X) = \rho_2(X)$.*

For the construction of the core, we just considered subsystems of type $\mathcal{P}$, i.e. a risk factor of a financial institution can either be accounted for completely or not at all. But especially in the context of Example 2.4 a subsystem can also be created by taking fractional parts of the profits and losses of the banks. Thus we will now characterize a subsystem by a fractional participation level $\lambda \in [0, 1]^d$ or $\lambda \in \mathbb{R}^d_+$. For this purpose we need to generalize the notion of a subsystem risk measure to a function $\rho : \mathcal{X}^d \times \mathbb{R}^d_+ \to \mathbb{R}$, where $\rho(X, \lambda)$ is the risk of the system $X \in \mathcal{X}^d$ where bank j participates with λ_j and $\rho(X, \mathbf{1}_d) = \rho(X)$. Here $\lambda_j = 1$ means full participation and $\lambda_j = 0$ is the absence of bank j.

Definition 2.6 (Fuzzy core)**.** *We say $k \in \mathbb{R}^d$ is in the fuzzy core $FC^-_\rho(X)$ if k is an allocation, i.e. $\rho(X, \mathbf{1}_d) = \mathbf{1}_d^\top k$ and for all $\lambda \in [0, 1]^d$ it holds that*

$$\rho(X, \lambda) \geq \lambda^\top k.$$

Since each subsystem risk measure with fractional participation ρ yields a subsystem risk measure $\widetilde{\rho} : \mathcal{X}^d \times \mathcal{P} \to \mathbb{R}$ via

$$\widetilde{\rho}(X, J) := \rho\left(X, \sum_{j \in J} e_j\right) \quad \text{for all } J \in \mathcal{P},$$

we have that $FC^-_\rho(X) \subseteq C^-_{\widetilde{\rho}}(X)$.

In the following theorem we recall the well-known result that under differentiability, convexity and positive homogeneity of a subsystem risk measure the fuzzy core is single-valued and equal to its gradient. For a proof see for instance Aubin (1979). In this case the fuzzy core is also called Euler allocation or Euler principle in the literature, c.f. Denault (2001); Tasche (2004).

Theorem 2.7. *Let $\rho : \mathcal{X}^d \times \mathbb{R}^d_+ \to \mathbb{R}$ be a subsystem risk measure which is positive homogeneous on the diagonal of its second argument, i.e. $\rho(\cdot, \alpha \mathbf{1}_d) = \alpha \rho(\cdot, \mathbf{1}_d)$ for all $\alpha \geq 0$. Then, the extended fuzzy core*

$$\overline{FC}^-_\rho(X) := \left\{k \in \mathbb{R}^d : \rho(X, \mathbf{1}_d) = \mathbf{1}_d^\top k \text{ and } \rho(X, \lambda) \geq \lambda^\top k, \forall \lambda \in \mathbb{R}^d_+\right\}$$

is equal to the subdifferential

$$\partial^{-}\rho(X,\mathbf{1}_d) := \left\{k \in \mathbb{R}^d : \rho(X,\lambda) \geq \rho(X,\mathbf{1}_d) + k^\top(\lambda - \mathbf{1}_d)\ \forall \lambda \in \mathbb{R}^d_+\right\}.$$

Thus, if the function $\lambda \mapsto \rho(X,\lambda)$ *is additionally convex and differentiable in* $\mathbf{1}_d$ *the extended fuzzy core*

$$\overline{FC}^{-}_{\rho}(X) = \nabla\rho(X,\mathbf{1}_d)$$

where $\nabla\rho(X,\cdot)$ *is the gradient of* ρ *in its second argument.*

Example 2.8 (Portfolio approach cont.)**.** A possible extension of (2.2) to allow for fractional participation is given by

$$\rho(X,\lambda) := \eta\left(\sum_{i=1}^{d}\lambda_i X_i\right) = \eta\left(\lambda^\top X\right), \tag{2.3}$$

where η is the coherent univariate risk measure from (2.2). Clearly, $\lambda \mapsto \rho(X,\lambda)$ is positively homogeneous and convex, thus the fuzzy core $FC^{-}_{\rho}(X)$ is non-empty. If $\lambda \mapsto \rho(X,\lambda)$ is also differentiable then the fuzzy core $FC^{-}_{\rho}(X)$ is even single valued. Therefore the fuzzy core $FC^{-}(X)$ or the larger core $C^{-}(X)$ seems to be a feasible allocation approach in the portfolio context.

3 Financial model with contagion

In this section we will investigate if the (fuzzy) core still yields fair allocations given that our financial network allows for feedback mechanisms among the financial institutions. For this purpose we need to alter the aggregation function in (2.2) and (2.3) respectively from a simple sum to a more complex aggregation function which allows for the inclusion of a channel of contagion. The aggregation function we will use traces back to the seminal paper of Eisenberg and Noe (2001) and has been extended in many directions, e.g. to include multiple sources of contagion, c.f. Awiszus and Weber (2015) for a survey. In order to focus on the impact of the feedback mechanism on the allocations, we will consider a deterministic risk $x \in \mathbb{R}^d$. The treatment of random risk factors is discussed in the appendix.
As before we assume that $\mathcal{I} := \{1,...,d\}$ represents a financial system. However, we now assume that only the first $d-1$ components are financial institutions and the last component represents the real economy. We suppose that the financial institutions have claims against each other which appear as interbank assets/liabilities on their balance sheets. The interbank assets/liabilities are summarized by the matrix $\mathbf{L} = (L_{i,j})_{i,j=1,...,d}$, where $L_{i,j}$ is the monetary amount of bank i which it owes to bank j. Furthermore the total amount of the interbank liabilities of bank i is denoted by $L_i := \sum_{j=1}^{d} L_{i,j}$ for all $i = 1,...,d$. Three standing assumptions on the liability matrix will be that each bank does not have claims against itself and against the real economy, however the real economy

has claims against each bank. Summarizing we assume that $L_{i,i} = 0, L_{d,i} = 0$ and $L_{i,d} > 0$ for all $i = 1, ..., d$. The first and the last assumption are more technical and not really restricting. The second assumption needs some further explanation. Of course, banks have claims against the real economy like households or industrial companies. However, we will not model these connections within the feedback mechanism, but the real economy can contribute losses to the banks via an initial shock. Another model assumptions is that in case of a default the debtors of the defaulting institution divide the remaining assets proportional to their claims, i.e. it will suffice to consider the relative liability matrix $\mathbf{\Pi} = (\Pi_{i,j})_{i,j=1,...,d}$ which is given by

$$\Pi_{i,j} := \begin{cases} \frac{L_{i,j}}{L_i} & , \text{if } L_i > 0 \\ 0 & , \text{if } L_i = 0 \end{cases}.$$

Moreover, the institutions are endowed with an initial capital/equity $c \in \mathbb{R}^d_+$. On the asset side the institutions have interbank assets as described above and some external assets, which also contains claims against the real economy. Therefore we have a full description of the balance sheet of each bank. Next we suppose that at a future point in time the external assets of each bank are hit by some adverse market event $y \leq \mathbf{0}_d$. Due to this market shock also the liability side of the balance sheet has to decrease by the same amount. As debt is senior to equity, the equity is used first to buffer the shock. However, if there is not a sufficient amount of equity to dampen the shock, the bank is in default and pays out the remaining assets proportionally to its creditors. Since the creditors are not paid in full this creates a further loss on their balance sheets which can result in a default of one or more of the creditors. Finally these defaults can trigger other defaults, so that a large fraction of the system might be affected. This contagion is modeled by the following aggregation function

$$\begin{aligned} \Lambda(x) := \min_{a \in \mathbb{R}^d_+} \mathbf{1}_d^\top a \\ \text{s.t. } a = \left(\mathbf{\Pi}^\top a - x\right)^+, \end{aligned} \tag{3.1}$$

where $x = c + y$ is the equity value of the financial institutions directly after the adverse market event y took place. Note that by monotonicity of the function $\mathbb{R}^d_+ \ni a \mapsto \left(\mathbf{\Pi}^\top a - x\right)^+$, we have that the optimal value for a in the optimization problem (3.1) of the aggregation function Λ can be found by iterating $a(n) = \left(\mathbf{\Pi}^\top a(n-1) - x\right)^+$ with $a(0) := \mathbf{0}_d$. The interpretation of this iteration procedure is as follows:

First, we suppose that no bank defaults which corresponds to $a(0) = \mathbf{0}_d$. In the first iteration we thus have that $a(1) = (-x)^+$ which is identical to the losses of the banks defaulting initially due to the adverse market event. Next $a(2) = \left(\mathbf{\Pi}^\top a(1) - x\right)^+ = \left(\mathbf{\Pi}^\top (-x)^+ - x\right)^+$, where $\mathbf{\Pi}^\top (-x)^+$ are the losses which the banks receive from the initially defaulting banks. Hence $a(2)$ contains the losses of the initially defaulting banks and of those banks which fail due to the losses transmitted by the defaulting banks. In each subsequent step these losses further spread into the system and we approach an equilibrium a fulfilling the constraint in (3.1).

