Supplement to “Liquidity based modeling of asset price bubbles via random matching”

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This is a supplement to the paper [1]. The supplement is organized as follows. First, we prove Theorem 3.13 in [1] which provides the existence of the dynamical system \( D \) introduced in Definition 3.6 in [1]. Second, we show some properties of \( D \) which are summarized in Theorem 3.14 in [1]. In the following, we only state the basic setting and refer to [1] for definitions.

1 Setting

Let \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) be a probability space and \((\hat{\Omega}, \hat{\mathcal{F}})\) another measurable space. We define the product space

\[
(\Omega, \mathcal{F}) := (\tilde{\Omega} \times \hat{\Omega}, \tilde{\mathcal{F}} \otimes \hat{\mathcal{F}}).
\]

(1.1)

Let \(\hat{\mathbb{P}}\) be a Markov kernel (or stochastic kernel) from \(\tilde{\Omega}\) to \(\hat{\Omega}\). Given \(\tilde{\omega} \in \tilde{\Omega}\), we set \(\hat{\mathbb{P}}(\tilde{\omega})\) with a slight notational abuse. We then introduce a probability measure \(P\) on \((\Omega, \mathcal{F})\) as the semidirect product of \(\tilde{\mathbb{P}}\) and \(\hat{\mathbb{P}}\), that is,

\[
P(\tilde{A} \times \hat{A}) := (\tilde{\mathbb{P}} \triangleright \hat{\mathbb{P}})(\tilde{A} \times \hat{A}) = \int_{\tilde{A}} \hat{\mathbb{P}}(\tilde{\omega}) \tilde{\mathbb{P}}(\tilde{\omega}) \hat{\mathbb{P}}(\hat{A}) d\tilde{\mathbb{P}}(\tilde{\omega}).
\]

(1.2)

We fix an atomless probability space \((I, \mathcal{I}, \lambda)\) representing the space of agents and let \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes \mathbb{P})\) be a rich Fubini extension of \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes \mathbb{P})\). All agents in \(I\) can be classified according to their type. In particular, we let \(S = \{1, 2, ..., K\}\) be a finite space of types and say that an agent has type \(J\) if he is not matched. We denote by \(\hat{S} := S \times (S \cup \{J\})\) the extended type space. Moreover, we call \(\hat{\Delta}\) the space of extended type distributions, which is the set of probability distributions \(p\) on \(\hat{S}\) satisfying \(p(k, l) = p(l, k)\) for any \(k\) and \(l\) in \(S\). This space is endowed with the topology \(\mathcal{T}\) induced by the topology of the space of matrices with \(|S|\) rows and \(|S| + 1\) columns. We consider \((n)_{n \geq 1}\) time periods and denote by \((\eta^n, \theta^n, \xi^n, \sigma^n, \varsigma^n)\) the matrix valued processes, with \((\eta^n_{kl}, \theta^n_{kl}, \xi^n_{kl}, \sigma^n_{kl}[r, s], \varsigma^n_{kl}[r])_{k,l,r \in S \times S \times S \times S}\) for \(n \geq 1\), on \((\Omega, \mathcal{F}, \mathbb{P})\). For a detailed introduction of these processes we refer to Section 3 in [1]. Moreover, let \(\hat{p} = (\hat{p}^n)_{n \geq 1}\) be a stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\) with values in \(\hat{\Delta}\), representing the evolution of the underlying extended type distribution. We assume that \(\hat{p}^0\) is deterministic.

Given the input processes \((\eta, \theta, \xi, \sigma, \varsigma)\) we denote by \(D\) a dynamical system on \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes \mathbb{P})\) and

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by $\Pi = (\alpha, \pi, g) = (\alpha^n, \pi^n, g^n)_{n \in \mathbb{N} \setminus \{0\}}$ the agent-type function, the random matching and the partner-type function, respectively, as introduced in Definition 3.6 in [1], which we recall in the following.

**Definition 1.1.** A dynamical system $\mathbb{D}$ defined on $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ is a triple $\Pi = (\alpha, \pi, g) = (\alpha^n, \pi^n, g^n)_{n \in \mathbb{N} \setminus \{0\}}$ such that for each integer period $n \geq 1$ we have:

1. $\alpha^n : I \times \Omega \to S$ is the $\mathcal{I} \otimes \mathcal{F}$-measurable agent type function. The corresponding end-of-period type of agent $i$ under the realization $\omega \in \Omega$ is given by $\alpha^n(i, \omega) \in S$.

2. A random matching $\pi^n : I \times \Omega \to I$, describing the end-of-period agent $\pi^n(i)$ to whom agent $i$ is currently matched, if agent $i$ is not matched, then $\pi^n(i) = i$. The associated $\mathcal{I} \otimes \mathcal{F}$-measurable partner-type function $g^n : I \times \Omega \to S \cup \{J\}$ is given by

$$g^n(i, \omega) = \begin{cases} \alpha^n(\pi^n(i, \omega), \omega) & \text{if } \pi^n(i, \omega) \neq i \\ J & \text{if } \pi^n(i, \omega) = i, \end{cases}$$

providing the type of the agent to whom agent $i$ is matched, if agent $i$ is matched, or $J$ if agent $i$ is not matched.

Let the initial condition $\Pi^0 = (\alpha^0, \beta^0)$ of $\mathbb{D}$ be given. We now construct a dynamical system $\mathbb{D}$ defined on $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ with input processes $(\eta^n, \theta^n, \xi^n, \sigma^n, \zeta^n)_{n \geq 1}$. We assume that $\Pi^{n-1} = (\alpha^{n-1}, \pi^{n-1}, g^{n-1})$ is given for some $n \geq 1$, and define $\Pi^n = (\alpha^n, \pi^n, g^n)$ by characterizing the three sub-steps of random change of types of agents, random matchings, break-ups and possible type changes after matchings and break-ups as follows.

**Mutation:** For $n \geq 1$ consider an $\mathcal{I} \otimes \mathcal{F}$-measurable post mutation function

$$\tilde{\alpha}^n : I \times \Omega \to S.$$ 

In particular, $\tilde{\alpha}^n_i(\omega) := \tilde{\alpha}^n(i, \omega)$ is the type of agent $i$ after the random mutation under the scenario $\omega \in \Omega$. The type of the agent to whom an agent is matched is identified by a $\mathcal{I} \otimes \mathcal{F}$-measurable function

$$\tilde{g}^n : I \times \Omega \to S \cup \{J\},$$

given by

$$\tilde{g}^n(i, \omega) = \tilde{\alpha}^n(\pi^{n-1}(i, \omega), \omega)$$

for any $\omega \in \Omega$. In particular, $\tilde{g}^n_i(\omega) := \tilde{g}^n(i, \omega)$ is the type of the agent to whom an agent is matched under the scenario $\omega \in \Omega$. Given $\tilde{p}^{n-1}$ and $\tilde{\omega} \in \tilde{\Omega}$, for any $k_1, k_2, l_1$ and $l_2$ in $S$, for any $r \in S \cup \{J\}$, for $\lambda$-almost every agent $i$, we set

$$\tilde{\tilde{p}}^\omega(n, \tilde{\omega}) := k_2, \tilde{g}^n_i(\tilde{\omega}, \cdot) = \tilde{\alpha}^n(\pi^{n-1}(\tilde{\omega}, \cdot), \omega) = k_1, \tilde{\tilde{g}}^n_i(\tilde{\omega}, \cdot) = l_1, \tilde{\tilde{p}}^{n-1}(\tilde{\omega}, \cdot)\tilde{g}^n_i(\tilde{\omega}, \cdot)\tilde{g}^n_i(\tilde{\omega}, \cdot)$$

$$= \eta_{k_1, k_2} \left(\tilde{\omega}, \cdot \right) \tilde{\omega}, \cdot \right) \eta_{l_1, l_2} \left(\tilde{\omega}, \cdot \right) \eta_{l_1, l_2} \left(\tilde{\omega}, \cdot \right) \delta_j(r), \quad (1.3)$$

