A dynamic version of the super-replication theorem under proportional transaction costs

Francesca Biagini* Thomas Reitsam*†

July 6, 2021

Abstract

We extend the super-replication theorems of [27] in a dynamic setting, both in the numéraire-based as well as in the numéraire-free setting. For this purpose, we generalize the notion of admissible strategies. In particular, we obtain a well-defined super-replication price process, which is right-continuous under some regularity assumptions.

Keywords: super-replication, proportional transaction costs, consistent price systems

Mathematics Subject Classification (2020): 91G15, 91G20, 60G07

JEL Classification: C65, G13

1 Introduction

In this paper, we provide a dynamic approach for super-replication dualities in market models with proportional transaction costs and finite time horizon T.
Since [11], the concept of super-replication has been thoroughly studied in a variety of market models. For the frictionless case, we refer to [1], [2], [4], [6], [11], [19], [21], [22], [23], [28]. Among other market models, super-replication dualities were established under proportional transaction costs, see e.g. [5], [7], [8], [15], [27], [28].
The aim of this paper is to extend the results from Theorem 4.1 of [5], and Theorems 1.4 and 1.5 of [27] to a dynamic version in order to obtain a well-defined super-replication price process in market models with transaction costs. In the frictionless case the super-replication price process is a supermartingale, see [11], [19]. In general, this is not the case in presence of transaction costs. In [3], Proposition 3.4 provides a dynamic version of the super-replication duality in the local case for a special case, where the liquidation value of the contingent claim is bounded from below by a random variable satisfying strong integrability conditions. Here, we formulate the dynamic super-replication dualities in the

*Workgroup Financial and Insurance Mathematics, Department of Mathematics, Ludwig-Maximilians Universität, Theresienstrae 39, 80333 Munich, Germany. Emails: biagini@math.lmu.de, reitsam@math.lmu.de.

†The financial support of the Verein zur Versicherungswissenschaft München e.V. is gratefully acknowledged

local and non-local case in a more general setting.

For this purpose, we follow the approach of [5] and [27]. In [5], the super-replication duality for $t = 0$ is proved for the numéraire-free scenario. In [27], the result of [5] is formulated in a one-dimensional setting and extended to a numéraire-based version. The proofs of Theorem 4.1 of [5] and Theorem 1.5 of [27] strongly rely on the bipolar theorem of [16].

In our approach, a fundamental role is played by the definition of admissible strategies. More precisely, we consider strategies on $[s,T]$, for $s \in [0,T]$, with random initial endowments depending on the information available at time s. The liquidation value of the corresponding portfolio is allowed to be be bounded from below by a random variable rather than by a constant, see Definition 2.5 and [3]. In this general setting, we then prove an analogous version of the bipolar theorem of [16], see Theorem 3.5, and the super-replication duality for the numéraire-free case, see Theorem 3.6. Finally, we show that the obtained super-replication price process is well-defined. Further, we provide sufficient conditions such that the super-replication price process is right-continuous. Our analysis is motivated by the study of asset price bubbles in the presence of transaction costs, see [3].

The paper is organized as follows. In Section 2, we present the setting, define admissible strategies and provide a dual representation for consistent local price systems. In Section 3, we provide an extended version of the bipolar theorem, see Theorem 3.5, and prove super-replication results for the dynamic setting in Theorem 3.6 and 3.7, respectively. Then, in Section 3.3, we elaborate further properties of the super-replication price process. In particular, we provide sufficient conditions such that the super-replication price process is right-continuous.

2 Setting

Let $T > 0$ describe a finite time horizon and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbf{P})$ be a filtered probability space, where the filtration $\mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}$ satisfies the usual conditions of right-continuity and saturatedness with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. We consider a financial market model consisting of a risk-free asset B, normalized to $B \equiv 1$, and a risky asset S. Throughout the paper we assume that $S = (S_t)_{0\leq t\leq T}$ is an $\mathbb{F}$-adapted stochastic process, with càdlàg and strictly positive paths such that $S_t \in L^1_+(\mathcal{F}_t, \mathbf{P})$ for all $t \in [0,T]$. For trading the risky asset in the market model, proportional transaction costs $0 < \lambda < 1$ are charged, i.e., to buy one share of S at time t the trader has to pay $(1+\lambda)S_t$ and for selling one share of S at time t the trader receives $(1-\lambda)S_t$. The interval $[(1-\lambda)S_t, (1+\lambda)S_t]$ is called *bid-ask-spread.*

Definition 2.1. For $0 \leq s < t \leq T$, we call $\mathrm{CPS}(s,t)$ (resp. $\mathrm{CPS}_{\mathrm{loc}}(s,t)$) the family of pairs $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ such that $\mathbf{Q}$ is a probability measure on $\mathcal{F}_t$, $\mathbf{Q} \sim \mathbf{P}|_{\mathcal{F}_t}$, $\tilde{S}^{\mathbf{Q}}$ is a martingale (resp. local martingale) under $\mathbf{Q}$ on $[s,t]$, and

$$(1-\lambda)S_u \leq \tilde{S}^{\mathbf{Q}}_u \leq (1+\lambda)S_u, \quad \text{for } s \leq u \leq t. \tag{2.1}$$

A pair $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ in $\mathrm{CPS}(s,t)$ (resp. $\mathrm{CPS}_{\mathrm{loc}}(s,t)$) is called *consistent price system* (resp. *consistent local price system*). By $\mathcal{Q}(s,T)$ (resp. $\mathcal{Q}_{\mathrm{loc}}(s,T)$) we denote the set of measures $\mathbf{Q}$ such that there exists a pair $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}(s,T)$ (resp. $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(s,T)$). Further, we write $L^p(\mathcal{F}_s, \mathcal{Q}) := \bigcap_{\mathbf{Q}\in\mathcal{Q}(s,T)} L^p(\mathcal{F}_s, \mathbf{Q})$ and $L^p(\mathcal{F}_s, \mathcal{Q}_{\mathrm{loc}}) := \bigcap_{\mathbf{Q}\in\mathcal{Q}_{\mathrm{loc}}(s,T)} L^p(\mathcal{F}_s, \mathbf{Q})$.

By $L^p_+(\mathcal{F}_s, \mathcal{Q})$ (resp. $L^p_+(\mathcal{F}_s, \mathcal{Q}_{\mathrm{loc}})$) we denote the space of $[0,\infty)$-valued random variables $X \in L^p(\mathcal{F}_s, \mathcal{Q})$ (resp. $X \in L^p(\mathcal{F}_s, \mathcal{Q}_{\mathrm{loc}})$).

It is well-known, that in the frictionless case the no-arbitrage condition no free lunch with vanishing risk (NFLVR) is equivalent to the existence of an equivalent local martingale measure, see [9]. In particular, the price process must be a semi-martingale and admit an equivalent local martingale measure. In contrast, under proportional transaction costs the existence of consistent (local) price systems guarantee the absence of arbitrage in the sense of Definition 4 of [14].
A consistent (local) price system can be thought as a frictionless market with better conditions for traders, see [13]. Considering consistent price systems in the non-local or local sense corresponds in the frictionless case to the characterization of no arbitrage using true martingales or local martingales. In both cases the difference lies in the choice of admissible trading strategies. If we fix a numéraire, we can control the portfolio in units of the numéraire, and we do not allow short positions in the risky asset. Without numéraire, we also admit portfolios with short positions in the assets. See Chapter 5 of [14] for a more detailed discussion. For the convenience of the reader, we here state the assumptions that we use through out the paper.

Assumption 2.2. We assume that S admits a consistent *local* price system for every $0 < \lambda' \le \lambda$.

Assumption 2.3. We assume that S admits a consistent price system for every $0 < \lambda' \le \lambda$.

We follow the approach of [3] and define admissible trading strategies as follows.

Definition 2.4. Fix $s \in [0,T]$. A *self-financing trading strategy* starting with initial endowment $(X^1_s, X^2_s) \in L^0_+(\mathcal{F}_s, \mathbf{P}) \times L^0_+(\mathcal{F}_s, \mathbf{P})$ is a pair of $\mathbb{F}$-predictable finite variation processes $(\varphi^1_t, \varphi^2_t)_{s\le t\le T}$ on $[s,T]$ such that

(i) $\varphi^1_s = X^1_s$ and $\varphi^2_s = X^2_s$,

(ii) denoting by $\varphi^1_t = \varphi^1_s + \varphi^{1,\uparrow}_t - \varphi^{1,\downarrow}_t$ and $\varphi^2_t = \varphi^{2,\uparrow}_t - \varphi^{2,\downarrow}_t$, the Jordan-Hahn decomposition of φ^1 and φ^2 into the difference of two non-decreasing processes, starting at $\varphi^{1,\uparrow}_s = \varphi^{1,\downarrow}_s = \varphi^{2,\uparrow}_s = \varphi^{2,\downarrow}_s = 0$, these processes satisfy

$$d\varphi^1_t \le (1-\lambda)S_t d\varphi^{2,\downarrow}_t - (1+\lambda)S_t d\varphi^{2,\uparrow}_t, \quad s \le t \le T. \tag{2.2}$$

Definition 2.5. Fix $s \in [0,T]$.

(i) Let $X_s \in L^1_+(\mathcal{F}_s, \mathcal{Q}_{\mathrm{loc}})$. A self-financing trading strategy $\varphi = (\varphi^1, \varphi^2)$ is called *admissible in a numéraire-based* sense on $[s,T]$ starting with initial endowment $\varphi_s = (X_s, 0)$ if there is $M_s \in L^1_+(\mathcal{F}_s, \mathcal{Q}_{\mathrm{loc}})$ such that the liquidation value V^{liq}_τ satisfies

$$V^{liq}_\tau(\varphi^1, \varphi^2) := \varphi^1_\tau + \left(\varphi^2_\tau\right)^+ (1-\lambda)S_\tau - \left(\varphi^2_\tau\right)^- (1+\lambda)S_\tau \ge -M_s, \tag{2.3}$$

for all $[s,T]$-valued stopping times τ.

(ii) Let $(X_s^1, X_s^2) \in L^1_+(\mathcal{F}_s, \mathcal{Q}) \times L^\infty_+(\mathcal{F}_s, \mathcal{Q})$. A self-financing trading strategy $\varphi = (\varphi^1, \varphi^2)$ is called *admissible in a numéraire-free* sense on $[s,T]$ starting with initial endowment $\varphi_s = (X_s^1, X_s^2)$ if there is $M_s := (M_s^1, M_s^2) \in L^1_+(\mathcal{F}_s, \mathcal{Q}) \times L^\infty_+(\mathcal{F}_s, \mathcal{Q})$ such that

$$V_\tau^{liq}(\varphi^1, \varphi^2) := \varphi_\tau^1 + \left(\varphi_\tau^2\right)^+ (1-\lambda) S_\tau - \left(\varphi_\tau^2\right)^- (1+\lambda) S_\tau \geq -M_s^1 - M_s^2 S_\tau, \tag{2.4}$$

for all $[s,T]$-valued stopping times τ.

