Stability, uniqueness and existence of solutions to McKean-Vlasov SDEs: a multidimensional Yamada-Watanabe approach

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Abstract

We establish stability and pathwise uniqueness of solutions to Wiener noise driven McKean-Vlasov equations with random coefficients, which are allowed to be non-Lipschitz continuous. In the case of deterministic coefficients we also obtain the existence of unique strong solutions. By using our approach, which is based on an extension of the Yamada-Watanabe ansatz to the multidimensional setting and which does not rely on the construction of Lyapunov functions, we prove first moment and pathwise α-exponential stability of solutions for $\alpha > 0$. Furthermore, we are able to compute Lyapunov exponents explicitly.

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1 Introduction

Let $d, m \in \mathbb{N}$ and $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, P)$ be a filtered probability space that satisfies the usual conditions and carries a standard d-dimensional $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-Brownian motion W. McKean-Vlasov stochastic differential equations (McKean-Vlasov SDEs), or alternatively mean-field SDEs, are integral equations of the form

$$X_t = X_0 + \int_0^t b(s, X_s, P_{X_s})\,ds + \int_0^t \sigma(s, X_s, P_{X_s})\,dW_s \quad \text{for } t \in \mathbb{R}_+ \text{ a.s.,} \tag{1.1}$$

where $b : \mathbb{R}_+ \times \mathbb{R}^m \times \mathscr{P}(\mathbb{R}^m) \to \mathbb{R}^m$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^m \times \mathscr{P}(\mathbb{R}^m) \to \mathbb{R}^{m\times d}$ are the measurable drift and the diffusion coefficients and P_X is the law map of the solution process X. Here, $\mathscr{P}(\mathbb{R}^m)$ is the convex space of all Borel probability measures on $\mathbb{R}^m$.

Originally motivated by Boltzmann's equation in kinetic gas theory, McKean-Vlasov equations were first studied by Kac [28], McKean [34] and Vlasov [36] and used to describe the stochastic dynamics of large interacting particle systems. In particular, Vlasov analysed the propagation of chaos of charged particles with long-range interaction in a plasma, where the interaction of particles can be modelled by means of a system of SDEs (N-particle system). It turns out that the convergence of an N-particle system as N tends to infinity, which is referred to as propagation of chaos, can be described by an equation of the form (1.1).

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Since the seminal papers [28], [34] and [36], mean-field SDEs have attracted much interest and sparked off new groundbreaking developments in both theory and applications in a variety of other research areas. For example, Lasry and Lions [32] employ mean-field techniques to study mean-field games in economic applications. There, the interaction of strategies of 'rational' players in an N-player differential game is described by a system of SDEs. Then, in the limiting case $N \to \infty$, the authors derive a system of partial differential equations that consists of a Hamilton-Jacobi and a Kolmogorov equation, which they use to examine the behaviour of agents in a vast network.

As for other works in this direction, but based on different methods, which rely on a probabilistic analysis, we refer to [14], [15], [16], [17] and [13]. Recent applications of mean-field methods include e.g. the modelling of systemic risk in financial networks, see [18], [19], [21], [22], [23], [24] and [30]. Further, extensions of (1.1) to the setting of mean-field SDEs driven by Lévy processes or backward mean-field SDEs were studied in [10], [11], [12] and [27]. More recently, a robust solution theory of mean-field SDEs from the perspective of rough path theory was developed in [4].

Mean-field equations of type (1.1) were also examined for non-Lipschitz vector fields b and σ. See e.g. [26], where the author invokes martingale techniques to construct unique weak solutions in the case of a bounded drift b, which is Lipschitz continuous in the law variable. We also refer to [20], [35] in the case of weak solutions. As for path-dependent coefficients see [31]. More recently, by using techniques based on Malliavin calculus, the existence of unique strong solutions to mean-field SDEs with additive Brownian noise and singular drift b were obtained in [8], [5] and [6], where the authors also derive a Bismut-Elworthy-Li formula for such equations. See also [7] in the case of a certain non-Markovian and 'rough' Gaussian driving noise in a Hilbert space setting.

Now let $t_0 \in \mathbb{R}_+$ and $\mathscr{P}$ be a separable metrisable topological space in $\mathscr{P}(\mathbb{R}^m)$. In this article we consider a *generalised McKean-Vlasov SDE* with *random drift* and *diffusion coefficients* of the form

$$dX_t = \mathrm{B}_t(X_t, P_{X_t})\,dt + \Sigma_t(X_t)\,dW_t \quad \text{for } t \in [t_0, \infty), \tag{1.2}$$

where $\mathrm{B} : [t_0,\infty) \times \Omega \times \mathbb{R}^m \times \mathscr{P} \to \mathbb{R}^m$ and $\Sigma : [t_0,\infty) \times \Omega \times \mathbb{R}^m \to \mathbb{R}^{m\times d}$ are measurable in an appropriate meaning. While the diffusion Σ does not depend on the measure variable $\mu \in \mathscr{P}$, both B and Σ are allowed to be non-Lipschitz as stated more precisely below.

The measure state space $\mathscr{P}$, whose topology may be finer than the topology of weak convergence, is required to be admissible in a suitable measurable sense, as introduced in Section 2. For instance, $\mathscr{P}$ may stand for $\mathscr{P}(\mathbb{R}^m)$, endowed with the Prokhorov metric, or the Polish space $\mathscr{P}_1(\mathbb{R}^m)$ of all measures in $\mathscr{P}(\mathbb{R}^m)$ with a finite first moment, equipped with the Wasserstein metric.

In this general framework, the main objective of our work is to establish stability, uniqueness and existence of solutions to McKean-Vlasov SDEs of type (1.2). Our methodology is based on a *multidimensional Yamada-Watanabe approach* and allows for irregular drift and diffusion coefficients. More precisely, our paper offers the following *novel contributions* compared to the existing literature, where on Σ we merely impose an Osgood condition on compact sets with random regularity coefficients and B satisfies the conditions as specified in the following:

(1) *Pathwise uniqueness* for (1.2) follows from Corollary 3.9 under a partial Osgood condition on B, and if the drift is independent of $\mu \in \mathscr{P}$, in which case (1.2) reduces to an SDE, then this condition is imposed on compact sets only.

(2) *(Asymptotic) moment stability* for (1.2) is inferred from Proposition 3.13, which yields a general L^1-*comparison estimate*, and stated in Corollary 3.15 under a partial mixed Hölder continuity condition on B and verifiable integrability conditions on the random partial Hölder coefficients with respect to $(x, \mu) \in \mathbb{R}^m \times \mathscr{P}$.

(3) *Exponential moment stability* is asserted by Corollary 3.16 if B satisfies a partial Lipschitz condition and the partial Lipschitz coefficients are bounded by a sum of power functions that *determines the moment Lyapunov exponent explicitly.*

(4) *Pathwise stability* is implied by Corollary 3.19 if the preceding conditions on B hold, the random Lipschitz coefficient of B relative to $\mu \in \mathscr{P}$ is of suitable growth and the Osgood condition on compacts sets on Σ is replaced by an 1/2-Hölder condition. In particular, the *pathwise Lyapunov exponent is half the moment Lyapunov exponent.*

(5) Thereby, we show that all these stability results can be obtained under verifiable assumptions *without resorting to the existence of Lyapunov functions.*

(6) *Existence of unique strong solutions* is established in Theorem 3.27 in the case that B and Σ are deterministic, B satisfies a partial affine growth and a partial Lipschitz condition, $\Sigma_s(0) = 0$ for all $s \in [t_0, \infty)$ and the Osgood condition on compact sets for Σ holds. In particular, B and Σ may fail to be of affine growth.