We then set

$$\tilde{\tilde{p}}^\omega(\omega) = \tilde{\tilde{p}}^n(\omega), \quad n \geq 1.$$
The post-mutation extended type distribution realized in the state of the world \( \omega \in \Omega \) is denoted by \( \tilde{p}(\omega) = (\tilde{p}^n(\omega)[k,l])_{k \in S, l \in S \cup J} \), where
\[
\tilde{p}^n(\omega)[k,l] := \lambda(\{i \in I : \alpha^n(i,\omega) = k, \tilde{g}^n(i,\omega) = l\}).
\] (1.5)

**Matching:** We introduce a random matching \( \tilde{\pi}^n : I \times \Omega \rightarrow I \) and the associated post-matching partner type function \( \tilde{g}^n \) given by
\[
\tilde{g}^n(i,\omega) = \begin{cases} 
\alpha^n(\tilde{\pi}^n(i,\omega),\omega) & \text{if } \tilde{\pi}^n(i,\omega) \neq i \\
J & \text{if } \tilde{\pi}^n(i,\omega) = i,
\end{cases}
\]
satisfying the following properties:

1. \( \tilde{g}^n \) is \( I \otimes \mathcal{F} \)-measurable.
2. For any \( \tilde{\omega} \in \tilde{\Omega} \), any \( k, l \in S \) and any \( r \in S \cup \{J\} \), it holds
\[
\tilde{P}^\tilde{\omega}(\tilde{g}^n(\tilde{\omega},\cdot) = r|\alpha^n(\tilde{\omega},\cdot) = k, \tilde{g}^n(\tilde{\omega},\cdot) = l)(\tilde{\omega}) = \delta_1(r).
\]
This means that
\[
\tilde{\pi}^n(i) = \pi^n-1(i) \quad \text{for any } i \in \{i : \pi^n-1(i,\omega) \neq i\}.
\]
3. Given \( \tilde{\omega} \in \tilde{\Omega} \) and the post-mutation extended type distribution \( \tilde{p}^n \) in (1.5), an unmatched agent of type \( k \) is matched to a unmatched agent of type \( l \) with conditional probability \( \theta_{kl}(\tilde{\omega},n,\tilde{p}^n) \), that is for \( \lambda \)-almost every agent \( i \) and \( \tilde{P}^\tilde{\omega} \)-almost every \( \tilde{\omega} \), we define
\[
\tilde{P}^\tilde{\omega}(\tilde{g}^n(\tilde{\omega},\cdot) = l|\alpha^n(\tilde{\omega},\cdot) = k, \tilde{g}^n(\tilde{\omega},\cdot) = J, \tilde{g}^n(\tilde{\omega},\cdot) = l)(\tilde{\omega}) = \theta^n_{kl}(\tilde{\omega},n,\tilde{p}^n(\tilde{\omega},\cdot)).
\] (1.6)
This also implies that
\[
\tilde{P}^\tilde{\omega}(\tilde{g}^n(\tilde{\omega},\cdot) = J|\alpha^n(\tilde{\omega},\cdot) = k, \tilde{g}^n(\tilde{\omega},\cdot) = J, \tilde{g}^n(\tilde{\omega},\cdot) = l) = 1 - \sum_{l \in S} \theta^n_{kl}(\tilde{\omega},n,\tilde{p}^n(\tilde{\omega},\cdot)) = b(\tilde{\omega},\tilde{p}^n(\tilde{\omega},\cdot)).
\] (1.7)
The extended type of agent \( i \) after the random matching step is
\[
\tilde{\beta}^n_i(\omega) = (\tilde{\alpha}^n_i(\omega), \tilde{g}^n_i(\omega)), \quad n \geq 1.
\]
We denote the post-matching extended type distribution realized in \( \omega \in \Omega \) by \( \tilde{p}^n(\omega) = (\tilde{p}^n(\omega)[k,l])_{k \in S, l \in S \cup J} \), where
\[
\tilde{p}^n(\omega)[k,l] := \lambda(\{i \in I : \tilde{\alpha}^n_i(i,\omega) = k, \tilde{g}^n(i,\omega) = l\}).
\] (1.8)

**Type changes of matched agents with break-up:** We now define a random matching \( \pi^n \) by
\[
\pi^n(i) = \begin{cases} 
\tilde{\pi}^n(i) & \text{if } \tilde{\pi}^n(i,\omega) \neq i \\
i & \text{if } \tilde{\pi}^n(i,\omega) = i.
\end{cases}
\] (1.9)
We then introduce an \( (I \otimes \mathcal{F}) \)-measurable agent type function \( \alpha^n \) and an \( (I \otimes \mathcal{F}) \)-measurable partner function \( \bar{g}^n \) with
\[
\bar{g}^n(i,\omega) = \alpha^n(\pi^n(i,\omega),\omega), \quad n \geq 1,
\]
for all \( (i,\omega) \in I \times \Omega \). Given \( \tilde{\omega} \in \tilde{\Omega} \), \( \tilde{p}^n \in \tilde{\Delta} \), for any \( k_1, k_2, l_1, l_2 \in S \) and \( r \in S \cup \{J\} \), for \( \lambda \)-almost every agent \( i \), and for \( \tilde{P}^\tilde{\omega} \)-almost every \( \tilde{\omega} \), we set
\[
\tilde{P}^\tilde{\omega}(\alpha^n_i(\tilde{\omega},\cdot) = l_1, \bar{g}^n_i(\tilde{\omega},\cdot) = r|\alpha^n_i(\tilde{\omega},\cdot) = k_1, \bar{g}^n_i(\tilde{\omega},\cdot) = J)(\tilde{\omega}) = \delta_{k_1,l_1}(l_1)\delta_J(r),
\] (1.10)
\[
\tilde{P}^\tilde{\omega}(\alpha^n_i(\tilde{\omega},\cdot) = l_1, \bar{g}^n_i(\tilde{\omega},\cdot) = l_2|\alpha^n_i(\tilde{\omega},\cdot) = k_1, \bar{g}^n_i(\tilde{\omega},\cdot) = k_2, \tilde{p}^n(\tilde{\omega},\cdot))(\tilde{\omega})
\]
We denote the extended type distribution at the end of period
\[\hat{\omega}, n, \hat{\psi}^n(\hat{\omega}, \hat{\psi})] = l_1, g^n_k(\hat{\omega}, \hat{\psi}) = k_1, g^n_k(\hat{\omega}, \hat{\psi}) = k_2, \hat{\psi}^n(\hat{\omega}, \hat{\psi}) (\hat{\omega}) = \xi_k^{(n)}(\hat{\omega}, n, \hat{\psi}^n(\hat{\omega}, \hat{\psi})), \tag{1.12}\]

The extended-type function at the end of the period is
\[\hat{\psi}^n(\omega) = (\alpha^n(\omega), \gamma^n(\omega)), \quad n \geq 1.\]

We denote the extended type distribution at the end of period \(n\) realized in \(\omega \in \Omega\) by \(\hat{\psi}^n(\omega) = (\hat{\psi}^n(\omega)[k, l])_{k \in S, l \in S \cup J}\), where
\[\hat{\psi}^n(\omega)[k, l] := \lambda\{i \in I : \alpha^n(i, \omega) = k, \gamma^n(i, \omega) = l\}. \tag{1.13}\]

Furthermore, the definition of Markov conditionally independent (MCI) dynamical system is provided in Definition 3.8 in [1]. We work under the following assumption, which is Assumption 3.9 in [1].

**Assumption 1.2.** Let \((\hat{\Omega}, \hat{F}, \hat{P})\) be the probability space introduced. We assume that there exists its corresponding hyperfinite internal probability space, which we denote from now on also by \(\hat{\Omega}, \hat{F}, \hat{P}\) by a slight notational abuse.