We call a strategy satisfying (2.3) or (2.4) M_s-admissible in a numéraire-based or numéraire-free sense, respectively.

The numéraire-based definition of admissibility corresponds to the setting of consistent local price systems. The portfolio is bounded from below, i.e., can be hedged in units of the numéraire. In particular, no short positions in the risky asset are admissible.
On the other side, the numéraire-free version of admissible trading strategy is used in the context of consistent price systems in the non-local sense. In this case no natural numéraire is needed and the portfolio is bounded from below by a position which depends on each of the assets. Thus, also short positions in the risky asset are admissible.

Definition 2.6. A contingent claim $X_T = (X_T^1, X_T^2)$ is an $\mathcal{F}_T$-measurable random variable in $L^0(\mathbb{R}^2; \mathcal{F}_T, \mathbf{P})$ which pays X_T^1 units of the bond and X_T^2 units of the risky asset at time T.

Note that by Definition 2.6 a contingent claim is not assumed to be strictly positive. However, in the sequel we will require some lower bound properties depending on the setting, see (2.3) and (2.4).

Definition 2.7. For $M_s := (M_s^1, M_s^2) \in L^1_+(\mathcal{F}_s, \mathcal{Q}) \times L^\infty_+(\mathcal{F}_s, \mathcal{Q})$ (resp. $M_s \in L^1_+(\mathcal{F}_s, \mathcal{Q}_{\text{loc}})$) we denote by $\mathcal{A}_{s,T}^{M_s}$ (resp. $\mathcal{A}_{s,T}^{M_s,\text{loc}}$) the set of pairs $(\varphi_T^1, \varphi_T^1) \in L^0(\mathbb{R}^2; \mathcal{F}_T, \mathbf{P})$ of terminal values of self-financing trading strategies φ, starting at $\varphi_s = (0,0)$, which are M_s-admissible in the numéraire-free sense (resp. numéraire-based sense). Further, we denote

$$\mathcal{A}_{s,T} := \left\{\varphi_T : \varphi_T \in \mathcal{A}_{s,T}^{M_s} \text{ for some } M_s = (M_s^1, M_s^2) \in L^1_+(\mathcal{F}_s, \mathcal{Q}) \times L^\infty_+(\mathcal{F}_s, \mathcal{Q})\right\}. \tag{2.5}$$

We now introduce a dual theory for consistent (local) price systems, see also [5], [14], [16]. For fixed $\lambda > 0$ we denote by K_t the solvency cone at time t, defined as

$$K_t(\omega) = \text{cone}\left\{(1+\lambda)S_t(\omega)e_1 - e_2, -e_1 + \frac{1}{(1-\lambda)S_t(\omega)}e_2\right\}, \tag{2.6}$$

where $e_1 = (1,0), e_2 = (0,1)$ are the unit vectors in $\mathbb{R}^2$, and by $K_t^* = (-K_t)^\circ$ the corresponding polar cone, given by

$$\begin{aligned} K_t^*(\omega) = (-K_t)^\circ(\omega) &= \left\{(y_1, y_2) \in \mathbb{R}^2_+ : (1-\lambda)S_t(\omega) \leq \frac{y_2}{y_1} \leq (1+\lambda)S_t(\omega)\right\} \cup \{0\} \\ &= \left\{y \in \mathbb{R}^2 : \langle x, y\rangle \leq 0, \forall x \in (-K_t(\omega))\right\} \\ &= \left\{y \in \mathbb{R}^2 : \langle x, y\rangle \geq 0, \ \forall x \in K_t(\omega)\right\}. \end{aligned} \tag{2.7}$$

Definition 2.8. Let $s \in [0,T]$. We define $\mathcal{Z}(s,T)$ (resp. $\mathcal{Z}_{\text{loc}}(s,T)$) as the set of processes $Z = (Z_t^1, Z_t^2)_{s \leq t \leq T}$ such that Z^1 is a $\mathbf{P}$-martingale and Z^2 is a $\mathbf{P}$-martingale (resp. local $\mathbf{P}$-martingale) and such that $Z_t \in K_t^* \backslash \{0\}$ a.s. for all $t \in [s,T]$.

The following proposition from [14] provides a useful representation of consistent (local) price systems by elements in $\mathcal{Z}$ (resp. $\mathcal{Z}_{\text{loc}}$) and follows directly from the definition of K_t^* in (2.7).

Proposition 2.9 (Proposition 3, [14]). *Let $s \in [0,T]$ and $Z = (Z_t^1, Z_t^2)_{s \le t \le T}$ be a 2-dimensional stochastic process with $Z_T^1 \in L^1(\mathcal{F}_T, \boldsymbol{P})$. Define the measure $\boldsymbol{Q}(Z) \ll \boldsymbol{P}$ by $d\boldsymbol{Q}(Z)/d\boldsymbol{P} := Z_T^1/\mathbb{E}[Z_T^1]$. Then $Z \in \mathcal{Z}(s,T)$ (resp. $Z \in \mathcal{Z}_{loc}(s,T)$) if and only if $(\boldsymbol{Q}(Z), (Z^2/Z^1))$ is a consistent price system (resp. consistent local price system) on $[s,T]$.*

3 Dynamic super-replication

3.1 A Bipolar Theorem in a dynamic setting

In this section we aim to provide a bipolar theorem which will then be used to apply techniques from [5] and [27] in the proof of the super-replication theorems. Theorem 3.5 extends the bipolar theorem of [16], see also [17].

Definition 3.1. We define the partial order $\geq$ on $L^0(\mathbb{R}^2; \mathcal{F}_T, \mathbf{P})$ by letting $\varphi \geq \psi$ if and only if $V_T^{liq}(\varphi^1 - \psi^1, \varphi^2 - \psi^2) \geq 0$, i.e., if the portfolio $\varphi - \psi$ can be liquidated to the zero portfolio.
We say that a set $\Phi \subset L^0(\mathbb{R}^2; \mathcal{F}_T, \mathbf{P})$ is directed upwards if for $\psi_1, \psi_2 \in \Phi$ there exists $\psi \in \Phi$ with $\psi \geq \psi_1 \vee \psi_2$.

We denote by $L^0_{1,\infty}$ the cone in $L^0(\mathbb{R}^2; \mathcal{F}_T, \mathbf{P})$ given by the random variables $\xi = (\xi^1, \xi^2)$ such that $(\xi^1, \xi^2) \geq (-M_s^1, -M_s^2)$ for some $(M_s^1, M_s^2) \in L^1_+(\mathcal{F}_s, \mathcal{Q}) \times L^\infty_+(\mathcal{F}_s, \mathcal{Q})$. Further, we denote by $L^0_b \subset L^0(\mathbb{R}^2; \mathcal{F}_T, \mathbf{P})$ the cone formed by the random variables ξ such that $(\xi^1, \xi^2) \geq (-M, -M)$ for some $M > 0$, following [17].

Remark 3.2. *Note that the conditional expectation*

$$\mathbb{E}_{\boldsymbol{P}}[\xi \cdot Z_T \mid \mathcal{F}_s], \quad Z_T \in L^1(K_T^*),\ \xi \in L^0_{1,\infty} \tag{3.1}$$

is well-defined. In fact, by the definition of $L^0_{1,\infty}$ there exists $M_s = (M_s^1, M_s^2) \in L^1(\mathbb{R}_+; \mathcal{F}_s, \mathcal{Q}) \times L^\infty(\mathbb{R}_+; \mathcal{F}_s, \mathcal{Q})$ such that $(\xi + M_s) \in K_T$ and hence $(\xi + M_s) \cdot Z_T \geq 0$. In particular, we get

$$\mathbb{E}_{\boldsymbol{P}}[\xi \cdot Z_T \mid \mathcal{F}_s] = \mathbb{E}_{\boldsymbol{P}}[(\xi + M_s) \cdot Z_T - M_s \cdot Z_T \mid \mathcal{F}_s].$$

For non-negative random variables the conditional expectation is always well-defined, although it might be infinity. Thus,

$$\mathbb{E}_{\boldsymbol{P}}[(\xi + M_s) \cdot Z_T \mid \mathcal{F}_s]$$

is well-defined. Furthermore, we need

$$\mathbb{E}_{\boldsymbol{P}}[M_s \cdot Z_T \mid \mathcal{F}_s] < \infty. \tag{3.2}$$

First, note that $M_s \cdot Z_T \geq 0$ and hence (3.2) is well-defined. Following Section 27 of [20], we use that M_s is $\mathcal{F}_s$-measurable and $Z_T \in L^1(K_T^*, \mathcal{F}_T, \boldsymbol{P})$ to conclude

$$\mathbb{E}_{\boldsymbol{P}}[M_s \cdot Z_T \mid \mathcal{F}_s] = M_s \mathbb{E}_{\boldsymbol{P}}[Z_T \mid \mathcal{F}_s] < \infty.$$

Therefore,

$$\mathbb{E}_{\boldsymbol{P}}[\xi \cdot Z_T \mid \mathcal{F}_s] \geq -M_s \mathbb{E}_{\boldsymbol{P}}[Z_T \mid \mathcal{F}_s] > -\infty$$

is well-defined. However, the expectation

$$\mathbb{E}_{\boldsymbol{P}}[M_s \cdot Z_T]$$

is in general not well-defined. In contrast, for $\eta \in L_b^0$ and $Z_T \in L^1(K_T^*; \mathcal{F}_T, \boldsymbol{P})$ as in the bipolar theorem of [16] there exists $M > 0$ such that

$$\mathbb{E}_{\boldsymbol{P}}[\eta \cdot Z_T] \geq -M \mathbb{E}_{\boldsymbol{P}}[Z_T] > -\infty$$

is well-defined.

We now extend the definition of Fatou convergence of [5], [17], [27].

Definition 3.3. Consider a sequence $(X_n)_{n\in\mathbb{N}} = (X_n^1, X_n^2)_{n\in\mathbb{N}} \subset L^0(\mathbb{R}^2; \mathcal{F}_T, \mathbf{P})$. We say that $(X_n)_{n\in\mathbb{N}}$ is $L^0(\mathcal{F}_s)$-Fatou converging to $X = (X^1, X^2)$ if $X_n \xrightarrow{a.s.} X$ and $(X_n^1, X_n^2) \geq (-M_s^1, -M_s^2)$ for all $n \in \mathbb{N}$ and some $(M_s^1, M_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$.

If $(-M_s^1, -M_s^2) = (-M, -M)$ for some $M \in \mathbb{R}_+$, $L^0(\mathcal{F}_s)$-Fatou convergence coincides with the Fatou convergence[1] as defined in [5], [17], [27].