As mentioned above, our method for proving these results rests on a pathwise uniqueness approach of Yamada and Watanabe [37], which we extend in this paper to the multidimensional setting. In this context, let us mention two articles which employ the Yamada-Watanabe ansatz to show pathwise uniqueness and stability of solutions:

In [1] the authors verify pathwise uniqueness of solutions to a multidimensional SDE with deterministic coefficients. However, the Yamada-Watanabe condition given in the multidimensional case is rather restrictive and essentially reduces to a Lipschitz condition on the diffusion.

Further, the article [3] pertains to the study of one-dimensional mean-field SDEs with bounded drift and diffusion coefficients by means of the Yamada-Watanabe approach. Here, in addition it is essentially assumed that the drift coefficient is Lipschitz, both in the spatial and law variable, while the diffusion coefficient satisfies a global Osgood condition. In our paper, however, the conditions on the coefficients, even in the one-dimensional and deterministic case, are weaker than in [3], since no growth conditions on those are imposed. Further, it is only required that B and Σ satisfy a partial Osgood condition and an Osgood condition on compact sets, respectively.

Finally, in the context of stability results for SDEs with irregular coefficients, we also mention the recent work [2], where the authors prove moment exponential stability of solutions to (non-mean-field) SDEs driven by a drift vector field with discontinuities on a hyperplane.

Our paper is organised as follows. In Section 2 we introduce the mathematical setting and prove some auxiliary results which are necessary for establishing our main results. In Section 3 our main results are presented, whose proofs are given in the last section (Section 5). Section 4 is devoted to a discussion of a priori estimates and a pathwise asymptotic analysis for random Itô processes. The results in Section 4 then serve as basis for the derivation of our main results in Section 3.

2 Preliminaries

Throughout the paper, for any interval I in $\mathbb{R}$ and each monotone function $f : I \to \mathbb{R}$, we set $f(a) := \lim_{v \downarrow a} f(v)$ for $a := \inf I$, if $a \notin I$. Similarly, $f(b) := \lim_{v \uparrow b} f(v)$ for $b := \sup I$, if $b \notin I$. Further, $|\cdot|$ is used as absolute value function, Euclidean norm or Hilbert-Schmidt norm and the transpose of a matrix $A \in \mathbb{R}^{m \times d}$ is denoted by A'.

2.1 Processes with locally integrable components

For $p \in [1,\infty)$ let $\mathscr{L}_+^p(\Omega,\mathscr{F},P)$ and $\mathscr{L}_+^\infty(\Omega,\mathscr{F},P)$ denote the convex cones of all random variables X such that $E[(X^+)^p] < \infty$ and $X^+ \leq c$ a.s. for some $c \in \mathbb{R}_+$, respectively. We define a pseudonorm $[\cdot]_p$ on $\mathscr{L}_+^p(\Omega,\mathscr{F},P)$ and a sublinear functional $[\cdot]_\infty$ on $\mathscr{L}_+^\infty(\Omega,\mathscr{F},P)$ by

$$[X]_p := E\big[(X^+)^p\big]^{\frac{1}{p}} \quad \text{and} \quad [X]_\infty := \operatorname{ess\,sup} X. \tag{2.1}$$

For $\alpha \in [0,1]$ we set $q_\alpha := (1-\alpha)^{-1}$, which is the dual exponent of α^{-1}. Despite that $[\cdot]_\infty$ may take negative values, it follows for any $\beta \in [0,1]$ with $\alpha + \beta \leq 1$, each $X \in \mathscr{L}_+^{q_\alpha}(\Omega,\mathscr{F},P)$ and any $\mathbb{R}_+$-valued random variable Y that

$$E[XY^\alpha]E[Y]^\beta \leq [X]_{q_\alpha}\big(1 - (\alpha+\beta) + (\alpha+\beta)E[Y]\big), \tag{2.2}$$

by Hölder's and Young's inequality. This bound leads to the quantitative L^1-estimates of Theorems 4.5 and 4.6, on which our main results are based. By allowing infinite values, we extend the definitions of $[\cdot]_p$ and $[\cdot]_\infty$ in (2.1) to any random variable X.

Next, let us write $\mathscr{L}_{loc}^p(\mathbb{R}^{m\times d})$ for the linear space of all $\mathbb{R}^{m\times d}$-valued Borel measurable locally p-fold integrable maps on $[t_0,\infty)$ and $\mathscr{L}_{loc}^p(\mathbb{R}_+^{m\times d})$ be the convex cone of all maps in $\mathscr{L}_{loc}^p(\mathbb{R}^{m\times d})$ with non-negative entries.

By $\mathscr{S}(\mathbb{R}^{m\times d})$ we denote the linear space of all $\mathbb{R}^{m\times d}$-valued $(\mathscr{F}_t)_{t\in[t_0,\infty)}$-progressively measurable processes on $[t_0,\infty)\times\Omega$ and for $p \in [1,\infty]$ let $\mathscr{S}_{loc}^p(\mathbb{R}^{m\times d})$ be the subspace of all $\kappa \in \mathscr{S}(\mathbb{R}^{m\times d})$ with locally integrable paths such that

$$\int_{t_0}^t \big|\big[\kappa_s^{(i,j)}\big]_p\big|\,ds < \infty \quad \text{for any } t \in [t_0,\infty)$$

and all $(i,j) \in \{1,\dots,m\}\times\{1,\dots,d\}$, where $\kappa^{(i,j)}$ is the (i,j)-entry of κ. For example, if X is an $\mathscr{F}_{t_0}$-measurable random variable and $\hat{\kappa}$ is an $\mathbb{R}^{m\times d}$-valued measurable function on $[t_0,\infty)$, then $\kappa \in \mathscr{S}(\mathbb{R}^{m\times d})$ given by $\kappa_s(\omega) := \hat{\kappa}(s)X(\omega)$ belongs to $\mathscr{S}_{loc}^p(\mathbb{R}^{m\times d})$ if and only if $X \in \mathscr{L}_+^p(\Omega,\mathscr{F},P)$ and $\hat{\kappa} \in \mathscr{L}_{loc}^1(\mathbb{R}^{m\times d})$.

We let $\mathscr{S}(\mathbb{R}_+^{m\times d})$ and $\mathscr{S}_{loc}^p(\mathbb{R}_+^{m\times d})$ be the convex cones of all processes in $\mathscr{S}(\mathbb{R}^{m\times d})$ and $\mathscr{S}_{loc}^p(\mathbb{R}^{m\times d})$ with non-negative coordinates, respectively. Finally, for any $\kappa \in \mathscr{L}_{loc}^1(\mathbb{R}^m)$ and each $\hat{\kappa} \in \mathscr{S}_{loc}^1(\mathbb{R}^m)$, we define

$$\kappa_0 := \sum_{i=1}^m \kappa_i^+ \quad \text{and} \quad \hat{\kappa}^{(0)} := \sum_{i=1}^m (\hat{\kappa}^{(i)})^+. \tag{2.3}$$

Then κ_0 and $E[\hat{\kappa}^{(0)}]$ are locally integrable, by construction. That is, $\kappa_0 \in \mathscr{L}_{loc}^1(\mathbb{R}_+)$ and $\hat{\kappa}^{(0)} \in \mathscr{S}_{loc}^1(\mathbb{R}_+)$. As we shall see, these concepts allow for sharp L^1-estimates.

