STOCHASTIC FEYNMAN-KAC EQUATIONS ASSOCIATED TO LÉVY-ITÔ DIFFUSIONS

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Abstract. We consider linear parabolic stochastic integro-PDE's of Feynman-Kac type associated to Lévy-Itô diffusions. The solution of such equations can be represented as certain Feynman-Kac functionals of the associated diffusion such that taking expectation yields the deterministic Feynamn-Kac formula. We interpret the problem in the framework of white noise analysis and consider differentiation in the sense of stochastic distributions. This concept allows for relaxed assumptions on the equation coefficients, identically to those required in problems of similar deterministic integro-PDE's.

1. Introduction

We examine solutions $u(t,x) = u(\omega,t,x)$ of linear parabolic stochastic integro-PDE's of the following type

$$(1.1)\quad \begin{cases} -du(t,y) = \mathcal{L}u(t,y)dt + \int_{\mathbb{R}_0} \mathcal{B}u(t,y)\nu(d\zeta)dt + g(t,y)dt \\ \qquad + \{\mathcal{L}'u(t,y) + f(t,y)\}\, dB_t + \int_{\mathbb{R}_0} \{\mathcal{B}'u(t,y) + k(t,y,\zeta)\}\, \widetilde{N}(dt,d\zeta), \\ \\ u(T,y) = \varphi(y), \quad (t,y) \in [0,T) \times \mathbb{R}, \end{cases}$$

where

$$\begin{aligned} \mathcal{L}u(t,y) &= \frac{1}{2}(\sigma^2(t,y) + \hat{\sigma}^2(t,y))\,\partial_{yy}u(t,y) + b(t,y)\,\partial_y u(t,y) + c(t,y)u(t,y) \\ \mathcal{B}u(t,y) &= u(t,y+\gamma(t,y,\zeta)) + u(t,y+\hat{\gamma}(t,y,\zeta)) - 2u(t,y) \\ \mathcal{L}'u(t,y) &= \sigma(t,y)\,\partial_y u(t,y) + p(t,y)u(t,y) \\ \mathcal{B}'u(t,y) &= u(t,y+\gamma(t,y,\zeta)) - u(t,y) + q(t,y,\zeta)u(t,y), \end{aligned}$$

and where B_t is Brownian motion and $\widetilde{N}(dt,d\zeta) = N(dt,d\zeta) - dt\nu(d\zeta)$ is the compensated jump measure of a pure jump Lévy process L_t. Without the jump parts $\int_{\mathbb{R}_0} \mathcal{B}u(t,y)\nu(d\zeta)dt$ and $\int_{\mathbb{R}_0} \{\mathcal{B}'u(t,y) + k(t,y,\zeta)\}\, \widetilde{N}(dt,d\zeta)$ these equations have been studied by many authors, see for example [K3], [Pa], [KR2]. Solutions of such equations can be represented as Feynman-Kac functionals of certain associated Itô diffusions such that taking expectation (which makes the stochastic integrals

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in equation (1.1) disappear) yields the usual deterministic Feynman-Kac formula. However, if one only considered classical solutions of (1.1), the smoothness assumptions required on the coefficients would be much stronger than those which guarantee the existence of the corresponding Feynman-Kac solution candidate. In the Gaussian case, existing Sobolev L^p theory and Hölder theory for stochastic PDE's closes this gap, and regularity conditions for uniformly non-degenerate SPDE's (i.e. $\hat{\sigma}^2 \geq \delta > 0$) are the same on $\mathcal{L}$ and g and higher by 1 on $\mathcal{L}'$ and f as in similar deterministic problems. In the work of [MR2] the authors introduce the notion of so called soft solutions (see also [MR1]) which allows to further relax conditions on $\mathcal{L}'$ and $\hat{\sigma}^2$. This notion is an extended concept of solution to stochastic partial differential equations and the main tool used here is the Cameron-Martin-Itô theory of Wiener chaos (see [CM], [K1]).

In this paper we add an integro part to the equation and interpret equation (1.1) in the white noise framework for Lévy processes developed in [ØP], [LP] and [LØP], which is the analogue for Lévy processes of the white noise theory for Brownian motion in [HØUZ]. Using white noise notations one can rewrite equation (1.1) as equation (3.1) in Section 3. In the white noise framework the notion of a solution is extended to the concept of generalized solutions that take values in the *Kondratiev distribution space* $(\mathcal{S})_{-1}$ (see Section 2 for definitions). To the best of our knowledge, solution theory for stochastic integro-PDE's is very little developed. The concept of solutions in stochastic distribution spaces allows to assume regularity conditions on the coefficients in equation (1.1) equal to those for classical solutions of similar deterministic integro-PDE problems. In particular uniformly non-degeneracy is relaxed. It is shown that the explicit solution can be represented as a Feynman-Kac type functional of a certain associated Lévy-Itô diffusion (where however the integration is with respect to a two parameter Lévy process).

Further, concerning the time direction in (1.1) the following has to be mentioned. In the existing literature stochastic integration in (1.1) is Itô integration, which requires that SPDE's with terminal condition run backward in time (i.e. Brownian motion starts in time T). In our case, the white noise equation (3.1) together with relation (2.12) allows us to consider forward running time, i.e. Brownian motion and Poisson jump measure start in time 0. Stochastic integrals in (1.1) are then Skorohod integrals, provided they exist.

The study of stochastic integro-PDE's of the type (1.1) is as in the continuous case motivated by its appearance in different applications. One important example is the *Zakai equation* occurring in non-linear filtering problems for Cox processes. Assume a partially observable two dimensional process (X_t, Y_t), $0 \leq t \leq T$. X_t stands for the unobservable component of the process, referred to as the *signal process*, whereas Y_t is the observable part, called *observation process.* Suppose that the dynamics of the process is described by the following SDE:

$$dX_t \;=\; b(X_t)dt + \sigma(X_t)dB_t^X$$

$$dY_t = h(t,X_t)dt + dB_t^Y + \int_{\mathbb{R}_0} \varsigma N_\lambda(dt,d\varsigma),$$

where (B_t^X, B_t^Y) is a Wiener process, and N_λ is an integer valued random measure with predictable compensator

$$\hat{\mu}(dt,d\varsigma,\omega) = \lambda(t,X_t,\varsigma)dt\nu(d\varsigma)$$

for a Lévy measure ν and a function $\lambda(t,x,\varsigma)$, such that the increments of N_λ are conditionally independent with respect to the filtration generated by B_t^X. The interest is now to estimate the (possibly transformed) signal $f(X_t)$ for a function f given the observations Y_t, $0 \le s \le t$. The filter theory proposes the least square estimate of $f(X_t)$ at time t given the history of the observation process up to time t, that is the optimal filter is given through the conditional expectation

$$E[f(X_t)\,|\mathcal{F}_t^Y] \,.$$

Given some smoothness conditions on the coefficients and that the initial condition X_0 has a density p_0, one can then show that

$$E[f(X_t)\,|\mathcal{F}_t^Y] = \frac{\int f(x)u(\omega,t,x)\,dx}{\int u(\omega,t,x)\,dx},$$

where the unnormalized conditional density $u(t,x) = u(\omega,t,x)$ is a solution of the Zakai equation

$$\begin{aligned} du(t,x) &= \mathcal{L}u(t,x)dt + h(t,x)u(t,x)dB_t \\ &\quad + \int_{\mathbb{R}_0} (\lambda(t,x,\varsigma)-1)u(t,x)\widetilde{N}(dt,d\varsigma) \\ u(0,x) &= p_0(x). \end{aligned}$$