In contrast to Eisenberg and Noe (2001) we do not cap the transmission of losses to other banks by the corresponding interbank liability. We did so in order to keep the model simple and thus for a better understanding of the contagion effects later on and second the inclusion of the real economy makes the events where the losses exceed the interbank liabilities rather unlikely. As there is not an upper bound for the transmitted losses, it could be that for a finite shock the contagion effects wind each other up more and more. However, we will see in Lemma 3.1 below that this is not possible in our framework, since a certain percentage of the losses is always transfered to the real economy, where the channel of contagion ends.

Lemma 3.1. *For each $x \in \mathbb{R}^d$ the aggregation function $\Lambda(x)$ is finite.*

Proof. Firstly, we observe that Λ is monotonically decreasing and that $\Lambda(x) = 0$ for all $x \geq \mathbf{0}_d$. Thus it suffices to consider $x \leq \mathbf{0}_d$, i.e. all institutions default initially. Then the constraint in (3.1) can be simplified to

$$a = \left(\mathbf{\Pi}^\top a - x\right)^+ = \mathbf{\Pi}^\top a - x$$

and thus if the matrix $\mathbf{I}_d - \mathbf{\Pi}^\top$ is invertible, then there exists a unique solution

$$a = -(\mathbf{I}_d - \mathbf{\Pi}^\top)^{-1}x,$$

where $\mathbf{I}_d$ the $d \times d$ dimensional identity matrix.
We denote by $(A_{j,k})_{j,k=1,...,d} = \mathbf{A} := \mathbf{I}_d - \mathbf{\Pi}^\top$. Moreover by $\widetilde{\mathbf{\Pi}}$ and $\widetilde{\mathbf{A}}$ we denote the $(d-1)\times(d-1)$ matrices which are obtained by erasing the last row and column from $\mathbf{\Pi}$ and $\mathbf{A}$ resp. Note that, since we assumed that every institution has liabilities to the real economy, i.e. $\Pi_{j,d} > 0$ for all $j = 1, ..., d-1$, the row sums of $\widetilde{\mathbf{\Pi}}$ are less then 1 and thus the operator norm $\|\widetilde{\mathbf{\Pi}}^\top\|_1 = \max_{i=1,...,d-1} \sum_{j=1}^{d-1} |\Pi_{ij}| < 1$. Hence a classic result from functional analysis, see e.g. Werner (2011) Satz II.1.11, yields that the Neumann series $\sum_{i=0}^n (\widetilde{\mathbf{\Pi}}^\top)^i, n \in \mathbb{N}$, converges and that the inverse of $\widetilde{\mathbf{A}} = \mathbf{I}_{d-1} - \widetilde{\mathbf{\Pi}}^\top$ exists and is equal to the limit of the series. Furthermore, a Laplace expansion along the last column of $\mathbf{A}$ yields $\det(\mathbf{A}) = \det(\widetilde{\mathbf{A}}) \neq 0$ and thus $\mathbf{A}$ is invertible.

□

In the next lemma we derive an element in the (fuzzy) core for the subsystem construction scheme $\Lambda^0(x, \lambda) := \Lambda(\lambda * x)$ by making use of Theorem 2.7. As already pointed out earlier this subsystem construction scheme parallels the portfolio approach (2.3), where the sum as aggregation function is replaced by Λ. For the result we need to identify the institutions which default after all possible contagion has taken place. We denote the set of these institutions for a given $x \in \mathbb{R}^d$ by

$$\mathcal{D}(x) = \{p_1, ..., p_{|\mathcal{D}|}\} := \{i \in \mathcal{I} : (\mathbf{\Pi}^\top a - x)_i \geq 0\},$$

where a is the limit of the sequence $a(n) = (\mathbf{\Pi}^\top a(n-1) - x)^+$ with $a(0) = \mathbf{0}_d$. If it is clear from the context we will mostly drop the reference to the risk factor x.

Lemma 3.2. *Let $x \in \mathbb{R}^d$ and define $k \in \mathbb{R}^d$ by*

$$k_{p_i} := -\sum_{j=1}^{|\mathcal{D}|} \left(\left(\mathbf{I}_{|\mathcal{D}|} - \mathbf{\Pi}_{\mathcal{D},\mathcal{D}}^\top \right)^{-1} \right)_{j,i} x_{p_i}, \quad i = 1, ..., |\mathcal{D}|,$$

and $k_i := 0$ for $i \notin \mathcal{D}$. Here $\mathbf{\Pi}_{\mathcal{D},\mathcal{D}} = (\Pi_{i,j})_{i,j \in \mathcal{D}}$. Then

$$k \in FC^-_{\Lambda^0}(x).$$

Proof. That the matrix $\mathbf{I}_{|\mathcal{D}|} - \mathbf{\Pi}_{\mathcal{D},\mathcal{D}}^\top$ is invertible can be shown analogously to the considerations made in the proof of Lemma 3.1.
Denote by a is the limit of the sequence $a(n) = (\mathbf{\Pi}^\top a(n-1) - x)^+$ with $a(0) = \mathbf{0}_d$. Since $(\mathbf{\Pi}^\top a - x)_i < 0$ for all non-defaulting institutions $i \notin \mathcal{D}$ we have that $a_i = 0$. Therefore we obtain for the vector of losses of the defaulting institutions $a_{\mathcal{D}} := (a_j)_{j \in \mathcal{D}}$ that

$$a_{\mathcal{D}} = \left((\mathbf{\Pi}^\top a - x)_j^+ \right)_{j \in \mathcal{D}} = \mathbf{\Pi}_{\mathcal{D},\mathcal{D}}^\top a_{\mathcal{D}} - x_{\mathcal{D}},$$

where and $x_{\mathcal{D}} := (x_j)_{j \in \mathcal{D}}$. Finally, we obtain that

$$\Lambda(x) = \sum_{i \in \mathcal{D}} a_i = \sum_{i=1}^{|\mathcal{D}|} -\mathbf{1}_{|\mathcal{D}|}^\top \left(\mathbf{I}_{|\mathcal{D}|} - \mathbf{\Pi}_{\mathcal{D},\mathcal{D}}^\top \right)^{-1} \widetilde{e}_i x_{p_i} = \sum_{i=1}^{|\mathcal{D}|} k_{p_i} = \sum_{i=1}^{d} k_i, \qquad (3.2)$$

where $\widetilde{e}_i$ is the i-th unit vector in $\mathbb{R}^{|\mathcal{D}|}$ and thus k is an allocation.
Moreover, suppose that $a \in \mathbb{R}_+^d$ is such that $a = (\mathbf{\Pi}^\top a - x)^+$ and $\Lambda(x) = \mathbf{1}_d^\top a$. Then we have for each $\lambda > 0$ that $\lambda a = (\mathbf{\Pi}^\top (\lambda a) - \lambda x)^+$. Hence

$$\Lambda^0(x, \lambda \mathbf{1}_d) = \Lambda(\lambda x) \leq \mathbf{1}_d^\top (\lambda a) = \lambda \Lambda(x) = \lambda \Lambda^0(x, \mathbf{1}_d).$$

On the other hand we obtain by a similar argumentation for λx that

$$\begin{aligned} \lambda \Lambda^0(x, \mathbf{1}_d) &= \lambda \Lambda^0 \left(x, \frac{1}{\lambda} \lambda \mathbf{1}_d \right) = \lambda \Lambda^0 \left(\lambda x, \frac{1}{\lambda} \mathbf{1}_d \right) \\ &\leq \Lambda^0(\lambda x, \mathbf{1}_d) = \Lambda^0(x, \lambda \mathbf{1}_d). \end{aligned}$$

Combining both results yields positive homogeneity on the diagonal of the second argument of Λ^0.

Furthermore, it can be easily shown that Λ is a convex function, from which it immediately follows that the function $z \mapsto \Lambda^0(x, z)$ is also convex. Hence it follows from Theorem 2.7 that $FC^-_{\Lambda^0}(x)$ is equal to the subdifferential of $z \mapsto \Lambda^0(x, z)$ at $\mathbf{1}_d$.