As already pointed out in [1], the proofs of the results below follow by analogous arguments as in [2] which is possible due to the product structure of the space \(\Omega\) in [1.1], and the Markov kernel \(P\) in [1.2]. As in [2], we use some concepts and notations from nonstandard analysis. Note here that an object with an upper left star means the transfer of a standard object to the nonstandard universe. For a detailed overview of the necessary tools of nonstandard analysis, we refer to Appendix D.2. in [2].

## 2 Proof of Theorem 3.13 in [1]

From now on, we fix the hyperfinite internal space \((\hat{\Omega}, \hat{F}, \hat{P})\), along with the input functions \((\eta_{kl}, \theta_{kl}, \xi_{kl}, \sigma_{kl}[r, s], \varsigma_{kl}[r])_{k, l, r, s \in S \times S \times S \times S}\) from \(\hat{\Omega} \times \mathbb{N} \times \Delta\) to \([0, 1]\) introduced above. Given this framework, we prove the existence of a rich Fubini extension \((I \times \Omega, I \otimes F, \lambda \otimes P)\), on which a dynamical system \(D\) described in Definition 1.1 for such input probabilities is defined. More specifically, we are going to construct the space \(\hat{\Omega}\) and the probability measure \(\hat{P}\) such that \(\Omega = \hat{\Omega} \times \hat{\Omega}\) and \(P = \hat{P} \times \hat{P}\) is a Markov kernel from \(\hat{\Omega}\) to \(\hat{\Omega}\).

We now present and prove Theorem 3.13 in [1]. The proof is based on Proposition 3.12 in [1], which focuses on the random matching step and shows the existence of a suitable hyperfinite probability space and partial matching, generalizing Lemma 7 in [2].

**Theorem 2.1.** Let Assumption 3.9 in [1] hold and \((\eta_{kl}, \theta_{kl}, \xi_{kl}, \sigma_{kl}[r, s], \varsigma_{kl}[r])_{k, l, r, s \in S \times S \times S \times S}\) be the input functions from \(\Omega \times \mathbb{N} \times \Delta\) defined in Section 3 in [1]. Then for any extended type distribution \(\hat{p} \in \hat{\Delta}\) and any deterministic initial condition \(\Pi^0 = (\alpha^0, \pi^0)\) there exists a rich Fubini extension \((I \times \Omega, I \otimes \mathcal{F}, \lambda \otimes P)\) on which a discrete dynamical system \(\hat{D} = (\Pi^n)_{n=0}^{\infty}\) as in Definition 3.6 in [1] can be constructed with discrete time input processes \((\eta^n, \theta^n, \xi^n, \sigma^n, \varsigma^n)_{n \geq 1}\) coming from \((\eta_{kl}, \theta_{kl}, \xi_{kl}, \sigma_{kl}[r, s], \varsigma_{kl}[r])_{k, l, r, s \in S \times S \times S \times S}\) as stated in Section 2 in [1]. In particular,
\[\Omega = \hat{\Omega} \times \hat{\Omega}, \quad \mathcal{F} = \hat{F} \otimes \hat{F}, \quad P = \hat{P} \times \hat{P}.\]
where \((\tilde{\Omega}, \tilde{F})\) is a measurable space and \(\hat{P}\) a Markov kernel from \(\tilde{\Omega}\) to \(\hat{\Omega}\). The dynamical system \(\mathcal{D}\) is also MCI according to Definition 3.8 in \([1]\) and with initial cross-sectional extended type distribution \(\hat{p}^0\) equal to \(\hat{p}^0\) with probability one.

**Proof.** At each time period we construct three internal measurable spaces with internal transition probabilities taking into account the following steps:

1. random mutation
2. random matching
3. random type changing with break-up.

Let \(M\) be a limited hyperfinite number in \(\mathcal{N}_\infty\). Let \(\{n\}_{n=0}^M\) be the hyperfinite discrete time line and \((I, \mathcal{I}_0, \lambda_0)\) the agent space, where \(I = \{1, \ldots, M\}\), \(\mathcal{I}_0\) is the internal power set on \(I\), \(\lambda_0\) is the internal counting probability measure on \(\mathcal{I}_0\), and \(\hat{M}\) is an unlimited hyperfinite number in \(\mathcal{N}_\infty\).

We start by transferring the deterministic functions \(\theta(0, \cdot), \xi(0, \cdot), \sigma(0, \cdot), \varsigma(0, \cdot) : \hat{\Delta} \to [0, 1]\) to \(\hat{\mathcal{D}}\) to \([0, 1]\) to the nonstandard universe. In particular, we denote by \(*\theta^0_{kl}\) for any \(k, l \in S\) and by \(*f^0\) for \(f = \eta, \xi, \sigma, \varsigma\) the internal functions from \(\hat{\mathcal{D}}\) to \([0, 1]\).

We also let \(\hat{\theta}^0_{kl}(\hat{\rho}) = *\theta^0_{kl}(\hat{\rho})\) and \(\hat{\rho}^0 = 1 - \sum_{i \in S} \theta^0_{kl}(\hat{\rho})\) for any \(k, l \in S\) and \(\hat{\rho} \in \hat{\mathcal{D}}\), with \(1 \in \mathcal{N}\).

We start at \(n = 0\). To do so, we introduce the trivial probability space over the single set \(\{0\}\) denoted by \((\hat{\Omega}_0, \hat{\mathcal{F}}_0, \hat{Q}_0)\). Let \(\{A_{kl}\}_{(k, l) \in S}\) be an internal partition of \(I\) such that \(|A_{kl}| \geq \frac{|\hat{\mathcal{D}}_N|}{M}\) for any \(k \in S\) and \(l \in S \cup \{J\}\), such that \(|A_{kk}|\) is even for any \(k, l \in S\) and \(|A_{kl}| = |A_{lk}|\) for any \(k, l \in S\).

Let \(\alpha^0\) be an internal function from \((I, \mathcal{I}_0, \lambda_0)\) to \(S\) such that \(\alpha^0(i) = k\) if \(i \in \bigcup_{l \in S \cup \{J\}} A_{kl}\). Let \(\pi^0\) be an internal partial matching from \(I\) to \(S\) such that \(\pi^0(i) = k\) on \(\bigcup_{k \in S} A_{k,l}\), and the restriction \(\pi^0|A_{kl}\) is an internal bijection from \(A_{kl}\) to \(A_{lk}\) for any \(k, l \in S\).

Let \((\hat{\Omega}, \hat{F}, \hat{P})\) be the hyperfinite internal space. Since the intensities are supposed to be deterministic at initial time, the Markov kernel from \(\hat{\Omega}\) is trivial and we define the initial internal product probability space as

\[
(\Omega_0, \mathcal{F}_0, Q_0) := (\hat{\Omega} \times \hat{\Omega}, \hat{\mathcal{F}} \otimes \hat{\mathcal{F}}_0, \hat{P} \otimes \hat{Q}_0).
\]

Suppose now that the dynamical system \(\mathcal{D}\) has been constructed up to time \(n - 1 \in \mathcal{N}\) for \(n \geq 1\), i.e., that the sequences \(\{(\Omega_m, \mathcal{F}_m, Q_m)\}_{m=0}^{3m-3}\) and \(\{\alpha^t, \pi^t\}_{l=0}^{n-1}\) have been constructed. In particular, we assume to have introduced the spaces \((\hat{\Omega}_m, \hat{\mathcal{F}}_m)\) and the Markov kernel \(\hat{P}_m\) from \(\hat{\Omega}\) to \(\hat{\Omega}_m\) for any \(m = 1, \ldots, n - 3\), so that we can define \(\Omega_m := \hat{\Omega} \times \hat{\Omega}_m\) as a hyperfinite internal set with internal power set \(\mathcal{F}_m := \hat{\mathcal{F}} \otimes \hat{\mathcal{F}}_m\) and \(Q_m := \hat{P} \times \hat{P}_m\) as an internal transition probability from \(\Omega^{m-1}\) to \((\Omega_m, \mathcal{F}_m)\), where

\[
\Omega^m := \hat{\Omega} \times \hat{\Omega}^m, \quad \hat{\Omega}^m := \hat{\Omega}_0 \times \prod_{j=1}^{m} \hat{\Omega}_j, \quad \hat{\mathcal{F}}^m := \hat{\mathcal{F}}_0 \otimes \left(\otimes_{j=1}^{m} \hat{\mathcal{F}}_j\right) \quad \text{and} \quad \hat{\mathcal{F}}^m = \hat{\mathcal{F}} \otimes \hat{\mathcal{F}}^m. \tag{2.1}
\]