Here $\mathcal{L}$ is the adjoint operator of the generator of X_t and $\widetilde{N}(ds,d\varsigma)$ is the compensated jump measure of a Lévy process. As can be seen, this equation is a special case of equation (1.1) (modulo a time reversed terminal condition). For more details regarding the Zakai equation for jump diffusions and its solution see for example [G1], [G2] and [MP] and the references therein.

A second important example of equation (1.1) is referred to as *backward diffusion equation.* The continuous version goes back to [Za] and is further treated in e.g. [KR1], [K2] and [R]. It states that if $Y_s^{t,y}$ solves the Itô diffusion described by

$$\begin{cases} dY_s = b(Y_s)ds + \sigma(Y_s)dB_s \\ Y_t = y,\ t \le s \le T, \end{cases}$$

then under appropriate smoothness conditions on the coefficients $u(t,y) = Y_T^{T-t,y}$ is a solution of

$$\begin{cases} du(t,y) = \left\{\frac{1}{2}\sigma^2(y)\,\partial_{yy}u(t,y) + b(y)\,\partial_y u(t,y)\right\}dt + \sigma(y)\,\partial_y u(t,y)d\overleftarrow{B}_t \\ u(0,y) = y. \end{cases}$$

Here $\overleftarrow{B_t}$ is the Brownian motion $B_T - B_{T-t}$. In Section 3 we derive for the jump case the analogue *backward diffusion equation for Lévy-Itô diffusions* as a corollary.

In the remaining parts of the paper we recapitulate the essential concepts of white noise theory for Lévy processes in Section 2 before these are used to state and solve the stochastic Feynman-Kac problem in Section 3.

2. White Noise Framework

In this Section we provide a brief review of some concepts of a white noise theory for Lévy processes, developed in [ØP], [LP] and [LØP]. In the next Section we will use this theory as a basic tool to determine the solution of a SPDE in the Lévy-Hida space. For general information about white noise theory the reader is referred to the excellent accounts of [HKPS], [HØUZ], [Ku] and [O].

Let us recall that a *Lévy process* $L(t)$ is a stochastic process on $\mathbb{R}_+$, which has independent and stationary increments starting at zero, i.e. $L(0) = 0$. The process $L(t)$ is by its nature a càdlàg semimartingale, which is uniquely determined by the characteristic triplet

$$(B_t, C_t, \hat{\mu}) = (a \cdot t, \sigma \cdot t, \nu(d\zeta)dt), \tag{2.1}$$

where a, σ are constants and where ν is the *Lévy measure* on $\mathbb{R}_0 := \mathbb{R} - \{0\}$. For more information about Lévy processes we refer to e.g. [B], [Sa] or [JS].

We will first concentrate on the less familiar white noise framework in the case of a pure jump Lévy processes. i.e. $L(t)$ has no Brownian motion part. At the end of this section, we will then quickly recapitulate how to extend the setting to general Lévy processes.

We denote by $\mathcal{S}(\mathbb{R}^2)$ the Schwartz space on $\mathbb{R}^2$. The space $\mathcal{S}'(\mathbb{R}^2)$ is the dual of $\mathcal{S}(\mathbb{R}^2)$, that is the space of tempered distributions. Set $\Pi := \mathbb{R} \times \mathbb{R}_0$. We want to work with a white noise measure, which is constructed on the nuclear algebra $\widetilde{\mathcal{S}}'(\Pi)$, introduced in [LØP] as follows. The space $\widetilde{\mathcal{S}}(\Pi)$ is defined as the quotient algebra

$$\widetilde{\mathcal{S}}(\Pi) = \mathcal{S}(\Pi)/\mathcal{N}_\pi, \tag{2.1}$$

where $\mathcal{S}(\Pi)$ is a subspace of $\mathcal{S}(\mathbb{R}^2)$, given by

$$\mathcal{S}(\Pi) := \left\{ \varphi(t,\zeta) \in \mathcal{S}(\mathbb{R}^2) : \varphi(t,0) = (\frac{\partial}{\partial \zeta}\varphi)(t,0) = 0 \right\} \tag{2.2}$$

and where the closed ideal $\mathcal{N}_\pi$ in $\mathcal{S}(\Pi)$ is defined as

$$\mathcal{N}_\pi := \{\phi \in \mathcal{S}(\Pi) : \|\phi\|_{L^2(\pi)} = 0\} \tag{2.3}$$

with $\pi = \nu(d\zeta)dt$. The space $\widetilde{\mathcal{S}}(\Pi)$ is a (countably Hilbertian) nuclear algebra. We indicate by $\widetilde{\mathcal{S}}'(\Pi)$ its dual.

From the Bochner-Minlos theorem we deduce that there exists a unique probability measure μ on the Borel sets of $\widetilde{\mathcal{S}}'(\Pi)$ such that

$$\int_{\widetilde{\mathcal{S}}'(\Pi)} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = \exp\left(\int_{\Pi}(e^{i\phi}-1)d\pi\right) \tag{2.4}$$

for all $\phi \in \widetilde{\mathcal{S}}(\Pi)$, where $\langle\omega,\phi\rangle := \omega(\phi)$ denotes the action of $\omega \in \widetilde{\mathcal{S}}'(\Pi)$ on $\phi \in \widetilde{\mathcal{S}}(\Pi)$. The measure μ on $\Omega = \widetilde{\mathcal{S}}'(\Pi)$ is called *(pure jump) Lévy white noise probability measure.*

In the sequel we consider the compensated Poisson random measure induced through relation (2.4)

$$\widetilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)dt$$

associated with a Lévy process $L(t)$, which is defined on the *white noise probability space*

$$(\Omega, \mathcal{F}, P) = \left(\widetilde{\mathcal{S}}'(\Pi), \mathcal{B}(\widetilde{\mathcal{S}}'(\Pi)), \mu\right).$$