2.2 Admissible Polish spaces of Borel probability measures

In this section we consider a tractable class of spaces of Borel probability measures on $\mathbb{R}^m$, which serve as part of the domain of the drift and the diffusion coefficient of the McKean-Vlasov equation (1.2).

Definition 2.1. A metrisable topological space $\mathscr{P}$ in $\mathscr{P}(\mathbb{R}^m)$ is called *admissible* if for any metric space S, each probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{P})$ and every process $X : S\times\tilde{\Omega} \to \mathbb{R}^m$ with continuous paths satisfying

$$\tilde{P}_{X_s} \in \mathscr{P} \quad \text{for all } s \in S,$$

the map $S \to \mathscr{P}$, $s \mapsto \tilde{P}_{X_s}$ is Borel measurable, where $\tilde{P}_{\tilde{X}}$ denotes the law $\tilde{P}\circ\tilde{X}^{-1}$ of any random vector $\tilde{X} : \tilde{\Omega} \to \mathbb{R}^m$ under $\tilde{P}$.

For $\mu, \nu \in \mathscr{P}(\mathbb{R}^m)$ let $\mathscr{P}(\mu, \nu)$ be the convex set of all Borel probability measures θ on $\mathbb{R}^m \times \mathbb{R}^m$ with first and second marginal distributions μ and ν, respectively, that is, $\theta(B \times \mathbb{R}^m) = \mu(B)$ and $\theta(\mathbb{R}^m \times B) = \nu(B)$ for all Borel sets $B \in \mathscr{B}(\mathbb{R}^m)$.

To give sufficient conditions for the admissibility of a metrisable topological space $\mathscr{P}$ in $\mathscr{P}(\mathbb{R}^m)$, we introduce *stochastic convergence for probability measures*. Namely, a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathscr{P}(\mathbb{R}^m)$ converges stochastically to some $\mu \in \mathscr{P}(\mathbb{R}^m)$ if

$$\lim_{n\uparrow\infty} \theta_n(\{(x,y) \in \mathbb{R}^m \times \mathbb{R}^m \,|\, |x-y| \geq \delta\}) = 0 \tag{2.4}$$

is satisfied for every $\delta > 0$ by some sequence $(\theta_n)_{n\in\mathbb{N}}$ of Borel measures on $\mathbb{R}^m \times \mathbb{R}^m$ such that $\theta_n \in \mathscr{P}(\mu_n, \mu)$ for each $n \in \mathbb{N}$.

Proposition 2.2. *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{R}^m$-valued bounded uniformly continuous maps on $\mathbb{R}^m$ such that*

$$|\varphi_k(x)| \leq |\varphi_{k+1}(x)| \quad \text{and} \quad \lim_{n\uparrow\infty} |\varphi_n(x) - x| = 0$$

for all $k \in \mathbb{N}$ and any $x \in \mathbb{R}^m$. Assume that the following two conditions hold:

(i) Each $\mu \in \mathscr{P}$ satisfies $\mu \circ \varphi_n^{-1} \in \mathscr{P}$ for all $n \in \mathbb{N}$ and $\lim_{n\uparrow\infty} \mu \circ \varphi_n^{-1} = \mu$ in $\mathscr{P}$.

(ii) For any sequence $(\mu_k)_{k\in\mathbb{N}}$ in $\mathscr{P}$ that converges stochastically to some $\mu \in \mathscr{P}$ we have $\lim_{k\uparrow\infty} \mu_k \circ \varphi_n^{-1} = \mu \circ \varphi_n^{-1}$ for any $n \in \mathbb{N}$.

Then $\mathscr{P}$ is admissible.

Example 2.3. For any $n \in \mathbb{N}$ we may let φ_n be the *radial retraction* of the closed ball $\{x \in \mathbb{R}^m \,|\, |x| \leq n\}$ with center $0 \in \mathbb{R}^m$ and radius n. That is, for any $x \in \mathbb{R}^m$ we have

$$\varphi_n(x) = x, \quad \text{if } |x| \leq n, \quad \text{and} \quad \varphi_n(x) = \frac{n}{|x|}x, \quad \text{if } |x| > n.$$

Indeed, φ_n is Lipschitz continuous and it holds that $|\varphi_n(x)| \leq n \wedge \varphi_{n+1}(x)$ and $|\varphi_n(x) - x| = (|x| - n)^+$ for every $x \in \mathbb{R}^m$.

To give more concrete conditions, let $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing and $\mathscr{P}_\rho(\mathbb{R}^m)$ denote the set of all $\mu \in \mathscr{P}(\mathbb{R}^m)$ for which $\int_{\mathbb{R}^m} \rho(|x|)\,\mu(dx)$ is finite.

Corollary 2.4. *Assume that $\mathscr{P} \subset \mathscr{P}_\rho(\mathbb{R}^m)$ and ρ is continuous. Then the admissibility of $\mathscr{P}$ holds under the following two conditions:*

(i) If $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ is bounded and uniformly continuous and $|\varphi(x)| \leq |x|$ for all $x \in \mathbb{R}^m$, then $\mu \circ \varphi^{-1} \in \mathscr{P}$ for any $\mu \in \mathscr{P}$.

(ii) A sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathscr{P}$ converges to some $\mu \in \mathscr{P}$ if there is a sequence $(\theta_n)_{n\in\mathbb{N}}$ of Borel measures on $\mathbb{R}^m \times \mathbb{R}^m$ with $\theta_n \in \mathscr{P}(\mu_n, \mu)$ for all $n \in \mathbb{N}$ and

$$\lim_{n\uparrow\infty} \int_{\mathbb{R}^m\times\mathbb{R}^m} \rho(|x-y|)\,\theta_n(dx,dy) = 0.$$

By the measure transformation formula, any measurable map $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ satisfies $\int_{\mathbb{R}^m} \rho(|x|)\,(\mu \circ \varphi^{-1})(dx) = \int_{\mathbb{R}^m} \rho(|\varphi(x)|)\,\mu(dx)$. Thus, the first condition of the corollary holds for $\mathscr{P}_\rho(\mathbb{R}^m)$. All these considerations lead to two admissible metric spaces.

Examples 2.5. (i) The *Prokhorov metric* ϑ_P turns $\mathscr{P}(\mathbb{R}^m)$ into a Polish space and can be represented as follows: For $\varepsilon > 0$ let $N_\varepsilon(B)$ be the ε-neighbourhood of a set B in $\mathbb{R}^m$. Then the $[0,1]$-valued functional on $\mathscr{P}(\mathbb{R}^m) \times \mathscr{P}(\mathbb{R}^m)$ given by

$$\vartheta_0(\mu,\nu) := \inf\{\varepsilon > 0 \,|\, \forall B \in \mathscr{B}(\mathbb{R}^m) : \mu(B) \leq \nu(N_\varepsilon(B)) + \varepsilon\}$$

satisfies the triangle inequality and $\vartheta_P(\mu,\nu) = \vartheta_0(\mu,\nu) \vee \vartheta_0(\nu,\mu)$ for all $\mu,\nu \in \mathscr{P}(\mathbb{R}^m)$. As convergence with respect to ϑ_P is equivalent to weak convergence, all requirements of Proposition 2.2 are met by $\mathscr{P}(\mathbb{R}^m)$ if equipped with ϑ_P.