Firstly, we suppose that there exists a neighborhood N of x such that $\mathcal{D}(z) = \mathcal{D}(x)$ for all $z \in N$. Then $\Lambda(x)$ is the linear function (3.2) on N and thus differentiable in x. Therefore k is the gradient of $\Lambda^0(x,\cdot)$ at $\mathbf{1}_d$ and hence $k = FC^-_{\Lambda^0}(x)$.
If no such neighborhood N exists the subdifferential might not be single valued. However, it can still be shown that k is a member of the subdifferential.
Since the function $z \mapsto |\mathcal{D}(z)|$ is left-continuous with values in $\{0,...,d\}$, we can always find $\widetilde{x} \leq x$ with $\|x-\widetilde{x}\|_\infty > \varepsilon$ for some $\varepsilon > 0$ such that $\mathcal{D}(\widetilde{x}) = \mathcal{D}(x)$ and $\lambda \mapsto \Lambda^0(\widetilde{x},\lambda)$ is differentiable in $\mathbf{1}_d$. Note that $\widetilde{x}$ can also be chosen such that $\widetilde{x}_i \neq 0$ for all $i = 1,...,d$ and thus the componentwise quotient $u \in \mathbb{R}^d_+$ of x and $\widetilde{x}$, i.e. $u_i = \frac{x_i}{\widetilde{x}_i}$ for all $i = 1,...,d$, is well-defined. Since Λ is linear between $\widetilde{x}$ and x, we have that

$$\Lambda^0(\widetilde{x},\mathbf{1}_d) + \nabla\Lambda^0(\widetilde{x},\mathbf{1}_d)^\top(u-\mathbf{1}_d) = \Lambda^0(\widetilde{x},u) = \Lambda^0(x,\mathbf{1}_d), \tag{3.3}$$

where $\nabla\Lambda^0(\widetilde{x},\mathbf{1}_d)$ denotes the gradient of the function $\lambda \mapsto \Lambda^0(\widetilde{x},\lambda)$ at $\mathbf{1}_d$. From this we can immediately infer that for all $\lambda \in \mathbb{R}^d_+$

$$\begin{aligned}\Lambda^0(x,\lambda) &= \Lambda^0(\widetilde{x},\lambda * u)\\ &\geq \Lambda^0(\widetilde{x},\mathbf{1}_d) + \nabla\Lambda^0(\widetilde{x},\mathbf{1}_d)^\top(\lambda * u - \mathbf{1}_d)\\ &= \Lambda^0(x,\mathbf{1}_d) + \left(\nabla\Lambda^0(\widetilde{x},\mathbf{1}_d) * u\right)^\top(\lambda - \mathbf{1}_d),\end{aligned}$$

where we used (3.3) in the last step. Hence $\nabla\Lambda^0(\widetilde{x},\mathbf{1}_d) * u$ is a subdifferential of $\lambda \mapsto \Lambda^0(x,\lambda)$ at $\mathbf{1}_d$. By using (3.2) a simple calculation shows that $k = \nabla\Lambda^0(\widetilde{x},\mathbf{1}_d) * u$ and the result follows. □

Lemma 3.3. *Let $x \in \mathbb{R}^d$ and $k \in FC^-_{\Lambda^0}(x)$ be the allocation from Lemma 3.2. Moreover, denote by $\mathcal{D}_0 := \{i \in \mathcal{I} : x_i \leq 0\}$ the set of initially defaulting institutions. Then we have that the allocations $k_i, i \in \mathcal{D}\backslash\mathcal{D}_0$ of the institutions which default due to contagion are non-positive.*

Proof. First, we prove that $\left(\mathbf{I}_{|\mathcal{D}|} - \mathbf{\Pi}^\top_{\mathcal{D},\mathcal{D}}\right)^{-1} = \sum_{i=0}^\infty \left(\mathbf{\Pi}^\top_{\mathcal{D},\mathcal{D}}\right)^i$. We have already seen in the proof of Lemma 3.1 that this holds true if $\|\mathbf{\Pi}^\top_{\mathcal{D},\mathcal{D}}\|_1 < 1$. Therefore we assume that $\|\mathbf{\Pi}^\top_{\mathcal{D},\mathcal{D}}\|_1 = 1$. In particular this implies that the real economy $d \in \mathcal{D}$. We consider the matrix $\widetilde{\mathbf{\Pi}} \in \mathbb{R}^{(|\mathcal{D}|-1)\times(|\mathcal{D}|-1)}$ which we obtain from $\mathbf{\Pi}_{\mathcal{D},\mathcal{D}}$ by erasing the last row and column and the vector $\mathbf{\Pi}_d = (\Pi_{p_i,d})_{i=1,...,|\mathcal{D}|-1} \in \mathbb{R}^{(|\mathcal{D}|-1)}$ containing the relative liabilities of the defaulting banks to the real economy. Then

$$\mathbf{\Pi}_{\mathcal{D},\mathcal{D}} = \begin{pmatrix} \widetilde{\mathbf{\Pi}} & \mathbf{\Pi}_d \\ \mathbf{0}^\top_{|\mathcal{D}|-1} & 0 \end{pmatrix}$$

and it can be easily shown that for all $n \in \mathbb{N}$

$$\sum_{i=0}^n \left(\mathbf{\Pi}^\top_{\mathcal{D},\mathcal{D}}\right)^i = \begin{pmatrix} \sum_{i=0}^n (\widetilde{\mathbf{\Pi}}^\top)^i & \mathbf{0}_{|\mathcal{D}|-1} \\ \mathbf{\Pi}_d^\top \sum_{i=0}^{n-1} (\widetilde{\mathbf{\Pi}}^\top)^i & 0 \end{pmatrix}.$$

Since $\|\widetilde{\mathbf{\Pi}}\|_1 < 1$ the Neumann series $\sum_{i=0}^{n}(\widetilde{\mathbf{\Pi}}^\top)^i$ converges and hence also $\sum_{i=0}^{n}(\mathbf{\Pi}_{\mathcal{D},\mathcal{D}}^\top)^i$. From Werner (2011) it thus follows that the limit $\sum_{i=0}^{\infty}(\mathbf{\Pi}_{\mathcal{D},\mathcal{D}}^\top)^i = \left(\mathbf{I}_{|\mathcal{D}|} - \mathbf{\Pi}_{\mathcal{D},\mathcal{D}}^\top\right)^{-1}$. Therefore all entries of the inverse of $\mathbf{I}_{|\mathcal{D}|} - \mathbf{\Pi}_{\mathcal{D},\mathcal{D}}^\top$ must be positive. Finally this implies that for all $i \in \mathcal{D}$ the allocation k from Lemma 3.2 can be rewritten as $k_i = -w_i x_i$ for the positive weighting factor $w_i := -\sum_{j=1}^{|\mathcal{D}|}\left(\left(\mathbf{I}_{|\mathcal{D}|} - \mathbf{\Pi}_{\mathcal{D},\mathcal{D}}^\top\right)^{-1}\right)_{j,i}$. Therefore, we have for all $i \in \mathcal{D}\backslash\mathcal{D}_0$ that $k_i \leq 0$, since $x_i > 0$. □

In summary the allocation k from Lemma 3.2 seems to be reasonable for the initially defaulting banks, since it is a combination of the severity of the loss x_i and of how much the loss propagates further into the system which is specified by the weighting factor w_i. Moreover, those banks which do not default at all get an allocation of zero which could also be declared as fair. However, those banks which have enough equity at the beginning $x_i > 0$ but which default due to contagion, get an allocation which is strictly negative. Compendiously, this allocation creates an incentive to control the standalone risk factor x_i, but to ignore (or even to increase) the systemic risk which originates from the network effects w_i.

The major problem with this allocation is that it is based on the subsystem construction scheme Λ^0. Whilst in the portfolio framework the entities outside of a subsystem had no influence on the risk evaluation of the subsystem, the subsystem construction scheme Λ^0 just sets the equity of the neighboring entities to zero. However, this does not imply that they have no impact on the subsystem anymore, since the network linkages have not changed at all. Even worse the banks outside of the subsystem are assumed to be already in default which means that they transmit all the losses. Thus this construction scheme corresponds in some sense to a worst-case view on how much a subsystem is able to spread its losses within the whole system. This interpretation is also in line with the definition of the core, i.e. that the construction scheme is always an upper bound of the subsystems allocation.

Another problem with the core allocations in this interaction model is that each entity do not only act as a spreader of risk as in the portfolio approach, but can also function as a transmitter of the losses of some other entities. This perspective is exactly the crucial part for the fuzzy core element from above. Namely the banks which are in $\mathcal{D}\backslash\mathcal{D}_0$, do not contribute losses to the system and thus any core allocation must be bounded by zero. Nonetheless in the complete system they face losses from other institutions and transmit them further into the system. However, they can not be charged for this loss transmission as their share is already capped by zero.

Contrarily, we now want to find an appropriate subsystem construction scheme such that the causality of the risk of a subsystem can solely be explained by the subsystem itself. Moreover, we also want that the feedback effects within the subsystem still remain intact. The most intuitive choice for such a subsystem construction scheme is equipping all banks outside of the subsystem with a very high amount of capital such that these banks can never face a default, i.e.

$$\Lambda^b(x,\lambda) = \Lambda(\lambda * x + (\mathbf{1}_d - \lambda) * b), \tag{3.4}$$

with $b \in \mathbb{R}^d_+$ sufficiently large.

Note that, whilst in the financial network without feedback mechanisms we had that joining two subgroups of banks always resulted in a risk reduction compared to the sum of the single risks, it might now happen that two single subsystems are not able to trigger a default of a bank but they can in a combined subsystem. That is the diversification benefit can turn into a cost. Moreover, in contrast to the prior subsystem construction scheme Λ^0, we now have a best-case view as we suppose that the external system is capable of covering all losses.

4 The reverse core

Because of the change of the perspective towards a best-case view, we also need to change the definition of a fair allocation in a way that the allocation of a subsystem should at least cover the risk of the subsystem. Moreover, for this new subsystem construction, we have that the risk of a single financial institution is just a measure of the adverse market event, since we assume that no loss can spread to the other institutions. This is in line with the current market practice of measuring the risk on a standalone basis. Therefore we should demand that the allocation of the systemic risk to this bank does not fall below this threshold in order to cover it own losses. For this reason we introduce the notion of the reverse (fuzzy) core.