By using generalized Charlier polynomials $C_n(\omega) \in \left(\widetilde{\mathcal{S}}(\Pi)^{\widehat{\otimes} n}\right)'$ (dual of the n-th completed symmetric tensor product of $\widetilde{\mathcal{S}}(\Pi)$ with itself) it is possible to construct an orthogonal $L^2(\mu)-$basis $\{\mathcal{K}_\alpha(\omega)\}_{\alpha\in\mathcal{J}}$ defined by

$$\mathcal{K}_\alpha(\omega) = \left\langle C_{|\alpha|}(\omega), \delta^{\widehat{\otimes}\alpha}\right\rangle, \tag{2.5}$$

where $\mathcal{J}$ is the multiindex set of all $\alpha = (\alpha_1, \alpha_2, ...)$ with finitely many non-zero components $\alpha_i \in \mathbb{N}_0$. The symbol $\delta^{\widehat{\otimes}\alpha}$ denotes the symmetrization of $\delta_1^{\otimes\alpha_1} \otimes ... \otimes \delta_j^{\otimes\alpha_j}$, where $\{\delta_j\}_{j\geq 1} \subset \widetilde{\mathcal{S}}(\Pi)$ is an orthonormal basis of $L^2(\mathbb{R}\times\mathbb{R}_0, dt\nu(d\zeta))$. We assume the basis elements δ_j to be of the form $\delta_j = \xi_l\gamma_k$ where $(\xi_l)_{l\in\mathbb{N}}$ are the Hermite functions and $(\gamma_k)_{k\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}_0, \nu(d\zeta))$.

Then every $X \in L^2(\mu)$ has the unique representation

$$X = \sum_{\alpha\in\mathcal{J}} c_\alpha \mathcal{K}_\alpha$$

with Fourier coefficients $c_\alpha \in \mathbb{R}$. Moreover we have the isometry

$$\|X\|^2_{L^2(\mu)} = \sum_{\alpha\in\mathcal{J}} \alpha! c_\alpha^2 \tag{2.6}$$

with $\alpha! := \alpha_1!\ \alpha_1!...$ for $\alpha \in \mathcal{J}$. The *Kondratiev test function space* $(\mathcal{S})_1$ consists of all $f = \sum_{\alpha\in\mathcal{J}} c_\alpha\mathcal{K}_\alpha \in L^2(\mu)$ such that

$$\|f\|^2_{1,k} := \sum_{\gamma\in\mathcal{J}^m} (\alpha!)^2 c_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty \tag{2.7}$$

holds for all $k \in \mathbb{N}_0$ with weights $(2\mathbb{N})^{k\alpha} = (2\cdot1)^{k\alpha_1}(2\cdot2)^{k\alpha_2}...(2\cdot l)^{k\alpha_l}$, if $Index(\alpha) := l$. The space $(\mathcal{S})_1$ is given the projective topology, induced by the norms $(\|\cdot\|_{1,k})_{k\in\mathbb{N}_0}$

in (2.7). The *Kondratiev distribution space*, denoted by $(\mathcal{S})_{-1}$ is the topological dual of $(\mathcal{S})_1$. So we obtain the following Gel'fand triple

$$(\mathcal{S})_1 \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})_{-1}. \tag{2.8}$$

We can endow $(\mathcal{S})_{-1}$ with the structure of a topological algebra by introducing the *Wick product* $\diamond$, defined by

$$(\mathcal{K}_\alpha \diamond \mathcal{K}_\beta)(\omega) = (\mathcal{K}_{\alpha+\beta})(\omega),\ \ \alpha, \beta \in \mathcal{J}. \tag{2.9}$$

The product is linearly extensible to $(\mathcal{S})_{-1} \times (\mathcal{S})_{-1}$. It can be proven that

$$\langle C_n(\omega), f_n \rangle \diamond \langle C_m(\omega), g_m \rangle = \left\langle C_{n+m}(\omega), f_n \widehat{\otimes} g_m \right\rangle \tag{2.10}$$

for $f_n \in \widetilde{\mathcal{S}}(\Pi)^{\widehat{\otimes} n}$and $g_m \in \widetilde{\mathcal{S}}(\Pi)^{\widehat{\otimes} m}$ (see [LØP]).

A nice feature of the Lévy-Hida distribution space is that it carries the white noise $\overset{\bullet}{\widetilde{N}}(t, \zeta)$ of the compensated Poisson random measure $\widetilde{N}(dt, d\zeta)$. That is the formal Radon-Nikodym derivative of $\widetilde{N}(dt, d\zeta)$ defined as

$$\overset{\bullet}{\widetilde{N}}(t, \zeta) = \sum_{k \geq 1} \delta_k(t, \zeta) \mathcal{K}_{\epsilon_k}(\omega) \tag{2.11}$$

is in $(\mathcal{S})_{-1}$ $dt\nu(d\zeta)-$a.e.. Here ϵ_k is the multiindex with 1 on its k'*th* place and zero otherwise. The Wick product relates to Skorohod integrals with respect to $\widetilde{N}(dt, d\zeta)$ (see for example [ØP] for the definition of the Skorohod integral) in the following way: If $Y(t, \zeta, \omega)$ is a Skorohod integrable process, fulfilling the condition

$$E\left[\int_0^T \int_{\mathbb{R}_0} Y^2(t, \zeta, \omega) dt\nu(d\zeta)\right] < \infty,$$

then $Y(t, \zeta, \omega) \diamond \overset{\bullet}{\widetilde{N}}(t, \zeta)$ is $\lambda \times \nu$-Bochner integrable in $(\mathcal{S})_{-1}$ and (see [LØP] or [ØP])

$$\int_0^T \int_{\mathbb{R}_0} Y(t, \zeta, \omega)\ \widetilde{N}(\delta t, d\zeta) = \int_0^T \int_{\mathbb{R}_0} \left(Y(t, \zeta, \omega) \diamond \overset{\bullet}{\widetilde{N}}(t, \zeta)\right)\ dt\nu(d\zeta), \tag{2.12}$$

where the left hand side denotes the Skorohod integral.

One of our main tools in the study of Lévy-Itô diffusions is the *Lévy Hermite transform* $\mathcal{H}$, which is used to give a characterization of distributions in $(\mathcal{S})_{-1}$ (see characterization Theorem 2.3.8 in [LØP]). Similar to the Gaussian case the definition of $\mathcal{H}$ rests on the basis $\{\mathcal{K}_\alpha(\omega)\}_{\alpha \in \mathcal{J}}$ in (2.6). The *Lévy Hermite transform* of $X(\omega) = \sum_\alpha c_\alpha \mathcal{K}_\alpha(\omega) \in (\mathcal{S})_{-1}$, denoted by $\mathcal{H}X$ or shorter by $\widetilde{X}$, is defined by

$$\mathcal{H}X(z) = \widetilde{X}(z) = \sum_\alpha c_\alpha z^\alpha \in \mathbb{C}\ , \tag{2.13}$$

where $z = (z_1, z_2, ...) \in \mathbb{C}^{\mathbb{N}}$, i.e. in the space of $\mathbb{C}-$valued sequences, and where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}....$ We have that $\mathcal{H}X(z)$ in (2.13) is absolutely convergent on the infinite dimensional neighborhood

$$\mathbb{K}_q(R) := \left\{ (z_1, z_2, ...) \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha \neq 0} |z^\alpha|^2 (2\mathbb{N})^{q\alpha} < R^2 \right\} \tag{2.14}$$

for some $0 < q \leq R < \infty$. For example, the Hermite transform of $\overset{\bullet}{\widetilde{N}}(t, \zeta)$ can be evaluated as

$$\mathcal{H}(\overset{\bullet}{\widetilde{N}}(t, \zeta))(z) = \sum_{k \geq 1} \delta_k(t, \zeta) z_k. \tag{2.15}$$