(ii) For $p \in [1,\infty)$ consider the Polish space $\mathscr{P}_p(\mathbb{R}^m)$ of all $\mu \in \mathscr{P}(\mathbb{R}^m)$ with finite p-th moment $\int_{\mathbb{R}^m} |x|^p \,\mu(dx)$, equipped with the *p-th Wasserstein metric* given by

$$\vartheta_p(\mu,\nu) := \bigg(\inf_{\theta \in \mathscr{P}(\mu,\nu)} \int_{\mathbb{R}^m \times \mathbb{R}^m} |x-y|^p \,\theta(dx,dy)\bigg)^{\frac{1}{p}}. \tag{2.5}$$

As the definition of ϑ_p entails that the second condition of Corollary 2.4 is valid for the choice $\rho(v) = v^p$ for all $v \in \mathbb{R}_+$, we see that $\mathscr{P}_p(\mathbb{R}^m)$ is admissible.

2.3 Notions of solutions, pathwise uniqueness and stability

In what follows, let $\mathscr{P}$ be an admissible separable metrisable topological space in $\mathscr{P}(\mathbb{R}^m)$ and $\mathscr{A}$ denote the progressive σ-field on $[t_0,\infty) \times \Omega$. Thus, a set A in $[t_0,\infty) \times \Omega$ lies in $\mathscr{A}$ if and only if $\mathbb{1}_A$ is progressively measurable. We shall call a map

$$F : [t_0,\infty) \times \Omega \times \mathbb{R}^m \times \mathscr{P} \to \mathbb{R}^{m \times d}, \quad (s,\omega,x,\mu) \mapsto F_s(x,\mu)(\omega)$$

admissible if it is measurable relative to the product σ-field $\mathscr{A} \otimes \mathscr{B}(\mathbb{R}^m) \otimes \mathscr{B}(\mathscr{P})$. In this case, for any process $X \in \mathscr{S}(\mathbb{R}^m)$ and each Borel measurable map $\mu : [t_0,\infty) \to \mathscr{P}$, the process

$$[t_0,\infty) \times \Omega \to \mathbb{R}^{m \times d}, \quad (s,\omega) \mapsto F_s(X_s(\omega), \mu(s))(\omega)$$

is progressively measurable. In particular, if X is continuous and satisfies $P_{X_t} \in \mathscr{P}$ for all $t \in [t_0,\infty)$, then its law map $P_X : [t_0,\infty) \to \mathscr{P}$, $t \mapsto P_{X_t}$ is a feasible choice for μ, by the admissibility of $\mathscr{P}$. In this sense, we require the drift B and the diffusion Σ of the McKean-Vlasov equation (1.2) to be admissible.

Definition 2.6. A *solution* to (1.2) is an $\mathbb{R}^m$-valued adapted continuous process X such that P_X is $\mathscr{P}$-valued, $\int_{t_0}^t |\mathrm{B}_s(X_s,P_{X_s})| + |\Sigma_s(X_s)|^2 \,ds < \infty$ for any $t \in [t_0,\infty)$ and

$$X_t = X_{t_0} + \int_{t_0}^t \mathrm{B}_s(X_s,P_{X_s})\,ds + \int_{t_0}^t \Sigma_s(X_s)\,dW_s$$

for every $t \in [t_0,\infty)$ a.s.

We observe that B and Σ are independent of $\omega \in \Omega$ if and only if there are two Borel measurable maps b and σ on $[t_0,\infty) \times \mathbb{R}^m \times \mathscr{P}$ and $[t_0,\infty) \times \mathbb{R}^m$, taking their values in $\mathbb{R}^m$ and $\mathbb{R}^{m \times d}$, respectively, such that

$$\mathrm{B}_s(x,\mu) = b(s,x,\mu) \quad \text{and} \quad \Sigma_s(x) = \sigma(s,x) \tag{2.6}$$

for all $(s,x,\mu) \in [t_0,\infty) \times \mathbb{R}^m \times \mathscr{P}$. For the deterministic coefficients b and σ we may also consider *solutions in the strong and the weak sense* and write (1.2) formally in the form

$$dX_t = b(t,X_t,P_{X_t})\,dt + \sigma(t,X_t)\,dW_t \quad \text{for } t \in [t_0,\infty). \tag{2.7}$$

Namely, for a fixed $\mathbb{R}^m$-valued $\mathscr{F}_{t_0}$-measurable random vector ξ we let $({}_\xi\mathscr{E}_t^0)_{t\in[t_0,\infty)}$ denote the natural filtration of the adapted continuous process $[t_0,\infty)\times\Omega\to\mathbb{R}^m\times\mathbb{R}^d$, $(t,\omega)\mapsto(\xi,W_t-W_{t_0})(\omega)$. That means,

$$ {}_\xi\mathscr{E}_t^0 = \sigma(\xi)\vee\sigma(W_s-W_{t_0}:s\in[t_0,t]) \quad \text{for all } t\in[t_0,\infty). $$

Then a solution X to (2.7) satisfying $X_{t_0}=\xi$ a.s. is called *strong* if it is adapted to the right-continuous filtration of the augmented filtration of $({}_\xi\mathscr{E}_t^0)_{t\in[t_0,\infty)}$.

A *weak solution* to (2.7) is a solution X defined on some filtered probability space $(\tilde{\Omega},\tilde{\mathscr{F}},(\tilde{\mathscr{F}}_t)_{t\in\mathbb{R}_+},\tilde{P})$ satisfying the usual conditions and on which there exists a standard d-dimensional $(\tilde{\mathscr{F}}_t)_{t\in\mathbb{R}_+}$-Brownian motion $\tilde{W}$.

That means, X is an $\mathbb{R}^m$-valued $(\tilde{\mathscr{F}}_t)_{t\in[t_0,\infty)}$-adapted continuous process satisfying $\tilde{P}_X\in\mathscr{P}$, $\int_{t_0}^t|b(s,X_s,\tilde{P}_{X_s})|+|\sigma(s,X_s)|^2\,ds<\infty$ for all $t\in[t_0,\infty)$ and

$$ X_t = X_{t_0} + \int_{t_0}^t b(s,X_s,\tilde{P}_{X_s})\,ds + \int_{t_0}^t \sigma(s,X_s)\,d\tilde{W}_s $$

for each $t\in[t_0,\infty)$ a.s. In this case, we will also say that X solves (2.7) weakly on $(\tilde{\Omega},\tilde{\mathscr{F}},(\tilde{\mathscr{F}}_t)_{t\in\mathbb{R}_+},\tilde{P})$ relative to $\tilde{W}$. Let us now return to stochastic coefficients.

The regularity of the drift B relative to the variable $\mu\in\mathscr{P}$ will be stated in terms of an $\mathbb{R}_+$-valued Borel measurable functional ϑ on $\mathscr{P}\times\mathscr{P}$ for which there is $c_\mathscr{P}>0$ such that

$$ \vartheta(P_X,P_{\tilde{X}}) \leq c_\mathscr{P} E[|X-\tilde{X}|] \tag{2.8} $$

for any two $\mathbb{R}^m$-valued random vectors $X,\tilde{X}$ with $P_X,P_{\tilde{X}}\in\mathscr{P}$. For instance, this estimate holds if ϑ is *dominated* by the Wasserstein metric ϑ_1, introduced in Examples 2.5, in the sense that

$$ \vartheta(\mu,\nu) \leq c_\mathscr{P}\vartheta_1(\mu,\nu) \tag{2.9} $$

for all $\mu,\nu\in\mathscr{P}$, where the definition of ν_1 in (2.5) is extended for any $\mu,\nu\in\mathscr{P}(\mathbb{R}^m)$, by allowing infinite values. If in fact $\mathscr{P}\subset\mathscr{P}_1(\mathbb{R}^m)$, then this extension is not used.

