Construction of Strong Solutions of SDE's via Malliavin Calculus

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Abstract

In this paper we develop a new method for the construction of strong solutions of stochastic equations with discontinuous coefficients. We illustrate this approach by studying stochastic differential equations driven by the Wiener process. Using Malliavin calculus we derive the result of A.K. Zvonkin [Zv] for bounded and measurable drift coefficients as a special case of our analysis of SDE's. Moreover, our approach yields the important insight that the solutions obtained by Zvonkin are even Malliavin differentiable. The latter indicates that the "nature" of strong solutions of SDE's is tightly linked to the property of Malliavin differentiability. We also stress that our method does not involve a pathwise uniqueness argument but provides a direct construction of strong solutions.

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1 Introduction

Consider the stochastic differential equation (SDE) given by

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t\,, \quad 0 \le t \le T\,, \quad X_0 = \ x \in \mathbb{R}^d, \tag{1}$$

where B_t is a $d-$dimensional Brownian motion with respect to a filtration $\mathcal{F}_t$, generated by B_t on a probability space $(\Omega, \mathcal{F}, \pi)$. Further the drift coefficient $b : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and dispersion coefficient $\sigma : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ are assumed to be Borel measurable functions.

It is well known that if

$$\|b(t, x)\| + \|\sigma(t, x)\| \le C(1 + \|x\|)\,, \quad x \in \mathbb{R}^d, \quad 0 \le t \le T\,, \tag{2}$$

and

$$\|b(t, x) - b(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \le D(\|x - y\|), \quad x, y \in \mathbb{R}^d, \quad 0 \le t \le T, \tag{3}$$

for constants $C, D \ge 0$, then there exists a unique global strong solution X_t of (1), that is a continuous $\mathcal{F}_t-$adapted process X_t solving (1). Moreover we have that

$$E\left[\int_0^T X_t^2 dt\right] < \infty.$$

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Stochastic differential equations were first studied by Itô, K (see [I]). Such types of equations can be e.g. used to model the motion of diffusing particles in a liquid. Many other important applications pertain to physics, biology, social sciences and mathematical finance. From a theoretical point of view these processes play a central role in the theory of Markov processes. The importance of SED's is emphasized by the fact that a very broad class of continuous Markov processes can be reduced to diffusion or quasidiffusion processes through certain transformations. See e.g. [Sk].

Stochastic differential equations are natural generalizations of ordinary differential equations, that is of equations of the type

$$\frac{dX_t}{dt} = b(t, X_t)\,, \quad X_0 = x\,. \tag{4}$$

However, there is a peculiar difference between these two equations if the diffusion coefficient of the SDE is non-degenerate: It is known that a solution to (4) may not be unique or even not exist, if b is non-Lipschitzian. On the other hand, if the right hand side of (4) is superposed by a (small) noise, that is

$$X_t = x + \int_0^t b(s, X_s)ds + \varepsilon B_t\,,$$

then the regularization effect of the Brownian motion B_t enforces the existence of a unique global strong solution for all $\varepsilon > 0$, when b is e.g. bounded and measurable. This remarkable fact was first observed by Zvonkin in his celebrated paper [Zv]. An extension of this important result to the multidimensional case was given by Veretennikov [V]. The authors first employ estimates of solutions of parabolic partial differential equations to construct weak solutions. Then a pathwise uniqueness argument is applied to ensure a unique strong solution.

We mention that the latter results, which can be considered a milestone of the theory of SDE's, have been generalized by Gyöngy, Martínez [GM] and Krylov, Röckner [KR], who impose deterministic integrability conditions on the drift coefficient to guarantee existence and uniqueness of strong solutions. As in [Zv] and [V], the authors first derive weak solutions. Here the approach of Gyöngy and Martínez rests on the Skorohod embedding, whereas the method of Krylov and Röckner is based on an argument of Portenko [Po]. Finally, the authors resort to pathwise uniqueness and obtain strong solutions. Another technique which relies on an Euler scheme of approximation and pathwise uniqueness can be found in Gyöngy, Krylov [GK]. See also [FZ], where a modified version of Gronwall's Lemma is invoked.

We repeat that the common idea in all the references above is the construction of weak solutions together with the subsequent use of the celebrated pathwise uniqueness argument obtained by Yamada, Watanabe [YW] which can be formulated as

$$\text{weak existence } + \text{ pathwise uniqueness} \Longrightarrow \text{ strong uniqueness.} \tag{5}$$

In this paper we devise a new method to study strong solutions of SDEs of type (1) with irregular coefficients. Our method does not rely on the pathwise uniqueness argument but gives a direct construction of strong solutions. Further, to conclude strong uniqueness, we derive the following result that in some sense is diametrically opposed to (5):

$$\text{strong existence } + \text{ uniqueness in law} \Longrightarrow \text{ strong uniqueness.}$$

Our technique is mainly based on Malliavin calculus. More precisely, we use a compactness criterion based on Malliavin calculus combined with "local time variational calculus" and an approximation argument to show that a certain generalized process in the Hida distribution space satisfies the SDE. As our main result we derive stochastic (and deterministic) integrability conditions on the coefficients to guarantee the existence of strong solutions. Moreover, our method yields the insight that these strong solutions are Malliavin differentiable. As a special case of our technique we obtain the result of Zvonkin [Zv] for SDE's with merely bounded and measurable drift coefficients together with the additional and rather surprising information that these solutions are Malliavin differentiable. The latter sheds a new light on solutions of SDE's and raises the question to which extent the "nature" of strong solutions is tied to the property of Malliavin differentiability.

We emphasize that, although in this paper we focus on SDEs driven by Brownian motion, our technique exhibits the capacity to cover stochastic equations for a broader class of driving noises. In particular, it can be applied to SDE's driven by Lévy processes, fractional Brownian motion or more general by fractional Lévy processes. Other applications are evolution equations on Hilbert spaces, SPDE's and anticipative SDE's. In particular, the applicability of our technique to infinite dimensional equations on Hilbert spaces seems very interesting since very little is known in this case. Such equations cannot be captured by the framework of the above authors. The latter is due to the fact that the authors' techniques utilize specific estimates, which work in the Euclidean space, but fail in infinite dimensional spaces. See e.g. [GM, Lemma 3.1], where an estimate of Krylov [K2] for semimartingales is used.

Our paper is inspired by ideas initiated in [M-BP1] and [P]. In [P] the author constructs strong solutions of SDE's with functional discontinuous coefficients by using certain estimates for the $n-$th homogeneous chaos of chaos expansions of solutions. The paper [M-BP1] exploits a comparison theorem to derive solutions of SDE's.

The paper is organized as follows: In Section 2 we give the framework of our paper. Here we review basic concepts of Malliavin calculus and Gaussian white noise theory. Then in Section 3 we illustrate our method by studying the SDE (1). Our main results are Theorem 4, 5 and 22. Section 4 concludes with a discussion of our method.

2 Framework

In this section we recall some facts from Gaussian white noise analysis and Malliavin calculus, which we aim at employing in Section 3 to construct strong solutions of SDE's. See [HKPS], [O], [Ku] for more information on white noise theory. As for Malliavin calculus the reader is referred to [N], [M1], [M2] and [DOP].

2.1 Basic Elements of Gaussian White Noise Theory

As mentioned in the Introduction we want to use generalized stochastic processes on a certain stochastic distribution space to analyze strong solutions of SDE's. In the sequel we give the construction of the stochastic distribution space, which goes back to T. Hida (see [HKPS]).

From now on we fix a time horizon $0 < T < \infty$. Consider a (positive) self-adjoint operator A on $L^2([0,T])$ with $Spec(A) > 1$. Let us require that A^{-r} is of Hilbert-Schmidt type for some $r > 0$. Denote by $\{e_j\}_{j\geq 0}$ a complete orthonormal basis of $L^2([0,T])$ in $Dom(A)$ and let $\lambda_j > 0$, $j \geq 0$ be the eigenvalues of A such that

$$1 < \lambda_0 \leq \lambda_1 \leq \ldots \longrightarrow \infty.$$

Let us assume that each basis element e_j is a continuous function on $[0, T]$. Further let $O_\lambda, \lambda \in \Gamma$ be an open covering of $[0, T]$ such that

$$\sup_{j \geq 0} \lambda_j^{-\alpha(\lambda)} \sup_{t \in O_\lambda} |e_j(t)| < \infty$$

for $\alpha(\lambda) \geq 0$.

In what follows let $\mathcal{S}([0, T])$ denote the standard countably Hilbertian space constructed from $(L^2([0, T]), A)$. See [O]. Then $\mathcal{S}([0, T])$ is a nuclear subspace of $L^2([0, T])$. We denote by $\mathcal{S}'([0, T])$ the corresponding conuclear space, that is the topological dual of $\mathcal{S}([0, T])$. Then the Bochner-Minlos theorem provides the existence of a unique probability measure π on $\mathcal{B}(\mathcal{S}'([0, T]))$ (Borel $\sigma-$algebra of $\mathcal{S}'([0, T])$) such that

$$\int_{\mathcal{S}'([0,T])} e^{i\langle \omega, \phi \rangle} \pi(d\omega) = e^{-\frac{1}{2} \|\phi\|^2_{L^2([0,T])}}$$

holds for all $\phi \in \mathcal{S}([0, T])$, where $\langle \omega, \phi \rangle$ is the action of $\omega \in \mathcal{S}'([0, T])$ on $\phi \in \mathcal{S}([0, T])$. Set

$$\Omega_i = \mathcal{S}'([0, T]), \quad \mathcal{F}_i = \mathcal{B}(\mathcal{S}'([0, T])), \quad \mu_i = \pi,$$

for $i = 1, ..., d$. Then the product measure

$$\mu = \times_{i=1}^d \mu_i \tag{6}$$

on the measurable space

$$(\Omega, \mathcal{F}) := \left(\prod_{i=1}^d \Omega_i, \otimes_{i=1}^d \mathcal{F}_i \right) \tag{7}$$

is referred to as *d-dimensional white noise probability measure*.

Consider the Doleans-Dade exponential

$$\widetilde{e}(\phi, \omega) = \exp\left(\langle \omega, \phi \rangle - \frac{1}{2} \|\phi\|^2_{L^2([0,T];\mathbb{R}^d)} \right),$$

for $\omega = (\omega_1, ..., \omega_d) \in (\mathcal{S}')^d$ and $\phi = (\phi^{(1)}, ..., \phi^{(d)}) \in (\mathcal{S}([0, T]))^d$, where $\langle \omega, \phi \rangle := \sum_{i=1}^d \langle \omega_i, \phi_i \rangle$.