The Hermite transform translates the Wick product into an ordinary (complex) product, that is

$$\mathcal{H}(X \diamond Y)(z) = \mathcal{H}(X)(z) \cdot \mathcal{H}(Y)(z). \tag{2.16}$$

As a consequence of Theorem 2.3.8 in [LØP] the last relation can be generalized to Wick versions of complex analytical functions g: If the function $g : \mathbb{C} \longrightarrow \mathbb{C}$ can be expanded into a Taylor series around $\xi_0 = \mathcal{H}(X)(0)$ with real valued coefficients, then there exists a unique distribution $Y \in (\mathcal{S})_{-1}$ such that

$$\mathcal{H}(Y)(z) = g\left(\mathcal{H}(X)(z)\right) \tag{2.17}$$

on $\mathbb{K}_q(R)$ for some $0 < q \leq R < \infty$. We set $g^\diamond(X) = Y$.

For example, the Wick version of the exponential function can be written as

$$\exp^\diamond X = \sum_{n \geq 0} \frac{1}{n!} X^{\diamond n}. \tag{2.18}$$

Let us now shortly outline how the preceding concepts and results can be generalized to capture the case of Lévy processes with Brownian motion and pure jump part (see [P]). Indicate by μ_G the Gaussian white noise measure on the measurable space

$$(\Omega_G, \mathcal{F}_G) = (\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R}))).$$

Further recall the construction of the orthogonal $L^2(\mu_G)$ basis $\{H_\alpha(\omega)\}_{\alpha \in J}$, given by

$$H_\alpha(\omega) = \prod_{j \geq 1} h_{\alpha_j}(\langle \omega, \xi_j \rangle),$$

where $\langle \omega, \cdot \rangle = \omega(\cdot)$ and where ξ_j (respectively h_j), $j = 1, 2, ...$ are the Hermite functions (respectively Hermite polynomials). Using μ_J to denote the pure jump white noise measure on $(\Omega_J, \mathcal{F}_J) = (\widetilde{\mathcal{S}}'(\Pi), \mathcal{B}(\widetilde{\mathcal{S}}'(\Pi)))$ as introduced above, we can define the *Lévy white noise measure* μ as the product measure $\mu_G \times \mu_J$ on

$$(\Omega, \mathcal{F}) = (\Omega_G \times \Omega_J, \mathcal{F}_G \otimes \mathcal{F}_J). \tag{2.19}$$

Set

$$\mathcal{L}_\gamma(\omega) = \mathcal{L}_\gamma(\omega_1, \omega_2) = H_\alpha(\omega_1)\mathcal{K}_\beta(\omega_2), \tag{2.20}$$

if $\gamma = (\alpha, \beta) \in \mathcal{I} := \mathcal{J}^2$. Thus $(\mathcal{L}_\gamma(\omega))_{\gamma \in \mathcal{I}}$ constitutes an $L^2(\mu)-$basis with norm expression

$$\|\mathcal{L}_\gamma\|^2_{L^2(\mu)} = \gamma!,$$

where $\gamma! := \alpha!\beta!$ for $\gamma = (\alpha, \beta) \in \mathcal{I}$.

As in the pure jump setting, we employ the basis $(\mathcal{L}_\gamma(\omega))_{\gamma \in \mathcal{I}}$ to establish the concepts of Hida space, Wick product and Hermite transform to the mixture of Gaussian and pure jump Lévy noise. As in [HØUZ], the white noise $\overset{\bullet}{B}_t$ of Brownian motion is defined as an element in the Hida distribution space

$$\overset{\bullet}{B}_t := \sum_k \xi_k(t) H_{\epsilon_k}. \tag{2.19}$$

In particular, the analogous relation of (2.12) is also valid for Brownian motion and its white noise.

We conclude this Section with two remarks.

Remark 2.1. Note that by choosing an appropriate basis the above described white noise theory can be established on any time interval $[0, T]$ instead of the complete time line $\mathbb{R}$ (which is used in the next section).

Remark 2.2. Due to notational convenience we have chosen to present the white noise framework only for Lévy processes with one dimensional time parameter. The generalization to d-parameter Lévy processes (d-dimensional time parameter), which are used in the beginning of the next Section, is straight forward and can be found in [P].

3. Generalized Solutions of Stochastic Feynman-Kac Equations associated to Lévy-Itô Diffusions

Let $(\Omega, \mathcal{F}, \mu)$ be a white noise space corresponding to a two dimensional fixed time interval $[0, T] \times [0, U]$ with associated Brownian motion B_t and 2-parameter pure jump Lévy process $P(t, u)$ with Lévy measure $\nu'(d\zeta)$. For convenience we suppose that $P(t, u)$ is a square integrable 2-parameter martingale, i.e. we have the representation

$$P(t, u) = \int_0^t \int_0^u \int_{\mathbb{R}_0} \zeta \, \widetilde{M}(dt, du, d\zeta),$$

where $\widetilde{M}(dt, du, d\zeta) = M(dt, du, d\zeta) - dt du \nu'(d\zeta)$ is the compensated jump measure of $P(t, u)$. The image measure of $M(dt, du, d\zeta)$ under the projection

$$pr : [0, T] \times [0, U] \times \mathbb{R}_0 \longrightarrow [0, T] \times \mathbb{R}_0,$$

denoted by $N(dt, d\zeta)$, is the jump measure of a square integrable Lévy process L_t with Lévy measure $\nu(d\zeta) = U \cdot \nu'(d\zeta)$. This is easily seen from the characteristic function of μ. We have the Lévy-Itô representation

$$L_t = \int_0^t \int_{\mathbb{R}_0} \zeta \, \widetilde{N}(dt, d\zeta),$$

where $\widetilde{N}(dt, d\zeta) = N(dt, d\zeta) - dt\nu(d\zeta)$ is the compensated Poisson random measure of L_t. Let $\{\mathcal{F}_t\}_{0\le t\le T}$ be the completion of the filtration generated by B_t and the Poisson random measure $N(dt, d\zeta)$. Restriction to $\mathcal{F}_T$-functionals in the white noise setting on $(\Omega, \mathcal{F}, \mu)$ leads in a natural way to a white noise space $(\Omega, \mathcal{F}_T, \mu)$ corresponding to the time interval $[0, T]$ with associated Brownian motion B_t and pure jump Lévy process L_t. In the sequel this will be our underlying white noise probability space.