In this case, $\mathscr{P}$ is automatically admissible as soon as ϑ is a pseudometric that generates its topology and condition (i) of Corollary 2.4 holds.

Example 2.7. Let $\phi:\mathbb{R}\to\mathbb{R}_+$ and $\varphi:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ be measurable, $\rho:\mathbb{R}_+\to\mathbb{R}_+$ be increasing and $c>0$ be such that

$$ |\varphi(x,y)|\leq\rho(|x-y|) \quad \text{and} \quad \rho(v+w)/c\leq\rho(v)+\rho(w) $$

for all $x,y\in\mathbb{R}^m$ and any $v,w\in\mathbb{R}_+$. For instance, we may take $\rho(v)=\lambda v^p$ for all $v\in\mathbb{R}_+$ and some $\lambda,p>0$. Suppose that $\mathscr{P}\subset\mathscr{P}_\rho(\mathbb{R}^m)$ and

$$ \vartheta(\mu,\nu) = \phi\Big(\inf_{\theta\in\mathscr{P}(\mu,\nu)}\int_{\mathbb{R}^m\times\mathbb{R}^m}\varphi(x,y)\,\theta(dx,dy)\Big) $$

for each $\mu,\nu\in\mathscr{P}$. Then ϑ is well-defined and the following three assertions are readily checked:

(1) If $\varphi(x,y)=\psi(x)-\psi(y)$ for all $x,y\in\mathbb{R}^m$ and some uniformly continuous function $\psi:\mathbb{R}^m\to\mathbb{R}$ that admits ρ as modulus of continuity, then ϑ is of the form

$$ \vartheta(\mu,\nu) = \phi\Big(\int_{\mathbb{R}^m}\psi(x)\,\mu(dx) - \int_{\mathbb{R}^m}\psi(y)\,\nu(dy)\Big) \quad \text{for any } \mu,\nu\in\mathscr{P}. $$

(2) In the case $\phi(v)=v^+$ for all $v\in\mathbb{R}$, $\varphi(x,y)=|x-y|$ for any $x,y\in\mathbb{R}^m$ and $\rho(v)=v$ for each $v\in\mathbb{R}_+$, we obtain that $\mathscr{P}_\rho(\mathbb{R}^m)=\mathscr{P}_1(\mathbb{R}^m)$ and $\vartheta=\vartheta_1$.

(3) Assume that $\phi(v) \leq v^+$ and $c_{\mathscr{P}} v \leq \rho(v)$ for every $v \in \mathbb{R}_+$. Then $\mathscr{P} \subset \mathscr{P}_1(\mathbb{R}^m)$ and the general domination estimate (2.9) holds.

Let for the moment $\mathscr{P} = \mathscr{P}_1(\mathbb{R}^m)$, $\vartheta = \vartheta_1$ and the following affine growth condition be valid: $|\mathrm{B}(\cdot, \mu)| \leq c(1 + \vartheta_1(\mu, \delta_0))$ for all $\mu \in \mathscr{P}_1(\mathbb{R}^m)$ and some $c > 0$, where δ_0 is the Dirac measure in $0 \in \mathbb{R}^m$. Then any $\mathbb{R}^m$-valued continuous process X satisfies

$$\int_r^t |\mathrm{B}_s(X_s, P_{X_s})|\, ds \leq c \int_r^t 1 + E[|X_s|]\, ds$$

for all $t \in [t_0, \infty)$, by using that $\vartheta_1(\mu, \delta_0) = \int_{\mathbb{R}^m} |x|\, \mu(dx)$ for any $\mu \in \mathscr{P}_1(\mathbb{R}^m)$. While the left-hand integral is finite if X solves (1.2), the right-hand integral may be infinite, since the function $[t_0, \infty) \to \mathbb{R}_+$, $s \mapsto E[|X_s|]$ is not necessarily locally integrable.

Remark 2.8. Lower semicontinuity of $E[|X|]$ follows directly from Fatou's lemma, and $E[|X|]$ is continuous if and only if $(X_{t_n})_{n \in \mathbb{N}}$ is uniformly integrable for any convergent sequence $(t_n)_{n \in \mathbb{N}}$ in $[t_0, \infty)$.

If $\mathrm{B}_s(x, \cdot)$ is bounded for all $(s, x) \in [t_0, \infty) \times \mathbb{R}^m$, then $E[|X|]$ is locally bounded for any solution X to (1.2), by Lemmas 3.22 and 3.23. However, to allow for *growth in the measure variable*, we study uniqueness, stability and existence under a local integrability condition, which leads to more generality.

Until the end of this section, let Θ be an $[0, \infty]$-valued Borel measurable functional on $[t_0, \infty) \times \mathscr{P}^2 \times \mathscr{P}(\mathbb{R}^m)$ when $\mathscr{P}(\mathbb{R}^m)$ is equipped with the Prokhorov metric and the induced Borel σ-field.

Definition 2.9. We say that *pathwise uniqueness* holds for (1.2) (relative to Θ) if any two solutions X and $\tilde{X}$ with $X_{t_0} = \tilde{X}_{t_0}$ a.s. (for which the measurable function

$$[t_0, \infty) \to [0, \infty], \quad s \mapsto \Theta(s, P_{X_s}, P_{\tilde{X}_s}, P_{X_s - \tilde{X}_s}) \tag{2.10}$$

is locally integrable) are indistinguishable.

For our main application of Θ let us write R_c for the cone of all $\mathbb{R}_+$-valued continuous functions on $\mathbb{R}_+$ that are positive on $(0, \infty)$ and vanish at 0.

Example 2.10. Let $\eta, \lambda \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$ and $\rho, \varrho \in \mathrm{R}_c$ be such that ρ is concave and ϱ is increasing. We suppose that

$$\Theta(s, \mu, \tilde{\mu}, \nu) = \lambda(s)\varrho(\vartheta(\mu, \tilde{\mu})) + \eta(s) \int_{\mathbb{R}^m} \rho(|x|)\, \nu(dx)$$

for all $s \in [t_0, \infty)$, any $\mu, \tilde{\mu} \in \mathscr{P}$ and each $\nu \in \mathscr{P}(\mathbb{R}^m)$. Then $\Theta(s, \mu, \tilde{\mu}, \nu)$ is finite if $\nu \in \mathscr{P}_1(\mathbb{R}^m)$, by Jensen's inequality, and the following two facts are valid:

(1) For any two continuous processes X and $\tilde{X}$, the function (2.10) is of the form

$$\Theta(\cdot, P_X, P_{\tilde{X}}, P_{X - \tilde{X}}) = \eta\varrho(\vartheta(P_X, P_{\tilde{X}})) + \lambda E[\rho(|X - \tilde{X}|)]$$

and it is locally integrable if the product of $(\eta \vee \lambda)$ with $E[|X - \tilde{X}|]$ satisfies this property. This is the case if $E[|X - \tilde{X}|]$ is locally bounded, for instance.