In the following let $\left((\mathcal{S}([0, T]))^d\right)^{\widehat{\otimes} n}$ be the $n-$th completed symmetric tensor product of $(\mathcal{S}([0, T]))^d$ with itself. One verifies that $\widetilde{e}(\phi, \omega)$ is holomorphic in ϕ around zero. Hence there exist generalized Hermite polynomials $H_n(\omega) \in \left(\left((\mathcal{S}([0, T]))^d\right)^{\widehat{\otimes} n}\right)'$ such that

$$\widetilde{e}(\phi, \omega) = \sum_{n \geq 0} \frac{1}{n!} \left\langle H_n(\omega), \phi^{(n)} \right\rangle \tag{8}$$

for ϕ in a certain neighbourhood of zero in $(\mathcal{S}([0, T]))^d$. It can be shown that

$$\left\{ \left\langle H_n(\omega), \phi^{(n)} \right\rangle : \phi^{(n)} \in \left((\mathcal{S}([0, T]))^d\right)^{\widehat{\otimes} n}, \ n \in \mathbb{N}_0 \right\} \tag{9}$$

is a total set of $L^2(\mu)$. Further one finds that the orthogonality relation

$$\int_{S'} \left\langle H_n(\omega), \phi^{(n)} \right\rangle \left\langle H_m(\omega), \psi^{(m)} \right\rangle \mu(d\omega) = \delta_{n,m} n! \left(\phi^{(n)}, \psi^{(n)} \right)_{L^2([0,T]^n;(\mathbb{R}^d)^{\otimes n})} \tag{10}$$

is valid for all $n, m \in \mathbb{N}_0$, $\phi^{(n)} \in \left((\mathcal{S}([0,T]))^d\right)^{\widehat{\otimes} n}$, $\psi^{(m)} \in \left((\mathcal{S}([0,T]))^d\right)^{\widehat{\otimes} m}$ where

$$\delta_{n,m} \begin{cases} 1 & \text{if } n = m \\ 0 & \text{else} \end{cases} .$$

Define $\widehat{L}^2([0,T]^n; (\mathbb{R}^d)^{\otimes n})$ as the space of square integrable symmetric functions $f(x_1, ..., x_n)$ with values in $(\mathbb{R}^d)^{\otimes n}$. Then the orthogonality relation (10) implies that the mappings

$$\phi^{(n)} \longmapsto \left\langle H_n(\omega), \phi^{(n)} \right\rangle$$

from $\left(\mathcal{S}([0,T])^d\right)^{\widehat{\otimes} n}$ to $L^2(\mu)$ possess unique continuous extensions

$$I_n : \widehat{L}^2([0,T]^n; (\mathbb{R}^d)^{\otimes n}) \longrightarrow L^2(\mu)$$

for all $n \in \mathbb{N}$. We remark that $I_n(\phi^{(n)})$ can be viewed as an $n-$fold iterated Itô integral of $\phi^{(n)} \in \widehat{L}^2([0,T]^n; (\mathbb{R}^d)^{\otimes n})$ with respect to a $d-$dimensional Wiener process

$$B_t = \left(B_t^{(1)}, ..., B_t^{(d)} \right) \tag{11}$$

on the white noise space

$$(\Omega, \mathcal{F}, \mu) \ . \tag{12}$$

It turns out that square integrable functionals of B_t admit a Wiener-Itô chaos representation which can be regarded as an infinite-dimensional Taylor expansion, that is

$$L^2(\mu) = \bigoplus_{n \geq 0} I_n(\widehat{L}^2([0,T]^n; (\mathbb{R}^d)^{\otimes n})). \tag{13}$$

We construct the Hida stochastic test function and distribution space by using the Wiener-Itô chaos decomposition (13). For this purpose let

$$A^d := (A, ..., A) \, , \tag{14}$$

where A was the operator introduced in the beginning of the section. We define the *Hida stochastic test function space* $(\mathcal{S})$ via a second quantization argument, that is we introduce $(\mathcal{S})$ as the space of all $f = \sum_{n \geq 0} \left\langle H_n(\cdot), \phi^{(n)} \right\rangle \in L^2(\mu)$ such that

$$\|f\|_{0,p}^2 := \sum_{n \geq 0} n! \left\| \left((A^d)^{\otimes n} \right)^p \phi^{(n)} \right\|^2_{L^2([0,T]^n;(\mathbb{R}^d)^{\otimes n})} < \infty \tag{15}$$

for all $p \geq 0$. It turns out that the space $(\mathcal{S})$ is a nuclear Fréchet algebra with respect to multiplication of functions and its topology is given by the seminorms $\|\cdot\|_{0,p}$, $p \geq 0$. Further one observes that

$$\widetilde{e}(\phi, \omega) \in (\mathcal{S}) \tag{16}$$

for all $\phi \in (\mathcal{S}([0,T]))^d$.

In the sequel we refer to the topological dual of $(\mathcal{S})$ as *Hida stochastic distribution space* $(\mathcal{S})^*$. Thus we have constructed the Gel'fand triple

$$(\mathcal{S}) \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})^*.$$

The Hida distribution space $(\mathcal{S})^*$ exhibits the crucial property that it contains the *white noise* of the coordinates of the $d-$dimensional Wiener process B_t, that is the time derivatives

$$W_t^i := \frac{d}{dt} B_t^i, \; i = 1, ..., d, \tag{17}$$

belong to $(\mathcal{S})^*$.

We shall also recall the definition of the S-transform which is an important tool to characterize elements of the Hida test function and distribution space. See [PS]. The $S-$transform of a $\Phi \in (\mathcal{S})^*$, denoted by $S(\Phi)$, is defined by the dual pairing

$$S(\Phi)(\phi) = \langle \Phi, \widetilde{e}(\phi, \omega) \rangle \tag{18}$$

for $\phi \in (\mathcal{S}_{\mathbb{C}}([0,T]))^d$. Here $\mathcal{S}_{\mathbb{C}}([0,T])$ the complexification of $\mathcal{S}([0,T])$. We mention that the $S-$transform is a monomorphism from $(\mathcal{S})^*$ to $\mathbb{C}$. In particular, if

$$S(\Phi) = S(\Psi) \text{ for } \Phi, \Psi \in (\mathcal{S})^*$$

then

$$\Phi = \Psi.$$

One checks that

$$S(W_t^i)(\phi) = \phi^i(t), \; i = 1, ..., d \tag{19}$$

for $\phi = (\phi^{(1)}, ..., \phi^{(d)}) \in (\mathcal{S}_{\mathbb{C}}([0,T]))^d$.

Finally, we need the important concept of the *Wick* or *Wick-Grassmann product*, which we want to use in Section 3 to represent solutions of SDE's. The Wick product can be regarded as a tensor algebra multiplication on the Fock space and can be defined as follows: The Wick product of two distributions $\Phi, \Psi \in (\mathcal{S})^*$, denoted by $\Phi \diamond \Psi$, is the unique element in $(\mathcal{S})^*$ such that

$$S(\Phi \diamond \Psi)(\phi) = S(\Phi)(\phi) S(\Psi)(\phi) \tag{20}$$

for all $\phi \in (\mathcal{S}_{\mathbb{C}}([0,T]))^d$. As an example we find that

$$\left\langle H_n(\omega), \phi^{(n)} \right\rangle \diamond \left\langle H_m(\omega), \psi^{(m)} \right\rangle = \left\langle H_{n+m}(\omega), \phi^{(n)} \widehat{\otimes} \psi^{(m)} \right\rangle \tag{21}$$

for $\phi^{(n)} \in \left((\mathcal{S}([0,T]))^d\right)^{\widehat{\otimes} n}$ and $\psi^{(m)} \in \left((\mathcal{S}([0,T]))^d\right)^{\widehat{\otimes} m}$. The latter in connection with (8) shows that

$$\widetilde{e}(\phi, \omega) = \exp^{\diamond}(\langle \omega, \phi \rangle) \tag{22}$$

for $\phi \in (\mathcal{S}([0,T]))^d$. Here the Wick exponential $\exp^{\diamond}(X)$ of a $X \in (\mathcal{S})^*$ is defined as

$$\exp^{\diamond}(X) = \sum_{n \geq 0} \frac{1}{n!} X^{\diamond n}, \tag{23}$$

where $X^{\diamond n} = X \diamond ... \diamond X$, if the sum on the right hand side converges in $(\mathcal{S})^*$.

2.2 Some Definitions and Notions from Malliavin Calculus

In this Section we briefly elaborate a framework for Malliavin calculus.

Without loss of generality we consider the case $d = 1$. Let $F \in L^2(\mu)$. Then it follows from (13) that

$$F = \sum_{n \geq 0} \left\langle H_n(\cdot), \phi^{(n)} \right\rangle \tag{24}$$

for unique $\phi^{(n)} \in \widehat{L}^2([0,T]^n)$. Assume that

$$\sum_{n \geq 1} n n! \left\| \phi^{(n)} \right\|^2_{L^2([0,T]^n)} < \infty \,. \tag{25}$$

Then the *Malliavin derivative* D_t of F in the direction B_t is defined by

$$D_t F = \sum_{n \geq 1} n \left\langle H_{n-1}(\cdot), \phi^{(n)}(\cdot, t) \right\rangle . \tag{26}$$

We introduce the stochastic Sobolev space $\mathbb{D}_{1,2}$ as the space of all $F \in L^2(\mu)$ such that (25) is fulfilled. The Malliavin derivative $D.$ is a linear operator from $\mathbb{D}_{1,2}$ to $L^2(\lambda \times \mu)$, where λ denotes the Lebesgue measure. We mention that $\mathbb{D}_{1,2}$ is a Hilbert space with the norm $\|\cdot\|_{1,2}$ given by

$$\|F\|^2_{1,2} := \|F\|^2_{L^2(\mu)} + \|D.F\|^2_{L^2([0,T] \times \Omega, \lambda \times \mu)} \,. \tag{27}$$

We obtain the following chain of continuous inclusions:

$$(\mathcal{S}) \hookrightarrow \mathbb{D}_{1,2} \hookrightarrow L^2(\mu) \hookrightarrow \mathbb{D}_{-1,2} \hookrightarrow (\mathcal{S})^*, \tag{28}$$

where $\mathbb{D}_{-1,2}$ is the dual of $\mathbb{D}_{1,2}$.

3 Main Results

In this section we want to introduce a new technique to analyze strong solutions of stochastic differential equations. We will focus on the analysis of strong solutions of equations driven by Brownian motion with merely measurable drift $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, that is

$$dX_t = b(t, X_t)dt + dB_t, \ 0 \leq t \leq T, \ X_0 = x \in \mathbb{R}^d, \tag{29}$$

where B_t is a d-dimensional Brownian motion with respect to the stochastic basis

$$(\Omega, \mathcal{F}, \mu) \,, \{\mathcal{F}_t\}_{0 \leq t \leq T} \tag{30}$$

for a $\mu-$augmented filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ generated by B_t. At the end of the section we show how equations with more general diffusion coefficients can be reduced to equations of type (29).

While in this paper we are focusing on SDEs driven by Brownian motion, we want to point out that our approach relies on a general principle that exhibits potential to study strong solutions of equations driven by a broader class of processes including the infinite dimensional case.