\(^1\)Note that at initial time, the functions are supposed to be deterministic and in particular independent of \(\hat{\Omega}\).
In this setting, $\alpha^I$ is an internal type function from $I \times \Omega^{3l-1}$ to the space $S$, and $\pi^I$ an internal random matching from $I \times \Omega^{3l}$ to $I$, such that

$$\alpha^I(i, (\hat{\omega}, \hat{\omega}^{3l-1})) = \alpha^I(i, \hat{\omega}^{3l-1}), \quad \text{for any } (\hat{\omega}, \hat{\omega}^{3l-1}) \in \Omega^{3l-1}$$

and

$$\pi^I(i, (\hat{\omega}, \hat{\omega}^{3l})) = \pi^I(i, \hat{\omega}^{3l}), \quad \text{for any } (\hat{\omega}, \hat{\omega}^{3l}) \in \Omega^{3l}.$$ 

Given $\omega^{3l} \in \Omega^{3l}$ we denote by $\pi_3^{I, \omega} : I \to I$ the function given by

$$\pi_3^{I, \omega}(i) := \pi^I(i, (\hat{\omega}, \hat{\omega}^{3l})).$$

A similar notation will be used for $\alpha_3^{I, \omega} : I \to S$. We now have the following.

(i) Random mutation step:
We let $\hat{\Omega}_{3n-2} := \hat{S}^I$, which is the space of all internal functions from $I$ to $S$, and denote its internal power set by $\hat{\mathcal{F}}_{3n-2}$. For each $i \in I$ and $\omega^{3n-3} = (\hat{\omega}, \hat{\omega}^{3n-3}) \in \Omega^{3n-3}$, if $\alpha^{n-1}(i, \omega^{3n-3}) = \alpha^{n-1}(i, \hat{\omega}^{3n-3}) = k$, define a probability measure $\lambda_i^{\omega^{3n-3}}$ on $S$ by letting $\lambda_i^{\omega^{3n-3}}(l) := \theta_k(\hat{\omega}, \omega, \rho^{n-1}_{3n-3})$ for each $l \in S$ with

$$\rho^{n-1}_{3n-3}[k, r] := \lambda_0(\{i \in I : \alpha_{3n-3}^{n}(i) = k, \alpha_{3n-3}^{n}(\pi_{3n-3}^{n}(i)) = r\}), \quad k, r \in S$$

and

$$\rho^{n-1}_{3n-3}[k, J] := \lambda_0(\{i \in I : \alpha_{3n-3}^{n}(i) = k, \pi_{3n-3}^{n}(i) = i\}), \quad k \in S.$$

Define a Markov kernel $\tilde{P}_{3n-2}$ from $\hat{\Omega}$ to $\hat{\Omega}_{3n-2}$ by letting $\tilde{P}_{3n-2}(\hat{\omega})$ be the internal product measure $\prod_{i \in I} \lambda_i^{\omega^{3n-3}}$. Define $\tilde{\alpha}^n : (I \times \Omega^{3n-2}) \to S$ by

$$\tilde{\alpha}^n(i, (\hat{\omega}, \hat{\omega}^{3n-2})) := \tilde{\alpha}^{n}(i, \hat{\omega}^{3n-2}) = : \hat{\omega}^{3n-2}(i)$$

and $\tilde{g}^n : (I \times \Omega^{3n-2}) \to S \cup \{J\}$ by

$$\tilde{g}^n(i, (\hat{\omega}, \hat{\omega}^{3n-2})) := \tilde{g}^n(i, \hat{\omega}^{3n-2}) := \begin{cases} \tilde{\alpha}^{n}(\pi_{3n-3}^{n-1}(i, \hat{\omega}^{3n-3}), \hat{\omega}^{3n-2}) & \text{if } \pi_{3n-3}^{n-1}(i, \hat{\omega}^{3n-3}) \neq i \\ J & \text{if } \pi_{3n-3}^{n-1}(i, \hat{\omega}^{3n-3}) = i. \end{cases}$$

Moreover, we introduce the notation

$$\tilde{\alpha}_{3n-2}^{n}(\cdot) : I \to S, \quad \tilde{\alpha}_{3n-2}^{n}(i) := \tilde{\alpha}^{n}(i, (\hat{\omega}, \hat{\omega}^{3n-2})): = \tilde{\alpha}^{n}(i, \hat{\omega}^{3n-2})$$

for the type function. We then define $\pi_{3n-3}^{n-1}(\cdot) : I \to I$ and $\tilde{g}_{3n-2}^{n} : I \to S \cup \{J\}$ analogously. Finally, we define the cross-internal extended type distribution after random mutation $\tilde{\rho}_{3n-2}^{n}$ by

$$\tilde{\rho}_{3n-2}^{n}[k, l] := \lambda_0(\{i \in I : \tilde{\alpha}_{3n-2}^{n}(i) = k, \tilde{g}_{3n-2}^{n}(i) = l\}), \quad k, l \in S.$$

(ii) Directed random matching:
Let $(\Omega_{3n-1}, \mathcal{F}_{3n-1})$ and $P_{3n-1}^{1}$ be the measurable space and the Markov kernel, respectively, provided by Proposition 3.12 in [1], with type function $\tilde{\alpha}_{3n-2}^{n}(\cdot)$ and partial matching function $\pi_{3n-3}^{n-1}(\cdot)$, for fixed matching probability function $\theta(\cdot, n, \tilde{\rho}_{3n-2}^{n})$. Proposition 3.12 in [1] also provides the directed random matching

$$\pi_{\theta^{n}(\cdot, \tilde{\rho}_{3n-2}^{n}), \tilde{\alpha}_{3n-2}^{n}; \pi_{3n-3}^{n-1}(\cdot)}.$$
which is a function defined on \((\Omega_{3n-1}, \mathcal{F}_{3n-1})\) by

\[
\pi^n(\tilde{\omega}_{3n-2}, \alpha_{3n-2}, \pi_{3n-3}^{-1}(i, \tilde{\omega}_{3n-1})) := \pi^n(\tilde{\omega}_{3n-2}, \alpha_{3n-2}, \pi_{3n-3}^{-1}(i, \tilde{\omega}_{3n-1}).
\]

We then define \(\tilde{\pi}^n : I \times \Omega^{3n-1} \rightarrow I\) by

\[
\tilde{\pi}^n(i, (\tilde{\omega}, \tilde{\omega}_{3n-1})) := \pi^n(i, \tilde{\omega}_{3n-1}) := \pi^n(\tilde{\omega}_{3n-2}, \alpha_{3n-2}, \pi_{3n-3}^{-1}(i, \tilde{\omega}_{3n-1})
\]

and

\[
\tilde{\theta}^n(i, (\tilde{\omega}, \tilde{\omega}_{3n-1})) = \tilde{\theta}^n(i, \tilde{\omega}_{3n-1}) := \begin{cases} 
\alpha^n(\tilde{\pi}^n(i, \tilde{\omega}_{3n-1}), \tilde{\omega}_{3n-2}) & \text{if } \tilde{\pi}^n(i, \tilde{\omega}_{3n-1}) \neq i \\
J & \text{if } \tilde{\pi}^n(i, \tilde{\omega}_{3n-1}) = i.
\end{cases}
\]