Definition 4.1 (Reverse (fuzzy) core)**.** *Let $X \in \mathcal{X}^d$ and $\widetilde{\rho} : \mathcal{X}^d \times \mathcal{P} \to \mathbb{R}$ a subsystem risk measure. We say that $k \in \mathbb{R}^d$ is in the reverse core $C^+_{\widetilde{\rho}}(X)$ if $\sum_{i=1}^d k_j = \rho(X,\mathcal{I})$ and for all $J \in \mathcal{P}$*

$$\widetilde{\rho}(X,J) \le \sum_{j\in J} k_j.$$

Similarly, we say that k is in the reverse fuzzy core $FC^+_\rho(X)$ for a subsystem risk measure $\rho : \mathcal{X}^d \times \mathbb{R}^d_+ \to \mathbb{R}$ if $\sum_{i=1}^d k_i = \rho(X,\mathbf{1}_d)$ and for all $\lambda \in [0,1]^d$ we have that

$$\rho(X,\lambda) \le \lambda^\top k.$$

Next we investigate the relationship between the core and the reverse core which are generated by the same subsystem risk measure. Note that a similar result also holds for the fuzzy cores.

Lemma 4.2. *Let $\widetilde{\rho} : \mathcal{X}^d \times \mathcal{P} \to \mathbb{R}$ be a subsystem risk measure. The core and the reverse core are related in one of the following ways*

- $C^-_{\widetilde{\rho}}(X) = C^+_{\widetilde{\rho}}(X) = \emptyset$;

- *$C^-_{\widetilde{\rho}}(X) = C^+_{\widetilde{\rho}}(X) = \{k\}$ with $\widetilde{\rho}(X,J) = \sum_{j\in J} k_j$ for all $J \in \mathcal{P}$;*
- *One core contains only allocations where the inequality is strict for at least one $J \in \mathcal{P}$ and the other core is empty.*

In particular, if $C^{\pm}_{\widetilde{\rho}}(X) \neq \emptyset$, then $|C^{\mp}_{\widetilde{\rho}}(X)| \leq 1$.

Proof. Suppose $C^-_{\widetilde{\rho}}(X)$ as well as $C^+_{\widetilde{\rho}}(X)$ are non-empty. Let $k^- \in C^-_{\widetilde{\rho}}(X)$ and $k^+ \in C^+_{\widetilde{\rho}}(X)$ from which it follows that

$$\sum_{i=1}^d k_i^- = \sum_{i=1}^d k_i^+ \quad \text{and} \quad \sum_{j\in J} k_j^- \leq \sum_{j\in J} k_j^+ \text{ for all } J \in \mathcal{P}.$$

We define further $K := \{i \in \mathcal{I} : k_i^- < k_i^+\}$ and $G := \{i \in \mathcal{I} : k_i^- > k_i^+\}$. We assume that $G \cup K \neq \emptyset$. If $K \neq \emptyset$, then $\sum_{i=1}^d k_i^- = \sum_{i=1}^d k_i^+$ implies that

$$\sum_{i\in G} k_i^+ - k_i^- = \sum_{i\in K} k_i^- - k_i^+ < 0.$$

Thus also $G \neq \emptyset$, which contradicts $\sum_{j\in J} k_j^- \leq \sum_{j\in J} k_j^+$ for all $J \in \mathcal{P}$. Hence $k^- = k^+$ and we can deduce that both cores are equal and single valued.
Finally, if one core is empty, we clearly have that

$$\left\{ k \in \mathbb{R}^d : \widetilde{\rho}(X,J) = \sum_{j\in J} k_j, \forall J \in \mathcal{P} \right\} = \emptyset$$

and thus the other core has to be empty as well or the inequality has to be strict for at least one $J \in \mathcal{P}$ for each allocation. □

In the prior lemma we studied the connection of the two core concepts for the same subsystem risk measure. In contrast to this, we will see in the next lemma that we can also translate one core concept to the other by changing the underlying subsystem risk measure.

Lemma 4.3. *Let $\widetilde{\rho} : \mathcal{X}^d \times \mathcal{P} \to \mathbb{R}$ be a subsystem risk measure with $\widetilde{\rho}(X,\emptyset) = 0$ for all $X \in \mathcal{X}^d$. Then*

$$C^+_{\widetilde{\rho}}(X) \subseteq \left\{ k \in \mathbb{R}^d : \sum_{j\in J} k_j \leq \rho(X) - \widetilde{\rho}\left(X, J^C\right) \ \forall J \in \mathcal{P} \right\},$$

where $J^C := \mathcal{I} \backslash J$ is again the complement w.r.t. the complete system. The interpretation is that each element of the reverse core must "undercut" the "with and without risk". In particular by defining the subsystem risk measure $\bar{\rho}$ via

$$\bar{\rho}(X,J) := \rho(X) - \widetilde{\rho}(X,J^C)$$

we obtain from the result above and from $\rho(X) - \bar{\rho}(X,J^C) = \widetilde{\rho}(X,J)$ that

$$C^+_{\widetilde{\rho}}(X) = C^-_{\bar{\rho}}(X).$$

Proof. Let $k \in C^{+}_{\widetilde{\rho}}(X)$. Then for each $J \in \mathcal{P}$ it holds that

$$\widetilde{\rho}(X, J^C) - \sum_{j \in J^C} k_j \leq 0 = \rho(X, \mathcal{I}) - \sum_{i=1}^{d} k_i,$$

which is equivalent to

$$\sum_{j \in J} k_j \leq \rho(X) - \widetilde{\rho}\left(X, J^C\right).$$

□

We remark that a similar result also holds for the fuzzy and reverse fuzzy cores.

5 The reverse core in the financial model with contagion

Equipped with this new core concept, we come back to our interaction model from section 3. Recall that we are interested in a risk factor $x = c + y$, where $c \in \mathbb{R}^d_+$ is the vector of some initial capital endowments of the financial institutions and $-y \in \mathbb{R}^d_+$ is a negative shock. For the subsystem construction scheme Λ^b defined in (3.4) we vaguely demanded that $b \in \mathbb{R}^d_+$ should be sufficiently large. In Lemma 5.2 we will show that it is already sufficient to consider Λ^c in order that the reverse fuzzy core $FC^{+}_{\Lambda^c}(x)$ is non-empty. Moreover, similar to Lemma 2.5 we derive that $FC^{+}_{\Lambda^c}(x) \subseteq FC^{+}_{\Lambda^b}(x)$ for all $b \geq c$.
Note that the subsystem construction scheme Λ^b is independent of the specific decomposition of the risk factor x into a positive capital amount c and a shock y. Therefore, if we allow also for positive shocks, i.e. $x = c + y$ with $y \in \mathbb{R}^d$, then we can choose the decomposition $x = \widetilde{c} + \widetilde{y}$ with $\widetilde{c} = c + \max\{y, \mathbf{0}_d\}$ and $\widetilde{y} = \min\{y, \mathbf{0}_d\}$. Since $\widetilde{c}$ and $-\widetilde{y}$ are again positive, we have that $FC^{+}_{\Lambda^{\widetilde{c}}}(x) \subseteq FC^{+}_{\Lambda^b}(x)$. Thus, if we are interested in the non-emptiness of the reverse core of Λ^b, assuming a negative shock is essentially not a restriction.
Before we identify an element of the reverse fuzzy core of Λ^c, we need the following preparatory lemma:

Lemma 5.1. *Let* $\mathbf{A} = (A_{i,j})_{i,j=1,\dots,d} \in \mathbb{R}^{d\times d}_+$ *and* $b \in \mathbb{R}^d_+$. *Then there exists a* $\mathbf{B} = (B_{i,j})_{i,j=1,\dots,d} \in \mathbb{R}^{d\times d}_+$ *such that*

$$\left(\sum_{j=1}^{d} A_{\cdot,j} - b\right)^{+} = \sum_{j=1}^{d} \left(A_{\cdot,j} - B_{\cdot,j}\right)^{+}, \tag{5.1}$$

where $\sum_{j=1}^{d} B_{i,j} = b_i$ *and* $B_{i,i}$ *is either equal to* $A_{i,i}$ *or* b_i *for all* $i = 1, \dots, d$.

Proof. Let $i \in \mathcal{I}$ be fixed. We denote by $\pi : \mathcal{I} \to \mathcal{I}$ the permutation which exchanges the first and the i-th entry, i.e. $\pi(1) = i$, $\pi(i) = 1$ and $\pi(j) = j$ for all $j \notin \{1, i\}$. We distinguish the following two cases:

- If $\sum_{j=1}^{d} A_{i,j} \leq b_i$, then set $B_{i,\pi(j)} := A_{i,\pi(j)}$ for all $j = 1, ..., d-1$ and $B_{i,\pi(d)} := b_i - \sum_{j=1}^{d-1} A_{i,\pi(j)}$.
- If $\sum_{j=1}^{d} A_{i,j} > b_i$, then define for $j = 1, ..., d$

$$
\begin{aligned}
B_{i,\pi(j)} := {} & A_{i,\pi(j)} \mathbb{1}_{\{\sum_{k=1}^{j} A_{i,\pi(k)} \leq b_i\}} \\
& + \left(b_i - \sum_{k=1}^{j-1} A_{i,\pi(k)} \right) \mathbb{1}_{\{\sum_{k=1}^{j-1} A_{i,\pi(k)} \leq b_i, \sum_{k=1}^{j} A_{i,\pi(k)} > b_i\}}.
\end{aligned}
$$

Obviously, (5.1) is fulfilled and we have for this choice of $B_{i,\cdot}$ that $\sum_{j=1}^{d} B_{i,j} = b_i$. Depending on the size of $A_{i,i}$, we have either $B_{i,i} = A_{i,i}$ or $B_{i,i} = b_i$. □

Lemma 5.2. *Let $x = y + c$ with $y \leq \mathbf{0}_d$ and $c \in \mathbb{R}_+^d$. Then the reverse fuzzy core $FC_{\Lambda^c}^{+}(x)$ with the subsystem construction scheme*

$$
\Lambda^c(x, \lambda) := \Lambda \left(\lambda * x + (\mathbf{1}_d - \lambda) * c \right) = \Lambda(\lambda * y + c),
$$

is non-empty. Moreover, there exists an allocation $k \in FC_{\Lambda^c}^{+}(x)$ such that for all $i = 1, ..., d$

$$
k_i \leq \Lambda(x_i e_i),
$$

that is k also fulfills the property of the core for the single financial institutions and the subsystem construction scheme $\Lambda^0(x, \lambda) = \Lambda(\lambda * x)$.