On this space, we now consider the following linear parabolic stochastic integro-PDE in $(\mathcal{S})_{-1}$ which is the interpretation of equation (1.1) in the white noise framework

$$\text{(3.1)} \quad \begin{cases} 0 = \partial_t u(t,y) + \mathcal{L}u(t,y) + \int_{\mathbb{R}_0} \mathcal{B}u(t,y)\nu(d\zeta) + g(t,y) \\ + \{\mathcal{L}'u(t,y) + f(t,y)\} \diamond \overset{\bullet}{B}_t + \int_{\mathbb{R}_0} \{\mathcal{B}'u(t,y) + k(t,y,\zeta)\} \diamond \overset{\bullet}{\widetilde{N}}(t,\zeta)\nu(d\zeta) \\ u(T,y) = \varphi(y), \quad (t,y) \in [0,T] \times \mathbb{R}, \end{cases}$$

where

$$\begin{aligned} \mathcal{L}u(t,y) &= \frac{1}{2}(\sigma^2(t,y) + \hat{\sigma}^2(t,y))\, \partial_{yy}u(t,y) + b(t,y)\, \partial_y u(t,y) + c(t,y)u(t,y) \\ \mathcal{B}u(t,y) &= u(t, y + \gamma(t,y,\zeta)) + u(t, y + \hat{\gamma}(t,y,\zeta)) - 2u(t,y) \\ \mathcal{L}'u(t,y) &= \sigma(t,y)\, \partial_y u(t,y) + p(t,y)u(t,y) \\ \mathcal{B}'u(t,y) &= u(t, y + \gamma(t,y,\zeta)) - u(t,y) + q(t,y,\zeta)u(t,y). \end{aligned}$$

The aim of the paper is to identify and represent a generalized solution of (3.1) in the stochastic distribution space $(\mathcal{S})_{-1}$. A generalized solution of (3.1) is defined as a process $u(t, y)$ such that equation (3.1) is fulfilled with differentiation and integration taken in $(\mathcal{S})_{-1}$. One main tool in achieving this aim will be the Hermite transform that enables the transformation of the stochastic problem to a similar deterministic problem. Also, as already mentioned in the introduction, even if we impose a terminal condition we can consider forward running time in the white noise framework, i.e. Brownian motion and Poisson jump measure start in time 0. Integrating equation (3.1) in t then leads by relation (2.12) to Skorohod integrals with respect to Brownian motion and Poisson jump measure, provided they exist. Further, for notational convenience we focus in this paper on the one-dimensional case of equation (3.1), but note that the analog techniques and results go through in the n-dimensional case.

Concerning the coefficients in equation (3.1) we impose three sets of conditions ensuring the three essential requirements for our solution concept.

The first set regards the stochastic differential equation (3.15), and we suppose the following boundedness and Lipschitz conditions. We assume that there exist $K > 0$ and $\beta : \mathbb{R} \longrightarrow \mathbb{R}_+$ with $\int_{\mathbb{R}_0} \beta(\zeta)\nu(d\zeta) < +\infty$ such that for all $s \in [0,T]$ and $v, y \in \mathbb{R}$

$$\begin{aligned} |b(s,y)| + |\sigma(s,y)| + |\hat{\sigma}(s,y)| + |c(s,y)| \\ + |p(s,y)| + |g(s,y)| + |f(s,y)| &\leq K \end{aligned} \tag{3.2}$$

$$|\gamma(s,y,\zeta)| + |\hat{\gamma}(s,y,\zeta)| + |q(s,y,\zeta)| + |k(s,y,\zeta)| \leq \beta(\zeta) \cdot K \tag{3.3}$$

and

$$\begin{aligned} |b(s,v) - b(s,y)| + |\sigma(s,v) - \sigma(s,y)| + |\hat{\sigma}(s,v) - \hat{\sigma}(s,y)| \\ + |p(s,v) - p(s,y)| + |g(s,v) - g(s,y)| \\ + |f(s,v) - f(s,y)| + |\varphi(v) - \varphi(y)| &\leq K\,|v-y| \end{aligned} \tag{3.4}$$

$$\begin{aligned} |\gamma(s,v,\zeta) - \gamma(s,y,\zeta)| + |\hat{\gamma}(s,v,\zeta) - \hat{\gamma}(s,y,\zeta)| \\ + |q(s,v,\zeta) - q(s,y,\zeta)| + |k(s,v,\zeta) - k(s,y,\zeta)| &\leq \beta(\zeta)\,|v-y|\,. \end{aligned} \tag{3.5}$$

In addition we set $0 \leq q(s,y,\zeta)$ for all $(s,y,\zeta) \in [0,T] \times \mathbb{R} \times \mathbb{R}_0$.

The second set regards the solution of a deterministic integro-PDE problem. More precisely, we assume the coefficients in equation (3.1) to be such that with

$$\phi'_z(t) = \mathcal{H}(\overset{\bullet}{B}(t))(z) = \sum_k z_k \xi_k(t)$$

and

$$\phi''_z(t,\zeta) = \mathcal{H}(\overset{\bullet}{\tilde{N}}(t,\zeta))(z) = \sum_k z_k \delta_k(t,\zeta)$$

for a given $z \in \mathbb{K}_q(R) \cap \mathbb{R}^{\mathbb{N}}$ the deterministic integro-PDE

$$\begin{cases} 0 = \partial_t u(t,y) + \mathcal{L}u(t,y) + \int_{\mathbb{R}_0} \mathcal{B}u(t,y)\nu(d\zeta) + g(t,y) \\ + \{\mathcal{L}'u(t,y) + f(t,y)\}\,\phi'_z(t) + \int_{\mathbb{R}_0} \{\mathcal{B}'u(t,y) + k(t,y,\zeta)\}\,\phi''_z(t,\zeta)\nu(d\zeta) \\ \\ u(T,y) = \varphi(y), \quad (t,y) \in [0,T] \times \mathbb{R}, \end{cases} \tag{3.6}$$

has a solution u^* in $C^{1,2}([0,T],\mathbb{R})$. For example, one sufficient set of conditions on the coefficients additionally to (3.2)-(3.5) would be as follows (see [PH], Section 5.2). For $\beta(\zeta)$ as before and $\beta' : \mathbb{R} \longrightarrow \mathbb{R}_+$ such that $\int_{\mathbb{R}_0} (\beta')^2(\zeta)\nu(d\zeta) < +\infty$ we require

(3.7) The diffusion coefficient $(\hat{\sigma}^2 + \sigma^2)$ is bounded away from zero.

(3.8) The coefficients $b(s,y)$, $\sigma(s,y)$, $\hat{\sigma}(s,y)$, $\gamma(s,y,\zeta)\beta'^{-1}(\zeta)$, $\hat{\gamma}(s,y,\zeta)\beta^{-1}(\zeta)$ are locally Lipschitz continuous in (s,y) uniformly in ζ.