Remark 3.4. *Note that Lemma 3.1 of [27] also holds in our setting. We omit the proof since it holds in our setting with minor modifications. The lemma ensures that the total variation of strategies, which are M_s-admissible in the numéraire-free sense, remains bounded in $L^0(\mathcal{F}_T, \boldsymbol{P})$. Furthermore, this implies that Theorem 3.4 and Theorem 3.6 of [27] hold true as well. Theorem 3.4 (resp. Theorem 3.6) of [27] guarantee that $A_{s,T}^{M_s} \subset L^0(\mathcal{F}_s, \boldsymbol{P})$ (resp. $A_{s,T}^{M_s,loc} \subset L^0(\mathcal{F}_s, \boldsymbol{P})$) is closed with respect to the topology of convergence in measure. For more details we refer to [25].*

Theorem 3.5. *Let $s \in [0, T]$. It holds that*

$$\mathcal{A}_{s,T} = \left\{ \xi \in L_{1,\infty}^0 : \mathbb{E}_{\boldsymbol{P}}[\xi \cdot Z_T \mid \mathcal{F}_s] \leq \operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T}} \mathbb{E}_{\boldsymbol{P}}[\eta \cdot Z_T \mid \mathcal{F}_s] \ \ \forall Z_T \in L^1(K_T^*; \mathcal{F}_T, \boldsymbol{P}) \right\}, \quad (3.3)$$

where $A_{s,T}$ is defined in (2.5).

Proof. The inclusion " $\subseteq$ " is trivial.
For the reverse inclusion we make use of the bipolar theorem of [16], Theorem 4.2, see

[1] Following [16], [27], let $(X_n)_{n\in\mathbb{N}} \subset L^0(\mathbb{R}^2; \mathcal{F}_T, \mathbf{P})$. We say that $(X_n)_{n\in\mathbb{N}}$ is Fatou converging to $X = (X^1, X^2)$ if $X_n \xrightarrow{a.s.} X$ and $(X_n^1, X_n^2) \geq (-M, -M)$ for all $n \in \mathbb{N}$ and some $M > 0$.

also Theorem 5.5.3 of [17]. Suppose the conditions of Theorem 4.2 of [16] are satisfied for $\mathcal{A}_{s,T} \cap L_b^0$. Then we obtain

$$\mathcal{A}_{s,T} \cap L_b^0 = \left\{ \xi \in L_b^0 : \mathbb{E}_{\mathbf{P}}\left[\xi \cdot Z_T\right] \le \sup_{\eta \in \mathcal{A}_{s,T} \cap L_b^0} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T\right] \;\; \forall Z_T \in L^1(K_T^*; \mathcal{F}_T, \mathbf{P}) \right\}. \tag{3.4}$$

First, we show

$$\mathbb{E}_{\mathbf{P}}\left[\operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T} \cap L_b^0} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T \mid \mathcal{F}_s\right] \right] = \sup_{\eta \in \mathcal{A}_{s,T} \cap L_b^0} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T\right]. \tag{3.5}$$

By monotonicity we have that

$$\mathbb{E}_{\mathbf{P}}\left[\operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T} \cap L_b^0} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T \mid \mathcal{F}_s\right] \right] \ge \sup_{\eta \in \mathcal{A}_{s,T} \cap L_b^0} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T\right]. \tag{3.6}$$

For $Z_T \in L^1(K_T^*; \mathcal{F}_T, \mathbf{P})$ we define $\Phi_{Z_T} := \{\mathbb{E}_{\mathbf{P}}[\eta Z_T \mid \mathcal{F}_s] : \eta \in L_b^0\}$. It is easy to see that Φ_{Z_T} is directed upwards, since for $\eta, \tilde{\eta} \in L_b^0$ we get for

$$D_s := \{\mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T \mid \mathcal{F}_s\right] \ge \mathbb{E}_{\mathbf{P}}\left[\tilde{\eta} \cdot Z_T \mid \mathcal{F}_s\right]\} \in \mathcal{F}_s \tag{3.7}$$

that $\psi := \eta \mathbb{1}_{D_s} + \tilde{\eta} \mathbb{1}_{D_s^c} \in L_b^0$ and

$$\mathbb{E}_{\mathbf{P}}\left[\psi \cdot Z_T \mid \mathcal{F}_s\right] \ge \left(\mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T \mid \mathcal{F}_s\right] \vee \mathbb{E}_{\mathbf{P}}\left[\tilde{\eta} \cdot Z_T \mid \mathcal{F}_s\right]\right). \tag{3.8}$$

By Theorem A.33 of [12] there exists $(\eta_n)_{n \in \mathbb{N}} \subset L_b^0$ such that

$$\mathbb{E}_{\mathbf{P}}\left[\eta_n \cdot Z_T \mid \mathcal{F}_s\right] \uparrow \operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T} \cap L_b^0} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T \mid \mathcal{F}_s\right], \text{ as } n \to \infty. \tag{3.9}$$

Hence, we get

$$\mathbb{E}_{\mathbf{P}}\left[\operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T} \cap L_b^0} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T \mid \mathcal{F}_s\right] \right] = \mathbb{E}_{\mathbf{P}}\left[\lim_{n \to \infty} \mathbb{E}_{\mathbf{P}}\left[\eta_n \cdot Z_T \mid \mathcal{F}_s\right] \right]. \tag{3.10}$$

Note that $0 \in \mathcal{A}_{s,T} \cap L_b^0$ and thus $\operatorname{ess\,sup}_{\eta \in \mathcal{A}_{s,T} \cap L_b^0} \mathbb{E}_{\mathbf{P}}[\eta Z_T \mid \mathcal{F}_s] \ge 0$. In particular we can assume without loss of generality that $\mathbb{E}_{\mathbf{P}}[\eta_n \cdot Z_T \mid \mathcal{F}_s] \ge 0$ a.s. for all $n \in \mathbb{N}$. Thus we obtain by monotone convergence

$$\mathbb{E}_{\mathbf{P}}\left[\lim_{n \to \infty} \mathbb{E}_{\mathbf{P}}\left[\eta_n \cdot Z_T \mid \mathcal{F}_s\right] \right] = \lim_{n \to \infty} \mathbb{E}_{\mathbf{P}}\left[\eta_n \cdot Z_T\right] \le \sup_{\eta \in \mathcal{A}_{s,T} \cap L_b^0} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T\right], \tag{3.11}$$

and thus (3.5) is fulfilled. Therefore, we have that

$$\left\{ \xi \in L_b^0 : \mathbb{E}_{\mathbf{P}}\left[\xi \cdot Z_T \mid \mathcal{F}_s\right] \le \operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T} \cap L_b^0} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T \mid \mathcal{F}_s\right], \; \forall Z_T \in L^1(K_T^*; \mathcal{F}_T, \mathbf{P}) \right\} \tag{3.12}$$

$$\subseteq \left\{ \xi \in L_b^0 : \mathbb{E}_{\mathbf{P}}\left[\xi \cdot Z_T\right] \le \sup_{\eta \in \mathcal{A}_{s,T} \cap L_b^0} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T\right], \; \forall Z_T \in L^1(K_T^*; \mathcal{F}_T, \mathbf{P}) \right\}. \tag{3.13}$$

In particular, under the assumption that the conditions for Theorem 4.2 of [16] are fulfilled we get

$$I := \left\{ \xi \in L_b^0 : \mathbb{E}_{\mathbf{P}}\left[\xi \cdot Z_T \mid \mathcal{F}_s\right] \le \operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T} \cap L_b^0} \mathbb{E}_{\mathbf{P}}\left[\eta \cdot Z_T \mid \mathcal{F}_s\right] \;\; \forall Z_T \in L^1(K_T^*; \mathcal{F}_T, \mathbf{P}) \right\} \subseteq \mathcal{A}_{s,T} \cap L_b^0. \tag{3.14}$$

We now prove that the assumptions of Theorem 4.2 of [16] are indeed fulfilled for $\mathcal{A}_{s,T} \cap L_b^0$. The arguments are similar to the proof of Theorem 4.1 of [5] and the proof of Theorem 1.5 of [27]. First, Theorem 3.6 of [27] and Remark 3.4 imply directly that $\mathcal{A}_{s,T} \cap L_b^0$ is Fatou-closed. By standard arguments $\mathcal{A}_{s,T} \cap L^\infty(\mathcal{F}_T, \mathbf{P})$ is dense in $\mathcal{A}_{s,T} \cap L_b^0$ with respect to Fatou-convergence. In fact, let $\varphi = (\varphi^1, \varphi^2) \in \mathcal{A}_{s,T} \cap L_b^0$, i.e. $\varphi \ge (-M, -M)$ for some $M > 0$. We define the sequence $\varphi_n := \varphi \mathbb{1}_{\{|\varphi| \le n\}} - (M, M)\mathbb{1}_{\{|\varphi| > n\}}$. Then $(\varphi_n)_{n \in \mathbb{N}} \subset (\mathcal{A}_{s,T} \cap L_b^0) \cap L^\infty(\mathcal{F}_T, \mathbf{P})$ and $\varphi_n \xrightarrow{a.s.} \varphi$. Furthermore, $\varphi_n \ge (-M, -M)$ for all $n \in \mathbb{N}$ which guarantees that φ_n Fatou-converges to φ and that $(\mathcal{A}_{s,T} \cap L_b^0) \cap L^\infty(\mathcal{F}_T, \mathbf{P})$ is dense with respect to Fatou-convergence in $\mathcal{A}_{s,T} \cap L_b^0$. From this construction we also observe that $-L^\infty(K_T; \mathcal{F}_T, \mathbf{P}) \subset \mathcal{A}_{s,T} \cap L_b^0$. Therefore, the conditions of Theorem 4.2 of [16] are satisfied for $\mathcal{A}_{s,T} \cap L_b^0$.
It is left to show that L_b^0 is dense in $L_{1,\infty}^0$ with respect to $L^0(\mathcal{F}_s)$-Fatou-convergence. To see this, let $\xi \in L_{1,\infty}^0$. Then $\xi \ge (-M_s^1, -M_s^2)$ for some $(M_s^1, M_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$. If we set $\xi_n := \xi^+ - (\xi^- \wedge n)$, then $\xi_n \in L_b^0$ for all $n \in \mathbb{N}$, $\xi_n \ge (-M_s^1, -M_s^2)$ and $\xi_n \xrightarrow{a.s.} \xi$. Furthermore, $\mathcal{A}_{s,T}$ is $L^0(\mathcal{F}_s)$-Fatou closed by Theorem 3.6 and Remark 3.4 and hence

$$\operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T} \cap L_b^0} \mathbb{E}_{\mathbf{P}}\left[\eta Z_T \mid \mathcal{F}_s\right] = \operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T}} \mathbb{E}_{\mathbf{P}}\left[\eta Z_T \mid \mathcal{F}_s\right],$$

and

$$\bar{I} = \left\{ \xi \in L_{1,\infty}^0 : \mathbb{E}_{\mathbf{P}}\left[\xi Z_T \mid \mathcal{F}_s\right] \le \operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T} \cap L_b^0} \mathbb{E}_{\mathbf{P}}\left[\eta Z_T \mid \mathcal{F}_s\right] \;\; \forall Z_T \in L^1(K_T^*; \mathcal{F}_T, \mathbf{P}) \right\},$$

where the closure is taken with respect to $L^0(\mathcal{F}_s)$-Fatou convergence. Then

$$\bar{I} \subset \overline{\mathcal{A}_{s,T} \cap L_b^0} = \mathcal{A}_{s,T}.$$

This concludes the proof. □

3.2 Super-replication theorems in a dynamic setting

In this section we prove dynamic super-replication results in the context of local and non-local consistent price systems, respectively. We start with the non-local version.