Proof. As in Lemma 3.2, we will use the fact that the optimal value for a in the optimization problem (3.1) of the aggregation function Λ can be found by iterating $a(n) = \left(\mathbf{\Pi}^\top a(n-1) - x \right)^+, n \in \mathbb{N}$ with $a(0) := \mathbf{0}_d$.
First, we iteratively define a non-negative partition $A_i(n) \in \mathbb{R}_+^d, i = 1, ..., d$ of $a(n)$, that is $\sum_{i=1}^{d} A_i(n) = a(n)$. Clearly, $A_i(0) := \mathbf{0}_d, i = 1, ..., d$ is a partition of $a(0) = \mathbf{0}_d$. Note that, if we have found a partition of $a(n-1)$ for some $n \in \mathbb{N}$, then

$$
\begin{aligned}
a(n) &= \left(\mathbf{\Pi}^\top a(n-1) - x \right)^+ \\
&= \left(\sum_{i=1}^{d} \mathbf{\Pi}^\top A_i(n-1) - y - c \right)^+ \\
&= \left(\sum_{i=1}^{d} \left(\mathbf{\Pi}^\top A_i(n-1) - y_i e_i \right) - c \right)^+.
\end{aligned}
$$

Since $\mathbf{\Pi}^\top A_i(n-1) - y_i e_i \geq \mathbf{0}_d$ for all $i = 1, ..., d$ and $c \geq \mathbf{0}_d$, we can apply Lemma 5.1 in order to obtain the existence of $C_i(n) \in \mathbb{R}^d_+, i = 1, ..., d$ with $\sum_{i=1}^d C_i(n) = c$ and

$$a(n) = \sum_{i=1}^{d} \left(\mathbf{\Pi}^\top A_i(n-1) - y_i e_i - C_i(n)\right)^+ .$$

Hence, $A_i(n) := \left(\mathbf{\Pi}^\top A_i(n-1) - y_i e_i - C_i(n)\right)^+ \in \mathbb{R}^d_+, i = 1, ..., d$ is a non-negative partition of $a(n)$.

Since $(a(n))_{n\in\mathbb{N}}$ is an increasing sequence, we have that $A_i(n)$ is also bounded from above by $a = \lim_{n\to\infty} a(n)$ for all $i = 1, ..., d$. Recall that a is finite by Lemma 3.1. Thus $((A_i(n))_{i=1,...,d})_{n\in\mathbb{N}}$ is a bounded sequence and we obtain by applying the Bolzano-Weierstrass theorem, that there exists a converging subsequence $\left((A_i(n_k))_{i=1,...,d}\right)_{k\in\mathbb{N}}$. We denote the limit of this subsequence by $(A_i)_{i=1,...,d}$. Thus, we have that

$$a = \lim_{k\to\infty} a(n_k) = \lim_{k\to\infty} \sum_{i=1}^{d} A_i(n_k) = \sum_{i=1}^{d} A_i$$

and that $A_i \geq \mathbf{0}_d$ for all $i = 1, ..., d$.

Next we define for a level of participation $\lambda \in [0, 1]^d$ the sequence

$$a(\lambda, n) := \left(\mathbf{\Pi}^\top a(\lambda, n-1) - \lambda * y - c\right)^+ \text{ for all } n \in \mathbb{N}$$

and $a(\lambda, 0) = \mathbf{0}_d$ which corresponds to the fixpoint iteration for $\Lambda^c(x, \lambda)$. In the following we prove that

$$\sum_{i=1}^{d} \lambda_i A_i(n) \geq a(\lambda, n), \qquad \text{for all } n \in \mathbb{N}. \tag{5.2}$$

Obviously (5.2) holds true for $n = 0$. Suppose now that (5.2) is valid for $n-1$, with $n \in \mathbb{N}$. Then, we have that

$$\begin{aligned}
\sum_{i=1}^{d} \lambda_i A_i(n) &= \sum_{i=1}^{d} \lambda_i \left(\mathbf{\Pi}^\top A_i(n-1) - y_i e_i - C_i(n)\right)^+ \\
&\geq \left(\mathbf{\Pi}^\top \sum_{i=1}^{d} \lambda_i A_i(n-1) - \sum_{i=1}^{d} \lambda_i y_i e_i - \sum_{i=1}^{d} \lambda_i C_i(n)\right)^+ \\
&\geq \left(\mathbf{\Pi}^\top \sum_{i=1}^{d} \lambda_i A_i(n-1) - \lambda * y - c\right)^+ \\
&\geq \left(\mathbf{\Pi}^\top a(\lambda, n-1) - \lambda * y - c\right)^+ \\
&= a(\lambda, n),
\end{aligned}$$

where we used the induction hypothesis in the penultimate step and that $\sum_{i=1}^{d} \lambda_i C_i(n) \leq c$ in the third. By taking the limit we obtain

$$\sum_{i=1}^{d} \lambda_i A_i = \lim_{k\to\infty} \sum_{i=1}^{d} \lambda_i A_i(n_k) \geq \lim_{k\to\infty} a(\lambda, n_k) =: a(\lambda).$$

Finally, by defining $k = (k_1, ..., k_d)$ with $k_i := \mathbf{1}_d^\top A_i, i = 1, ..., d$, we have for all $\lambda \in [0,1]^d$ that

$$\lambda^\top k = \mathbf{1}_d^\top \sum_{i=1}^{d} \lambda_i A_i \geq \mathbf{1}_d^\top a(\lambda) = \Lambda^c(x, \lambda)$$

and

$$\mathbf{1}_d^\top k = \mathbf{1}_d^\top \sum_{i=1}^{d} A_i = \mathbf{1}_d^\top a = \Lambda^c(x, \mathbf{1}_d) = \Lambda(x).$$

Thus $k = (k_1, ..., k_d) \in FC^+_{\Lambda^c}(x)$.

In the end we still have to show that $k_i \leq \Lambda^0(x, e_i)$ for all $i = 1, ..., d$. For this purpose denote by

$$a(i, n) := \left(\mathbf{\Pi}^\top a(i, n-1) - x_i e_i\right)^+, \quad a(i, 0) = \mathbf{0}_d,$$

the fixpoint iteration for $\Lambda^0(x, e_i)$. Again we will use induction to show that

$$A_i(n) \leq a(i, n), \quad \text{for all } n \in \mathbb{N}. \tag{5.3}$$

Then, obviously $A_i(0) \leq a(i, 0)$. Thus suppose that (5.3) holds for $n-1$. Before we proceed with the induction step recall that by construction of $\big(C_i(n)\big)_i = (\mathbf{\Pi}^\top A_i(n-1) - y_i e_i)_i$ or $\big(C_i(n)\big)_i = c_i$ and in both cases we obtain

$$\left(\left(\mathbf{\Pi}^\top A_i(n-1)\right)_i - y_i - \left(C_i(n)\right)_i\right)^+ \leq \left(\left(\mathbf{\Pi}^\top A_i(n-1)\right)_i - y_i - c_i\right)^+.$$

Moreover, since $C_i(n) \geq \mathbf{0}_d$ for all $i = 1, ..., d$ we have that

$$\begin{aligned} A_i(n) &= \left(\mathbf{\Pi}^\top A_i(n-1) - y_i e_i - C_i(n)\right)^+ \\ &\leq \left(\mathbf{\Pi}^\top A_i(n-1) - y_i e_i - c_i e_i\right)^+ \\ &\leq \left(\mathbf{\Pi}^\top a(i, n-1) - y_i e_i - c_i e_i\right)^+ = a(i, n). \end{aligned}$$

Hence (5.3) holds and we can conclude that

$$k_i = \mathbf{1}_d^\top A_i \leq \mathbf{1}_d^\top a(i) = \Lambda^0(x, e_i),$$

where $a(i) := \lim_{k\to\infty} a(i, n_k)$.