(2) If in fact $\eta, \lambda \in \mathscr{L}^q_{loc}(\mathbb{R}_+)$ for some $q \in (1, \infty)$ and p is the dual exponent of q, then local p-fold integrability of $E[|X - \tilde{X}|]$ suffices, by Hölder's inequality.

Now we present *generalised notions of stability* for (1.2) in a *global sense*, which apply directly without shifting the stochastic drift and diffusion. Namely, in the literature for stability of SDEs it is a convenient assumption that drift and diffusion vanish at all times at the origin of $\mathbb{R}^m$, ensuring that the constant zero process is a solution.

If, however, a reader seeks to use stability results for McKean-Vlasov SDEs and the normalisations $\mathrm{B}(0,\delta_0) = 0$ and $\Sigma(0) = 0$ fail, then there should exist at least one solution $\hat{X}$ to (1.2). In this case, the maps $\hat{\mathrm{B}}$ and $\hat{\Sigma}$ on $[t_0,\infty)\times\Omega\times\mathbb{R}^m\times\mathscr{P}$ and $[t_0,\infty)\times\Omega\times\mathbb{R}^m$ with values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively, given by

$$\hat{\mathrm{B}}_t(x,\mu) := \mathrm{B}_t(\hat{X}_t + x, \hat{l}(t,\mu)) - \mathrm{B}_t(\hat{X}_t, P_{\hat{X}_t}) \quad \text{and} \quad \hat{\Sigma}_t(x) := \Sigma_t(\hat{X}_t + x) - \Sigma_t(\hat{X}_t)$$

are admissible and satisfy $\hat{\mathrm{B}}(0,\delta_0) = 0$ and $\hat{\Sigma}(0) = 0$ for any Borel measurable map $\hat{l} : [t_0,\infty)\times\mathscr{P}\to\mathscr{P}$ such that $\hat{l}(t,\delta_0) = P_{\hat{X}_t}$ for all $t\in[t_0,\infty)$. So, in effect the reader is forced to replace the drift B and the diffusion Σ by $\hat{\mathrm{B}}$ and $\hat{\Sigma}$, respectively, and use stability concepts that are stated in terms of the particular solution $\hat{X}$.

Further, even if B and Σ were deterministic as in (2.6), the coefficients $\hat{\mathrm{B}}$ and $\hat{\Sigma}$ would in general become random, unless $\hat{X}_t$ is constant for each $t\in[t_0,\infty)$. This translation procedure may certainly have its justification for a local stability analysis, but, as it is not necessary for *global comparisons of solutions*, we do not apply it.

Definition 2.11. Let $\alpha > 0$.

(i) We call (1.2) *stable in moment* (with respect to Θ) if any two solutions X and $\tilde{X}$ (for which the function (2.10) is locally integrable) satisfy

$$\sup_{t\in[t_0,\infty)} E\big[|X_t - \tilde{X}_t|\big] < \infty$$

under the condition that $E[|X_{t_0} - \tilde{X}_{t_0}|] < \infty$. If in addition $\lim_{t\uparrow\infty} E[|X_t - \tilde{X}_t|] = 0$, then we speak about *asymptotic stability in moment.*

(ii) Equation (1.2) is said to be *α-exponentially stable in moment* (relative to Θ) if there are $\lambda < 0$ and $c\in\mathbb{R}_+$ such that for any two solutions X and $\tilde{X}$ to (1.2),

$$E\big[|X_t - \tilde{X}_t|\big] \leq c e^{\lambda(t-t_0)^\alpha} E\big[|X_{t_0} - \tilde{X}_{t_0}|\big] \tag{2.11}$$

for all $t\in[t_0,\infty)$, provided $E[|X_{t_0} - \tilde{X}_{t_0}|] < \infty$ (and (2.10) is locally integrable). In this case, λ is said to be a *moment α-Lyapunov exponent* for (1.2).

(iii) We say that (1.2) is *pathwise α-exponentially stable* (relative to an initial moment and Θ) if there is $\lambda < 0$ such that for any two solutions X and $\tilde{X}$ we have

$$\limsup_{t\uparrow\infty} \frac{1}{t^\alpha} \log\big(|X_t - \tilde{X}_t|\big) \leq \lambda \quad \text{a.s.}$$

(as soon as $E[|X_{t_0} - \tilde{X}_{t_0}|] < \infty$ and (2.10) is locally integrable). If this is the case, then λ is called a *pathwise α-Lyapunov exponent* for (1.2).

Remark 2.12. Let $\psi : [t_0,\infty)\to\mathbb{R}_+$ be positive on (t_0,∞). If $u : [t_0,\infty)\to\mathbb{R}_+$ is locally bounded and satisfies

$$\limsup_{t\uparrow\infty} \frac{1}{\psi(t)} \log(u(t)) \leq \lambda \quad \text{for some } \lambda\in\mathbb{R},$$

then for any $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that $u(t) \leq c_\varepsilon \exp(\psi(t)(\lambda+\varepsilon))$ for each $t\in[t_0,\infty)$. Thus, while (2.11) readily implies

$$\limsup_{t\uparrow\infty} \frac{1}{t^\alpha} \log\big(E\big[|X_t - \tilde{X}_t|\big]\big) \leq \lambda,$$

the latter bound entails the former only when λ is replaced by $\lambda+\varepsilon$ for any $\varepsilon > 0$, provided $E[|X - \tilde{X}|]$ is locally bounded.

Note that for our general equation (1.2) with random coefficients B and Σ we have to formulate Definitions 2.9 and 2.11 for uniqueness and stability with respect to the underlying probability space.

These concepts directly carry over to the case (2.6) of deterministic coefficients, applied to each filtered probability space satisfying the usual conditions and on which there is a standard d-dimensional Brownian motion.

To be precise, *pathwise uniqueness* holds for (2.7) in the usual sense (relative to Θ) if any two weak solutions X and $\tilde{X}$ on a common filtered probability space relative to one standard d-dimensional Brownian motion with $X_{t_0} = \tilde{X}_{t_0}$ a.s. (and for which the measurable function

$$[t_0, \infty) \to [0, \infty], \quad s \mapsto \Theta\big(s, \mathscr{L}(X_s), \mathscr{L}(\tilde{X}_s), \mathscr{L}(X_s - \tilde{X}_s)\big)$$

is locally integrable) are indistinguishable. Similarly, by considering weak solutions on common filtered probability spaces in Definition 2.11 instead of solutions on the underlying space, *each notion of stability applies to* (2.7).

3 Main results

3.1 A quantitative first moment estimate and pathwise uniqueness

We seek to compare solutions to (1.2) with varying drifts and thereby show pathwise uniqueness. To this end, let $\tilde{\mathrm{B}} : [t_0, \infty) \times \Omega \times \mathbb{R}^m \times \mathscr{P} \to \mathbb{R}^m$ be admissible and U be an $\mathbb{R}^{m\times m}$-valued adapted locally absolutely continuous process that is orthonormal.

That is, $\{{}_1U_s, \ldots, {}_mU_s\}$ yields an orthonormal basis of $\mathbb{R}^m$ for all $s \in [t_0, \infty)$, where ${}_iU$ is the i-th column of U for any $i \in \{1, \ldots, m\}$. The dependence of Σ with respect to the space variable $x \in \mathbb{R}^m$ will be measured in terms of this random basis.