Theorem 3.6. *Let Assumption 2.3 hold and $s \in [0, T]$. Let $X_T = (X_T^1, X_T^2)$ be a contingent claim such that*

$$X_T^1 + (X_T^2)^+(1 - \lambda)S_T - (X_T^2)^-(1 + \lambda)S_T \ge -M_s^1 - M_s^2 S_T, \tag{3.15}$$

for some $(M_s^1, M_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$. For a random variable $X_s = (X_s^1, X_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$ the following assertions are equivalent:

(i) *There is a self-financing trading strategy* $\varphi = (\varphi_t^1, \varphi_t^2)_{s \le t \le T}$ *with* $\varphi_s = (X_s^1, X_s^2)$ *and* $\varphi_T = (X_T^1, X_T^2)$ *which is admissible in a numéraire-free sense on the interval* $[s, T]$, *see* (2.4).

(ii) *For every consistent price system* $(\boldsymbol{Q}, \tilde{S}^{\boldsymbol{Q}}) \in CPS(s, T)$ *we have*

$$\mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 - X_s^1 + (X_T^2 - X_s^2)\tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right] \le 0. \tag{3.16}$$

Proof of Theorem 3.6. "$i) \Rightarrow ii)$" Let $\varphi = (\varphi_t^1, \varphi_t^2)_{s \le t \le T}$ be an admissible strategy in the numéraire-free sense such that $\varphi_s = (X_s^1, X_s^2)$ and $\varphi_T = (X_T^1, X_T^2)$. By Proposition 3 of [26] and Remark 2.8 of [3], $(\varphi_t^1 + \varphi_t^2 \tilde{S}_t^{\mathbf{Q}})_{s \le t \le T}$ is an optional strong supermartingale for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}(s, T)$. In particular, we obtain

$$\mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] = \mathbb{E}_{\mathbf{Q}}\left[\varphi_T^1 + \varphi_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] \le \varphi_s^1 + \varphi_s^2 \tilde{S}_s^{\mathbf{Q}} = X_s^1 + X_s^2 \tilde{S}_s^{\mathbf{Q}}, \tag{3.17}$$

for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}(s, T)$.
"$ii) \Rightarrow i)$" Note that for any contingent claim X_T, there is an admissible strategy in the numéraire-free sense φ with $\varphi_s = (X_s^1, X_s^2)$ and $\varphi_T = (X_T^1, X_T^2)$ if and only if $\tilde{X}_T := (X_T^1 - X_s^1, X_T^2 - X_s^2) \in \mathcal{A}_{s,T}$, i.e., if there is an admissible strategy in the numéraire-free sense $\tilde{\varphi}$ with $\tilde{\varphi}_s = (0, 0)$ and $\tilde{\varphi}_T = (\tilde{X}_T^1, \tilde{X}_T^2)$. Thus, it is enough to show that for $\tilde{X}_T \notin \mathcal{A}_{s,T}$ there exists a consistent price system in the non-local sense $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}(s, T)$ such that

$$\mathbf{P}(\mathbb{E}_{\mathbf{Q}}[\tilde{X}_T^1 + \tilde{X}_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s] > 0) > 0. \tag{3.18}$$

Let $\tilde{X}_T \notin \mathcal{A}_{s,T}$, then by Theorem 3.5 there exists $Y = (Y^1, Y^2) \in L^1(K_T^*; \mathcal{F}_T, \mathbf{P})$ such that

$$\mathbf{P}\left(B^Y\right) > 0, \tag{3.19}$$

where

$$B^Y := \left\{\omega \in \Omega : \ \mathbb{E}_{\mathbf{P}}\left[Y^1 \tilde{X}_T^1 + Y^2 \tilde{X}_T^2 \mid \mathcal{F}_s\right](\omega) > \operatorname*{ess\,sup}_{\eta \in \mathcal{A}_{s,T}} \mathbb{E}_{\mathbf{P}}\left[\eta^1 Y^1 + \eta^2 Y^2 \mid \mathcal{F}_s\right](\omega)\right\} \in \mathcal{F}_s. \tag{3.20}$$

We now construct a consistent price system $\hat{Z} = (\hat{Z}_t^1, \hat{Z}_t^2)_{s \le t \le T}$ as follows. By Proposition 2.9 we can represent any consistent price system in the non-local sense $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ by a pair (Z^1, Z^2) such that Z^i is a $\mathbf{P}$-martingale for $i = 1, 2$ and Z_t^2 / Z_t^1 takes values in the bid-ask spread, i.e., $(Z_t^1, Z_t^2) \in K_t^* \backslash \{0\}$ almost surely. Conversely, if for a process (Z^1, Z^2) we have that $(Z_t^1, Z_t^2) \in K_t^* \backslash \{0\}$ and Z^i is a $\mathbf{P}$-martingale for $i = 1, 2$, then $d\mathbf{Q}(Z)/d\mathbf{P} := Z_T^1 / \mathbb{E}_{\mathbf{P}}[Z_T^1]$ and $\tilde{S}^{\mathbf{Q}}(Z) := Z^2 / Z^1$ defines a consistent price system in the non-local sense. We start by defining the process $Z = (Z_t^1, Z_t^2)_{s \le t \le T}$ by $Z_t^i := \mathbb{E}_{\mathbf{P}}\left[Y^i \mathbb{1}_{B^Y} \mid \mathcal{F}_t\right]$, $s \le t \le T$, $i = 1, 2$. Note that $0 \in \mathcal{A}_{s,T}$. Therefore we obtain by (3.19) and (3.20), that

$$\mathbf{P}(\mathbb{E}_{\mathbf{P}}[Z_T^1 \tilde{X}_T^1 + Z_T^2 \tilde{X}_T^2 \mid \mathcal{F}_s] > 0) \ge \mathbf{P}(B^Y) > 0.$$

We show that $Z_t \in K_t^*$ a.s. for $t \in [s, T]$. In particular, if Z was $\mathbb{R}_+ \backslash \{0\}$-valued it is a consistent price system in the non-local sense.
Consider the process

$$\psi_u := -\nu \gamma \mathbb{1}_{[t,T]}(u), \quad u \in [s, T], \tag{3.21}$$

for some $t \in [s,T]$ and arbitrary random variables $\nu \in L^\infty(\mathbb{R}_+;\mathcal{F}_t,\mathbf{P})$ and $\gamma \in L^\infty(K_T;\mathcal{F}_t,\mathbf{P})$. As ψ is almost surely bounded, ψ is an admissible strategy in the numéraire-free sense and

$$\mathbb{E}_{\mathbf{P}}\left[Y\cdot\psi_T \mathbb{1}_{B^Y} \mid \mathcal{F}_s\right] = \mathbb{E}_{\mathbf{P}}\left[Z_T\cdot\psi_T \mid \mathcal{F}_s\right] = -\mathbb{E}_{\mathbf{P}}\left[\nu\gamma\cdot Z_t \mid \mathcal{F}_s\right]. \tag{3.22}$$

Because $\psi \in \mathcal{A}_{s,T}$ (3.20) and (3.22) imply for $\omega \in B^Y$ that

$$\mathbb{E}_{\mathbf{P}}\left[\nu\gamma Z_t \mid \mathcal{F}_s\right](\omega) > -\mathbb{E}_{\mathbf{P}}\left[Y\tilde{X}_T\mathbb{1}_{B^Y} \mid \mathcal{F}_s\right](\omega) = -\left(\mathbb{E}_{\mathbf{P}}\left[Y\cdot\tilde{X}_T \mid \mathcal{F}_s\right]\mathbb{1}_{B^Y}\right)(\omega),$$

where we used that $B^Y \in \mathcal{F}_s$. For $\omega \in (B^Y)^c$ we have

$$\mathbb{E}_{\mathbf{P}}\left[\nu\gamma Z_t \mid \mathcal{F}_s\right](\omega) = -\left(\mathbb{E}_{\mathbf{P}}\left[Y\cdot\tilde{X}_T \mid \mathcal{F}_s\right]\mathbb{1}_{B^Y}\right)(\omega) = 0.$$

Because ν is arbitrary, we can deduce that $Z_t\gamma \geq 0$ for all $\gamma \in L^\infty(K_t;\mathcal{F}_t,\mathbf{P})$, yielding $Z_t \in K_t^*$ a.s.. Thus Z is a $\mathbf{P}$-martingale satisfying $Z_t \in K_t^*$ for all $t \in [s,T]$.
It is still possible that $\mathbf{P}(Z_t = 0) > 0$ for some $t \in [s,T]$ and thus Z is not necessarily a consistent price systems. We now construct the desired consistent price system $\hat{Z} = (\hat{Z}_t^1,\hat{Z}_t^2)_{t\in[s,T]}$ as follows. Take any consistent price system in the non-local sense $\tilde{Z} = (\tilde{Z}_t^1,\tilde{Z}_t^2)_{s\leq t\leq T}$. Then for a suitable[2] $\beta \in L^\infty((0,1);\mathcal{F}_s,\mathbf{P})$ we define $\hat{Z}_t^i := (\beta\tilde{Z}_t^i + (1-\beta)Z_t^i)$, $i = 1,2$. Then $\hat{Z}_t \in K_t^*\backslash\{0\}$, $t \in [s,T]$, and

$$\mathbf{P}\left(\mathbb{E}_{\mathbf{P}}\left[\hat{Z}_T^1\tilde{X}_T^1 + \hat{Z}_T^2\tilde{X}_T^2 \mid \mathcal{F}_s\right] > 0\right) > 0. \tag{3.23}$$

Clearly, $\hat{Z}$ is strictly positive and hence a consistent price system in the non-local sense on $[s,T]$. For $\tilde{X}_T \notin \mathcal{A}_{s,T}$ we have constructed a consistent price system in the non-local sense $\hat{Z} = (\hat{Z}_t^1,\hat{Z}_t^2)_{s\leq t\leq T}$ satisfying $\mathbf{P}(\mathbb{E}_{\mathbf{P}}[\hat{Z}_T^1\tilde{X}_T^1 + \hat{Z}_T^2\tilde{X}_T^2 \mid \mathcal{F}_s] > 0) > 0$. This concludes the proof. □

Now we can also prove the local or numéraire-based version of the super-replication theorem.