Our study of strong solutions is inspired by the following observation, which was made in [LP]. See also [M-BP2] for an analogue result on equations driven by Lévy processes including the infinite dimensional case.

Proposition 1 *Suppose that the drift coefficient $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ in (29) is bounded and Lipschitz continuous. Then there exists a unique strong solution X_t of (29), which allows for the explicit representation*

$$\varphi\left(t, X_t^{(i)}(\omega)\right) = E_{\widetilde{\mu}}\left[\varphi\left(t, \widetilde{B}_t^{(i)}(\widetilde{\omega})\right) \mathcal{E}_T^{\diamond}(b)\right] \tag{31}$$

for all $\varphi : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $\varphi\left(t, B_t^{(i)}\right) \in L^2(\mu)$ for all $0 \leq t \leq T$, $i = 1, ..., d,$. The object $\mathcal{E}^{\diamond}(b)$ is given by

$$\begin{aligned} \mathcal{E}_T^{\diamond}(b)(\omega, \widetilde{\omega}) \quad := \quad & \exp^{\diamond}\left(\sum_{j=1}^{d} \int_0^T \left(W_s^{(j)}(\omega) + b^{(j)}(s,)\right) d\widetilde{B}_s^{(j)}(\widetilde{\omega})\right. \\ & \left. -\frac{1}{2} \int_0^T \left(W_s^{(j)}(\omega) + b^{(j)}(s, \widetilde{B}_s(\widetilde{\omega}))\right)^{\diamond 2} ds\right). \end{aligned} \tag{32}$$

Here $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mu}\right)$, $\left(\widetilde{B}_t\right)_{t \geq 0}$ is a copy of the quadruple $(\Omega, \mathcal{F}, \mu)$, $(B_t)_{t \geq 0}$ in (30). Further $E_{\widetilde{\mu}}$ denotes a Pettis integral of random elements $\Phi : \widetilde{\Omega} \longrightarrow (\mathcal{S})^*$ with respect to the measure $\widetilde{\mu}$. The Wick product $\diamond$ in the Wick exponential of (32) is taken with respect to μ and $W_t^{(j)}$ is the white noise of $B_t^{(j)}$ in the Hida space $(\mathcal{S})^*$ (see (17)). The stochastic integrals $\int_0^T \phi(t, \widetilde{\omega}) d\widetilde{B}_s^{(j)}(\widetilde{\omega})$ in (32) are defined for predictable integrands ϕ with values in the conuclear space $(\mathcal{S})^*$. See [KX] for definitions. The other integral type in (32) is to be understood in the sense of Pettis.

Remark 2 *Let $0 = t_1^n < t_2^n < ... < t_{m_n}^n = T$ be a sequence of partitions of the interval $[0,T]$ with $\max_{i=1}^{m_n - 1} \left|t_{i+1}^n - t_i^n\right| \longrightarrow 0$. Then the stochastic integral of the white noise $W^{(j)}$ can be approximated as follows:*

$$\int_0^T W_s^{(j)}(\omega) d\widetilde{B}_s^{(j)}(\widetilde{\omega}) = \lim_{n \longrightarrow \infty} \sum_{i=1}^{m_n} (\widetilde{B}_{t_{i+1}^n}^{(j)}(\widetilde{\omega}) - \widetilde{B}_{t_i^n}^{(j)}(\widetilde{\omega})) W_{t_i^n}^{(j)}(\omega)$$

in $L^2(\lambda \times \widetilde{\mu}; (\mathcal{S})^*)$. For more information about stochastic integration on conuclear spaces the reader may consult [KX].

In what follows let us denote the expression on the right hand side of (31) for $\varphi(t,x) = x$ by $Y_t^{i,b}$, that is

$$Y_t^{i,b} := E_{\widetilde{\mu}}\left[\widetilde{B}_t^{(i)} \mathcal{E}_T^{\diamond}(b)\right]$$

for $i = 1, ..., d$. Further set

$$Y_t^b = \left(Y_t^{1,b}, ..., Y_t^{d,b}\right) \tag{33}$$

In particular, under the assumptions of Proposition 1, the solution is explicitly given as $X_t = Y_t^b$. Further, for notational convenience, we also will from now on assume that

$$T = 1.$$

The appearance of the representation formula (31) naturally raises the following questions:

(i) Proposition 1 provokes to conjecture that Y_t^b in (33) also represents strong solutions of (29) with discontinuous drift coefficients b. So, if b is non-Lipschitzian, will it be possible to interprete Y_t^b under some conditions as a generalized stochastic process in the Hida distribution space which could be verified as a strong solution of (29) ?

(ii) The result also suggests that one could formally apply the Malliavin derivative D_t (see Section 2.2) to the generalized process Y_t^b without affecting the coefficient b. The latter gives rise to the following questions: Are strong solutions of (29) for a larger class of non-Lipschtzian drift coefficients Malliavin differentiable ? Or is even Malliavin differentiability a property which is intrinsically linked to the "nature" of strong solutions?

It turns out that the above questions can be positively affirmed as our main result shows (see Theorem 4).

Before we state this theorem we need to introduce some notation and definitions. Assume that $B_t = \left(B_t^1, ..., B_t^d\right), 0 \le t \le 1$ is a Brownian motion starting in $x \in \mathbb{R}^d$, whose components $B_t^{(i)}$ are defined on $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, ..., d$ (see (7)). In the sequel we denote by

$$\widehat{B}_t := (\widehat{B}_t^{(1)}, ..., \widehat{B}_t^{(d)}) := B_{1-t}\,, \quad 0 \le t \le 1\,, \tag{34}$$

the time-reversed Brownian motion. Note that $\widehat{B}_t^{(i)}$ satisfies for each $i = 1, ..., d$ the equation

$$\widehat{B}_t^{(i)} = B_1^{(i)} + \widetilde{W}_t^{(i)} - \int_0^t \frac{\widehat{B}_s^{(i)}}{1-s} ds\,, \quad 0 \le t \le 1\,, \quad \text{a.e.}, \tag{35}$$

where $\widetilde{W}_t^{(i)}$, $0 \le t \le 1$ are independent μ_i-Brownian motions with respect to the filtrations $\mathcal{F}_t^{\widehat{B}^{(i)}}$ generated by $\widehat{B}_t^{(i)}$, $i = 1, ..., d$. See [E].

Definition 3 *Let $L(t) \in \mathbb{R}^{d\times d}, 0 \le t \le 1$ be a continuous matrix function. We shall say that L belongs to the class $\mathcal{L} \subset \mathbb{R}^{d\times d}$ iff $L(t)$ commutes with $\int_s^t L(u)du$ for all $0 \le s \le t \le 1$.*

We are coming to our main result:

Theorem 4 *Suppose that $B_t = \left(B_t^{(1)}, ..., B_t^{(d)}\right)$, $0 \le t \le 1$ is a Brownian motion starting in $x \in \mathbb{R}^d$. Let $b : [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a Borel measurable function. Set $b_0 = b$ and let $b_n : [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $n \ge 1$ be a sequence of continuous functions with compact support such that $b_n(t,\cdot)$ is continuously differentiable, $0 \le t \le 1$. Further assume that $b_n'(\cdot, X_\cdot^{(n)}) \in \mathcal{L}$ for all $n \ge 1$, where $X_t^{(n)}$ is the solution to (29) associated with the drift b_n and $'$ the derivative with respect to the space variable. Denoting by $(\cdot)_{1\le i,j \le d}$ $\mathbb{R}^{d\times d}-$matrices require that the following conditions are satisfied:*
(i)

$$\sup_{n\ge 0} E\left[\exp\left\{512 \int_0^1 \|b_n(s, B_s)\|^2 ds\right\}\right] < \infty. \tag{36}$$

(ii)

$$\sup_{n\ge1}\sup_{0\le t\le t'\le 1} E\left[\sup_{0\le\lambda\le1}\left\|\exp\left\{-\lambda\left(\left(\int_t^{t'} b_n^{(j)}(s, B_s) dB_s^{(i)}\right)_{1\le i,j\le d}\right\}\right\|^{36}\right] < \infty. \tag{37}$$

(iii)

$$\sup_{n\geq 1}\sup_{0\leq t\leq t'\leq 1} E\left[\sup_{0\leq\lambda\leq 1}\left\|\exp\left\{-\lambda\left(\int_t^{t'} b_n^{(j)}(1-s,\widehat{B}_s)d\widetilde{W}_s^{(i)}\right)_{1\leq i,j\leq d}\right\}\right\|^{36}\right]<\infty. \tag{38}$$

(iv)

$$\sup_{n\geq 1}\sup_{0\leq t\leq t'\leq 1} E\left[\sup_{0\leq\lambda\leq 1}\left\|\exp\left\{\lambda\left(\int_t^{t'} b_n^{(j)}(1-s,\widehat{B}_s)\frac{\widehat{B}_s^{(i)}}{1-s}ds\right)_{1\leq i,j\leq d}\right\}\right\|^{36}\right]<\infty. \tag{39}$$

(v)

$$\sup_{n\geq 1} E\left[\left(\int_0^1 \left|b_n^{(j)}(s,B_s)\right|^{2+\varepsilon} ds\right)^{12/(2+\varepsilon)}\right]<\infty \tag{40}$$

for all $j=1,...,d$ for some $\varepsilon>2$.

(vi)

$$\sup_{n\geq 1} E\left[\left(\int_0^1 \left|b_n^{(j)}(1-s,\widehat{B}_s)\frac{\widehat{B}_s^{(i)}}{1-s}\right|^{1+\varepsilon/2} ds\right)^{24/(2+\varepsilon)}\right]<\infty \tag{41}$$

for all $j=1,...,d$ for some $\varepsilon>2$.
(vii)

$$R_n \underset{n\longrightarrow\infty}{\longrightarrow} 0, \tag{42}$$

where the factor R_n is defined as

$$R_n=(E\left[J_n\right])^{1/2}$$

with

$$\begin{aligned} J_n &= \sum_{j=1}^d \left(2\int_0^1 (b_n^{(j)}(s,B_s)-b^{(j)}(s,B_s))^2 ds\right. \\ &\quad \left. +\left(\int_0^1 \left|(b_n^{(j)}(s,B_s))^2-(b^{(j)}(s,B_s))^2\right| ds\right)^2\right). \end{aligned}$$

Then there exists a global strong solution X_t to

$$dX_t=b(t,X_t)dt+dB_t,\ X_0=x\in\mathbb{R}^d,$$

which is explicitly represented by (33). Moreover,

$$X_t^{(i)}\in\mathbb{D}_{1,2}$$

for all $i=1,...,d$, $0\leq t\leq 1$.
If in addition X_t is unique in law then strong uniqueness holds.

As a consequence of Theorem 4 we now obtain the following result in the one-dimensional case.