Define now the cross-internal extended type distribution after the random matching \(\tilde{\rho}^n_{\omega_{3n-1}}\) by

\[
\tilde{\rho}^n_{\omega_{3n-1}[k, l]} := \lambda_0(\{i \in I : \tilde{\alpha}^n_{\omega_{3n-1}}(i) = k, \tilde{\theta}^n_{\omega_{3n-1}}(i) = l\}).
\]

(iii) Random type changing with break-up for matched agents:
Introduce \(\tilde{\Omega}_{3n} := (S \times \{0, 1\})^I\) with internal power set \(\tilde{\mathcal{F}}_{3n}\), where 0 represents “unmatched” and 1 represents “paired”; each point \(\tilde{\omega}_{3n} = (\tilde{\omega}_{3n}^1, \tilde{\omega}_{3n}^2) \in \tilde{\Omega}_{3n}\) represents an internal function from \(I\) to \(S \times \{0, 1\}\). Define a new type function \(\alpha^n : (I \times \Omega^{3n}) \rightarrow S\) by letting \(\alpha^n(i, (\tilde{\omega}, \tilde{\omega}_{3n})) := \alpha^n(i, \tilde{\omega}_{3n}) = \tilde{\omega}_{3n}(i)\). Fix \((\tilde{\omega}, \tilde{\omega}_{3n}) \in \Omega^{3n-1}\).

For each \(i \in I\), we proceed in the following way.

1. If \(\tilde{\pi}^n(i, \tilde{\omega}_{3n-1}) = i\) (\(i\) is not paired after the matching step at time \(n\)), let \(\tau^\tilde{\omega}_{3n-1}_i\) be the probability measure on the type space \(S \times \{0, 1\}\) that assigns probability one to the type \((\tilde{\alpha}^n(i, (\tilde{\omega}, \tilde{\omega}_{3n-2})), 0) = (\tilde{\alpha}^n(i, \tilde{\omega}_{3n-2}), 0)\) and zero to the rest.

2. If \(\tilde{\pi}^n(i, (\tilde{\omega}, \tilde{\omega}_{3n-1})) = \tilde{\pi}^n(i, \tilde{\omega}_{3n-1}) \neq i\) (\(i\) is paired after the matching step at time \(n\)), \(\tilde{\alpha}^n(i, (\tilde{\omega}, \tilde{\omega}_{3n-2})) = \tilde{\alpha}^n(i, \tilde{\omega}_{3n-2}) = k, \tilde{\theta}^n(i, (\tilde{\omega}, \tilde{\omega}_{3n-1})) = \tilde{\pi}^n(i, \tilde{\omega}_{3n-1}) = j\) and \(\tilde{\alpha}^n(j, (\tilde{\omega}, \tilde{\omega}_{3n-1})) = \alpha^n(j, \tilde{\omega}_{3n-1}) = l\), define a probability measure \(\tau^\tilde{\omega}_{3n-1}_{ij}\) on \((S \times \{0, 1\}) \times (S \times \{0, 1\})\) as

\[
\tau^\tilde{\omega}_{3n-1}_{ij}((k', 0), (l', 0)) := (1 - \xi_{kl}(\tilde{\omega}, n, \tilde{\rho}^n_{\omega_{3n-1}})) \xi_{kl}[k'] \tilde{\omega}_n \tilde{\rho}^n_{\omega_{3n-1}} \xi_{lk}[l'] \tilde{\omega}_n \tilde{\rho}^n_{\omega_{3n-1}}
\]

and

\[
\tau^\tilde{\omega}_{3n-1}_{ij}((k', 1), (l', 1)) := \xi_{kl}(\tilde{\omega}, n, \tilde{\rho}^n_{\omega_{3n-1}}) \sigma_{kl}[k'] \tilde{\omega}_n \tilde{\rho}^n_{\omega_{3n-1}}
\]

for \(k', l' \in S\), and zero for the rest.

Let \(A^n_{\omega_{3n-1}} = \{(i, j) \in I \times I : i < j, \tilde{\pi}^n(i, (\tilde{\omega}, \tilde{\omega}_{3n-1})) = \tilde{\pi}^n(i, \tilde{\omega}_{3n-1}) = j\}\) and \(B^n_{\omega_{3n-1}} = \{(i) \in I : \tilde{\pi}^n(i, (\tilde{\omega}, \tilde{\omega}_{3n-1})) = \tilde{\pi}^n(i, \tilde{\omega}_{3n-1}) = i\}\). Define a Markov kernel \(\tilde{\rho}^n_{\omega_{3n-1}}\) from \(\tilde{\Omega}_{3n}\) to \(\tilde{\Omega}_{3n}\) by

\[
\tilde{\rho}^n_{\omega_{3n-1}}(\tilde{\omega}) := \prod_{i \in I \times \omega_{3n-1}} \tau^\tilde{\omega}_{3n-1}_{ii} \otimes \prod_{(i, j) \in A^n_{\omega_{3n-1}}} \tau^\tilde{\omega}_{3n-1}_{ij}.
\]

Let

\[
\tilde{\pi}^n(i, (\tilde{\omega}, \tilde{\omega}_{3n})) = \tilde{\pi}^n(i, \tilde{\omega}_{3n}) := \begin{cases} 
J & \text{if } \tilde{\pi}^n(i, \tilde{\omega}_{3n-1}) = J \text{ or } \tilde{\omega}_{3n}(\tilde{\pi}^n(i, \tilde{\omega}_{3n-1})) = 0 \\
\tilde{\pi}^n(i, \tilde{\omega}_{3n-1}) & \text{otherwise},
\end{cases}
\]
and
\[ g^n(i, (\tilde{\omega}, \tilde{\omega}^3n)) = g^n(i, \tilde{\omega}^3n) := \begin{cases} 
\alpha^n(\pi^n(i, \tilde{\omega}^3n), \tilde{\omega}^3n) & \text{if } \pi^n(i, \tilde{\omega}^3n) \neq i \\
J & \text{if } \pi^n(i, \tilde{\omega}^3n) = i.
\end{cases} \]

Define \( \hat{\rho}_n = \lambda_0(\alpha_{\tilde{\omega}^3n}, \pi_{\tilde{\omega}^3n})^{-1} \).

By repeating this procedure, we construct a hyperfinite sequence of internal transition probability spaces \( \{(\Omega_m, \mathcal{F}_m, Q_m)\}_{m=0}^{\mathcal{M}} \) and a hyperfinite sequence of internal type functions and internal random matchings \( \{ (\alpha^n, \pi^n) \}_{n=0}^{\mathcal{M}} \). Moreover, define \( (\Omega^n, \mathcal{F}^m) \) as in (2.1), and
\[ \hat{P}^m := \prod_{i=1}^m \hat{P}_i, \quad Q^m := \hat{P} \times \hat{P}^m, \]
where the product of the Markov kernels is \( \tilde{\omega}^{-}\)-wise.

Let \( (I \times \Omega^{3M}, \mathcal{I}_0 \otimes \mathcal{F}^{3M}, \lambda_0 \otimes Q^{3M}) \) be the internal product probability space of \( (I, \mathcal{I}_0, \lambda_0) \) and \( (\Omega^{3M}, \mathcal{F}^{3M}, Q^{3M}) \). Denote the Loeb spaces of \( (\Omega^{3M}, \mathcal{F}^{3M}, Q^{3M}) \) and the internal product \( (I \times \Omega^{3M}, \mathcal{I}_0 \otimes \mathcal{F}^{3M}, \lambda_0 \otimes Q^{3M}) \) by \( (\hat{\Omega}^{3M}, \hat{\mathcal{F}}, \hat{P}) \) and \( (I \times \hat{\Omega}^{3M}, \hat{\mathcal{I}} \otimes \hat{\mathcal{F}}, \hat{\lambda} \otimes \hat{P}) \), respectively. For simplicity, let \( \hat{\Omega}^{3M} \) be denoted by \( \hat{\Omega} \) and \( \hat{\Omega}^{3M} \) by \( \hat{\Omega} \). Denote now \( Q^{3M} \) by \( \hat{P} \) and the Markov kernel \( P^{3M} \) by \( \hat{P} \).