Theorem 3.7. *Let Assumption 2.2 hold and $s \in [0,T]$. Let $X_T = (X_T^1, X_T^2)$ be a contingent claim such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \geq -M_s \tag{3.24}$$

for some $M_s \in L_+^1(\mathcal{F}_s,\mathcal{Q}_{loc})$. For a random variable $X_s = (X_s^1,0) \in L^1(\mathcal{F}_s,\mathcal{Q}_{loc})$ the following assertions are equivalent:

(i) *There is a self-financing trading strategy $\varphi = (\varphi_t^1,\varphi_t^2)_{s\leq t\leq T}$ with $\varphi_s = (X_s^1,0)$ and $\varphi_T = (X_T^1,X_T^2)$ which is admissible in a numéraire-based sense on the interval $[s,T]$, see* (2.3).

[2]For instance:

$$\beta(\omega) := \begin{cases} 1, & \omega \in (B^Y)^c, \\ \frac{\mathbb{E}_{\mathbf{P}}[Z_T\cdot\tilde{X}_T|\mathcal{F}_s]}{|\mathbb{E}_{\mathbf{P}}[\tilde{Z}_T\cdot X_T|\mathcal{F}_s]|+\mathbb{E}_{\mathbf{P}}[Z_T\cdot\tilde{X}_T|\mathcal{F}_s]}, & \omega \in B^Y. \end{cases}$$

(ii) For every consistent price system $(\boldsymbol{Q}, \tilde{S}^{\boldsymbol{Q}}) \in CPS_{loc}(s,T)$ *we have*

$$\mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 - X_s^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right] \le 0. \tag{3.25}$$

Proof. The arguments are identical to the proof Theorem 1.4 of [27]. Note that Theorem 3.4 of [27] as well as Theorem 2 of [26] are also valid for our setting. For the proof and further details we refer to [25]. □

These duality results can also be formulated in the following way.

Proposition 3.8. *Let Assumption 2.2 hold and* $s \in [0,T]$. *Let* $X_T = (X_T^1, X_T^2)$ *be a contingent claim such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \ge -M_s \tag{3.26}$$

for some $M_s \in L_+^1(\mathcal{F}_s, \mathcal{Q}_{loc})$. *If*

$$\operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS_{loc}(s,T)} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right] \in L^1(\mathcal{F}_s, \mathcal{Q}_{loc}),$$

then we have

$$\begin{aligned} &\operatorname{ess\,inf}\left\{\xi_s^1 \in L_+^1(\mathcal{F}_s, \mathcal{Q}_{loc}) : \exists \varphi \in \mathcal{V}_{s,T}^{loc}(\xi_s, \lambda) \text{ with } \varphi_s = (\xi_s, 0),\ \varphi_T = X_T\right\} \\ = &\operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS_{loc}(s,T)} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right]. \end{aligned} \tag{3.27}$$

Proof. Since X_T satisfies the conditions of Theorem 3.7, we have

$$\begin{aligned} &\left\{X_s \in L^1(\mathcal{F}_s, \mathcal{Q}_{\text{loc}}) :\ \exists \varphi \in \mathcal{V}_{s,T}^{\text{loc}}(X_s, \lambda) \text{ with } \varphi_s = (X_s, 0) \text{ and } \varphi_T = (X_T^1, X_T^2)\right\} \\ = &\underbrace{\left\{X_s \in L^1(\mathcal{F}_s, \mathcal{Q}_{\text{loc}}) :\ \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right] \le X_s,\ \forall (\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \text{CPS}_{\text{loc}}(s,T)\right\}}_{=:D_s} \end{aligned} \tag{3.28}$$

It is left to show that

$$\operatorname{ess\,inf} D_s = \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\text{CPS}_{\text{loc}}(s,T)} \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s\right]. \tag{3.29}$$

For the first direction "≤" we get that $\operatorname{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\text{CPS}_{\text{loc}}(s,T)} \mathbb{E}_{\mathbf{Q}}[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s] \in D_s$, because $\operatorname{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\text{CPS}_{\text{loc}}(s,T)} \mathbb{E}_{\mathbf{Q}}[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s] \in L^1(\mathcal{F}_s, \mathcal{Q}_{\text{loc}})$.
For the reverse direction "≥" we have that $\operatorname{ess\,inf} D_s \ge \mathbb{E}_{\mathbf{Q}}[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s]$ for all $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \text{CPS}_{\text{loc}}(s,T)$, which implies by the definition of the essential supremum that

$$\operatorname{ess\,inf} D_s \ge \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\text{CPS}_{\text{loc}}(s,T)} \mathbb{E}_{\mathbf{Q}}[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_s].$$

□

Proposition 3.9. *Let Assumption 2.3 hold. Let $X_T = (X_T^1, X_T^2)$ be a contingent claim such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \geq -M_s^1 - M_s^2 S_\tau \tag{3.30}$$

for some $(M_s^1, M_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$. If

$$\operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS_{loc}(s,T)} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right] \in L^1(\mathcal{F}_s, \mathcal{Q}),$$

then we have

$$\begin{aligned} &\operatorname{ess\,inf}\{\xi_s \in L_+(\mathcal{F}_s, \mathcal{Q}) : \exists \varphi \in \mathcal{V}_{s,T}(\xi,\lambda) \text{ with } \varphi_s = (\xi_s, 0)\ \varphi_T = X_T\} \\ = &\operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS(s,T)} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right]. \end{aligned} \tag{3.31}$$

Proof. We obtain (3.31) with the same arguments as in the proof of Proposition 3.8. ☐

Theorem 3.10. *Let Assumption 2.2 hold and $s \in [0,T]$. Let $X_T = (X_T^1, X_T^2)$ be a contingent claim such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \geq -M_s \tag{3.32}$$

for some $M_s \in L_+^1(\mathcal{F}_s, \mathcal{Q}_{loc})$. Then the following identity holds:

$$\operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS_{loc}(s,T)} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right] = \operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS_{loc}(0,T)} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right].$$

Proof. The proof is analogous to the one of Theorem 3.9 of [3]. For further details, see [25]. ☐

Theorem 3.11. *Let Assumption 2.3 hold and $s \in [0,T]$. Let $X_T = (X_T^1, X_T^2)$ be a contingent claim such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \geq -M_s^1 - M_s^2 S_T \tag{3.33}$$

for some $(M_s^1, M_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$. Then the following identity holds:

$$\operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS(s,T)} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right] = \operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS(0,T)} \mathbb{E}_{\boldsymbol{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\boldsymbol{Q}} \mid \mathcal{F}_s\right].$$

Proof. The proof is analogous to the one of Theorem 3.9 of [3]. For further details, see [25]. ☐

From now on, we set $\mathrm{CPS_{loc}} := \mathrm{CPS_{loc}}(s,T,\lambda)$ and $\mathrm{CPS} := \mathrm{CPS}(s,T,\lambda)$, respectively. For sake of convenience, we use the following notation for a fixed claim $X_T = (X_T^1, X_T^2)$. We denote

$$F_t := \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in \mathrm{CPS_{loc}}} \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right], \quad t \in [0,T], \tag{3.34}$$

and

$$F_t^{\mathbf{Q}} := \mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right], \quad t \in [0,T]. \tag{3.35}$$

Next, we provide sufficient conditions such that $F = (F_t)_{t\in[0,T]}$ admits a right-continuous modification.

3.3 Right continuity

In this section we study the right-continuity of the process F defined in (3.34). For this purpose we need some preliminary result and further assumptions.

Lemma 3.12. *Let Assumption 2.2 hold and $s \in [0,T]$. Let $X_T = (X_T^1, X_T^2) \in L^0(\mathbb{R}^2; \mathcal{F}_T, \boldsymbol{P})$ such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \geq -M_s \tag{3.36}$$

for some $M_s \in L_+^1(\mathcal{F}_s, \mathcal{Q}_{loc})$. Then for any $\boldsymbol{Q}_0 \in \mathcal{Q}_{loc}$ the following identity holds

$$\mathbb{E}_{\boldsymbol{Q}_0}[F_t] = \sup_{(\boldsymbol{Q}, \tilde{S}^{\boldsymbol{Q}}) \in CPS_{loc}} \mathbb{E}_{\boldsymbol{Q}_0}\left[F_t^{\boldsymbol{Q}}\right], \quad t \in [s,T], \tag{3.37}$$

for F defined in (3.34).

Proof. Let $(\mathbf{Q}_0, \tilde{S}^{\mathbf{Q}_0}) \in \mathrm{CPS}_{\mathrm{loc}}(0,T)$ and $t \in [s,T]$. By monotonicity we immediately obtain

$$\mathbb{E}_{\mathbf{Q}_0}[F_t] \geq \sup_{(\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}} \mathbb{E}_{\mathbf{Q}_0}\left[F_t^{\mathbf{Q}}\right]. \tag{3.38}$$

For the reverse inequality we use Theorem 3.10 to show that

$$\Phi := \left\{F_t^{\mathbf{Q}} : (\mathbf{Q}, \tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS}_{\mathrm{loc}}(t,T)\right\}$$

is directed upwards, see Definition 3.1. Let $\mathbb{E}_{\mathbf{Q}}[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t], \mathbb{E}_{\bar{\mathbf{Q}}}[X_T^1 + X_T^2 \bar{S}_T^{\bar{\mathbf{Q}}} \mid \mathcal{F}_t] \in \Phi$. We construct $\mathbb{E}_{\widehat{\mathbf{Q}}}[X_T^1 + X_T^2 \widehat{S}^{\widehat{\mathbf{Q}}} \mid \mathcal{F}_t] \in \Phi$ such that $\mathbb{E}_{\widehat{\mathbf{Q}}}[X_T^1 + X_T^2 \widehat{S}_T^{\widehat{\mathbf{Q}}} \mid \mathcal{F}_t] \geq \mathbb{E}_{\mathbf{Q}}[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t] \vee \mathbb{E}_{\bar{\mathbf{Q}}}[X_T^1 + X_T^2 \bar{S}_T^{\bar{\mathbf{Q}}} \mid \mathcal{F}_t]$. Define

$$A_t := \left\{\mathbb{E}_{\mathbf{Q}}\left[X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t\right] \geq \mathbb{E}_{\bar{\mathbf{Q}}}\left[X_T^1 + X_T^2 \bar{S}_T^{\bar{\mathbf{Q}}} \mid \mathcal{F}_t\right]\right\} \in \mathcal{F}_t.$$

Let $Z = (Z^1, Z^2)$ and $\bar{Z} = (\bar{Z}^1, \bar{Z}^2)$ be the processes associated to $(\mathbf{Q}, \tilde{S}^{\mathbf{Q}})$ and $(\overline{\mathbf{Q}}, \bar{S}^{\bar{\mathbf{Q}}})$ respectively, as in Proposition 2.9. Then we define

$$\frac{d\widehat{\mathbf{Q}}}{d\mathbf{P}} = \frac{\widehat{Z}_T^1}{\mathbb{E}_{\mathbf{P}}\left[\widehat{Z}_T^1\right]} := \frac{\mathbb{1}_{A_t} Z_T^1 + \mathbb{1}_{A_t^c} \bar{Z}_T^1}{\mathbb{E}_{\mathbf{P}}\left[\mathbb{1}_{A_t} Z_T^1 + \mathbb{1}_{A_t^c} \bar{Z}_T^1\right]}, \tag{3.39}$$

and for $t \leq u \leq T$,

$$\widehat{Z}_u^2 := \mathbb{1}_{A_t} Z_u^2 + \mathbb{1}_{A_t^c} \bar{Z}_u^2 \tag{3.40}$$

with corresponding

$$\widehat{S}_u^{\widehat{\mathbf{Q}}} = \frac{\widehat{Z}_u^2}{\widehat{Z}_u^1}. \tag{3.41}$$