□

We have seen in Lemma 5.2 that the reverse fuzzy core for the subsystem construction scheme Λ^c is non-empty and that there exists an element in the reverse fuzzy core which additionally fulfills the essential property (2.1) of the core of Λ^0 at least for the single institutions. As we have seen that the usual core might not be a useful allocation in a financial model with contagion, we want to investigate if there is also an element in the intersection of the two cores $C^+_{\Lambda^c}$ and $C^-_{\Lambda^0}$. In Lemma 5.4 it will be shown that under a rather weak assumption on the risk factor x the intersection of the cores is empty. In order to put this assumption into context, we precede the following lemma.

Lemma 5.3. *We have for all $x \in \mathbb{R}^d$ that*

$$\sum_{i \in \mathcal{D}_0} \Lambda \left(\sum_{j \in \mathcal{D}_0^C \cup \{i\}} e_j x_j \right) \leq \Lambda(x), \tag{5.4}$$

where $\mathcal{D}_0 := \{i \in \mathcal{I} : x_i \leq 0\}$ denotes the set of institutes which default initially. Moreover, if (5.4) is strict, then there is at least one institution which defaults due to contagion, i.e. $\mathcal{D} \backslash \mathcal{D}_0 \neq \emptyset$.

Proof. Similar to Lemma 5.2 we consider the sequences

$$a(n) := \left(\mathbf{\Pi}^\top a(n-1) - x \right)^+, \quad a(0) = \mathbf{0}_d,$$

and for all $i \in \mathcal{D}_0$

$$a(i,n) := \left(\mathbf{\Pi}^\top a(i,n-1) - \sum_{j \in \mathcal{D}_0^C \cup \{i\}} e_j x_j \right)^+, \quad a(i,0) = \mathbf{0}_d.$$

By construction $a(0) = \sum_{i \in \mathcal{D}_0} a(i,0)$. Thus suppose that

$$a(n-1) \geq \sum_{i \in \mathcal{D}_0} a(i,n-1)$$

for some $n \in \mathbb{N}$, then

$$a(n) = \left(\mathbf{\Pi}^\top a(n-1) - x \right)^+ \geq \left(\mathbf{\Pi}^\top \sum_{i \in \mathcal{D}_0} a(i,n-1) - x \right)^+. \tag{5.5}$$

Now we look at the single entries of the vector on the right hand side of (5.5).

For $l \in \mathcal{D}_0$ we have

$$\begin{aligned}
\left(\mathbf{\Pi}^\top \sum_{i\in\mathcal{D}_0} a(i,n-1) - x\right)_l^+ &= \left(\sum_{i\in\mathcal{D}_0} \left(\mathbf{\Pi}^\top a(i,n-1)\right)_l - x_l\right)^+ \\
&= \sum_{i\in\mathcal{D}_0} \left(\mathbf{\Pi}^\top a(i,n-1)\right)_l - x_l \\
&= \left(\left(\mathbf{\Pi}^\top a(l,n-1)\right)_l - x_l\right)^+ + \sum_{i\in\mathcal{D}_0\setminus\{l\}} \left(\left(\mathbf{\Pi}^\top a(i,n-1)\right)_l\right)^+ \\
&= \sum_{i\in\mathcal{D}_0} \left(\mathbf{\Pi}^\top a(i,n-1) - \sum_{j\in\mathcal{D}_0^C\cup\{i\}} e_j x_j\right)_l^+
\end{aligned}$$

and for $l \in \mathcal{D}_0^C$

$$\begin{aligned}
\left(\mathbf{\Pi}^\top \sum_{i\in\mathcal{D}_0} a(i,n-1) - x\right)_l^+ &= \left(\sum_{i\in\mathcal{D}_0} \left(\mathbf{\Pi}^\top a(i,n-1)\right)_l - x_l\right)^+ \\
&= \sum_{i\in\mathcal{D}_0} \left(\left(\mathbf{\Pi}^\top a(i,n-1)\right)_l - X_{l,i}\right)^+ \\
&\geq \sum_{i\in\mathcal{D}_0} \left(\left(\mathbf{\Pi}^\top a(i,n-1)\right)_l - x_l\right)^+ \\
&= \sum_{i\in\mathcal{D}_0} \left(\mathbf{\Pi}^\top a(i,n-1) - x\right)_l^+ \\
&= \sum_{i\in\mathcal{D}_0} \left(\mathbf{\Pi}^\top a(i,n-1) - \sum_{j\in\mathcal{D}_0^C\cup\{i\}} e_j x_j - \sum_{j\in\mathcal{D}_0\setminus\{i\}} e_j x_j\right)_l^+ \\
&\geq \sum_{i\in\mathcal{D}_0} \left(\mathbf{\Pi}^\top a(i,n-1) - \sum_{j\in\mathcal{D}_0^C\cup\{i\}} e_j x_j\right)_l^+
\end{aligned}$$

where $(X_{l,i})_{l\in\mathcal{D}_0^C, i\in\mathcal{D}_0}$ is specified by Lemma 5.1. Note that, since $X_{l,i} \geq 0$ and $\sum_{i\in\mathcal{D}_0} X_{l,i} = x_l$ we have that $X_{l,i} \leq x_l$ for all $l \in \mathcal{D}_0^C$ and $i \in \mathcal{D}_0$ which we used in the third step. Now we can continue with (5.5)

$$a(n) \geq \sum_{i\in\mathcal{D}_0} \left(\mathbf{\Pi}^\top a(i,n-1) - \sum_{j\in\mathcal{D}_0^C\cup\{i\}} e_j x_j\right)^+ = \sum_{i\in\mathcal{D}_0} a(i,n)$$

and thus we have shown that $a(n) \geq \sum_{i\in\mathcal{D}_0} a(i,n)$ for all $n \in \mathbb{N}$. Finally by considering the limit for $n \to \infty$ we obtain (5.4).

Next we show the second claim. For this we will prove that $\mathcal{D}\setminus\mathcal{D}_0 = \emptyset$ implies that (5.4) holds with equality. Obviously this is true if $\mathcal{D}_0 = \emptyset$. Thus we suppose that $\mathcal{D}_0 \neq \emptyset$, i.e. at least one bank defaults initially. It can be readily seen that

$\mathcal{D}_0 \subseteq \mathcal{D}$ and thus, since $\mathcal{D}\backslash\mathcal{D}_0 = \emptyset$, we have that $\mathcal{D}_0 = \mathcal{D}$. For each $i \in \mathcal{D}$ we will also need the sets of banks which default initially and after all possible contagion took place for the subsystem with corresponding risk factor $\sum_{j\in\mathcal{D}_0^C\cup\{i\}} e_j x_j$. We denote these sets by $\mathcal{D}_0(i)$ and $\mathcal{D}(i)$ respectively. Since for all $i \in \mathcal{D}$ we have that $\sum_{j\in\mathcal{D}_0^C\cup\{i\}} e_j x_j \geq x$, it follows directly that

$$\mathcal{D}(i) \subseteq \mathcal{D}.$$

Contrarily, due to the fact that $\left(\sum_{j\in\mathcal{D}_0^C\cup\{i\}} e_j x_j\right)_l \leq 0$ for all $l \in \mathcal{D}_0$, we also have that

$$\mathcal{D}(i) \supseteq \mathcal{D}_0(i) = \mathcal{D}_0 = \mathcal{D}$$

and thus $\mathcal{D}(i) = \mathcal{D}$ for all $i \in \mathcal{D}$.

As in Lemma 3.2 let $\mathcal{D} = \{p_1, ..., p_{|\mathcal{D}|}\}$ and denote by $\mathbf{\Pi}_{\mathcal{D},\mathcal{D}} := (\Pi_{i,j})_{i,j\in\mathcal{D}} \in \mathbb{R}^{|\mathcal{D}|\times|\mathcal{D}|}$ the matrix $\mathbf{\Pi}$ where the rows and columns which are not in $\mathcal{D}$ have been erased. Similar to Lemma 3.2 we get that

$$\begin{aligned}
\Lambda(x) &= -\mathbf{1}_{|\mathcal{D}|}^\top \left(\mathbf{I}_{|\mathcal{D}|} - \mathbf{\Pi}_{\mathcal{D},\mathcal{D}}^\top\right)^{-1} \sum_{i=1}^{|\mathcal{D}|} \widetilde{e}_i x_{p_i} \\
&= \sum_{i=1}^{|\mathcal{D}|} -\mathbf{1}_{|\mathcal{D}|}^\top \left(\mathbf{I}_{|\mathcal{D}|} - \mathbf{\Pi}_{\mathcal{D},\mathcal{D}}^\top\right)^{-1} \widetilde{e}_i x_{p_i} \\
&= \sum_{i\in\mathcal{D}_0} \Lambda\left(\sum_{j\in\mathcal{D}_0^C\cup\{i\}} e_j x_j\right),
\end{aligned}$$

where $\widetilde{e}_i \in \mathbb{R}^{|\mathcal{D}|}$ is the i-th $|\mathcal{D}|$-dimensional unit vector. □

It is obvious that the reverse implication of Lemma 5.3 does not hold. As a counterexample take for instance a financial network comprising two banks, where the first bank defaults initially and the second fails as a consequence of this default. Then we have a contagious default, but (5.4) holds with equality. Therefore, (5.4) is strict if there is a contagious default and this default must be triggered by more than one defaulted bank. Thus (5.4) being strict can be interpreted as a scenario of a high level of interactions in the network.