Theorem 5 *Let $b : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ be bounded and Borel measurable. Then there exists a unique Malliavin differentiable solution to (29) which has the explicit form (33).*

Remark 6 *In the one-dimensional case the existence and uniqueness of strong solutions to (29) for bounded and measurable drift coefficients was first obtained by Zvonkin in his celebrated paper [Zv]. The extension to the multi-dimensional case was given by [V]. We point out that our solution technique grants the important additional insight that such solutions are Malliavin differentiable.*

We defer the proofs of the above two theorems to a later point. First, in order to facilitate the following reading, we give an overview of the structure of the proof of Theorem 4. The proof rests on the following three pillars:

(α) Compactness criterion for subsets of $L^2(\mu)$ via Malliavin calculus (see [DaPMN]).

(β) Local time-space calculus (see [E]).

(γ) The white noise representation (31).

Our programme to prove Theorem 4 essentially consists of two steps:

1. Step. We consider a drift coefficient $b : [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and an approximating sequence $b_n : [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$, $n \geq 1$, of functions in the sense of Theorem 4. Then we show that for each $0 \leq t \leq 1$ the sequence $X_t^{(n)} = Y_t^{b_n}$, $n \geq 1$, is relatively compact in $L^2(\mu; \mathbb{R}^d)$ by employing a compactness criterion for $L^2(\mu; \mathbb{R}^d)$ (see [DaPMN]). In applying this criterion we resort to a "local time variational calculus" argument which is based on a local time-space integration concept as developed in [E].

2. Step. We show that Y_t^b is a well-defined object in the Hida distribution space for each $0 \leq t \leq 1$, and prove by means of the S-transform (18) that

$$Y_t^{b_n} \underset{n \longrightarrow \infty}{\longrightarrow} Y_t^b$$

in $L^2(\mu; \mathbb{R}^d)$ for $0 \leq t \leq 1$. Using a certain transformation property for Y_t^b we can finally verify Y_t^b as a solution to (29).

In order to accomplish the first step of our programme we need some auxiliary results. The first result which is due to [DaPMN, Theorem 1] provides a compactness criterion for subsets of $L^2(\mu; \mathbb{R}^d)$.

Theorem 7 *Let $\{(\Omega, \mathcal{A}, P)\,; H\}$ be a Gaussian probability space, that is $(\Omega, \mathcal{A}, P)$ is a probability space and H a separable closed subspace of Gaussian random variables of $L^2(\Omega)$, which generate the σ-field $\mathcal{A}$. Denote by $\mathbf{D}$ the derivative operator acting on elementary smooth random variables in the sense that*

$$\mathbf{D}(f(h_1, ..., h_n)) = \sum_{i=1}^{n} \partial_i f(h_1, ..., h_n) h_i, \;\; h_i \in H, f \in C_b^\infty(\mathbb{R}^n).$$

Further let $\mathbf{D}_{1,2}$ be the closure of the family of elementary smooth random variables with respect to the norm

$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|\mathbf{D}F\|_{L^2(\Omega;H)} .$$

Assume that C is a self-adjoint compact operator on H with dense image. Then for any $c > 0$ the set

$$\mathcal{G} = \left\{ G \in \mathbf{D}_{1,2} : \|G\|_{L^2(\Omega)} + \|\mathbf{D}G\|_{L^2(\Omega;H)} \leq c \right\}$$

is relatively compact in $L^2(\Omega)$.

We also need the following technical result, which can be found in [DaPMN].

Lemma 8 *Let $v_s, s \geq 0$ be the Haar basis of $L^2([0,1])$. For any $0 < \alpha < 1/2$ define the operator A_α on $L^2([0,1])$ by*

$$A_\alpha v_s = 2^{k\alpha} v_s, \text{ if } s = 2^k + j$$

for $k \geq 0, 0 \leq j \leq 2^k$ and

$$A_\alpha 1 = 1.$$

Then for all β with $\alpha < \beta < (1/2)$, there exists a constant c_1 such that

$$\|A_\alpha f\| \leq c_1 \left\{ \|f\|_{L^2([0,1])} + \left(\int_0^1 \int_0^1 \frac{|f(t) - f(t')|^2}{|t - t'|^{1+2\beta}} dt dt' \right)^{1/2} \right\}.$$

In the sequel we will make use of the concept of stochastic integration over the plane with respect to Brownian local time. See [E]. Consider elementary functions $f_\Delta : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f_\Delta(s,x) = \sum_{(s_j,x_i) \in \Delta} f_{ij} \chi_{(s_j,s_{j+1}]}(s) . \chi_{(x_i,x_{i+1}]}(x) , \tag{43}$$

where $(x_i)_{1 \leq i \leq n}$, $(f_{ij})_{1 \leq i \leq n, 1 \leq j \leq m}$ are finite sequences of real numbers, $(s_j)_{1 \leq j \leq m}$ a partition of $[0,1]$ and $\Delta = \{(s_j, x_i), 1 \leq i \leq n, 1 \leq j \leq m\}$. Denote by $\{L(t,x)\}_{0 \leq t \leq 1, x \in \mathbb{R}}$ the local time of a $1-$dimensional Brownian motion B. Then integration of elementary functions with respect to L can be defined by

$$\int_0^1 \int_{\mathbb{R}} f_\Delta(s,x) L(ds,dx) = \sum_{(s_j,x_i) \in \Delta} f_{ij} (L(s_{j+1}, x_{i+1}) - L(s_j, x_{i+1}) - L(s_{j+1}, x_i) + L(s_j, x_i)). \tag{44}$$

The latter integral actually extends to integrands of the Banach space $(\mathcal{H}, \|\cdot\|)$ of measurable functions f such that

$$\|f\| = 2 \left(\int_0^1 \int_{\mathbb{R}} (f(s,x))^2 \exp(-\frac{x^2}{2s}) \frac{dsdx}{\sqrt{2\pi s}} \right)^{1/2} + \int_0^1 \int_{\mathbb{R}} |xf(s,x)| \exp(-\frac{x^2}{2s}) \frac{dsdx}{s\sqrt{2\pi s}} < \infty. \tag{45}$$

The proof of the following theorem is given in [E, Theorem 3.1, Corollary 3.2].

Theorem 9 *Let $f \in \mathcal{H}$.*
(i) Then

$$\int_0^t \int_{\mathbb{R}} f(s,x)L(ds,dx), \ 0 \le t \le 1$$

exists and

$$E\left[\left|\int_0^t \int_{\mathbb{R}} f(s,x)L(ds,dx)\right|\right] \le \|f\|$$

for $0 \le t \le 1$.
(ii) If f is such that $f(t,\cdot)$ is locally square integrable and $f(t,\cdot)$ continuous in t as a map from $[0,1]$ to $L^2_{loc}(\mathbb{R})$, then

$$\int_0^t \int_{\mathbb{R}} f(s,x)L(ds,dx) = -[f(\cdot,B),B]_t\,, \quad 0 \le t \le 1,$$

where $[\cdot,\cdot]$ stands for the quadratic covariation of processes.
(iii) In particular, if $f(t,x)$ is differentiable in x, then

$$\int_0^t \int_{\mathbb{R}} f(s,x)L(ds,dx) = -\int_0^t f'(s,B)\,ds\,, \quad 0 \le t \le 1\,,$$

where $f'(s,x)$ denotes the derivative in x.

The proof of Theorem 4 relies on the following lemma.

Lemma 10 *Retain the conditions of Theorem 4. Then the sequence $X_t^{(n)} = Y_t^{b_n}$, $n \ge 1$, (see (33)) is relatively compact in $L^2(\mu;\mathbb{R}^d)$, $0 \le t \le 1$.*

Proof. We want to prove the relative compactness of the sequence of SDE solutions $X_t^{(n)} = Y_t^{b_n}$, $n \ge 1$, by applying Theorem 7. Because of Lemma 8 it is sufficient to show that

$$\left\{E\left[\left\|X_1^{(n)}\right\|^2\right] + E\left[\left\|A_\alpha D.X_1^{(n)}\right\|^2_{L^2([0,1];\mathbb{R}^d)}\right]\right\} \le M \tag{46}$$

for all $n \ge 1$, where $0 \le M < \infty$.

Without loss of generality we study the sequence $X_1^{(n)}$, $n \ge 1$. Then by the conditions of Theorem 4 the solutions $X_t^{(n)}$, $n \ge 1$, are Malliavin differentiable (see [N], [DOP] or [W]). Moreover the chain rule with respect to the Malliavin derivative D_t implies that

$$D_t X_1^{(n)} = \int_t^1 b_n'(s, X_s^{(n)}) D_s X_s^{(n)} ds + I_d\,, \quad 0 \le t \le 1\,, \quad n \ge 1\,. \tag{47}$$

It follows from our assumptions that the solution to the linear equation (47) for each $n \ge 1$ is given by

$$D_t X_1^{(n)} = \exp\left\{\int_t^1 b_n'(s, X_s^{(n)}) ds\right\} \in \mathbb{R}^{d\times d}\,, \quad 0 \le t \le 1\,. \tag{48}$$

See e.g. [A]. Fix $0 \leq t \leq t' \leq 1$. Then we find that

$$
\begin{aligned}
& E\left[\left\| D_t X_1^{(n)} - D_{t'} X_1^{(n)} \right\|^2\right] \\
= \; & E\left[\left\| \exp\left\{ \int_t^1 b_n'(s, X_s^{(n)}) ds - \int_{t'}^1 b_n'(s, X_s^{(n)}) ds \right\} \right\|^2\right].
\end{aligned}
$$

So it follows from Girsanov's theorem, the properties of supermartingales and (36) that

$$
\begin{aligned}
& E\left[\left\| D_t X_1^{(n)} - D_{t'} X_1^{(n)} \right\|^2\right] \\
\leq \; & C \cdot E\left[\left\| \exp\left\{ \int_t^1 b_n'(s, B_s) ds \right\} - \exp\left\{ \int_{t'}^1 b_n'(s, B_s) ds \right\} \right\|^2\right]^{1/2}
\end{aligned}
$$

for a constant C which is independent of n. The latter in connection with the properties of evolution operators for linear systems of ODE's, the mean value theorem and Hölder's inequality gives

$$
\begin{aligned}
& E\left[\left\| D_t X_1^{(n)} - D_{t'} X_1^{(n)} \right\|^2\right] \\
\leq \; & const.E\left[\left\| \exp\left\{ \int_{t'}^1 b_n'(s, B_s) ds \right\} \right\|^4 \cdot \left\| \exp\left\{ \int_t^{t'} b_n'(s, B_s) ds \right\} - I_d \right\|^4\right]^{1/2} \\
\leq \; & const.E\left[\left\| \exp\left\{ \int_{t'}^1 b_n'(s, B_s) ds \right\} \right\|^4\right. \\
& \left. \left\| \int_t^{t'} b_n'(s, B_s) ds \right\|^4 \sup_{0 \leq \lambda \leq 1} \left\| \exp\left\{ \lambda \int_t^{t'} b_n'(s, B_s) ds \right\} \right\|^4\right]^{1/2} \\
\leq \; & const.E\left[\left\| \exp\left\{ \int_{t'}^1 b_n'(s, B_s) ds \right\} \right\|^{12}\right]^{1/6} \\
& \cdot E\left[\left\| \int_t^{t'} b_n'(s, B_s) ds \right\|^{12}\right]^{1/6} E\left[\sup_{0 \leq \lambda \leq 1} \left\| \exp\left\{ \lambda \int_t^{t'} b_n'(s, B_s) ds \right\} \right\|^{12}\right]^{1/6}. \qquad (49)
\end{aligned}
$$

In the next step of our proof we want to invoke a "local time variational calculus" argument based on Theorem9 to get rid of the derivatives b_n' in (49). In order to simplify notation let us assume that the initial value x of the SDE (29) is zero. In the sequel we introduce the notation $L_i(ds, dx)$ to indicate local time-space integration in the sense of Theorem9 with respect to $B_t^{(i)}$ (i.e. the $i-$th component of B_t) on $(\Omega_i, \mu_i), i = 1, ..., d$ (see (7)).