The properties of a dynamical system as well as the independence conditions follow now by applying similar arguments as in the proof of Theorem 5 in [2] for any fixed \( \tilde{\omega} \in \hat{\Omega} \). The only difference is that in our setting the input processes for the random mutation step and the break-up step also depend on the extended type distribution. Furthermore, these arguments are similar to the ones in the proof of Lemma 3.2 and can be found there with all details.

\[ \square \]

### 3 Proof of Theorem 3.14 in [1]

We now prove Theorem 3.14 in [1] which is a generalization of the results in Appendix C in [2]. For \( n \geq 1 \) we define the mapping \( \Gamma^n \) from \( \hat{\Omega} \times \tilde{\Delta} \) to \( \tilde{\Delta} \) by
\[ \Gamma^n_{kl}(\tilde{\omega}, \tilde{\nu}) = \sum_{k_1, l_1 \in S} (1 - \xi_{k_1, l_1}(\tilde{\omega}, n, \tilde{\nu}^n)) \sigma_{k_1, l_1} \hat{\rho}_{k_1, l_1} + \sum_{k_1, l_1 \in S} (1 - \xi_{k_1, l_1}(\tilde{\omega}, n, \tilde{\nu}^n)) \sigma_{k_1, l_1} \hat{\rho}_{k_1, l_1} \theta_{k_1, l_1}(\tilde{\omega}, n, \tilde{\nu}^n) \hat{\rho}^n_{k_1, l_1}, \quad (3.1) \]

and
\[ \Gamma^n_{k,l}(\tilde{\omega}, \tilde{\nu}) = b_k(\tilde{\omega}, n, \tilde{\nu}^n) \hat{\rho}^n_{k,l} + \sum_{k_1, l_1 \in S} \xi_{k_1, l_1}(\tilde{\omega}, n, \tilde{\nu}^n) \hat{\sigma}_{k_1, l_1} \hat{\rho}_{k_1, l_1} + \sum_{k_1, l_1 \in S} \xi_{k_1, l_1}(\tilde{\omega}, n, \tilde{\nu}^n) \hat{\sigma}_{k_1, l_1} \hat{\rho}_{k_1, l_1} \theta_{k_1, l_1}(\tilde{\omega}, n, \tilde{\nu}^n) \hat{\rho}^n_{k_1, l_1}, \quad (3.2) \]

with
\[ \hat{\rho}^n_{k,l} = \sum_{k_1, l_1 \in S} \eta_{k_1,k}(\tilde{\omega}, n, \tilde{\nu}) \eta_{l_1,l}(\tilde{\omega}, n, \tilde{\nu}) \hat{\rho}_{k_1, l_1}, \]
\[ \hat{\rho}_{k,l} = \sum_{l \in S} \hat{\rho}_l \eta_k(\tilde{\omega}, n, \tilde{\nu}) \]
Proof. Assume that the discrete dynamical system $\mathbb{D}$ defined in Definition 3.6 in [1] is Markov conditionally independent given $\hat{\omega}$ as defined in Definition 3.8 in [2]. Then given $\hat{\omega} \in \hat{\Omega}$, the discrete time processes $\{\beta_i^n\}_{n=0}^{\infty}, i \in I,$ are essentially pairwise independent on $(I \times \hat{\Omega}, \mathcal{I} \otimes \hat{\mathcal{F}}, \lambda \otimes \hat{P}^{\hat{\omega}}).$ Moreover, for fixed $n=1,\ldots,M$ also $(\tilde{\beta}_i^n)_{n=0}^{\infty}$ and $(\tilde{\beta}_i^n)_{n=0}^{\infty}, i \in I,$ are essentially pairwise independent on $(I \times \hat{\Omega}, \mathcal{I} \otimes \hat{\mathcal{F}}, \lambda \otimes \hat{P}^{\hat{\omega}}).$

Proof. This can be proven by the same arguments used in the proof of Lemma 3 in [2].

We now derive a result which shows how to compute for a fixed $\hat{\omega} \in \hat{\Omega}$ the expected cross-sectional distributions $\mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^n], \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\tilde{p}^n]$ and $\mathbb{E}^{\hat{P}^{\hat{\omega}}} [\tilde{p}^n]$. 

Lemma 3.2. The following holds for any fixed $\hat{\omega} \in \hat{\Omega}$.

1. For each $n \geq 1$, $\mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^n] = \Gamma^n(\hat{\omega}, \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^{n-1}])$, with $\Gamma$ defined in (3.1).

2. For each $n \geq 1$, we have

$$\mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^n_{k,l}] = \sum_{k_1,l_1 \in S} \eta_{k_1,k} (\hat{\omega}, n, \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^{n-1}]) \eta_{l_1,l} (\hat{\omega}, n, \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^{n-1}]) \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^{n-1}_{k_1,l_1}]$$

and

$$\mathbb{E}^{\hat{P}^{\hat{\omega}}} [\tilde{p}^n_{k,l}] = \sum_{k_1 \in S} \eta_{k_1,k} (\hat{\omega}, n, \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^{n-1}]) \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\tilde{p}^{n-1}_{k_1,l}].$$

3. For each $n \geq 1$, we have

$$\mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^n_{k,l}] = \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^n_{k,l}] + \theta_{k_1}(\hat{\omega}, n, \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^n]) \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^{n-1}_{k,l}].$$

and

$$\mathbb{E}^{\hat{P}^{\hat{\omega}}} [\tilde{p}^n_{k,l}] = b_k (\hat{\omega}, n, \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^n]) \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\tilde{p}^n_{k,l}].$$

Proof. Fix $\hat{\omega} \in \hat{\Omega}$ and $k,l \in S$. By Lemma 3.1, we know that the processes $(\beta_i^n)_{n=0}^{\infty}, i \in I,$ are essentially pairwise independent. Then the exact law of large numbers in Lemma 1 in [2] implies that $\hat{p}^n(\hat{\omega}) = \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\lambda(\beta^{n-1})^{-1}]$ for $\hat{P}$-almost all $\hat{\omega} \in \hat{\Omega}$. Thus equations (1.3) and (1.4) are equivalent to

$$\hat{P}^{\hat{\omega}} (\tilde{\alpha}_i^n = k_2, \tilde{g}_i^n = l_2 | \alpha_i^{n-1} = k_1, g_i^{n-1} = l_1) = \eta_{k_1,k_2} (\hat{\omega}, n, \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^{n-1}]) \eta_{l_1,l_2} (\hat{\omega}, n, \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^{n-1}]) \delta_J(r). \tag{3.3}$$

$$\hat{P}^{\hat{\omega}} (\tilde{\alpha}_i^n = k_2, \tilde{g}_i^n = r | \alpha_i^{n-1} = k_1, g_i^{n-1} = J) = \eta_{k_1,k_2} (\hat{\omega}, n, \mathbb{E}^{\hat{P}^{\hat{\omega}}} [\hat{p}^{n-1}]) \delta_J(r). \tag{3.4}$$

Therefore, for any $k_1,l_1 \in S$ we have

$$\hat{P}^{\hat{\omega}} (\tilde{\beta}_i^n = (k_1,l_1) | \beta_i^{n-1} = (k_1,l_1)) = 0 \tag{3.5}$$

$$\hat{P}^{\hat{\omega}} (\tilde{\beta}_i^n = (k_1,l) | \beta_i^{n-1} = (k_1,l)) = 0. \tag{3.6}$$