Lemma 5.4. *If* (5.4) *is strict for some $x \in \mathbb{R}^d$, i.e.*

$$\sum_{i\in\mathcal{D}_0} \Lambda\left(\sum_{j\in\mathcal{D}_0^C\cup\{i\}} e_j x_j\right) < \Lambda(x),$$

then

$$C^+_{\widetilde{\Lambda}^c}(x) \cap C^-_{\widetilde{\Lambda}^0}(x) = \emptyset,$$

where $\widetilde{\Lambda}^b(x, J) := \Lambda\left(\sum_{j\in J} x_j e_j + \sum_{j\in J^C} b_j e_j\right)$ for all $J \in \mathcal{P}$ and $b \in \mathbb{R}^d$.

Proof. Assume there exists an $k \in C^{+}_{\widetilde{\Lambda}^c}(x) \cap C^{-}_{\widetilde{\Lambda}^0}(x)$. We have for all $i \in \mathcal{D}_0^C$ that

$$0 \le \Lambda\left(e_i x_i + \sum_{j \ne i} e_j c_j\right) \le \Lambda(e_i x_i) = 0,$$

and thus the respective core properties imply that $k_i = 0$. Hence

$$\sum_{i=1}^{d} k_i = \sum_{i \in \mathcal{D}_0} k_i = \sum_{i \in \mathcal{D}_0} \sum_{j \in \mathcal{D}_0^C \cup \{i\}} k_i \le \sum_{i \in \mathcal{D}_0} \Lambda\left(\sum_{j \in \mathcal{D}_0^C \cup \{i\}} e_j x_j\right) < \Lambda(x) = \sum_{i=1}^{d} k_i,$$

which is a contradiction. $\square$

We finish this section with a small but concrete calculation of the core and the reverse core in order to exemplify their differences. We consider a financial network with the following specifications:

$$\mathbf{\Pi} = \begin{pmatrix} 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/4 & 0 & 1/2 & 1/4 \\ 0 & 1/3 & 0 & 0 & 2/3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } x = c + y = \begin{pmatrix} 0 - 5 \\ 5 - 3 \\ 10 - 12 \\ 5 - 3 \\ 0 - 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ -2 \\ 2 \\ 0 \end{pmatrix}.$$

The corresponding network is depicted in Figure 5.1 and the values of the subsystem construction schemes can be found in Table 5.1. Note that, since the inclusion of the real economy do not change the risk of a subsystem, we omitted the results in Table 5.1. Clearly, the initially defaulting banks are $\mathcal{D}_0 = \{1, 3\}$. Moreover, we observe that the initial default of bank 1 triggers a contagious default of bank 2 and that even after all possible defaults bank 4 is still solvent. Applying Lemma 3.2 yields that

$$(13.57, -4.86, 3.71, 0, 0)^\top = FC^{-}_{\widetilde{\Lambda}^0}(x)$$

and by using the partition from the proof of Lemma 5.2 we obtain that

$$(8.71, 0, 3.71, 0, 0)^\top \in FC^{+}_{\widetilde{\Lambda}^c}(x).$$

As bank 4 is not participating in the contagion process, it gets in both allocations a share of zero which can be considered as fair. However, here we see clearly that bank 2 is a transmitter of losses in the system and by the allocation of the fuzzy core it is rewarded for this position with a negative share compared to the solvent bank 4. Contrarily, bank 2 also gets a share of zero for the allocation which is in the reverse fuzzy core. Since, bank 2 does not default initially this allocation can barely be considered as fair. However, since bank 2 is also in a channel of contagion later on, it would also be fair that bank 2 gets a strictly positive share.

Using the Table 5.1, it can be readily seen that this holds for all other allocations in the reverse core, which is given by

$$C^{+}_{\widetilde{\Lambda}^c}(x) = \Big\{(k_1, k_2, k_3, 0, 0) \in \mathbb{R}^5_+ : \sum_{i=1}^{3} k_i = 12.43,$$
$$k_1 \geq 7.5, k_3 \geq 2.5, k_1 + k_2 \geq 8.25\Big\}.$$

The largest share of the systemic risk for bank 2 in the reverse core is attained for the allocation

$$(7.5, 2.43, 2.5, 0, 0)^\top.$$

Furthermore, since $\mathcal{D}_0 = \{1, 3\}$ and

$$\begin{aligned}\Lambda(x) &= 12.43 > 11.21 = 8.71 + 2.5 \\ &= \widetilde{\Lambda}^0(x, \{1, 2, 4, 5\}) + \widetilde{\Lambda}^0(x, \{2, 3, 4, 5\}) \\ &= \Lambda\left(\sum_{j \in \mathcal{D}_0^C \cup \{1\}} x_j e_j\right) + \Lambda\left(\sum_{j \in \mathcal{D}_0^C \cup \{3\}} x_j e_j\right),\end{aligned}$$

(5.4) is strict and thus the reverse core of $\widetilde{\Lambda}^c$ and the core of $\widetilde{\Lambda}^0$ do not have a common element.

Finally, we also observe that not only the fuzzy core, but also all core elements does not respect a fair ordering in the sense that $k_u \geq k_v \geq k_w$ for all $u \in \mathcal{D}_0, v \in \mathcal{D} \backslash \mathcal{D}_0$ and $w \in \mathcal{D}^C$. Recall that bank 2 defaults due to contagion, but not initially, and thus a core allocation k must fulfill that $k_2 \leq 0$. Since, this bank participates in the contagion process later on we want that its allocation should be non-negative. Now we assume that there exists an allocation $k \in C^{-}_{\widetilde{\Lambda}^0}(x)$ which respects our notion of a fair ordering, i.e. $k = (k_1, 0, k_3, k_4, 0)$ such that $k_1, k_3 \geq 0$ and $k_4 \leq 0$. Then,

$$\begin{aligned}k_4 &= (k_1 + k_4) + (k_3 + k_4) - \sum_{i=1}^{5} k_i \\ &\leq \widetilde{\Lambda}^0(x, \{1, 2, 4, 5\}) + \widetilde{\Lambda}^0(x, \{2, 3, 4, 5\}) - \Lambda(x) = -1.22.\end{aligned}$$

Moreover, we have that

$$12.43 = \Lambda(x) = \sum_{i=1}^{5} k_i \leq \widetilde{\Lambda}^0(x, \{1, 2\}) + \widetilde{\Lambda}^0(x, \{2, 3\}) + k_4 = 13.28 + k_4,$$

which immediately yields the contradiction that $k_4 \geq -0.85$. Hence there does not exist a core element which respects the fair ordering from above.

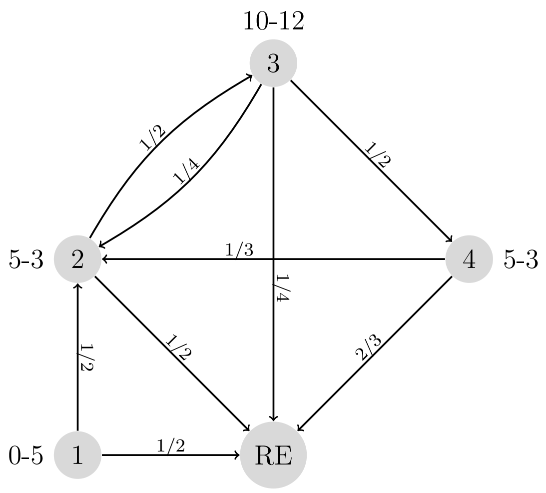

Figure 5.1: Exemplary system.

J	$\widetilde{\Lambda}^c(x,J)$	$\widetilde{\Lambda}^0(x,J)$
$\{1,2,3,4\}$	12.43	12.43
$\{1,2,3\}$	12.43	15.95
$\{1,2,4\}$	8.25	8.71
$\{1,3,4\}$	10.00	17.29
$\{2,3,4\}$	2.50	2.50
$\{1,2\}$	8.25	9.11
$\{1,3\}$	10.00	22.37
$\{1,4\}$	7.50	13.57
$\{2,3\}$	2.50	4.17
$\{2,4\}$	0	0
$\{3,4\}$	2.50	3.71
$\{1\}$	7.50	15.53
$\{2\}$	0	0
$\{3\}$	2.50	6.84
$\{4\}$	0	0

Table 5.1: Risks of the subsystems.

A Random risks

This section will be devoted to a discussion on how we can derive the non-emptiness of the core also for random risks. For this purpose, we first recall the well-known Bondareva-Shapley theorem which gives an alternative characterization of the non-emptiness of the core. For this we need the notion of a balanced collection of weights.

Definition A.1. *We say $(\alpha_J)_{J\in\mathcal{P}}$ is a balanced collection of weights if $\alpha_J \geq 0$ for all $J \in \mathcal{P}$ and $\sum_{J\in\mathcal{P}_i} \alpha_J = 1$ for all $i = 1, ..., d$. Here $\mathcal{P}_i := \{J \in \mathcal{P} : i \in J\}$ denotes the set of all subgroups containing the i-th financial institution.*

Theorem A.2 (Bondareva-Shapley)**.** *The core $C^-_{\widetilde{\rho}}(X)$ of the subsystem risk measure $\widetilde{\rho}$ is not empty if and only if for all balanced collections of weights $(\alpha_J)_{J\in\mathcal{P}}$ it holds that*

$$\rho(X) \leq \sum_{J\in\mathcal{P}} \alpha_J \widetilde{\rho}(X, J).$$

For a proof see for instance Shapley (1967).