Obviously, $\widehat{Z} = (\widehat{Z}^1_u, \widehat{Z}^2_u)_{t\le u\le T}$ satisfies all requirements from Definition 2.8, i.e., $\widehat{Z} \in \mathcal{Z}_{\mathrm{loc}}(t,T)$. Clearly, $(1-\lambda)S_u \le \widehat{S}^{\widehat{\mathbf{Q}}}_u \le (1+\lambda)S_u$ for all $u \in [t,T]$. For the local martingale property let $(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for $\tilde{S}^{\mathbf{Q}}$ and $\bar{S}^{\bar{\mathbf{Q}}}$. For $t \le u \le v \le T$ we get

$$
\begin{aligned}
\mathbb{E}_{\widehat{\mathbf{Q}}}\left[\left(\widehat{S}^{\widehat{\mathbf{Q}}}_v\right)^{\tau_n} \mid \mathcal{F}_u\right] &= \mathbb{E}_{\mathbf{P}}\left[\left(\frac{\widehat{Z}^2_v}{\widehat{Z}^1_v}\right)^{\tau_n} \frac{\widehat{Z}^1_T}{\mathbb{E}_{\mathbf{P}}\left[\widehat{Z}^1_T\right]} \mid \mathcal{F}_u\right]\frac{\mathbb{E}_{\mathbf{P}}\left[\widehat{Z}^1_T\right]}{\widehat{Z}^1_{u\wedge\tau_n}}\\
&=\mathbb{E}_{\mathbf{P}}\left[\left(\mathbb{1}_{A_t}Z^1_v + \mathbb{1}_{A_t^c}\bar{Z}^2_v\right)^{\tau_n} \mid \mathcal{F}_u\right]\frac{1}{\widehat{Z}^1_{u\wedge\tau_n}} = \left(\mathbb{1}_{A_t}\mathbb{E}_{\mathbf{P}}\left[(Z^2_v)^{\tau_n} \mid \mathcal{F}_u\right] + \mathbb{1}_{A_t^c}\mathbb{E}_{\mathbf{P}}\left[(\bar{Z}^2_v)^{\tau_n} \mid \mathcal{F}_u\right]\right)\frac{1}{\widehat{Z}^1_{u\wedge\tau_n}}\\
&= \left(\mathbb{1}_{A_t}Z^2_{u\wedge\tau_n} + \mathbb{1}_{A_t^c}\bar{Z}^2_{u\wedge\tau_n}\right)\frac{1}{\widehat{Z}^1_{u\wedge\tau_n}} = \left(\widehat{S}_u\right)^{\tau_n},
\end{aligned}
$$

where we used that $\mathbb{1}_{A_t}, \mathbb{1}_{A_t^c}$ are measurable for $\mathcal{F}_t \subset \mathcal{F}_u$. In particular, by Theorem A.33 of [12], there exists an increasing sequence $(\mathbb{E}_{\mathbf{Q}^n}[X^1_T + X^2_T\tilde{S}^n_T \mid \mathcal{F}_t])_{n\in\mathbb{N}} \subset \Phi$ such that

$$
F_t = \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}_{\mathrm{loc}}(t,T)} \mathbb{E}_{\mathbf{Q}}\left[X^1_T + X^2_T\tilde{S}^{\mathbf{Q}}_T \mid \mathcal{F}_t\right] = \lim_{n\to\infty}\mathbb{E}_{\mathbf{Q}^n}\left[X^1_T + X^2_T\tilde{S}^n_T \mid \mathcal{F}_t\right]. \tag{3.42}
$$

By the Theorem of Monotone Convergence we obtain

$$
\begin{aligned}
\mathbb{E}_{\mathbf{Q}_0}[F_t] &= \lim_{n\to\infty}\mathbb{E}_{\mathbf{Q}_0}\left[\mathbb{E}_{\mathbf{Q}^n}\left[X^1_T + X^2_T\tilde{S}^n_T \mid \mathcal{F}_t\right]\right]\\
&\le \sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}_{\mathrm{loc}}(t,T,\lambda)} \mathbb{E}_{\mathbf{Q}_0}\left[\mathbb{E}_{\mathbf{Q}}\left[X^1_T + X^2_T\tilde{S}^{\mathbf{Q}}_T \mid \mathcal{F}_t\right]\right]\\
&= \sup_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS}_{\mathrm{loc}}(0,T,\lambda)} \mathbb{E}_{\mathbf{Q}_0}\left[\mathbb{E}_{\mathbf{Q}}\left[X^1_T + X^2_T\tilde{S}^{\mathbf{Q}}_T \mid \mathcal{F}_t\right]\right].
\end{aligned} \tag{3.43}
$$

The last equality in (3.43) holds due to similar arguments as in the proof of Theorem 3.10. This concludes the proof of (3.37). ☐

Lemma 3.13. *Let Assumption 2.3 hold and* $s \in [0,T]$. *Let* $X_T = (X^1_T, X^2_T) \in L^0(\mathbb{R}^2; \mathcal{F}_T, \boldsymbol{P})$ *such that*

$$
X^1_T + (X^2_T)^+(1-\lambda)S_T - (X^2_T)^-(1+\lambda)S_T \ge -M^1_s - M^2_s S_T \tag{3.44}
$$

for some $(M^1_s, M^2_s) \in L^1_+(\mathcal{F}_s, \mathcal{Q}) \times L^\infty_+(\mathcal{F}_s, \mathcal{Q})$. *Then for any* $\boldsymbol{Q}_0 \in \mathcal{Q}$ *the following identity holds*

$$
\mathbb{E}_{\boldsymbol{Q}_0}\left[\operatorname*{ess\,sup}_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS}\mathbb{E}_{\boldsymbol{Q}}\left[X^1_T + X^2_T\tilde{S}^{\boldsymbol{Q}}_T \mid \mathcal{F}_t\right]\right] = \sup_{(\boldsymbol{Q},\tilde{S}^{\boldsymbol{Q}})\in CPS}\mathbb{E}_{\boldsymbol{Q}_0}\left[F^{\boldsymbol{Q}}_t\right], \quad t \in [s,T]. \tag{3.45}
$$

Proof. The arguments are identical to the proof of Lemma 3.12. ☐

Let $(\sigma_n)_{n\in\mathbb{N}}$ be a sequence of decreasing, $[0,T]$-valued stopping times with $\sigma_n \downarrow \sigma = \sigma_\infty$ as n tends to infinity. In the sequel we set $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$.
Lemma 3.12 shows that for any $n \in \bar{\mathbb{N}}$ there exists a sequence $(\mathbf{Q}(m_k(n)), \tilde{S}^{\mathbf{Q}}(m_k(n)))_{k\in\mathbb{N}} \subset \mathrm{CPS}_{\mathrm{loc}}(0,T)$ such that $\mathbb{E}_{\mathbf{Q}_0}[F^{\mathbf{Q}(m_k(n))}_{\sigma_n}] \uparrow \mathbb{E}_{\mathbf{Q}_0}[F_{\sigma_n}]$ as k tends to infinity. Further, it is

easy to see that these sequences can be taken uniformly over $n \in \bar{\mathbb{N}}$. For $n \in \bar{\mathbb{N}}$ take the subsequence $(m_{k_l}(n))_{l\in\mathbb{N}} \subset (m_k(n))_{k\in\mathbb{N}}$ defined by

$$m_{k_1}(n) := \inf\left\{k \geq 1 : \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right]\right| < 1\right\},$$
$$m_{k_l}(n) := \inf\left\{k > m_{k_{l-1}}(n) : \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right]\right| < \frac{1}{l}\right\}.$$

Assumption 3.14. We assume the existence of $\mathbf{Q}_0 \in \mathcal{Q}_{\text{loc}}$ such that for any decreasing sequence of stopping times $0 \leq (\sigma_n)_{n\in\mathbb{N}} \leq T$ with $\sigma_n \downarrow \sigma$ as n tends to infinity, there exists a sequence

$$(\mathbf{Q}(m_k(n)), \tilde{S}^{\mathbf{Q}}(m_k(n))_{k\in\mathbb{N}} \subset \text{CPS}_{\text{loc}}(0,T),\ n \in \bar{\mathbb{N}},$$

such that $(\mathbb{E}_{\mathbf{Q}_0}[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}])_{k\in\mathbb{N}}$ converges uniformly over all $n \in \bar{\mathbb{N}}$ to $\mathbb{E}_{\mathbf{Q}_0}[F_{\sigma_n}]$, i.e., for all $\epsilon > 0$ there exists $K = K(\epsilon) \in \mathbb{N}$ such that

$$\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right]\right| < \epsilon,\ \text{for all } k \geq K,\ \text{and for all } n \in \bar{\mathbb{N}}, \tag{3.46}$$

and that for all $k \in \mathbb{N}$, $\mathbf{Q}_0(\bigcup_{N\in\mathbb{N}} A_N^{\epsilon,k}) = 1$, where

$$A_N^{\epsilon,k} := \left\{\omega \in \Omega : |F_{\sigma_n}^{\mathbf{Q}(m_k(n_0))} - F_{\sigma}^{\mathbf{Q}(m_k(n_0))}|(\omega) < \epsilon,\ \forall n \geq N,\ \forall n_0 \in \bar{\mathbb{N}}\right\}. \tag{3.47}$$

Assumption 3.14 can be thought as equi-continuity in time at level k, of a family of approximating sequences of consistent price systems. A uniformly approximating sequence does always exist, see Lemma 3.12 and the comment thereafter, but in general it is not true that $\mathbf{Q}_0(\bigcup_{N\in\mathbb{N}} A_N^{\epsilon,k}) = 1$ for $A_N^{\epsilon,k}$ given in (3.48). This is the key feature of Assumption 3.14.