So using Theorem 9, point (iii), we infer from (49) that

$$\begin{aligned}
& E\left[\left\|D_t X_1^{(n)} - D_{t'} X_1^{(n)}\right\|^2\right] \\
\leq\ & const. E\left[\left\|\exp\left\{\left(-\int_{t'}^1 \int_{\mathbb{R}} b_n^{(j)}(s,x) L_i(ds,dx)\right)_{1\leq i,j\leq d}\right\}\right\|^{12}\right]^{1/6} \\
& \cdot E\left[\left\|\left(-\int_t^{t'} \int_{\mathbb{R}} b_n^{(j)}(s,x) L_i(ds,dx)\right)_{1\leq i,j\leq d}\right\|^{12}\right]^{1/6} \\
& \cdot E\left[\sup_{0\leq\lambda\leq 1}\left\|\exp\left\{\lambda\left(-\int_t^{t'} \int_{\mathbb{R}} b_n^{(j)}(s,x) L_i(ds,dx)\right)_{1\leq i,j\leq d}\right\}\right\|^{12}\right]^{1/6} \\
=\ & const.\ I_1 \cdot I_2 \cdot I_3,
\end{aligned} \tag{50}$$

where

$$I_1 := E\left[\left\|\exp\left\{\left(-\int_{t'}^1 \int_{\mathbb{R}} b_n^{(j)}(s,x) L_i(ds,dx)\right)_{1\leq i,j\leq d}\right\}\right\|^{12}\right]^{1/6},$$

$$I_2 := E\left[\left\|\left(-\int_t^{t'} \int_{\mathbb{R}} b_n^{(j)}(s,x) L_i(ds,dx)\right)_{1\leq i,j\leq d}\right\|^{12}\right]^{1/6}$$

and

$$I_3 := E\left[\sup_{0\leq\lambda\leq 1}\left\|\exp\left\{\lambda\left(-\int_t^{t'} \int_{\mathbb{R}} b_n^{(j)}(s,x) L_i(ds,dx)\right)_{1\leq i,j\leq d}\right\}\right\|^{12}\right]^{1/6}.$$

Now we want to use the following decomposition of local time-space integrals (see the proof of Theorem 3.1 in [E]):

$$\begin{aligned}
& \int_0^t f_i(s,x) L_i(ds,dx) \\
=\ & \int_0^t f_i(s, B_s^{(i)}) dB_s^{(i)} + \int_{1-t}^1 f_i(1-s, \widehat{B}_s^{(i)}) d\widetilde{W}_s^{(i)} - \int_{1-t}^1 f_i(1-s, \widehat{B}_s^{(i)}) \frac{\widehat{B}_s^{(i)}}{1-s} ds,
\end{aligned} \tag{51}$$

$0 \leq t \leq 1$, a.e. for $f_i \in \mathcal{H}$, $i = 1, ..., d$ (see Theorem 9) Here $\widehat{B}_t^{(i)}$ is the i-th component of the time-reversed Brownian motion and $\widetilde{W}_t^{(i)}$ is a Brownian motion on (Ω_i, μ_i) with respect to the filtration $\mathcal{F}_t^{\widehat{B}^{(i)}}$, $i = 1, ..., d$ (see (35)). Let us apply (51) to establish some upper bounds for the factors I_1, I_2, I_3.

(1) Estimate for I_1: (51) and Hölder's inequality entail that

$$\begin{aligned}
I_1 &= E\Bigg[\Bigg\|\exp\left\{\left(-\int_{t'}^{1} b_n^{(j)}(s,B_s)dB_s^{(i)}\right)_{1\le i,j\le d}\right\} \\
&\quad \cdot\exp\left\{\left(-\int_{0}^{1-t'} b_n^{(j)}(1-s,\widehat{B}_s)d\widetilde{W}_s^{(i)}\right)_{1\le i,j\le d}\right\} \\
&\quad \cdot\exp\left\{\left(\int_{0}^{1-t'} b_n^{(j)}(1-s,\widehat{B}_s)\frac{\widehat{B}_s^{(i)}}{1-s}ds\right)_{1\le i,j\le d}\right\}\Bigg\|^{12}\Bigg]^{1/6} \\
&\le E\left[\left\|\exp\left\{\left(-\int_{t'}^{1} b_n^{(j)}(s,B_s)dB_s^{(i)}\right)_{1\le i,j\le d}\right\}\right\|^{36}\right]^{1/18} \\
&\quad \cdot E\left[\left\|\exp\left\{\left(-\int_{0}^{1-t'} b_n^{(j)}(1-s,\widehat{B}_s)d\widetilde{W}_s^{(i)}\right)_{1\le i,j\le d}\right\}\right\|^{36}\right]^{1/18} \\
&\quad \cdot E\left[\left\|\exp\left\{\left(\int_{0}^{1-t'} b_n^{(j)}(1-s,\widehat{B}_s)\frac{\widehat{B}_s^{(i)}}{1-s}ds\right)_{1\le i,j\le d}\right\}\right\|^{36}\right]^{1/18}. \qquad (52)
\end{aligned}$$

(2) Estimate for I_3 : Similarly to **(1)** we find

$$\begin{aligned}
I_3 &\le E\left[\sup_{0\le\lambda\le1}\left\|\exp\left\{-\lambda\left(\int_{t}^{t'} b_n^{(j)}(s,B_s)dB_s^{(i)}\right)_{1\le i,j\le d}\right\}\right\|^{36}\right]^{1/18} \\
&\quad \cdot E\left[\sup_{0\le\lambda\le1}\left\|\exp\left\{-\lambda\left(\int_{1-t'}^{1-t} b_n^{(j)}(1-s,\widehat{B}_s)d\widetilde{W}_s^{(i)}\right)_{1\le i,j\le d}\right\}\right\|^{36}\right]^{1/18} \\
&\quad \cdot E\left[\sup_{0\le\lambda\le1}\left\|\exp\left\{\lambda\left(\int_{1-t'}^{1-t} b_n^{(j)}(1-s,\widehat{B}_s)\frac{\widehat{B}_s^{(i)}}{1-s}ds\right)_{1\le i,j\le d}\right\}\right\|^{36}\right]^{1/18}. \qquad (53)
\end{aligned}$$

(3) Estimate for I_2 : The decomposition (51), Burkholder's and Hölder's inequality imply that

$$
\begin{aligned}
I_2 &\leq const. \sum_{i,j=1}^{d} E\left[\left|\int_t^{t'}\int_{\mathbb{R}} b_n^{(j)}(s,x)L_i(ds,dx)\right|^{12}\right]^{1/6} \\
&\leq const. \sum_{i,j=1}^{d} \left\{ E\left[\left|\int_t^{t'} b_n^{(j)}(s,B_s)dB_s^{(i)}\right|^{12}\right]^{1/6}\right. \\
&\quad \left. E\left[\left|\int_{1-t'}^{1-t} b_n^{(j)}(1-s,\widehat{B}_s)d\widetilde{W}_s^{(i)}\right|^{12}\right]^{1/6} + E\left[\left|\int_{1-t'}^{1-t} b_n^{(j)}(1-s,\widehat{B}_s)\frac{\widehat{B}_s^{(i)}}{1-s}ds\right|^{12}\right]^{1/6}\right\} \\
&\leq const. \sum_{i,j=1}^{d} \left\{ E\left[\left(\int_t^{t'} \left(b_n^{(j)}(s,B_s)\right)^2 ds\right)^{6}\right]^{1/6}\right. \\
&\quad \left. E\left[\left(\int_{1-t'}^{1-t} \left(b_n^{(j)}(1-s,\widehat{B}_s)\right)^2 ds\right)^{6}\right]^{1/6} + E\left[\left|\int_{1-t'}^{1-t} b_n^{(j)}(1-s,\widehat{B}_s)\frac{\widehat{B}_s^{(i)}}{1-s}ds\right|^{12}\right]^{1/6}\right\} \\
&\leq const. \sum_{i,j=1}^{d} \left\{ \left|t'-t\right|^{\varepsilon/(2+\varepsilon)} E\left[\left(\int_t^{t'} \left(b_n^{(j)}(s,B_s)\right)^{2+\varepsilon} ds\right)^{12/(2+\varepsilon)}\right]^{1/6}\right. \\
&\quad \left|t'-t\right|^{\varepsilon/(2+\varepsilon)} E\left[\left(\int_{1-t'}^{1-t} \left(b_n^{(j)}(1-s,\widehat{B}_s)\right)^{2+\varepsilon} ds\right)^{12/(2+\varepsilon)}\right]^{1/6} \\
&\quad \left. \left|t'-t\right|^{2\varepsilon/(2+\varepsilon)} E\left[\left(\int_{1-t'}^{1-t} \left|b_n^{(j)}(1-s,\widehat{B}_s)\frac{\widehat{B}_s^{(i)}}{1-s}\right|^{1+\varepsilon/2} ds\right)^{24/(2+\varepsilon)}\right]^{1/6}\right\}
\end{aligned}
$$

for $\epsilon > 0$. Hence by the estimates in **(1), (2), (3)** we conclude from (50) that

$$
E\left[\left\|D_t X_1^{(n)} - D_{t'} X_1^{(n)}\right\|^2\right] \leq C\left|t'-t\right|^{\varepsilon/(2+\varepsilon)}
$$

for all $0 \leq t \leq t' \leq 1$, $n \geq 1$, where $C \geq 0$ is a universal constant. Then using Theorem 7 and Lemma 8 completes the proof. ■

We are coming to Step 2 of our programme. Under the conditions of Theorem 4 we want to gradually prove the following:

- Y_t^b in (33) is a well-defined object in the Hida distribution space $(\mathcal{S})^*$, $0 \leq t \leq 1$, (Lemma 11).
- $Y_t^b \in L^2(\mu)$, $0 \leq t \leq 1$, (Lemma 13).