(3.9) The coefficients $f(s,y)$, $g(s,y)$, $k(s,y,\zeta)\beta'^{-1}(\zeta)$ are globally Lipschitz continuous in (s,y) uniformly in ζ.

(3.10) The coefficients $c(s,y)$, $p(s,y)$, $q(s,y,\zeta)\beta'^{-1}(\zeta)$ are locally Hölder continuous in (s,y) uniformly in ζ.

(3.11) The Lévy measure ν is finite.

Note that for example for $k(s,y,\zeta)$ condition (3.9) together with the nature of $\phi''_z(t,\zeta)$ and Hölder's inequality imply the Lipschitz continuity in (s,y) of the function

$$\int_{\mathbb{R}_0} k(s,y,\zeta)\phi''_z(s,\zeta)\nu(d\zeta).$$

Analogue implications are valid for $\gamma(s,y,\zeta)$, $\hat{\gamma}(s,y,\zeta)$ and $q(s,y,\zeta)$. Also, we stress that by condition (3.7) uniformly non-degeneracy is relaxed.

Note further that other sets of sufficient conditions including infinite Lévy measures are possible (see for example [GS]).

Finally, in order to apply Hermite transform techniques as will be seen below, we suppose for a $\gamma \in (0,1)$ the following Hölder regularities. Let $C^{0,n+\gamma}$ (respectively $C^{n+\gamma}$), $n \in \mathbb{N}$, denote functions who are bounded and continuous in t and whose partial derivatives in y up to order n are γ-Hölder continuous (respectively whose partial derivatives up to order n are γ-Hölder continuous). Then we assume with β' as above uniformly in ζ:

(3.12) The coefficients $g(t,y)$, $f(t,y)$, $p(t,y)$, $c(t,y)$, $q(s,y,\zeta)\beta'^{-1}(\zeta)$, $k(s,y,\zeta)\beta'^{-1}(\zeta)$ are elements of $C^{0,0+\gamma}([0,T]\times\mathbb{R})$.

(3.13) The coefficient $\varphi(y)$ is element of $C^{2+\gamma}(\mathbb{R})$.

Then it follows from (3.12) and (3.13) together with (3.7) as in [MPr1] that

$$||u^*||_{C^{0,2+\gamma}([0,T]\times\mathbb{R})} \leq C\,, \tag{3.14}$$

where C is a constant depending on of the coefficients in (3.12) and (3.13).

We now want to specify the dynamics of a jump diffusion that can be associated with a stochastic Feynman-Kac solution of equation (3.1). For this purpose, define a copy $(\hat{\Omega}, \hat{\mathcal{F}}_T, \hat{\mu})$ of our underlying white noise space with corresponding Brownian motion $\hat{B}_s$ and independent pure jump Lévy process $\hat{L}_t$, also with Lévy measure $\nu(d\zeta)$ and jump measure denoted by $\hat{N}(dt,d\zeta)$. Then, consider the following SDE on the stochastic basis $(\Omega\times\hat{\Omega}, \mathcal{F}_T\otimes\hat{\mathcal{F}}_T, \mu\times\hat{\mu})$

$$\begin{aligned} dY_s &= [b(s,Y_{s-}) - \sigma(s,Y_{s-})p(t,Y_{s-})]\,ds + \sigma(s,Y_{s-})dB_s + \hat{\sigma}(s,Y_{s-})d\hat{B}_s \\ &\quad + \int_{[0,U]}\int_{\mathbb{R}_0} \gamma(s,Y_{s^-},\zeta)\mathbf{1}_{\left\{u\leq \frac{U}{1+q(s,Y_{s-},\zeta)}\right\}}\, M(ds,du,d\zeta) \\ &\quad + \int_{\mathbb{R}_0} \hat{\gamma}(s,Y_{s^-},\zeta)\; \hat{N}(ds,d\zeta), \\ Y_t &= y,\ t\leq s\leq T. \end{aligned} \tag{3.15}$$

By conditions (3.2)-(3.5) it can be proven similar to Proposition 1.1 in [F] that there exists a (unique) càdlàg square integrable solution of (3.15) for all $0 \leq t \leq s \leq T$, which we denote by $Y_s^{t,y} \in L^2$. Using this solution we define the following Feynman-Kac functional

$$
\begin{aligned}
u(t,y,\omega) &= E\Big[\varphi(Y_T^{t,y})\rho(t,T) + \int_t^T g(s,Y_s^{t,y})\rho(t,s)ds \\
&\qquad + \int_t^T f(s,Y_s^{t,y})\rho(t,s)dB_s \\
&\qquad + \int_t^T \int_{\mathbb{R}_0} k(s,Y_s^{t,y},\zeta)\rho(t,s)\,\tilde{N}(ds,d\zeta)\Big|\,\mathcal{F}_T\Big],
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
\rho(t,s) &= \exp\Big\{\int_t^s c(r,Y_r^{t,y})dr + \int_t^s p(r,Y_r^{t,y})dB_r - \frac{1}{2}\int_t^s p^2(r,Y_r^{t,y})dr \\
&\quad + \int_t^s \int_{\mathbb{R}_0} \log\left(1+q(r,Y_r^{t,y},\zeta)\right)\tilde{N}(ds,d\zeta) \\
&\quad + \int_t^s \int_{\mathbb{R}_0} \left(\log\left(1+q(r,Y_r^{t,y},\zeta)\right) - q(r,Y_r^{t,y},\zeta)\right)\nu(d\zeta)dr\Big\}.
\end{aligned}
$$

Our main result in this paper is then:

Theorem 3.1. *Under the conditions formulated in (3.2)-(3.13) we have that $u(t,y,\omega)$ as defined in (3.16) solves uniquely the stochastic integro-PDE (3.1) in $(\mathcal{S})_{-1}$.*

Before we give the proof of Theorem 3.1 we state the following help lemma as given in [P].