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By Lemma 3.1, \( \bar{\beta}^n \) is essentially pairwise independent. Again it follows by the exact law of large numbers that \( \tilde{p}^n(\tilde{\omega}) = \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n] \) for \( \hat{P}^\omega \)-almost all \( \tilde{\omega} \in \tilde{\Omega} \). Then (1.6) and (1.7) are equivalent to

\[
\hat{P}^{\omega}(\tilde{g}^n = l|\tilde{\omega}) = \theta_{kl}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n]) \quad (3.9)
\]

and

\[
\hat{P}^{\omega}(\tilde{g}^n = J|\tilde{\omega}) = b_k(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n]). \quad (3.10)
\]

By the same calculations as in the proof of Lemma 4 in [2] we have

\[
\mathbb{E}^{\hat{P}^\omega}[\hat{p}^n_{kl}] = \mathbb{E}^{\hat{P}^\omega}[\hat{p}^n_{kl}] = \theta_{kl}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n]) \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n_{kl}] \quad (3.11)
\]

and

\[
\mathbb{E}^{\hat{P}^\omega}[\hat{p}^n_{k,l}] = b_k(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n]) \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n_{k,l}]. \quad (3.12)
\]

By Lemma 3.1, \( \tilde{\beta}^n \) is essentially pairwise independent and thus \( \hat{p}^n(\hat{\omega}) = \mathbb{E}^{\hat{P}^\omega}[\hat{p}^n] \) for \( \hat{P}^\omega \)-almost all \( \hat{\omega} \in \hat{\Omega} \). Then (1.11) and (1.12) are equivalent to

\[
\hat{P}^{\omega}(\alpha^n_i = l_1, g^n_l = l_2|\alpha^n_i = k_1, g^n_l = k_2) = (1 - \xi_{k_1,k_2}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n])) \sigma_{k_1,k_2}[l_1, l_2] \left( \tilde{\omega}, n, \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n] \right) \quad (3.13)
\]

and

\[
\hat{P}^{\omega}(\alpha^n_i = l_1, g^n_l = J|\alpha^n_i = k_1, g^n_l = k_2) = \xi_{k_1,k_2}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n]) \sigma_{k_1,k_2}[l_1, l_2] \left( \tilde{\omega}, n, \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n] \right),
\]

respectively. Thus

\[
\mathbb{E}^{\hat{P}^\omega}[\hat{p}^n_{kl}] = \sum_{k_1,l_1 \in S} (1 - \xi_{k_1,l_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n])) \sigma_{k_1,l_1}[k,l] \left( \tilde{\omega}, n, \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n] \right) \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n_{kl}] \quad (3.13)
\]

and

\[
\mathbb{E}^{\hat{P}^\omega}[\hat{p}^n_{k,l}] = \mathbb{E}^{\hat{P}^\omega}[\hat{p}^n_{k,l}] + \sum_{k_1,l_1 \in S} \xi_{k_1,l_1}(\tilde{\omega}, n, \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n]) \sigma_{k_1,l_1}[k,l] \left( \tilde{\omega}, n, \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n] \right) \mathbb{E}^{\hat{P}^\omega}[\tilde{p}^n_{kl}], \quad (3.14)
\]

By plugging (3.8) in (3.13) we get

\[
\mathbb{E}^{\hat{P}^\omega}[\hat{p}^n_{kl}] \]

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Thus we have

\[ E^{\hat{\tilde{\omega}}}[\hat{p}^n_{k_{1,1}}] = \sum_{k_{1,1} \in S} \sum_{k_{1,1} \in S} (1 - \xi_{k_{1,1}}(\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n]))\sigma_{k_{1,1}}[k, l] (\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n]) E^{\hat{\tilde{\omega}}}[\hat{p}^n_{k_{1,1}}] \]

By using (3.12) and (3.13), it follows that

\[ E^{\hat{\tilde{\omega}}}[\hat{p}^n_{k_{1,1}}] = b_k(\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n])\]

\[ + \sum_{k_{1,1} \in S} \xi_{k_{1,1}}(\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n])\sigma_{k_{1,1}}[k] \left( \hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n] \right) \eta_{k_{1,1}}(\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n]) E^{\hat{\tilde{\omega}}}[\hat{p}^n_{k_{1,1}}] \]

\[ + \sum_{k_{1,1} \in S} \xi_{k_{1,1}}(\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n])\sigma_{k_{1,1}}[k] \left( \hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n] \right) \theta_{k_{1,1}}(\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n]) E^{\hat{\tilde{\omega}}}[\hat{p}^n_{k_{1,1}}]. \quad (3.15) \]

\[ \square \]

**Lemma 3.3.** Assume that the discrete dynamical system \( \mathbb{D} \) defined in Definition 3.6 in [1] is Markov conditionally independent given \( \hat{\omega} \in \hat{\Omega} \) according to Definition Definition 3.8 in [1]. Then for fixed \( \hat{\omega} \in \hat{\Omega} \) the following holds:

1. For \( \lambda \)-almost all \( i \in I \), the extended type process \( \{\beta^n_i\}_{n=0}^{\infty} \) for agent \( i \) is a Markov chain on \((I \times \hat{\Omega}, I \otimes \hat{\mathbb{F}}, \lambda \otimes \hat{P}^{\tilde{\omega}})\) with transition matrix \( z^n \) at time \( n - 1 \).

2. \( \{\beta^n_i\}_{n=0}^{\infty} \) is also a Markov chain with transition matrix \( z^n \) at time \( n - 1 \).

**Proof.** Fix \( \hat{\omega} \in \hat{\Omega} \).

1. The Markov property of \( \{\beta^n_i\}_{n=0}^{\infty} \) on \((I \times \hat{\Omega}, I \otimes \hat{\mathbb{F}}, \lambda \otimes \hat{P}^{\tilde{\omega}})\) follows by using the same arguments as in the proof of Lemma 5 in [2], for \( \lambda \)-almost all \( i \in I \). We now derive the transition matrix with similar arguments as in [2]. By putting together (3.7), (3.8) and (3.15), we get

\[ E^{\hat{\tilde{\omega}}}[\hat{p}^n_{k_{1,1}}] = \sum_{k_{1,1} \in S} (1 - \xi_{k_{1,1}}(\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n]))\sigma_{k_{1,1}}[k, l] (\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n]) \cdot \eta_{k_{1,1}}(\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n]) E^{\hat{\tilde{\omega}}}[\hat{p}^n_{k_{1,1}}] \]

Thus we have

\[ z^n_{(k', l')(k,l)}(\hat{\omega}) = \sum_{k_{1,1} \in S} (1 - \xi_{k_{1,1}}(\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n]))\sigma_{k_{1,1}}[k, l] (\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n]) \cdot \eta_{k_{1,1}}(\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n]) \quad (3.17) \]

and

\[ z^n_{(k', l')(k,l)}(\hat{\omega}) = \sum_{k_{1,1} \in S} (1 - \xi_{k_{1,1}}(\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n]))\sigma_{k_{1,1}}[k, l] (\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n]) \eta_{k'_{1,1}}(\hat{\omega}, n, E^{\hat{\tilde{\omega}}}[\hat{p}^n]) \]
\[ \text{Similarly, equations (3.7), (3.8) and (3.16) yield to } \]
\[
\mathbb{E}^{\hat{\tilde{P}}}[\hat{\tilde{P}}_{k,l}^{n}] = \sum_{k' \in S} b_k(\tilde{\omega}, n, \tilde{\omega}_l) \eta_{k,k'}(\tilde{\omega}, n, \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}_{k,l}^{n}]) \mathbb{E}^{\hat{\tilde{P}}}[\hat{\tilde{P}}_{k,l}^{n-1}] \\
+ \sum_{k_1, l_1, k', \tilde{P} \in S} \xi_{k_1,l_1}(\tilde{\omega}, n, \tilde{\omega}) \eta_{k_1,k'}(\tilde{\omega}, n, \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}_{k,l}^{n-1}]) \mathbb{E}^{\hat{\tilde{P}}}[\hat{\tilde{P}}_{k,l}^{n-1}] \\
+ \sum_{k_1, l_1, k'} \xi_{k_1,l_1}(\tilde{\omega}, n, \tilde{\omega}_l) \eta_{k_1,k'}(\tilde{\omega}, n, \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}_{k,l}^{n-1}]) \theta_{k_1,l_1}(\tilde{\omega}, n, \tilde{\omega}_l) \\
+ \sum_{k_1, l_1, k'} \xi_{k_1,l_1}(\tilde{\omega}, n, \tilde{\omega}_l) \eta_{k_1,k'}(\tilde{\omega}, n, \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}_{k,l}^{n-1}]) \theta_{k_1,l_1}(\tilde{\omega}, n, \tilde{\omega}_l).
\]