For $\hat{m} \in \mathbb{N}$ and $A \in \mathbb{R}^{m\times\hat{m}}$ of the form $A = (a_1, \ldots, a_{\hat{m}})$ with $a_1, \ldots, a_{\hat{m}} \in \mathbb{R}^m$, we define a pseudonorm on $\mathbb{R}^m$ by

$$|x|_A := \sum_{i=1}^{\hat{m}} |a_i' x|. \tag{3.1}$$

Then $|A'x| \leq |x|_A \leq \sqrt{\hat{m}}|A'x|$ for all $x \in \mathbb{R}^m$, as $|x|_A$ is simply the 1-norm of $A'x$. Thus, the *time-dependent random norm* $|\cdot|_U$ satisfies $|\cdot| \leq |\cdot|_U \leq \sqrt{m}|\cdot|$ and we impose a partial growth condition on the weak derivatives of the columns of U.

(C.1) There are $c_0 \in \mathbb{R}_+$ and $\zeta \in \mathscr{L}^1_{loc}(\mathbb{R}^m_+)$ with $\mathrm{sgn}({}_iU'x){}_i\dot{U}'x \leq c_0\zeta_i|x|_U$ for all $x \in \mathbb{R}^m$ a.e. on $[t_0, \infty)$ a.s. for any $i \in \{1, \ldots, m\}$.

Remark 3.1. By the orthonormality of U, for any $i \in \{1, \ldots, m\}$ there is a unique ${}_i\hat{\zeta} \in \mathscr{S}(\mathbb{R}^m)$ such that ${}_i\dot{U} = U{}_i\hat{\zeta}$ whenever the weak derivative ${}_i\dot{U}$ is given. Then

$$\mathrm{sgn}({}_iU'x){}_i\dot{U}'x \leq {}_i\hat{\zeta}^{(i)}|{}_iU'x| + \sum_{j=1,\, j\neq i}^{m} |{}_i\hat{\zeta}^{(j)}||{}_jU'x|$$

for any $x \in \mathbb{R}^m$. Thus, if there is $\zeta \in \mathscr{L}^1_{loc}(\mathbb{R}^m_+)$ with $\max_{j\in\{1,\ldots,m\}} |{}_i\hat{\zeta}^{(j)}| \leq \zeta_i$ a.e. on $[t_0, \infty)$ a.s. for all $i \in \{1, \ldots, m\}$, then (C.1) follows for any $c_0 \in [1, \infty)$.

Example 3.2. Let U be *deterministic*. That is, there is an $\mathbb{R}^{m\times m}$-valued orthonormal locally absolutely continuous map u on $[t_0, \infty)$ such that

$$U_s = u(s) \quad \text{for all } s \in [t_0, \infty). \tag{3.2}$$

Then (C.1) holds by the preceding remark. In fact, for all $i \in \{1, \ldots, m\}$ and any choice of $\dot{u}_i$, the measurable map $\hat{\zeta}_i := u'\dot{u}_i$ is locally integrable and satisfies $\dot{u}_i = u\hat{\zeta}_i$.

Let us introduce an *Osgood continuity condition on compact sets* for the coordinate processes of Σ relative to the underlying random basis and use the cone R_c of all continuous moduli of continuity, defined right before Example 2.10.

(C.2) For each $n \in \mathbb{N}$ there are ${}_n\hat{\eta} \in \mathscr{S}(\mathbb{R}_+)$ and an increasing function $\hat{\rho}_n \in \mathrm{R}_c$ such that $\int_{t_0}^t {}_n\hat{\eta}_s^2\,ds < \infty$ for all $t \in [t_0, \infty)$, $\int_0^1 \hat{\rho}_n(v)^{-2}\,dv = \infty$ and

$$\big|{}_iU'(\Sigma(x) - \Sigma(\tilde{x}))\big| \leq {}_n\hat{\eta}\hat{\rho}_n(|{}_iU'(x - \tilde{x})|)$$

for all $x, \tilde{x} \in \mathbb{R}^m$ with $|x| \vee |\tilde{x}| \leq n$ a.e. on $[t_0, \infty)$ a.s. for any $i \in \{1, \ldots, m\}$.

Remark 3.3. The condition forces ${}_iU'\Sigma$ to depend only on the i-th random coordinate ${}_iU'x$ of the space variable $x \in \mathbb{R}^m$ a.e. on $[t_0, \infty)$ a.s. for all $i \in \{1, \ldots, m\}$. This is the case if and only if the diffusion is of the form

$$\Sigma(x) = U \begin{pmatrix} \hat{\Sigma}_{1,1}({}_1U'x) & \cdots & \hat{\Sigma}_{1,d}({}_1U'x) \\ \vdots & \ddots & \vdots \\ \hat{\Sigma}_{m,1}({}_mU'x) & \cdots & \hat{\Sigma}_{m,d}({}_mU'x) \end{pmatrix} \quad \text{for all } x \in \mathbb{R}^m$$

a.e. on $[t_0, \infty)$ a.s. for some $\mathbb{R}^{m\times d}$-valued admissible map $\hat{\Sigma}$ on $[t_0, \infty) \times \Omega \times \mathbb{R}$. Further, for any $n \in \mathbb{N}$ we may take $\hat{\rho}_n(v) = v^{\alpha_n}$ for each $v \in \mathbb{R}_+$ and some $\alpha_n \in [1/2, 1]$ as modulus of continuity.

Example 3.4. For $m = 1$ let $l \in \mathbb{N}$ and $\varphi : \mathbb{R} \to \mathbb{R}^{d\times l}$ be locally 1/2-Hölder continuous. Suppose that there are ${}_0\hat{\eta}, \ldots, {}_k\hat{\eta} \in \mathscr{S}(\mathbb{R}^d)$ such that $\sum_{k=1}^l \int_{t_0}^t |{}_k\hat{\eta}_s|^2\,ds < \infty$ and

$$\Sigma_t^{(i)}(x) = {}_0\hat{\eta}_t^{(i)} + \sum_{k=1}^{l} {}_k\hat{\eta}_t^{(i)}\varphi_{i,k}(x) \tag{3.3}$$

for all $(t, x) \in [t_0, \infty) \times \mathbb{R}$ and any $i \in \{1, \ldots, d\}$. Then (C.2) holds, as $U \in \{-1, 1\}$ and $\dot{U} = 0$ a.e. In particular, $\varphi_{i,k}(x) = |x|^{\alpha_{i,k}}$ for all $(i, k) \in \{1, \ldots, d\} \times \{1, \ldots, l\}$, each $x \in \mathbb{R}$ and some $\alpha \in [1/2, \infty)^{d\times l}$ is possible.

We consider a *partial uniform error and continuity condition* on the random coordinate processes of B and $\tilde{\mathrm{B}}$, which uses the dual exponent given directly after (2.1) and allows for discontinuities in the space variable.

(C.3) There are $\alpha \in (0, 1]$, $\varepsilon, \lambda \in \mathscr{S}_{loc}^1(\mathbb{R}_+^m)$, $\eta \in \mathscr{S}_{loc}^{q_\alpha}(\mathbb{R}_+^m)$ and $\rho, \varrho \in \mathrm{R}_c$ such that $\rho^{1/\alpha}$ is concave, ϱ is increasing and

$$\mathrm{sgn}({}_iU'(x - \tilde{x})){}_iU'(\mathrm{B}(x, \mu) - \tilde{\mathrm{B}}(\tilde{x}, \tilde{\mu})) \leq \varepsilon^{(i)} + \eta^{(i)}\rho(|x - \tilde{x}|_U) + \lambda^{(i)}\varrho(\vartheta(\mu, \tilde{\mu}))$$

for all $x, \tilde{x} \in \mathbb{R}^m$ and any $\mu, \tilde{\mu} \in \mathscr{P}$ a.e. on $[t_0, \infty)$ a.s. for each $i \in \{1, \ldots, m\}$.