Lemma 3.2. *Let G be a bounded open subset of $\mathbb{R}_+ \times \mathbb{R}$. Assume a process $U : G \to (\mathcal{S})_{-1}$ with $\mathcal{H}U = u$ such that u and its partial derivatives $\frac{\partial u}{\partial t}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial y^2}$ are bounded on $G \times \mathbf{K}_q(R)$, continuous with respect to $(t,y) \in G$ for all $z \in \mathbf{K}_q(R)$, and analytic in $z \in \mathbf{K}_q(R)$ for all $(t,y) \in G$, $q < \infty$, $R > 0$. Then on $\mathbf{K}_q(R)$*

$$
\mathcal{H}\left(\frac{\partial U}{\partial t}\right) = \frac{\partial u}{\partial t}, \quad \mathcal{H}\left(\frac{\partial U}{\partial y}\right) = \frac{\partial u}{\partial y}, \quad \mathcal{H}\left(\frac{\partial^2 U}{\partial y^2}\right) = \frac{\partial u}{\partial y^2}.
$$

Proof. (Theorem 3.1) In this proof we only focus on the pure jump part of the problem, i.e. we set $\sigma(t,y)$, $\hat{\sigma}(t,y))$, $b(t,y)$, $f(t,y)$ and $p(t,y)$ identically to 0. The proof of the general case follows the same principle and doesn't add anything new to the existing literature. First note (see Theorem 2.7.10 in [HØUZ]) that since $u(t,y,\omega)$ is an $L^2(\mu)$ functional of L_t the Hermite transform can be expressed as

$$
\tilde{u}(t,y,z) := \mathcal{H}\left(u(t,y,\omega)\right)(z) = E\left[u(t,y,\omega)\cdot\mathcal{E}\left(\int_0^T \int_{\mathbb{R}_0} \phi_z''(t,\zeta)\widetilde{N}(dt,d\zeta)\right)\right],
\tag{3.17}
$$

where $\phi''_z(t,\zeta) = \sum_k z_k \delta_k(t,\zeta)$ is as in (3.6) for z in an infinite dimensional neighborhood $\mathbb{K}_q(R)$. Here, the exponential martingale is given by

$$\mathcal{E}\left(\int_0^T \int_{\mathbb{R}_0} \phi''_z(t,\zeta)\widetilde{N}(dt,d\zeta)\right) = \exp\left\{\int_0^T \int_{\mathbb{R}_0} \log(1+\phi''_z(t,\zeta))N(dt,d\zeta) - \int_0^T \int_{\mathbb{R}_0} \phi''_z(t,\zeta)\nu(d\zeta)dt\right\}. \tag{3.18}$$

In the following it is sufficient (see for example [P]) to consider the real part of (3.17), that is we assume $z \in \mathbb{K}_q(R) \cap \mathbb{R}^{\mathbb{N}}$. Further we observe that we can rewrite (3.18) in terms of integration with respect to $M(dt,du,d\zeta) - dtdu\nu'(d\zeta)$:

$$\mathcal{E}\left(\int_0^T \int_{\mathbb{R}_0} \phi''_z(t,\zeta)\widetilde{N}(dt,d\zeta)\right) = \exp\left\{\int_0^T \int_0^U \int_{\mathbb{R}_0} \log(1+\phi''_z(t,\zeta))M(dt,du,d\zeta) - \int_0^T \int_0^U \int_{\mathbb{R}_0} \phi''_z(t,\zeta)dtdu\nu'(d\zeta)\right\}.$$

If we now make a measure change to a measure Q induced by the Radon-Nikodym derivative given through the exponential martingale (3.18) then, by means of Girsanov's theorem for random measures (see for example [JS]), the jump measure $M(dt,du,d\zeta)$ (respectively $N(dt,d\zeta)$) has $(1+\phi''_z(s,\zeta))dsdu\nu'(d\zeta)$ (respectively $(1+\phi''_z(s,\zeta))ds\nu(d\zeta)$ as predictable compensator under Q. We thus get from the definition of $u(t,y,z)$ that the Hermite transform (3.17) can be written as

$$\begin{aligned}
\tilde{u}(t,y,z) &= E\Bigg[\Bigg\{\varphi(Y_T^{t,y})\rho(t,T) + \int_t^T g(s,Y_s^{t,y})\rho(t,s)ds \\
&\qquad + \int_t^T \int_{\mathbb{R}_0} k(s,Y_s^{t,y},\zeta)\rho(t,s)\,\tilde{N}(ds,d\zeta)\Bigg\} \\
&\qquad\qquad \mathcal{E}\left(\int_0^T \int_{\mathbb{R}_0} \phi''_z(t,\zeta)\widetilde{N}(dt,d\zeta)\right)\Bigg] \\
&= E_Q\Bigg[\varphi(Y_T^{t,y})\rho(t,T) + \int_t^T g(s,Y_s^{t,y})\rho(t,s)ds \\
&\qquad + \int_t^T \int_{\mathbb{R}_0} k(s,Y_s^{t,y},\zeta)\rho(t,s)\,\phi''_z(s,\zeta)\,ds\nu(d\zeta)\Bigg].
\end{aligned}$$

Now factorize $\rho(t,s)$ as

$$\rho(t,s) = \mathcal{E}\left(\int_t^s \int_{\mathbb{R}_0} q(r,Y_r^{t,y},\zeta)\left\{N(dr,d\zeta) - (1+\phi''_z(r,\zeta))\,dr\nu(d\zeta)\right\}\right) \cdot \rho'(t,s),$$

where

$$\rho'(t,s) = \exp\left\{ \int_t^s \left(c(r, Y_r^{t,y}) + \int_{\mathbb{R}_0} q(r, Y_r^{t,y}, \zeta)\phi_z''(r,\zeta)\nu(d\zeta) \right) dr \right\}.$$

Carrying out a second measure change to a measure Q' induced by the density process

$$\mathcal{E}\left(\int_t^{\cdot} \int_{\mathbb{R}_0} q(r,y,\zeta) \left(N(dr, d\zeta) - (1+\phi_z''(r,\zeta)) dr \nu(d\zeta) \right) \right),$$

we get

$$\begin{aligned} \tilde{u}(t,y,z) &= E_{Q'}\Bigg[\varphi(Y_T^{t,y})\rho'(t,T) + \int_t^T g(s, Y_s^{t,y})\rho'(t,s) ds \\ &\qquad + \int_t^T \int_{\mathbb{R}_0} k(s, Y_s^{t,y}, \zeta)\rho'(t,s)\, \phi_z''(s,\zeta) ds \nu(d\zeta) \Bigg], \end{aligned}$$

where now, again by means of Girsanov's theorem for random measures, the jump measure $M(dt, du, d\zeta)$ has the predictable compensator

$$(1+q(s,y,\zeta))\,(1+\phi_z''(s,\zeta)) ds du \nu'(d\zeta)$$

under Q'. Therefore the infinitesimal generator of $Y_s^{t,y}$ as diffusion under Q' applied to $\theta \in C_b^1(\mathbb{R})$ is given by

$$\begin{aligned} \mathcal{K}_s\theta(y) &= \int_{\mathbb{R}_0} \left\{ \theta(y+\hat{\gamma}(s,y,\zeta)) - \theta(y) \right\} \nu(d\zeta) \\ &\quad + \int_{[0,U]} \int_{\mathbb{R}_0} \left\{ \theta(y+\gamma(s,y,\zeta)) - \theta(y) \right\} \mathbf{1}_{\left\{ u \le \frac{U}{1+q(s,y,\zeta)} \right\}} \\ &\qquad\qquad (1+q(s,y,\zeta))\,(1+\phi_z''(s,\zeta)) du \nu'(d\zeta) \\ &= \int_{\mathbb{R}_0} \left\{ \theta(y+\gamma(s,y,\zeta)) - \theta(y) \right\} \phi_z''(s,\zeta)\nu(d\zeta) \\ &\quad + \int_{\mathbb{R}_0} \left\{ \theta(y+\hat{\gamma}(s,y,\zeta)) + \theta(y+\gamma(s,y,\zeta)) - 2\theta(y) \right\} \nu(d\zeta). \end{aligned}$$

So by our assumptions on the coefficients, the Feynman-Kac formula is applicable (see [PH]) and yields that $\tilde{u}(t,y,z)$ is the solution of (3.6).