Remark 3.15. *In the special case of $X_T = (0,1)$ Assumption 3.14 reads as follows: Fix an approximating sequence such that* (3.46) *holds. Then*

$$A_N^{\epsilon,k} := \left\{\omega \in \Omega : \begin{array}{l}\left|\mathbb{E}_{\mathbf{Q}(m_k(n_0))}\left[\tilde{S}_T^{\mathbf{Q}(m_k(n_0))} \mid \mathcal{F}_{\sigma_n}\right] - \mathbb{E}_{\mathbf{Q}(m_k(n_0))}\left[\tilde{S}_T^{\mathbf{Q}(m_k(n_0))} \mid \mathcal{F}_{\sigma}\right]\right|(\omega) < \epsilon, \\ \forall n \geq N,\ \forall n_0 \in \bar{\mathbb{N}}\end{array}\right\} \tag{3.48}$$

Theorem 3.16. *Suppose that Assumption 2.2 and 3.14 hold and $s \in [0,T]$. Let $X_T = (X_T^1, X_T^2) \in L^\infty(\mathcal{F}_T, \mathbf{P}) \times L^\infty(\mathcal{F}_T, \mathbf{P})$ such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \geq -M_s \tag{3.49}$$

for some $M_s \in L_+^1(\mathcal{F}_s, \mathcal{Q}_{loc})$. Then F in (3.34) *admits a right-continuous modification with respect to $\mathbf{P}$.*

Proof. Let $\mathbf{Q}_0 \in \mathcal{Q}_{\text{loc}}$ be the measure given by Assumption 3.14. We now prove that F in (3.34) admits a right-continuous modification with respect to $\mathbf{Q}_0$. Since all measure $\mathbf{Q} \in \mathcal{Q}_{\text{loc}}$ are equivalent to $\mathbf{P}$, this is equivalent to show that F admits a right-continuous modification with respect to $\mathbf{P}$.
By Theorem 48 in [10], the paths of F are right-continuous (outside an evanescent set), if $\lim_{n\to\infty} \mathbb{E}_{\mathbf{P}}[F_{\sigma_n}] = \mathbb{E}_{\mathbf{P}}[F_{\lim_{n\to\infty}\sigma_n}]$ for every decreasing sequence $(\sigma_n)_{n\in\mathbb{N}}$ of bounded

stopping times.
Let $(\sigma_n)_{n\in\mathbb{N}}$ be a decreasing sequence of stopping times with values in $[0,T]$ such that $\sigma_n \downarrow \sigma$ as n tends to infinity. We now prove that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right] = \mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma\right].$$

Because $(X_T^1, X_T^2) \in L^\infty(\mathcal{F}_T,\mathbf{P}) \times L^\infty(\mathcal{F}_T,\mathbf{P})$, there exists $C_1, C_2 \in \mathbb{R}$ such that $|X_T^1| \le C_1$ and $|X_T^2| \le C_2$. For $t \in [0,T]$ we thus have,

$$\begin{aligned}
|F_t| &\le \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS_{loc}}} \mathbb{E}_{\mathbf{Q}}\left[\left|X_T^1 + X_T^2 \tilde{S}_T^{\mathbf{Q}}\right| \mid \mathcal{F}_t\right] \\
&\le \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS_{loc}}} \mathbb{E}_{\mathbf{Q}}[C_1 + C_2\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t] \\
&= C_1 + C_2 \operatorname*{ess\,sup}_{(\mathbf{Q},\tilde{S}^{\mathbf{Q}})\in\mathrm{CPS_{loc}}} \mathbb{E}_{\mathbf{Q}}[\tilde{S}_T^{\mathbf{Q}} \mid \mathcal{F}_t] \\
&\le C_1 + C_2(1+\lambda)S_t \\
&\le C_1 + C_2\frac{1+\lambda}{1-\lambda}\tilde{S}_t^{\mathbf{Q}_0}.
\end{aligned} \tag{3.50}$$

We prove that the family

$$\mathcal{G} := \left\{|F_{\sigma_n}^{\mathbf{Q}} - F_\sigma^{\mathbf{Q}}| : n \in \mathbb{N},\ (\mathbf{Q},\tilde{S}^{\mathbf{Q}}) \in \mathrm{CPS_{loc}}(0,T)\right\}$$

is uniformly integrable with respect to $\mathbf{Q}_0$. First note, that

$$|F_{\sigma_n}^{\mathbf{Q}} - F_\sigma^{\mathbf{Q}}| \le |F_{\sigma_n}^{\mathbf{Q}}| + |F_\sigma^{\mathbf{Q}}| \le |F_{\sigma_n}| + |F_\sigma|. \tag{3.51}$$

It is easy to see that $F_\sigma \in L^1(\mathcal{F}_\sigma,\mathbf{Q}_0)$ because of (3.50). Equation (3.50) also implies

$$|F_{\sigma_n}| \le C_1 + C_2\frac{1+\lambda}{1-\lambda}\tilde{S}_{\sigma_n}^{\mathbf{Q}_0}, \quad \text{for all } n \in \mathbb{N}.$$

Therefore, it is enough to prove $\{\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} : n \in \mathbb{N}\}$ is uniformly integrable with respect to $\mathbf{Q}_0$. To this purpose, we first show that $\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} \xrightarrow{L^1} \tilde{S}_\sigma^{\mathbf{Q}_0}$. Because $\tilde{S}^{\mathbf{Q}_0}$ is a non-negative, càdlàg local $\mathbf{Q}_0$-martingale, $\tilde{S}^{\mathbf{Q}_0}$ is also a supermartingale under $\mathbf{Q}_0$ and we get by Theorem 9 of [24] that

$$\lim_{n\to\infty} \mathbb{E}_{\mathbf{Q}_0}[\tilde{S}_{\sigma_n}^{\mathbf{Q}_0}] = \mathbb{E}_{\mathbf{Q}_0}[\tilde{S}_\sigma^{\mathbf{Q}_0}]. \tag{3.52}$$

Thus, Scheffè's Lemma guarantees that $\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} \xrightarrow{L^1} \tilde{S}_\sigma^{\mathbf{Q}_0}$. Since $\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} \in L^1(\mathcal{F}_T,\mathbf{Q}_0)$ for all $n \in \mathbb{N}$ and $\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} \xrightarrow{a.s.} \tilde{S}_\sigma^{\mathbf{Q}_0}$ and $\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} \xrightarrow{L^1} \tilde{S}_\sigma^{\mathbf{Q}_0}$, Theorem 6.25 of [18] implies that $\{\tilde{S}_{\sigma_n}^{\mathbf{Q}_0} : n \in \mathbb{N}\}$ is uniformly integrable with respect to $\mathbf{Q}_0$ which yields that $\mathcal{G}$ is uniformly integrable with respect to $\mathbf{Q}_0$.
By Assumption 3.14 there exists for all $\epsilon > 0$ and for each $n \in \bar{\mathbb{N}}$ a sequence

$$(\mathbf{Q}(m_k(n)), \tilde{S}^{\mathbf{Q}}(m_k(n)))_{k\in\mathbb{N}} \subset \mathrm{CPS_{loc}}(0,T)$$

such that (3.46) is satisfied. Fix $\epsilon > 0$. By $(\mathbb{E}_{\mathbf{Q}_0}[F_\sigma^{\mathbf{Q}(m_k(\infty))}])_{k\in\mathbb{N}}$ we denote the sequence converging to $\mathbb{E}_{\mathbf{Q}_0}[F_\sigma]$. For $N \in \mathbb{N}$ and $k \in \mathbb{N}$ consider the set $A_N^k = A_N^{\epsilon/8,k}$ defined in (3.48), i.e.,

$$A_N^k = \left\{\omega \in \Omega : |F_{\sigma_n}^{\mathbf{Q}(m_k(n_0))} - F_\sigma^{\mathbf{Q}(m_k(n_0))}|(\omega) < \frac{\epsilon}{8},\ \forall n \geq N,\ \forall n_0 \in \bar{\mathbb{N}}\right\}. \tag{3.53}$$

By Assumption 3.14 we get that $\mathbf{Q}_0(\bigcup_{N\in\mathbb{N}} A_N^k) = 1$ for all $k \in \mathbb{N}$. As for fixed $k \in \mathbb{N}$ we get $A_N^k \subset A_{N+1}^k$ we can conclude that $\mathbf{Q}_0(A_N^k) \uparrow 1$ as N tends to infinity.
Fix $k \in \mathbb{N}$ such that

$$\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right]\right| < \frac{\epsilon}{8},\ \text{for all } n \in \bar{\mathbb{N}}. \tag{3.54}$$

Since $\mathcal{G}$ is uniformly integrable with respect to $\mathbf{Q}_0$, there exists $\delta = \delta(\epsilon)$ such that for all $\Lambda \in \mathcal{F}_T$ satisfying $\mathbf{Q}_0(\Lambda) < \delta$, we get

$$\mathbb{E}_{\mathbf{Q}_0}\left[\left|F_{\sigma_n}^{\mathbf{Q}(m_k(n_0))} - F_\sigma^{\mathbf{Q}(m_k(n_0))}\right| \mathbb{1}_\Lambda\right] < \frac{\epsilon}{8}, \tag{3.55}$$

for all $n, n_0 \in \bar{\mathbb{N}}$. Since $\mathbf{Q}_0(A_N^k) \uparrow 1$ as N tends to infinity, there exists $N_0 = N_0(\epsilon, k) \in \mathbb{N}$ such that $\mathbf{Q}_0((A_N^k)^c) < \delta$ for all $N \geq N_0$. Fix $N \geq N_0$ and let $n \geq N$. Then we have

$$\begin{aligned}\mathbb{E}_{\mathbf{Q}_0}\left[\left|F_{\sigma_n}^{\mathbf{Q}(m_k(n_0))} - F_\sigma^{\mathbf{Q}(m_k(n_0))}\right|\right] &= \mathbb{E}_{\mathbf{Q}_0}\left[\left|F_{\sigma_n}^{\mathbf{Q}(m_k(n_0))} - F_\sigma^{\mathbf{Q}(m_k(n_0))}\right| \mathbb{1}_{A_N^k}\right] \\ &\quad + \mathbb{E}_{\mathbf{Q}_0}\left[\left|F_{\sigma_n}^{\mathbf{Q}(m_k(n_0))} - F_\sigma^{\mathbf{Q}(m_k(n_0))}\right| \mathbb{1}_{(A_N^k)^c}\right] \\ &< \frac{\epsilon}{4},\end{aligned} \tag{3.56}$$

by the definition of the set A_N^k in (3.53) and by (3.55) because $\mathbf{Q}_0((A_N^k)^c) < \delta$. We consider three different cases.

Case 1: Assume

$$\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] \leq \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(\infty))}\right]. \tag{3.57}$$

Then we have

$$\begin{aligned}&\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma\right]\right| \\ &\leq \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(\infty))}\right]\right| + \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(\infty))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma^{\mathbf{Q}(m_k(\infty))}\right]\right| \\ &+ \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma^{\mathbf{Q}(m_k(\infty))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma\right]\right| \\ &< \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]\right| + \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(\infty))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma^{\mathbf{Q}(m_k(\infty))}\right]\right| + \frac{\epsilon}{8} \\ &< \frac{2\epsilon}{8} + \mathbb{E}_{\mathbf{Q}_0}\left[\left|F_{\sigma_n}^{\mathbf{Q}(m_k(\infty))} - F_\sigma^{\mathbf{Q}(m_k(\infty))}\right| \mathbb{1}_{A_N^k}\right] + \mathbb{E}_{\mathbf{Q}_0}\left[\left|F_{\sigma_n}^{\mathbf{Q}(m_k(\infty))} - F_\sigma^{\mathbf{Q}(m_k(\infty))}\right| \mathbb{1}_{(A_N^k)^c}\right] \end{aligned} \tag{3.58}$$

$$< \frac{4\epsilon}{8} < \epsilon$$

The second inequality holds due to (3.57) and to the fact that F_{σ_n} is the essential supremum over all consistent price systems, and because of (3.54). Also (3.58) holds due to (3.54). In the last step we applied (3.56).