- We invoke Theorem 9 and show that $X_t^{(n)} = Y_t^{b_n}$ converges to Y_t^b in $L^2(\mu;\mathbb{R}^d)$ for a subsequence, $0 \le t \le 1$.
- We apply a transformation property for Y_t^b (Lemma 15) and identify Y_t^b as a solution to (29).

The first lemma gives a criterion under which the process Y_t^b belongs to the Hida distribution space.

Lemma 11 *Suppose that*

$$E_\mu \left[\exp\left(36 \int_0^1 \|b(s,B_s)\|^2 \, ds \right) \right] < \infty, \tag{54}$$

where the drift $b : [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is measurable. Then the coordinates of the process Y_t^b, defined in (33), that is

$$Y_t^{i,b} = E_{\widetilde{\mu}} \left[\widetilde{B}_t^{(i)} \mathcal{E}_T^{\diamond}(b) \right] \tag{55}$$

are elements of the Hida distribution space.

Proof. Without loss of generality we consider the case $d = 1$. Set

$$\Phi(\widetilde{\omega}, \omega) = \varphi\left(\widetilde{B}_t(\widetilde{\omega}) \right) \mathcal{E}_T^{\diamond}(b)(\omega, \widetilde{\omega})$$

and

$$\mathcal{E}(M_t^\phi) = \exp\left(\int_0^1 \left(b(t,\widetilde{B}_t) + \phi(t) \right) d\widetilde{B}_t - \frac{1}{2} \int_0^1 \left(b(t,\widetilde{B}_t) + \phi(t) \right)^2 dt \right)$$

for $\phi \in \mathcal{S}_{\mathbb{C}}([0,1])$, where

$$M_s^\phi(\widetilde{\omega}) = \int_0^s \left(b(t,\widetilde{B}_t(\widetilde{\omega})) + \phi(t) \right) d\widetilde{B}_t(\widetilde{\omega}).$$

Using the Kubo-Yokoi delta function (see [Ku, Theorem 13.4]), the characterization theorem of Hida distributions and the concept of $G-$entire functions (see [PS]) we find that Φ is a well-defined map from $\widetilde{\Omega}$ to $(\mathcal{S})^*$ (up to equivalence). Further it follows from our assumption, Hölder's inequality and the supermartingale property of Doleans-Dade exponentials that

$$\begin{aligned} E_{\widetilde{\mu}}\left[|S(\Phi(\widetilde{\omega},\cdot))(\phi)| \right] &= E_{\widetilde{\mu}}\left[\left| \varphi\left(\widetilde{B}_t \right) \mathcal{E}(M_t^\phi) \right| \right] \\ &\le K \cdot E_{\widetilde{\mu}}\left[\mathcal{E}\left(\int_0^1 2\left(b(t,\widetilde{B}_t) + \operatorname{Re}\phi(t) \right) d\widetilde{B}_t \right) \right]^{\frac{1}{2}} \exp(a \int_0^1 |\phi(t)|^2 \, dt) \\ &\le K \exp(a \, |\phi|_0^2), \end{aligned}$$

for $\phi \in \mathcal{S}_{\mathbb{C}}([0,1])$, where a, $K \ge 0$ are constants and $|\phi|_0^2 := \int_0^1 |\phi(t)|^2 \, dt$. Then [Ku, Theorem 13.4] yields the result. ■

The next auxiliary result gives a justification for the factor R_n in (42) of Theorem 4.

Lemma 12 *Let* $b_n : [0,1]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ *be a sequence of Borel measurable functions with* $b_0 = b$ *such that the integrability condition (36) is valid. Then*

$$\left|S(Y_t^{i,b_n} - Y_t^{i,b})(\phi)\right| \leq const \cdot R_n \cdot \exp(34 \int_0^1 \|\phi(s)\|^2 \, ds)$$

for all $\phi \in (S_{\mathbb{C}}([0,1]))^d$, $i = 1,...,d$, *with the factor* R_n *as in (42).*

Proof. For $i = 1,...,d$ we obtain by Proposition 1 and (19) that

$$\begin{aligned} & \left|S(Y_t^{i,b_n} - Y_t^{i,b})(\phi)\right| \\ \leq E_{\widetilde{\mu}} & \left[\left(\left|\widetilde{B}_t^{(i)}\right| \exp\left\{\sum_{j=1}^d \mathrm{Re}\left[\int_0^1 \left(b^{(j)}(s,\widetilde{B}_s) + \phi^{(j)}(s)\right) d\widetilde{B}_s^{(j)} - \frac{1}{2}\int_0^1 \left(b^{(j)}(s,\widetilde{B}_s) + \phi^{(j)}(s)\right)^2 ds\right]\right\}\right)\right. \\ & \cdot\left|\exp\left\{\sum_{j=1}^d \int_0^1 \left(b_n^{(j)}(s,\widetilde{B}_s) - b^{(j)}(s,\widetilde{B}_s)\right) d\widetilde{B}_s^{(j)} + \frac{1}{2}\int_0^1 \left(b^{(j)}(s,\widetilde{B}_s)^2 - b_n^{(j)}(s,\widetilde{B}_s)\right)^2) ds\right.\right. \\ & \left.\left.+ \int_0^1 \phi^{(j)}(s)\left(b^{(j)}(s,\widetilde{B}_s) - b_n^{(j)}(s,\widetilde{B}_s)\right) ds\right\} - 1\right|\Bigg] \end{aligned}$$

Since

$$|\exp(z) - 1| \leq |z| \exp(|z|)$$

it follows from Hölder's inequality that

$$\begin{aligned} & \left|S(Y_t^{i,b_n} - Y_t^{i,b})(\phi)\right| \leq E_{\widetilde{\mu}}\left[|Q_n|^2\right]^{\frac{1}{2}} \\ & \cdot E_{\widetilde{\mu}}\left[\left(\left|\widetilde{B}_t^{(i)}\right| \exp\left\{\sum_{j=1}^d \mathrm{Re}\left[\int_0^1 \left(b^{(j)}(s,\widetilde{B}_s) + \phi^{(j)}(s)\right) d\widetilde{B}_s^{(j)} - \frac{1}{2}\int_0^1 \left(b^{(j)}(s,\widetilde{B}_s) + \phi^{(j)}(s)\right)^2 ds\right]\right\}\right)^2\right. \\ & \qquad \left.\exp\left\{2\left|Q_n\right|\right\}\right]^{\frac{1}{2}} \end{aligned}$$

where

$$\begin{aligned} Q_n &= \sum_{j=1}^d \int_0^1 \left(b_n^{(j)}(s,\widetilde{B}_s) - b^{(j)}(s,\widetilde{B}_s)\right) d\widetilde{B}_s^{(j)} + \frac{1}{2}\int_0^1 \left(b^{(j)}(s,\widetilde{B}_s)^2 - b_n^{(j)}(s,\widetilde{B}_s)^2\right) ds \\ & \qquad + \int_0^1 \phi^{(j)}(s)\left(b^{(j)}(s,\widetilde{B}_s) - b_n^{(j)}(s,\widetilde{B}_s)\right) ds. \end{aligned}$$

We find that

$$
\begin{aligned}
& E_{\widetilde{\mu}}\left[|Q_n|^2\right] \le 9d^2 \exp\left\{\int_0^1 \|\phi(s)\|^2\,ds\right\} \\
& \cdot E_{\widetilde{\mu}}\left[\sum_{j=1}^d \left\{\left(\int_0^1 \left(b_n^{(j)}(s,\widetilde{B}_s) - b^{(j)}(s,\widetilde{B}_s)\right) d\widetilde{B}_s^{(j)}\right)^2 + \left(\int_0^1 \left(b^{(j)}(s,\widetilde{B}_s)^2 - b_n^{(j)}(s,\widetilde{B}_s)^2\right) ds\right)^2\right.\right. \\
& \left.\left. + \int_0^1 \left(b^{(j)}(s,\widetilde{B}_s) - b_n^{(j)}(s,\widetilde{B}_s)\right)^2 ds\right\}\right] \\
= \;& 3\exp\left\{\int_0^1 \|\phi(s)\|^2\,ds\right\} E_{\widetilde{\mu}}\left[J_n\right],
\end{aligned}
$$

where

$$
J_n = \sum_{j=1}^d 2\int_0^1 \left(b_n^{(j)}(s,\widetilde{B}_s) - b^{(j)}(s,\widetilde{B}_s)\right)^2 ds + \left(\int_0^1 \left|b^{(j)}(s,\widetilde{B}_s)^2 - b_n^{(j)}(s,\widetilde{B}_s)^2\right| ds\right)^2 .
$$

Further we get that

$$
\begin{aligned}
& E_{\widetilde{\mu}}\left[\left(\left|\widetilde{B}_t^{(i)}\right| \exp\left\{\sum_{j=1}^d \mathrm{Re}\left[\int_0^1 \left(b^{(j)}(s,\widetilde{B}_s) + \phi^{(j)}(s)\right) d\widetilde{B}_s - \frac{1}{2}\int_0^1 \left(b^{(j)}(s,\widetilde{B}_s) + \phi^{(j)}(s)\right)^2 ds\right]\right\}\right)^2\right. \\
& \qquad \left. \exp\left\{2\left|Q_n\right|\right\}\right] \\
\le \;& E_{\widetilde{\mu}}\left[\left(\left|\widetilde{B}_t^{(i)}\right| \exp\left\{\sum_{j=1}^d \mathrm{Re}\left[\int_0^1 \left(b^{(j)}(s,\widetilde{B}_s) + \phi^{(j)}(s)\right) d\widetilde{B}_s^{(j)}\right.\right.\right.\right. \\
& \qquad\qquad \left.\left.\left.\left. - \frac{1}{2}\int_0^1 \left(b^{(j)}(s,\widetilde{B}_s) + \phi^{(j)}(s)\right)^2 ds\right]\right\}\right)^4\right]^{\frac{1}{2}} \\
& \cdot \frac{1}{\sqrt{2}}\left(E_{\widetilde{\mu}}\left[\exp\left\{-8\,\mathrm{Re}\,Q_n\right\}\right]^{\frac{1}{2}} + E_{\widetilde{\mu}}\left[\exp\left\{8\,\mathrm{Re}\,Q_n\right\}\right]^{\frac{1}{2}}\right. \\
& \qquad\qquad \left. + E_{\widetilde{\mu}}\left[\exp\left\{-8\,\mathrm{Im}\,Q_n\right\}\right]^{\frac{1}{2}} + E_{\widetilde{\mu}}\left[\exp\left\{8\,\mathrm{Im}\,Q_n\right\}\right]^{\frac{1}{2}}\right).
\end{aligned}
$$