The last step is to show that we can extract the Hermite transform in equation (3.6), that is to interchange Hermite transform and integration, and in this way end up with equation (3.1). To this end it is sufficient to show that on a neighborhood $\mathbf{K}_q(R)$

$$\begin{aligned} (3.19)\quad & \int_{\mathbb{R}_0} \sup_{z\in\mathbf{K}_q(R)} \left\{ \tilde{u}(t, y+\gamma(s,y,\zeta), z) + \tilde{u}(t, y+\hat{\gamma}(s,y,\zeta), z) - 2\tilde{u}(t,y,z) \right\} \nu(d\zeta) \\ & + \int_{\mathbb{R}_0} \sup_{z\in\mathbf{K}_q(R)} \left\{ \tilde{u}(t, y+\gamma(s,y,\zeta), z) - \tilde{u}(t,y,z) \right\} \phi_z''(s,\zeta)\nu(d\zeta) < \infty. \end{aligned}$$

It is not difficult to see with the help of conditions (3.4)-(3.5) and estimates like for example in ([PH], Lemma 3.1) that $\tilde{u}(t,y,z)$ is Lipschitz continuous in y uniformly on a neighborhood $\mathbf{K}_q(R)$, i.e. for all $t \in [0,T]$

$$\sup_{z\in\mathbf{K}_q(R)} |\tilde{u}(t,y,z) - \tilde{u}(t,v,z)| \le const \cdot |y-v| \ .$$

This together with the property of $\phi_z''(s,\zeta)$ as the Hermite transform of $\overset{\bullet}{\widetilde{N}}(t,\zeta)$ easily yields relation (3.19) which completes the proof. Note that in the presence of the diffusive parts in our operators one would at this point employ (3.14)) and Lemma 3.2 in order to exchange Hermite transform and differentiation as for example done in [P]. □

Remark 3.3. Note that in the diffusion equation (3.20) we integrate with respect to the jump measures only and not with respect to the compensated jump measures. As will be done in the Corollary 3.4, one can also set up an analogue problem where we integrate with respect to compensated jump measures, which would allow to relax conditions (3.3) and (3.5). In turn, we would have to sharpen the conditions on $q(t,y,\zeta)$, which would appear in some additional drift term in equation (3.1). In Corollary 3.4, however, we have $q(t,y,\zeta)=0$ which thus doesn't create any further problems.

As a corollary we now get the backward jump diffusion equation for Lévy-Itô diffusions whose Brownian motion version has its origin in [Za]. Let $Y_s^{t,y}$ denote the solution of

$$\begin{aligned} dY_s &= b(Y_{s-})ds + \sigma(Y_{s-})dB_s + \gamma(Y_{s-})dL_s \\ &= b(Y_{s-})ds + \sigma(Y_{s-})dB_s + \int_{\mathbb{R}_0} \gamma(Y_{s-})\,\zeta\,\widetilde{N}(ds,d\zeta), \\ Y_t &= y,\ t \le s \le T. \end{aligned} \tag{3.20}$$

Then we get the correspondence of $Y_s^{t,y}$ to the following SIPDE:

Corollary 3.4. *If we set $u(t,y,\omega) := Y_T^{T-t,y}$ then, under the assumptions on the coefficients specified in (3.2)-(3.13), $u(t,y,\omega)$ uniquely solves the following stochastic integro-PDE in $(\mathcal{S})_{-1}$*

(3.21)

$$\begin{cases} \partial_t u(t,y) = \frac{1}{2}\sigma^2(y)\,\partial_{yy}u(t,y) + b(y)\,\partial_y u(t,y) + \sigma(y)\,\partial_y u(t,y) \diamond \overset{\bullet}{B}_{T-t} \\ \qquad\qquad + \int_{\mathbb{R}_0} \{u(t,y+\gamma(y)\zeta) - u(t,y) - \partial_y u(t,y)\gamma(y)\zeta\}\,\nu(d\zeta) \\ \qquad\qquad\quad + \int_{\mathbb{R}_0} \{u(t,y+\gamma(y)\zeta) - u(t,y)\} \diamond \overset{\bullet}{\widetilde{N}}(T-t,\zeta)\nu(d\zeta), \\ u(0,y) = y, \quad (t,y) \in [0,T]\times\mathbb{R}. \end{cases}$$

Proof. The result is just a special case of Theorem 3.1 except for the following two modifications. First note that in (3.20) the jump integration is with respect to the compensated jump measure $\widetilde{N}(ds, d\zeta)$ in contrast to the integration with respect to the jump measure $M(ds, du, d\zeta)$ only in (3.15). As can be seen from the proof of Theorem 3.1 this doesn't yield any problem as long as $q(t, y, \zeta) = 0$ and just causes the extra term $\partial_y u(t, y)\gamma(y)\zeta$ under the integral with respect to $\nu(d\zeta)$ in (3.21). Further, in order to obtain an initial condition rather than a terminal condition of the corresponding SIPDE one has to revert the time which leads to the time reverted white noises. □

Note that under the appropriate smoothness conditions on the coefficients one can integrate equation (3.21) from 0 to t and interpret it in the Itô sense:

$$\begin{cases} du(t,y) = \left\{\frac{1}{2}\sigma^2(y)\,\partial_{yy}u(t,y) + b(y)\,\partial_y u(t,y)\right\} dt + \sigma(y)\,\partial_y u(t,y) d\overleftarrow{B_t} \\ \qquad\qquad + \int_{\mathbb{R}_0} \{u(t, y+\gamma(y)\zeta) - u(t,y) - \partial_y u(t,y)\}\, \nu(d\zeta)dt \\ \qquad\qquad + \int_{\mathbb{R}_0} (u(t^-, y+\gamma(y)\zeta) - u(t^-, y))\, \overleftarrow{\tilde{N}}(d\zeta, dt), \\ u(0,y) = y, \quad (t,y) \in [0,T) \times \mathbb{R}, \end{cases}$$

where $\overleftarrow{B}_t$ is the Brownian motion $B_T - B_{T-t}$ and $\overleftarrow{\tilde{N}}(d\zeta, dt)$ is the compensated jump measure associated to the pure jump Lévy process $\overleftarrow{L}_t = L_{T^-} - L_{(T-t)^-}$ for $0 \le t < T$.

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