In the following we suppose that $\widetilde{\rho} : \mathcal{X}^d \times \mathcal{P} \to \mathbb{R}$ is given by

$$\widetilde{\rho}(X, J) := \eta(-\widetilde{\Lambda}(X, J)),$$

where $\widetilde{\Lambda} : \mathbb{R}^d \times \mathcal{P} \to \mathbb{R}$ is a subsystem construction scheme and η is a univariate risk measure.

Lemma A.3. *If the subsystem risk measure $\widetilde{\rho}$ is given by $\widetilde{\rho}(X, J) = \eta(-\widetilde{\Lambda}(X, J))$ for all $X \in \mathcal{X}^d$ and $J \in \mathcal{P}$, where η is a positive homogeneous and subadditive* *univariate risk measure and $\widetilde{\Lambda}$ is a subsystem construction scheme which is additive with respect to the subsystems, i.e. for all disjoint sets $J_1, J_2 \in \mathcal{P}$ and $X \in \mathcal{X}^d$*

$$\widetilde{\Lambda}(X, J_1 \cup J_2) = \widetilde{\Lambda}(X, J_1) + \widetilde{\Lambda}(X, J_2), \tag{A.1}$$

then there exists a core allocation $k \in C^-_{\widetilde{\rho}}(X)$.

Proof. In order to prove the lemma we will utilize Theorem A.2. Let $(\alpha_J)_{J\in\mathcal{P}}$ be a balanced collection of weights, then we obtain by additivity of $\widetilde{\Lambda}$ that

$$\begin{aligned}\widetilde{\Lambda}(X) &= \sum_{i=1}^{d} \widetilde{\Lambda}\big(X, \{i\}\big) = \sum_{i=1}^{d} \sum_{J\in\mathcal{P}_i} \alpha_J \widetilde{\Lambda}\big(X, \{i\}\big) \\ &= \sum_{J\in\mathcal{P}} \alpha_J \sum_{i\in J} \widetilde{\Lambda}\big(X, \{i\}\big) = \sum_{J\in\mathcal{P}} \alpha_J \widetilde{\Lambda}\big(X, J\big)\end{aligned}$$

and thus by subadditivity and positive homogeneity that

$$\eta(-\widetilde{\Lambda}(X)) = \eta\left(-\sum_{J\in\mathcal{P}} \alpha_J \widetilde{\Lambda}(X, J)\right) \leq \sum_{J\in\mathcal{P}} \alpha_J \eta\big(-\widetilde{\Lambda}(X, J)\big).$$

□

Remark A.4. Note that in order to prove Lemma A.3 it would be sufficient to show that $\widetilde{\Lambda}(X) \leq \sum_{J\in\mathcal{P}} \alpha_J \widetilde{\Lambda}\big(X, J\big)$ for all balanced collection of weights $(\alpha_J)_{J\in\mathcal{P}}$.

Example A.5. The additivity over subsystems (A.1) is clearly satisfied by the subsystem construction scheme $\widetilde{\Lambda}(x, J) = -\sum_{j\in J} x_j$ which we already know as a suitable aggregation for financial systems without contagion. Furthermore, the additivity still holds if we just consider the losses of the financial institutions in this model, i.e. if $\widetilde{\Lambda}(x, J) = \sum_{j\in J} x_j^-$.

Note that the additivity property (A.1) in Lemma A.3 directly implies that the core of the subsystem construction scheme is always non-empty. In the following lemma we show that this weaker property is already sufficient for the core of the subsystem risk measure to be non-empty.

Lemma A.6. *Let $\widetilde{\rho}(X, J) = \eta\big(-\widetilde{\Lambda}(X, J)\big)$ be a subsystem risk measure where η is a positive homogeneous and subadditive univariate risk measure and $\widetilde{\Lambda}$ is a subsystem construction scheme such that the functions $x \mapsto \widetilde{\Lambda}(x, J)$ are continuous for all $J \in \mathcal{P}$. Then we have that $C^-_{\widetilde{\rho}}(X) \neq \emptyset$ for all $X \in \mathcal{X}^d$ with $C^-_{\widetilde{\Lambda}}(X(\omega)) \neq \emptyset$ for all $\omega \in \Omega$.*

Proof. Let $X \in \mathcal{X}^d$ such that $C^-_{\widetilde{\Lambda}}(X(\omega)) \neq \emptyset$ for all $\omega \in \Omega$. It is well-known that the set-valued function $\mathcal{C}^-$ mapping all possible $\nu : \mathcal{P} \to \mathbb{R}$ to its core is upper hemicontinuous, see for instance Delbaen (1974). Since $x \mapsto \widetilde{\Lambda}(x, \cdot)$ are continuous, we get that the set-valued composition $C^-_{\widetilde{\Lambda}}(x) = \mathcal{C}^- \circ \widetilde{\Lambda}(x, \cdot)$ is also upper-hemicontinuous, i.e. for all open $A \subset \mathbb{R}^d$, we have that $\{x \in \mathbb{R}^d : C^-_{\widetilde{\Lambda}}(x) \subset A\}$ is open. Moreover this implies that $C^-_{\widetilde{\Lambda}}$ is measurable and thus according to Theorem 8.1.3 in Aubin and Frankowska (2009) there exists a Borel measurable selection of $C^-_{\widetilde{\Lambda}}$. Therefore there also exists a measurable selection $K \in \mathcal{X}^d$ of $C^-_{\widetilde{\Lambda}}(X)$, i.e. $K(\omega) \in C^-_{\widetilde{\Lambda}}(X(\omega))$ for each $\omega \in \Omega$. Now, define the subsystem risk measure

$$\overline{\rho} : \mathcal{X}^d \times \mathcal{P} \to \mathbb{R}; (K, J) \mapsto \eta\left(\sum_{j \in J} K_j\right).$$

By applying Lemma $A.3$ we obtain that $C^-_{\overline{\rho}}(K) \neq \emptyset$. The monotonicity of η yields

$$\widetilde{\rho}(X) = \eta\big(-\widetilde{\Lambda}(X)\big) = \eta\left(\sum_{j=1}^{d} K_j\right) = \overline{\rho}(K)$$

as well as for all $J \in \mathcal{P}$

$$\widetilde{\rho}(X, J) = \eta\big(-\widetilde{\Lambda}(X, J)\big) \geq \eta\left(\sum_{j \in J} K_j\right) = \overline{\rho}(K, J)$$

and it immediately follows from Lemma 2.5 that also $C^+_{\widetilde{\rho}}(X) \neq \emptyset$ $\square$

In particular Lemma $A.6$ implies that for every coherent risk measure $\eta : \mathcal{X} \to \mathbb{R}$ the core of the subsystem risk measure $\eta \circ \Lambda^0$ is always non-empty. Note that this is possible since both the coherent risk measure η and the subsystem construction scheme Λ^0 share the same perspective towards diversification, that is joining to subgroups always results in a risk reduction. Unfortunately, for subsystem construction schemes for which the reverse core is non-empty like Λ^c this is no longer true. Thus it is more problematic to obtain a similar result as in Lemma $A.6$, i.e. that the reverse core of a random risk is non-empty if the reverse cores for the corresponding scenario-wise deterministic risks are non-empty. For instance this would hold if we ask that the univariate risk measure η is superadditive instead of subadditive, i.e.

$$\eta(F + G) \geq \eta(F) + \eta(G) \quad \text{for all } F, G \in \mathcal{X}.$$

However, the requirement of superadditivity is less clear on the level of the univariate risk measure compared to the level of aggregation. Clearly, a compromise in this context would be a linear risk measure.

Moreover, if we have a scenario-wise non-emptiness of the reverse core, but we insist upon a subadditive univariate risk measure, then a possible workaround is to consider the transition to the equivalent core. That is, if we suppose that there exists a $k(\omega) \in C^+_{\Lambda}(X(\omega))$ for all $\omega \in \Omega$, then it follows by Lemma 4.3 that $k(\omega) \in C^-_{\bar{\Lambda}}(X(\omega))$ for all $\omega \in \Omega$, where

$$\bar{\Lambda}(X(\omega), J) := \Lambda(X(\omega)) - \Lambda\left(X(\omega), J^C\right).$$

Now, define
$$\bar{\rho}(X, J) := \eta\left(-\bar{\Lambda}(X, J)\right),$$
where η is positive homogeneous and subadditive univariate risk measure. By Lemma $A.6$ we obtain that $C^{-}_{\bar{\rho}}(X) \neq \emptyset$. However, we remark that this is in general not equivalent to the reverse core of $\rho(X, J) := \eta(-\Lambda(X, J))$. To be more precise we only have that $C^{+}_{\hat{\rho}}(X) \neq \emptyset$ with

$$\hat{\rho}(X, J) := \bar{\rho}(X, \mathcal{I}) - \bar{\rho}(X, J^C) = \rho(X) - \eta\big(\Lambda(X, J) - \Lambda(X)\big).$$

In the special case of a linear univariate risk measure η, we also obtain that $\hat{\rho}(X, J) = \eta(-\Lambda(X, J)) = \rho(X, J)$.

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