Therefore, the transition probabilities from time \( n - 1 \) to time \( n \) can be written as
\[
\mathbb{E}^{\hat{\tilde{P}}}[\hat{\tilde{P}}_{k,l}^{n}] = \sum_{k_1, l_1 \in S} b_k(\tilde{\omega}, n, \tilde{\omega}_l) \eta_{k,k'}(\tilde{\omega}, n, \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}_{k,l}^{n-1}]) \mathbb{E}^{\hat{\tilde{P}}}[\hat{\tilde{P}}_{k,l}^{n-1}] \\
+ \sum_{k_1, l_1, k'} \xi_{k_1,l_1}(\tilde{\omega}, n, \tilde{\omega}_l) \eta_{k_1,k'}(\tilde{\omega}, n, \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}_{k,l}^{n-1}]) \mathbb{E}^{\hat{\tilde{P}}}[\hat{\tilde{P}}_{k,l}^{n-1}] \\
+ \sum_{k_1, l_1, k'} \xi_{k_1,l_1}(\tilde{\omega}, n, \tilde{\omega}_l) \eta_{k_1,k'}(\tilde{\omega}, n, \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}_{k,l}^{n-1}]) \theta_{k_1,l_1}(\tilde{\omega}, n, \tilde{\omega}_l).
\]

2. The transition matrix of \( \{\beta^n\}_{n=0}^{\infty} \) at time \( n - 1 \) can be derived by using (3.17)-(3.20) and the Fubini property applied to \( \lambda \otimes \hat{\tilde{P}} \hat{\tilde{P}} \) for every fixed \( \tilde{\omega} \in \tilde{\Omega} \) as in the proof of Lemma 6 in [2].

We are now able to prove Theorem 3.14 in [1], which we present here.

Theorem 3.4. Assume that the discrete dynamical system \( D \) introduced in Definition 3.6 in [3] is Markov conditionally independent given \( \tilde{\omega} \in \tilde{\Omega} \) according to Definition 3.8 in [3]. Given \( \tilde{\omega} \in \tilde{\Omega} \), the following holds:

1. For each \( n \geq 1 \), \( \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^n] = \Gamma^n(\tilde{\omega}, \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^{n-1}]) \).

2. For each \( n \geq 1 \), we have
\[
\mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^n_{k,l}] = \sum_{k_1, l_1 \in S} \eta_{k_1,k}(\tilde{\omega}, n, \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^{n-1}_{k,l}]) \eta_{l_1,l}(\tilde{\omega}, n, \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^{n-1}_{k,l}]) \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^n_{k,l}] \\
\]

and
\[
\mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^n_{k,l}] = \sum_{k_1 \in S} \eta_{k_1,k}(\tilde{\omega}, n, \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^{n-1}_{k_1,l}]) \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^n_{k_1,l}].
\]

3. For each \( n \geq 1 \), we have
\[
\mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^n_{k,l}] = \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^n_{k,l}] + \theta_{k,l}(\tilde{\omega}, n, \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^{n-1}]) \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^n_{k,l}]
\]

and
\[
\mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^n_{k,l}] = \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^n_{k,l}] + \theta_{k,l}(\tilde{\omega}, n, \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^{n-1}]) \mathbb{E}^{\hat{\tilde{P}}}[\tilde{P}^n_{k,l}].
\]
4. For $\lambda$-almost every agent $i$, the extended-type process $\{\beta^{n}_{i}\}_{n=0}^{\infty}$ is a Markov chain in $\hat{S}$ on $(I \times \hat{\Omega}, \mathcal{I} \otimes \hat{\mathcal{F}}, \lambda \otimes \hat{P}^{\omega})$, whose transition matrix $z^{n}$ at time $n - 1$ is given by

$$
z^{n}_{(k,l)(k,l)}(\tilde{\omega}) = \sum_{k, l, k', l' \in S} (1 - \xi_{k,l,i}(\tilde{\omega}, n, \tilde{p}^{\omega,n})) \eta_{k,l,i}(\tilde{\omega}, n, \tilde{p}^{\omega,n}) \theta_{k,l,i}(\tilde{\omega}, n, \tilde{p}^{\omega,n}) \\
\quad \cdot \eta_{k',k}(\tilde{\omega}, n, \hat{P}^{\omega}[p^{n-1}])
$$

(3.21)

$$
z^{n}_{(k,l')(k,l)}(\tilde{\omega}) = \sum_{k, l, k', l' \in S} (1 - \xi_{k,l,i}(\tilde{\omega}, n, \tilde{p}^{\omega,n})) \eta_{k,l,i}(\tilde{\omega}, n, \tilde{p}^{\omega,n}) \eta_{k',l}(\tilde{\omega}, n, \hat{P}^{\omega}[p^{n-1}])
$$

(3.22)

$$
z^{n}_{(k',l)(k,l)}(\tilde{\omega}) = \sum_{k, l, k', l' \in S} \xi_{k,l,i}(\tilde{\omega}, n, \tilde{p}^{\omega,n}) \eta_{k,l,i}(\tilde{\omega}, n, \tilde{p}^{\omega,n}) \theta_{k,l,i}(\tilde{\omega}, n, \tilde{p}^{\omega,n})
$$

(3.23)

$$
z^{n}_{(k',l')(k,l)}(\tilde{\omega}) = b_{k}(\tilde{\omega}, n, \tilde{p}^{\omega,n}) \eta_{k,k'}(\tilde{\omega}, n, \hat{P}^{\omega}[p^{n-1}])
$$

(3.24)

5. For $\lambda$-almost every $i$ and every $\lambda$-almost every $j$, the Markov chains $\{\beta^{n}_{i}\}_{n=0}^{\infty}$ and $\{\beta^{n}_{j}\}_{n=0}^{\infty}$ are independent on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}^{\omega})$.

6. For $\hat{P}^{\omega}$-almost every $\tilde{\omega} \in \hat{\Omega}$, the cross sectional extended type process $\{\beta^{n}_{\omega}\}_{n=0}^{\infty}$ is a Markov chain on $(I, \mathcal{I}, \lambda)$ with transition matrix $z^{n}$ at time $n - 1$, which is defined in (3.21)–(3.24).

7. We have $\hat{P}^{\omega}$-a.s. that

$$
\mathbb{E}^{\hat{P}^{\omega}}[\hat{p}_{kl}^{n}] = \hat{p}_{kl}^{n} \quad \text{and} \quad \mathbb{E}^{\hat{P}^{\omega}}[\hat{p}_{kl}^{\omega,n}] = \hat{p}_{kl}^{n} \quad \text{and} \quad \mathbb{E}^{\hat{P}^{\omega}}[\hat{p}_{kl}^{n}] = \hat{p}_{kl}^{n}.
$$

Proof. Fix $\tilde{\omega} \in \hat{\Omega}$. Points 1. to 5. of Theorem 3.14 in [1] follow directly by Lemma [3.1, 3.2 and 3.3]. Moreover, Points 6. and 7. can be proven by using the same arguments as in the proof of Theorem 4 in [2].

References