Case 2: Assume

$$\mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma^{\mathbf{Q}(m_k(n))}\right] \geq \mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma^{\mathbf{Q}(m_k(\infty))}\right]. \tag{3.59}$$

Then we have

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma\right]\right| \\
\leq &\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]\right| + \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(n))}\right]\right| \\
+ &\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma\right]\right| \\
< &\frac{\epsilon}{8} + \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(n))}\right]\right| + \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma^{\mathbf{Q}(m_k(\infty))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma\right]\right| \\
< &\frac{2\epsilon}{8} + \mathbb{E}_{\mathbf{Q}_0}\left[\left|F_{\sigma_n}^{\mathbf{Q}(m_k(n))} - F_{\sigma}^{\mathbf{Q}(m_k(n))}\right|\mathbb{1}_{A_N^k}\right] + \mathbb{E}_{\mathbf{Q}_0}\left[\left|F_{\sigma_n}^{\mathbf{Q}(m_k(n))} - F_{\sigma}^{\mathbf{Q}(m_k(n))}\right|\mathbb{1}_{(A_N^k)^c}\right] \\
< &\frac{4\epsilon}{8} < \epsilon
\end{aligned}$$

The steps in Case 2 are analogously to Case 1, replacing (3.57) by (3.59).

Case 3: Assume

$$\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] > \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(\infty))}\right] \text{ and } \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(n))}\right] < \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(\infty))}\right]. \tag{3.60}$$

Then we have

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma\right]\right| \\
\leq &\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right]\right| + \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(\infty))}\right]\right| \\
+ &\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma^{\mathbf{Q}(m_k(\infty))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma\right]\right| \\
< &\frac{\epsilon}{8} + \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(\infty))}\right]\right| + \frac{\epsilon}{8}
\end{aligned}$$

The second inequality holds due to (3.54). For the remaining part we obtain

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(\infty))}\right]\right| \\
\leq &\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(n))}\right]\right| + \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(\infty)}\right]\right| \\
< &\frac{2\epsilon}{8} + \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(\infty))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(n))}\right]\right|
\end{aligned}$$

by the triangle inequality and (3.56). Then (3.60), and (3.56) imply

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(\infty))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(n))}\right]\right| = \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(\infty))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(n))}\right] \\
< &\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(\infty))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] + \frac{2\epsilon}{8} \\
< &\mathbb{E}_{\mathbf{Q}}\left[F_{\sigma}^{\mathbf{Q}(m_k(\infty))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] + \frac{2\epsilon}{8} < \frac{4\epsilon}{8}.
\end{aligned}$$

Combining these estimations we obtain

$$\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_\sigma\right]\right| < \frac{4\epsilon}{8} + \left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(\infty))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}^{\mathbf{Q}(m_k(n))}\right]\right| < \epsilon.$$

Summarizing, we have

$$\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}\right]\right| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we can conclude that

$$\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right] \xrightarrow{n\to\infty} \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma}\right].$$

This implies that $(F_t)_{t\in[0,T]}$ admits a right-continuous modification with respect to $\mathbf{Q}_0$ and hence also with respect to $\mathbf{P}$. □

It is easy to see that Assumption 3.14 and Theorem 3.16 can be analogously formulated for the non-local case.

Assumption 3.17. We assume the existence of $\mathbf{Q}_0 \in \mathcal{Q}$ such that for any decreasing sequence of stopping times $0 \le (\sigma_n)_{n\in\mathbb{N}} \le T$ with $\sigma_n \downarrow \sigma$ as n tends to infinity, there exists a sequence

$$(\mathbf{Q}(m_k(n)), \tilde{S}^{\mathbf{Q}}(m_k(n))_{k\in\mathbb{N}} \subset \mathrm{CPS}(0,T),\ n \in \bar{\mathbb{N}},$$

such that $(\mathbb{E}_{\mathbf{Q}_0}[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}])_{k\in\mathbb{N}}$ converges uniformly over all $n \in \bar{\mathbb{N}}$ to $\mathbb{E}_{\mathbf{Q}_0}[F_{\sigma_n}]$, i.e., for all $\epsilon > 0$ there exists $K = K(\epsilon) \in \mathbb{N}$ such that

$$\left|\mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}^{\mathbf{Q}(m_k(n))}\right] - \mathbb{E}_{\mathbf{Q}_0}\left[F_{\sigma_n}\right]\right| < \epsilon, \text{ for all } k \ge K, \text{ and for all } n \in \bar{\mathbb{N}}, \tag{3.61}$$

and that for all $k \in \mathbb{N}$, $\mathbf{Q}_0(\bigcup_{N\in\mathbb{N}} A_N^{\epsilon,k}) = 1$, where

$$A_N^{\epsilon,k} := \left\{\omega \in \Omega : |F_{\sigma_n}^{\mathbf{Q}(m_k(n_0))} - F_{\sigma}^{\mathbf{Q}(m_k(n_0))}|(\omega) < \epsilon,\ \forall n \ge N,\ \forall n_0 \in \bar{\mathbb{N}}\right\}. \tag{3.62}$$

Theorem 3.18. *Let Assumption 2.3 and 3.17 hold and $s \in [0,T]$. Let $X_T = (X_T^1, X_T^2) \in L^\infty(\mathbb{R}^2; \mathcal{F}_T, \boldsymbol{P})$ such that*

$$X_T^1 + (X_T^2)^+(1-\lambda)S_T - (X_T^2)^-(1+\lambda)S_T \ge -M_s^1 - M_s^2 S_T \tag{3.63}$$

for some $(M_s^1, M_s^2) \in L_+^1(\mathcal{F}_s, \mathcal{Q}) \times L_+^\infty(\mathcal{F}_s, \mathcal{Q})$. Then F defined in (3.34) *admits a right-continuous modification with respect to $\boldsymbol{P}$.*

Proof. The proof of Theorem 3.16 carries over using Assumptions 2.3 and 3.17. □

Theorem 3.16 (resp. Theorem 3.18) guarantees that the process F, defined in (3.34), is well-defined. It is important to note the difference between F and the super-replication price in the frictionless setting, see [11], [19]. In a frictionless market model the wealth of a portfolio equals the liquidation value and the price to buy the portfolio.
Under the presence of transaction costs this does no longer hold. More precisely, the liquidation value of a portfolio and the buying price of a portfolio are in general not the same. The process F defines the capital that is needed to start a self-financing, admissible strategy in order to super-replicate the contingent claim. This price is usually higher than the liquidation value.

Example 3.19. Suppose Assumption 2.3 holds and let $(X_T^1, X_T^2) = (0,1)$. Then, for each $k \in \mathbb{N}$ such that $\frac{1}{k} \leq \lambda$ there exists $(\mathbf{Q}(k), \tilde{S}^{\mathbf{Q}}(k)) \in \mathrm{CPS}(0,T,\frac{1}{k})$. We get

$$(1-\lambda)S_t \leq \tilde{S}_t^{\mathbf{Q}}(k) \leq \frac{1+\lambda}{1+\frac{1}{k}}\tilde{S}_t^{\mathbf{Q}}(k) \leq (1+\lambda)S_t. \tag{3.64}$$

In particular, $(\mathbf{Q}(k), \mu_k \tilde{S}^{\mathbf{Q}}(k)) \in \mathrm{CPS}(0,T,\lambda)$ for $\mu_k := \frac{1+\lambda}{1+\frac{1}{k}}$. Furthermore, we have by Proposition 3.11 of [3] that

$$F_t = (1+\lambda)S_t, \quad t \in [0,T]. \tag{3.65}$$

By the martingale property of $\tilde{S}^{\mathbf{Q}}(k)$ we obtain

$$\begin{aligned}
&\left|(1+\lambda)S_t - \mathbb{E}_{\mathbf{Q}(k)}\left[\mu_k \tilde{S}_T^{\mathbf{Q}}(k) \mid \mathcal{F}_t\right]\right| = \left|(1+\lambda)S_t - \mu_k \tilde{S}_t^{\mathbf{Q}}(k)\right| \\
\leq &\left|(1+\lambda)S_t - \mu_k\left(1-\frac{1}{k}\right)S_t\right| = \left|(1+\lambda)S_t\left(1-\frac{1-\frac{1}{k}}{1+\frac{1}{k}}\right)\right| = (1+\lambda)S_t\frac{2}{k+1}.
\end{aligned} \tag{3.66}$$

Then we get for any $\mathbf{Q}_0 \in \mathcal{Q}$, and $t \in [0,T]$ that

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbf{Q}_0}[F_t] - \mathbb{E}_{\mathbf{Q}_0}\left[\mu_k F_t^{\mathbf{Q}(k)}\right]\right| \leq \mathbb{E}_{\mathbf{Q}_0}\left[\left|(1+\lambda)S_t - \mathbb{E}_{\mathbf{Q}(k)}\left[\mu_k \tilde{S}_T^{\mathbf{Q}}(k) \mid \mathcal{F}_t\right]\right|\right] \\
\leq &\mathbb{E}_{\mathbf{Q}_0}\left[(1+\lambda)S_t\frac{2}{k+1}\right] \leq \mathbb{E}_{\mathbf{Q}_0}\left[\frac{1+\lambda}{1-\lambda}\frac{2}{k+1}\tilde{S}_t^{\mathbf{Q}_0}\right] = \frac{1+\lambda}{1-\lambda}\frac{2}{k+1}\tilde{S}_0^{\mathbf{Q}_0} \\
\leq &\frac{(1+\lambda)^2}{1-\lambda}\frac{2}{k+1}S_0.
\end{aligned} \tag{3.67}$$

Therefore, we can easily see that for for every $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that (3.61) of Assumption 3.17 is fulfilled by the sequence $(\mathbf{Q}(k), \mu_k \tilde{S}^{\mathbf{Q}}(k))_{k \in \mathbb{N}} \subset \mathrm{CPS}(0,T)$ which is independent of $t \in [0,T]$. Let $(\sigma_n)_{n \in \mathbb{N}}$ be any decreasing sequence of stopping times. Note that in this case we get that A_N^k does not depend on $n_0 \in \mathbb{N}$ anymore, i.e.,

$$A_N^k = \left\{\omega \in \Omega : |F_{\sigma_n}^{\mathbf{Q}(k)} - F_{\sigma}^{\mathbf{Q}(k)}|(\omega) < \epsilon, \ \forall n \geq N\right\}. \tag{3.68}$$

By Definition 2.1, all consistent price systems are càdlàg which yields to $\mathbf{Q}_0(\bigcup_{N \in \mathbb{N}} A_N^k) = 1$. We conclude that under Assumption 2.3, Assumption 3.17 is fulfilled for $X_T = (0,1)$.

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