By Hölder's inequality again and the supermartingale property of Doléans-Dade exponentials we get the estimate

$$
\begin{aligned}
& E_{\widetilde{\mu}}\left[\exp\left\{-8\,\mathrm{Re}\,Q_n\right\}\right] \\
\le \;& E_{\widetilde{\mu}}\left[\exp\left\{\sum_{j=1}^d 128\int_0^1 \left(b^{(j)}(s,\widetilde{B}_s) - b_n^{(j)}(s,\widetilde{B}_s)\right)^2 ds - 8\int_0^1 \left(b^{(j)}(s,\widetilde{B}_s)^2 - b_n^{(j)}(s,\widetilde{B}_s)^2\right) ds\right.\right. \\
& \qquad \left.\left. + 8\int_0^1 \mathrm{Re}(\phi^{(j)}(s))^2 ds + \int_0^1 \left(b^{(j)}(s,\widetilde{B}_s) - b_n^{(j)}(s,\widetilde{B}_s)\right)^2 ds\right\}\right]^{\frac{1}{2}} \\
\le \;& L_n \exp\left\{4\int_0^1 \|\phi(s)\|^2\,ds\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
& L_n \\
= \; & E_{\widetilde{\mu}}\left[\exp\left\{\sum_{j=1}^{d} 128\int_0^1 \left(b^{(j)}(s,\widetilde{B}_s)-b_n^{(j)}(s,\widetilde{B}_s)\right)^2 ds + 8\int_0^1 \left|b^{(j)}(s,\widetilde{B}_s)^2-b_n^{(j)}(s,\widetilde{B}_s)^2\right| ds\right\}\right]^{\frac{1}{2}}.
\end{aligned}
$$

Similarly we conclude that

$$
E_{\widetilde{\mu}}\left[\exp\left\{8\operatorname{Re}Q_n\right\}\right]^{\frac{1}{2}} \leq L_n \exp\left\{4\int_0^1 \|\phi(s)\|^2\, ds\right\}.
$$

Also $E_{\widetilde{\mu}}\left[\exp\left\{-8\operatorname{Im}Q_n\right\}\right]^{\frac{1}{2}}$ and $E_{\widetilde{\mu}}\left[\exp\left\{8\operatorname{Im}Q_n\right\}\right]^{\frac{1}{2}}$ have the same upper bound as in the previous inequality.

Finally we see that

$$
\begin{aligned}
& E_{\widetilde{\mu}}\left[\left(\left|\widetilde{B}_t^{(i)}\right|\exp\left\{\sum_{j=1}^{d}\operatorname{Re}\left[\int_0^1\left(b^{(j)}(s,\widetilde{B}_s)+\phi^{(j)}(s)\right)d\widetilde{B}_s^{(j)}\right.\right.\right.\right. \\
& \qquad\qquad \left.\left.\left.\left. -\frac{1}{2}\int_0^1\left(b^{(j)}(s,\widetilde{B}_s)+\phi^{(j)}(s)\right)^2 ds\right]\right\}\right)^4\right]^{\frac{1}{2}} \\
\leq \; & E_{\widetilde{\mu}}\left[\left(\widetilde{B}_t^{(i)}\right)^8\right]^{\frac{1}{4}} E_{\widetilde{\mu}}^{\frac{1}{8}}\left[\exp\left\{512\int_0^1\left\|b(s,\widetilde{B}_s)\right\|^2 ds\right\}\right]\exp\left\{64\int_0^1\|\phi(s)\|^2\, ds\right\}.
\end{aligned}
$$

Altogether we get

$$
\left|S(Y_t^{i,b_n}-Y_t^{i,b})(\phi)\right| \leq const\cdot R_n\cdot\exp\left\{34\int_0^1\|\phi(s)\|^2\, ds\right\}
$$

with R_n as in (42). ■

The next lemma, which also is of independent interest, shows that under some conditions Y_t^b is square integrable.

Lemma 13 *Let $b_n : [0,1]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ with $b_0 = b$ be a sequence of Borel measurable functions such that the conditions (36) and (42) are fulfilled. Further impose on each b_n, $n \geq 1$, Lipschitz continuity and a linear growth condition. Then the process Y_t^b given by (55) is square integrable for all t.*

Proof. Because of our assumptions on b_n, $n \geq 1$, we conclude from Lemma 11 that $Y_t^{b_n}$ are square integrable unique solutions to (29). Further, Hölder's inequality and the supermartingale property of Doléans-Dade exponentials imply that

$$
\begin{aligned}
\left\|Y_t^{i,b_n}\right\|_{L^2(\mu)}^2 &= E_{\widetilde{\mu}}\left[\left(\widetilde{B}_t^{(i)}\right)^2\mathcal{E}\left(\int_0^1 b_n(s,\widetilde{B}_s)d\widehat{B}_s\right)\right] \\
&\leq const\cdot sup_{n\geq 1}E_{\widetilde{\mu}}\left[\exp\left\{6\int_0^1\left\|b_n(s,\widetilde{B}_s)\right\|^2 ds\right\}\right]^{\frac{1}{4}} \leq M < \infty. \qquad (56)
\end{aligned}
$$

Thus the sequence $Y_t^{b_n}$ is relatively compact in $L^2(\mu;\mathbb{R}^d)$ in the weak sense. From this we see that there exists a subsequence of $Y_t^{b_n}$ which converges to an element $Z_t \in L^2(\mu;\mathbb{R}^d)$ weakly. Without loss of generality we assume that

$$Y_t^{b_n} \longrightarrow Z_t \text{ weakly for } n \longrightarrow \infty\,.$$

In particular, since

$$\mathcal{E}\left(\int_0^1 \phi(s)dB_s\right) \in L^p(\mu),\ p > 0,$$

one finds that

$$E_\mu\left[Y_t^{i,b_n}\mathcal{E}\left(\int_0^1 \phi(s)dB_s\right)\right] \longrightarrow E_\mu\left[Z_t^{(i)}\mathcal{E}\left(\int_0^1 \phi(s)dB_s\right)\right] \text{ for } n \longrightarrow \infty.$$

On the other hand the estimate (**??**) in Lemma 12 yields

$$\begin{aligned} & E_\mu\left[Y_t^{i,b_n}\mathcal{E}\left(\int_0^1 \phi(s)dB_s\right)\right] = E_{\widetilde{\mu}}\left[\widetilde{B}_t^{(i)}\mathcal{E}\left(\int_0^1 \left(b_n(s,\widetilde{B}_s) + \phi(s)\right)d\widetilde{B}_s\right)\right] \\ \longrightarrow\ & E_{\widetilde{\mu}}\left[\widetilde{B}_t^{(i)}\mathcal{E}\left(\int_0^1 \left(b(s,\widetilde{B}_s) + \phi(s)\right)d\widetilde{B}_s\right)\right] \\ =\ & S(Y_t^b)(\phi),\ \phi \in (\mathcal{S}_{\mathbb{C}}([0,1]))^d. \end{aligned}$$

Hence

$$S(Y_t^{i,b})(\phi) = S(Z_t^{(i)})(\phi)\,,\quad \phi \in (\mathcal{S}_{\mathbb{C}}([0,1]))^d.$$

Since the S-transform is a monomorphism we get that $Y_t = Z_t \in L^2(\mu;\mathbb{R}^d)$. ■

We shall introduce classe $\mathcal{M}$ of approximating functions.

Definition 14 *We denote by $\mathcal{M}$ the class of Borel measurable functions $b : [0,1]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ for which there exists a sequence of approximating functions $b_n : [0,1]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ in the sense of $R_n \longrightarrow 0$ in (42) such that the conditions of Theorem 4 hold.*

We are ready to state the following "transformation property" for Y_t^b.

Lemma 15 *Assume that $b : [0,1]\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ belongs to the class $\mathcal{M}$. Then*

$$\varphi^{(i)}\left(t, Y_t^b\right) = E_{\widetilde{\mu}}\left[\varphi^{(i)}\left(t,\widetilde{B}_t\right)\mathcal{E}_T^{\diamond}(b)\right] \tag{57}$$

a.e. for all $0 \le t \le 1, i = 1,...,d$ and $\varphi = (\varphi^{(1)},...,\varphi^{(d)})$ such that $\varphi(B_t) \in L^2(\mu;\mathbb{R}^d)$.

Proof. See [P, Lemma 16] or [M-BP1] for a proof. ■

Using the above auxiliary results we can finally give the proof of Theorem 4.

Proof of Theorem 4. We aim at employing the transformation property (57) of Lemma 15 to verify that Y_t^b is a unique strong solution of the SDE (29). To shorten notation we set $\int_0^t \varphi(s,\omega)dB_s := \sum_{j=1}^d \int_0^t \varphi^{(j)}(s,\omega)dB_s^{(j)}$ and $x = 0$.

We first remark that $Y_\cdot^b$ has a continuous modification. The latter can be checked as follows: Since each $Y_t^{b_n}$ is a strong solution of the SDE (29) with respect to the drift b_n we obtain from Girsanov's theorem and (36) that

$$\begin{aligned} E_\mu\left[\left(Y_t^{i,b_n} - Y_u^{i,b_n}\right)^2\right] &= E_{\widetilde{\mu}}\left[\left(\widetilde{B}_t^{(i)} - \widetilde{B}_u^{(i)}\right)^2 \mathcal{E}\left(\int_0^1 b_n(s,\widetilde{B}_s)d\widetilde{B}_s\right)\right] \\ &\leq const \cdot |t-u| \end{aligned}$$

for all $0 \leq u,t \leq 1$, $n \geq 1$, $i = 1,...,d$. By Lemma 10 we know that

$$Y_t^{(b_n)} \longrightarrow Y_t^{(b)} \text{ in } L^2(\mu;\mathbb{R}^d)$$

for a subsequence, $0 \leq t \leq 1$. So we get that

$$E_\mu\left[\left(Y_t^{i,b} - Y_u^{i,b}\right)^2\right] \leq const \cdot |t-u| \tag{58}$$

for all $0 \leq u,t \leq 1$, $i = 1,...,d$. Then Kolmogorov's Lemma provides a continuous modification of Y_t^b.

Since $\widetilde{B}_t$ is a weak solution of (29) for the drift $b(s,x) + \phi(s)$ with respect to the measure $d\mu^* = \mathcal{E}\left(\int_0^1 \left(b(s,\widetilde{B}_s) + \phi(s)\right) d\widetilde{B}_s\right) d\mu$ we obtain that

$$\begin{aligned} S(Y_t^{i,b})(\phi) &= E_{\widetilde{\mu}}\left[\widetilde{B}_t^{(i)} \mathcal{E}\left(\int_0^1 \left(b(s,\widetilde{B}_s) + \phi(s)\right) d\widetilde{B}_s\right)\right] \\ &= E_{\mu^*}\left[\widetilde{B}_t^{(i)}\right] \\ &= E_{\mu^*}\left[\int_0^1 \left(b^{(i)}(s,\widetilde{B}_s) + \phi^{(i)}(s)\right) ds\right] \\ &= \int_0^t E_{\widetilde{\mu}}\left[b^{(i)}(s,\widetilde{B}_s)\mathcal{E}\left(\int_0^1 \left(b(u,\widetilde{B}_u) + \phi(u)\right) d\widetilde{B}_u\right)\right] ds + S\left(B_t^{(i)}\right)(\phi). \end{aligned}$$

Hence the transformation property (57) applied to b gives

$$S(Y_t^{i,b})(\phi) = S(\int_0^t b^{(i)}(u, Y_u^{i,b})du)(\phi) + S(B_t^{(i)})(\phi).$$

Then the injectivity of S implies that

$$Y_t^b = \int_0^t b(s, Y_u^b)ds + B_t\,.$$

The Malliavin differentiability of Y_t^b follows from the fact that

$$\sup_{n\geq 1} \left\|Y_t^{i,b_n}\right\|_{1,2} \leq M < \infty$$

for all $i = 1, .., d$ and $0 \leq t \leq 1$. See e.g. [N].

Finally, if the solution $X_t = Y_t^b$ is unique in law then our conditions allow the application of Girsanov's theorem to any other strong solution. Then the proof of Proposition 1 (see e.g. [P, Proposition 1]) shows that any other solution necessarily takes the form Y_t^b. ■

As a consequence of Theorem 4 we obtain the following result.

Corollary 16 *Replace in Theorem 4 the conditions (36),...,(39) by*

$$\sup_{n \geq 0} E\left[\exp\left\{2592 d^5 \int_0^1 \|b_n(s, B_s)\|^2 \, ds\right\}\right] < \infty \tag{59}$$

and

$$\sup_{n \geq 1} E\left[\exp\left\{36 \int_0^1 \left| b_n^{(j)}(s, B_s) \frac{B_s^{(i)}}{s} \right| ds\right\}\right] < \infty \tag{60}$$

for all $1 \leq i, j \leq d$ and retain all the other assumptions of Theorem 4. Then there exists a Malliavin differentiable strong solution to (29) which has the explicit representation (33). Moreover, if the solution is unique in law, then strong uniqueness holds.

Proof. The proof directly follows from Girsanov's theorem and Hölder's inequality. ■

Remark 17 *(i) A sufficient condition for uniqueness in law in Corollary 16 is e.g. the assumption that*

$$\int_0^1 \|b(t, Y_t)\|^2 \, dt < \infty \ \ a.e.$$

for all weak solutions Y_t to (29). See [KS, Proposition 5.3.10].
(ii) It is worth mentioning that the Malliavin differentiability of the solution in Corollary 16 implies its quasi-continuity on the sample space. See [BH].
(iii) We shall mention that for $d = 2$ the commutativity requirement $b_n'(\cdot, X_\cdot^{(n)}) \in \mathcal{L}$ (see Definition 3) is fulfilled if e.g.

$$\begin{cases} \frac{\partial}{\partial x_1} b_n^{(2)}(t, x) = -\alpha_n \beta_n \frac{\partial}{\partial x_2} b_n^{(1)}(t, x) \\ \frac{\partial}{\partial x_2} b_n^{(2)}(t, x) = \frac{\partial}{\partial x_1} b_n^{(1)}(t, x) - (\alpha_n + \beta_n) \frac{\partial}{\partial x_2} b_n^{(1)}(t, x) \end{cases} \tag{61}$$

holds for all $n \geq 1$, $0 \leq t \leq 1$, $x \in \mathbb{R}^2$, where (α_n) and (β_n) are sequences of real numbers such that $\alpha_n \neq \beta_n$, $n \geq 1$.

Now let us turn to the proof of Theorem 5.

Proof of Theorem 5. Note that in the 1-dimensional case, the commutativity requirement $b_n'(\cdot, X_\cdot^{(n)}) \in \mathcal{L}$ is always fulfilled. Then the proof follows from Corollary 16 and Lemma 18 (see below). ■

Lemma 18

$$E\left[\exp\left\{k \int_0^1 \frac{\|B_t\|}{t} dt\right\}\right] < \infty$$

for all $k \geq 0$.

Proof. See [KSY]. ■

Remark 19 *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded Borel measurable function. Then, if one combines Lemma 18 for $d = 2$ with Corollary 16 and the relation (61) one obtains that drift coefficients $b : [0,1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ of the form*

$$b(t,x) = \begin{pmatrix} f(x_1 + x_2) \\ f(x_1 + x_2) \end{pmatrix}$$

admit unique Malliavin differentiable solutions Y_t^b to (29). We remark that many other examples of this type- also for general dimensions- can be constructed.

In the next result, integrability conditions imposed on the drift coefficient are of non-exponential type.

Corollary 20 *Let us assume that the sequence of functions b_n, $n \geq 0$, with $b_0 = b$ satisfies all conditions of Theorem 4 aside from (36),...,(41). Set $b_n(t,x) = 0$ if $t \notin [0,1]$. Further let $S, R \in [0, \infty)$ and $S < R$. Suppose that*

$$\lim_{r \searrow 0} \sup_{n \geq 0} \sup_{(s,x) \in [S,S+R] \times \mathbb{R}^d} E\left[\int_s^{s+r} \|b_n(t, x + B_{t-s})\|^{2+\varepsilon} dt\right] = 0 \tag{62}$$

and

$$\lim_{r \searrow 0} \sup_{n \geq 1} \sup_{(s,x) \in [S,S+R] \times \mathbb{R}^d} E\left[\int_s^{s+r} \left| b_n^{(j)}(t, x + B_{t-s}) \frac{(x + B_{t-s}^{(j)})}{t} \right|^{1+\varepsilon} dt\right] = 0 \tag{63}$$

for all $j = 1,..,d$ and some $\varepsilon > 2$. Then uniqueness in law entails the existence of a unique Malliavin diffrentiable solution to (29).

Proof. The proof follows from an argument of N. I. Portenko which exploits the Markovianity of the Wiener process ([Po]). ■

Remark 21 *Using the $m-$dimensional Gaussian kernel $p(t; x, y)$ the conditions (62) and (63) can be recast in the deterministic form*

$$\lim_{r \searrow 0} \sup_{n \geq 0} \sup_{(s,y) \in [S,S+R] \times \mathbb{R}^d} \int_s^{s+r} \int_{\mathbb{R}^d} \|b_n(t,x)\|^{2+\varepsilon} p(t-s; x, y) dx dt = 0 \tag{64}$$

and

$$\lim_{r \searrow 0} \sup_{n \geq 1} \sup_{(s,y) \in [S,S+R] \times \mathbb{R}^d} \int_s^{s+r} \int_{\mathbb{R}^d} \left| b_n^{(j)}(t,x) \frac{x_j}{t} \right|^{1+\varepsilon} p(t-s; x, y) dx dt = 0 \tag{65}$$

for all $j = 1,..,d$ and some $\varepsilon > 2$.

Finally, we give an extension of Theorem 4 to a class of non-degenerate $d-$dimensional Itô-diffusions.

Theorem 22 *Consider the time-homogeneous $\mathbb{R}^d-$valued SDE*

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t,\ X_0 = x \in \mathbb{R}^d,\ 0 \le t \le T, \tag{66}$$

where the coefficients $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \longrightarrow \mathbb{R}^d \times \mathbb{R}^d$ are Borel measurable. Require that there exists a bijection $\Lambda : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, which is twice continuously differentiable. Let $\Lambda_x : \mathbb{R}^d \longrightarrow L(\mathbb{R}^d, \mathbb{R}^d)$ and $\Lambda_{xx} : \mathbb{R}^d \longrightarrow L(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ be the corresponding derivatives of Λ and assume that

$$\Lambda_x(y)\sigma(y) = id_{\mathbb{R}^d} \text{ for } y \text{ a.e.}$$

as well as

$$\Lambda^{-1} \text{ is Lipschitz continuous.}$$

Suppose that the function $b_* : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ given by

$$\begin{aligned} & b_*(x) := \Lambda_x\left(\Lambda^{-1}(x)\right)\left[b(\Lambda^{-1}(x))\right] \\ & +\frac{1}{2}\Lambda_{xx}\left(\Lambda^{-1}(x)\right)\left[\sum_{i=1}^{d}\sigma(\Lambda^{-1}(x))[e_i], \sum_{i=1}^{d}\sigma(\Lambda^{-1}(x))[e_i]\right] \end{aligned}$$

satisfies the conditions of Theorem 4, where e_i, $i = 1, ..., d$, is a basis of $\mathbb{R}^d$. Then there exists a Malliavin differentiable solution X_t to (66).

Proof. Itô's Lemma applied to (66) implies that

$$\begin{aligned} dY_t &= \Lambda_x\left(\Lambda^{-1}(Y_t)\right)\left[b(\Lambda^{-1}(Y_t))\right] \\ &\quad +\frac{1}{2}\Lambda_{xx}\left(\Lambda^{-1}(Y_t)\right)\left[\sum_{i=1}^{d}\sigma(\Lambda^{-1}(Y_t))[e_i], \sum_{i=1}^{d}\sigma(\Lambda^{-1}(Y_t))[e_i]\right] dt + dB_t, \\ Y_0 &= \Lambda(x),\ 0 \le t \le T, \end{aligned}$$

where $Y_t = \Lambda(X_t)$. Because of Theorem 4 there exists a Malliavin differentiable solution Y_t to the last equation. Thus $X_t = \Lambda^{-1}(Y_t)$ solves (66). Finally, since Λ^{-1} is Lipschitz continuous, X_t is Malliavin differentiable. See e.g. [N, Proposition 1.2.3]. ■

4 Conclusion

In the previous sections we have presented a new method for the construction of strong solutions of SDE's with discontinuous coefficents. This approach is more direct and more "natural" compared to other techniques in the literature in so far as it evades a pathwise uniqueness argument. Actually, using Malliavin calculus and "local time variational calculus" we directly verify a generalized process to be a strong solution. Our approach covers the result of Zvonkin ([Zv]) as a special case. Furthermore, our method gives the additional valuable insight that the solutions derived are Malliavin differentiable. The latter suggests that the nature of solutions of SDE's is closely linked to the property of Malliavin differentiabilty.

Finally, we mention that our technique relies on a general principle which seems flexible enough to study strong solutions of a broader class of stochastic equations, for example:

1. Infinite dimensional Brownian motion with drift:

$$dX_t = b(t, X_t)dt + dB_t^Q, \ X_0 = x \in H, \tag{67}$$

where B_t^Q is a Q-cylindrical Brownian motion on a Hilbert space H and Q a positive symmetric trace class operator. This nessesitates an infinite dimensional version of Theorem 7. Furthermore, it is possible to study equation (67) with an additional densely defined operator in the drift term.

2 Infinite dimensional jump SDE's:

$$dX_t = \gamma(X_{t-})dL_t, \ L_0 = x \in H, \tag{68}$$

where L_t is a $H-$valued additive process.

3 Certain types of anticipative SDE's, that is e.g. Brownian motion with non-adapted drift.

4 Similar equations for fractional Brownian motion or fractional Lévy processes.

5 Certain types of forward-backward SDE's driven by Brownian motion.

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