Note that the preceding estimate follows as soon as the random norm $|\cdot|_U$ is replaced by the Euclidean norm $|\cdot|$ and the modulus of continuity ρ is increasing. If $\mathrm{B} = \tilde{\mathrm{B}}$ and $\varepsilon = 0$, then (C.3) reduces to a *partial uniform continuity condition* on the coordinates of B. In particular, in the case that there are $\alpha_0, \beta_0 \in (0, 1]$ such that

$$\alpha_0 \leq \alpha, \quad \rho(v) = v^{\alpha_0} \quad \text{and} \quad \varrho(v) = v^{\beta_0} \quad \text{for all } v \in \mathbb{R}_+,$$

we get a *partial Hölder condition*. Following this reasoning, the term ε provides an estimate for $\mathrm{B} - \tilde{\mathrm{B}}$. That is, if (C.3) holds for $\mathrm{B} = \tilde{\mathrm{B}}$ and $\varepsilon = 0$, then it is valid in the general case as soon as $|{}_iU'(\mathrm{B} - \tilde{\mathrm{B}})| \leq \varepsilon^{(i)}$ for any $i \in \{1, \ldots, m\}$.

As our first estimation result is based on Bihari's inequality, we recall that for any $\rho \in \mathrm{R}_c$ the function $\Phi_\rho \in C^1((0,\infty))$ defined via

$$\Phi_\rho(w) := \int_1^w \frac{1}{\rho(v)}\, dv \tag{3.4}$$

is a strictly increasing C^1-diffeomorphism onto the interval $(\Phi_\rho(0), \Phi_\rho(\infty))$. Let D_ρ denote the set of all $(v,w) \in \mathbb{R}_+^2$ with $\Phi_\rho(v) + w < \Phi_\rho(\infty)$. Then $\Psi_\rho : D_\rho \to \mathbb{R}$ given by

$$\Psi_\rho(v,w) := \Phi_\rho^{-1}(\Phi_\rho(v) + w) \tag{3.5}$$

is continuous as continuous extension of a locally Lipschitz continuous function and it is increasing in each variable. In particular, $(v,0) \in D_\rho^\circ$ for any $v \in [0, \Phi_\rho(\infty))$.

Under (C.1) and (C.3), for fixed $\beta \in (0,1]$ we introduce $\gamma, \delta \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$ by

$$\gamma := c_0\zeta_0 + \alpha[\eta^{(0)}]_{q_\alpha} + \beta E[\lambda^{(0)}] \quad \text{and} \quad \delta := (1-\alpha)[\eta^{(0)}]_{q_\alpha} + (1-\beta)E[\lambda^{(0)}]$$

and obtain a *quantitative* L^1*-bound*, by using the constant appearing in the L^1-norm estimate (2.8) for the functional ϑ.

Proposition 3.5. *Let* (C.1)-(C.3) *hold,* X *and* $\tilde{X}$ *be solutions to* (1.2) *with respective drifts* B *and* $\tilde{\mathrm{B}}$ *so that* $E[|Y_{t_0}|] < \infty$ *for* $Y := X - \tilde{X}$ *and* $E[\lambda^{(0)}]\varrho(\vartheta(P_X, P_{\tilde{X}}))$ *is locally integrable. Define* $\rho_0, \varrho_0 \in C(\mathbb{R}_+)$ *by*

$$\rho_0(v) := (c_0 v) \vee \rho(v)^{\frac{1}{\alpha}} \quad \text{and} \quad \varrho_0(v) := \rho_0(v) \vee \varrho(c_{\mathscr{P}} v)^{\frac{1}{\beta}} \tag{3.6}$$

and suppose that $\Phi_{\rho_0}(\infty) = \infty$ *or* $E[\eta^{(0)}\rho(|Y|_U)] + c_0\zeta_0 E[|Y|_U]$ *is locally integrable. Then* $E[|Y|]$ *is locally bounded and*

$$\sup_{s\in[t_0,t]} E[|Y_s|_{U_s}] \leq \Psi_{\varrho_0}\bigg(E[|Y_{t_0}|_{U_{t_0}}] + \int_{t_0}^t E[\varepsilon_s^{(0)}] + \delta(s)\, ds, \int_{t_0}^t \gamma(s)\, ds\bigg)$$

for any $t \in [t_0, t_0^+)$*, where* t_0^+ *stands for the supremum over all* $t \in [t_0, \infty)$ *for which*

$$\bigg(E[|Y_{t_0}|_{U_{t_0}}] + \int_{t_0}^t E[\varepsilon_s^{(0)}] + \delta(s)\, ds, \int_{t_0}^t \gamma(s)\, ds\bigg) \in D_{\varrho_0}.$$

Remark 3.6. If in fact $\Phi_{\varrho_0}(\infty) = \infty$, then $\Phi_{\rho_0}(\infty) = \infty$ and $D_{\varrho_0} = \mathbb{R}_+^2$. In this case, Y is bounded in $L^1(\Omega, \mathscr{F}, P)$ as soon as $E[\varepsilon^{(0)}]$, γ and δ are integrable. Further, if

$$\Phi_{\varrho_0}(0) = -\infty, \quad Y_{t_0} = 0 \quad \text{a.s.} \quad \text{and} \quad E[\varepsilon^{(0)}] = \delta = 0 \quad \text{a.e.},$$

then $t_0^+ = \infty$ and $Y = 0$ a.s. This implication will be exploited to deduce pathwise uniqueness.

Example 3.7. Let U be independent of time and $\rho(v) = \varrho(v) = v^{\alpha_0}$ for all $v \in \mathbb{R}_+$ and some $\alpha_0 \in (0, \alpha]$. Then $\Phi_{\varrho_0}(\infty) = \infty$ and for $\alpha_0 < \alpha$ and $\beta = \alpha$ we get that

$$\Psi_{\varrho_0}(v,w) = \Big(v^{1-\hat{\alpha}} + (1-\hat{\alpha})\hat{c}w\Big)^{\frac{1}{1-\hat{\alpha}}} \quad \text{for all } v, w \in \mathbb{R}_+$$

with $\hat{\alpha} := \alpha_0/\alpha$ and $\hat{c} := 1 \vee c_{\mathscr{P}}$, as $c_0 = 0$ is feasible in (C.1). If instead $\alpha_0 = \alpha$, then $\Psi_{\varrho_0}(v,w) = v\exp(\hat{c}w)$ for any $v, w \in \mathbb{R}_+$. Thus, Proposition 3.5 provides an estimate for any choice of the parameter $\hat{\alpha} \in (0,1]$.

To infer pathwise uniqueness for (1.2) from the comparison, we specify (C.3) for $\mathrm{B} = \tilde{\mathrm{B}}$, $\varepsilon = 0$, $\alpha = 1$ and a deterministic choice of η. Further, if B does not depend on $\mu \in \mathscr{P}$, then this condition can be localised.

(C.4) There are $\eta \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$, $\lambda \in \mathscr{S}^1_{loc}(\mathbb{R}^m_+)$ and $\rho, \varrho \in \mathrm{R}_c$ so that ρ is concave, ϱ is increasing and

$$\mathrm{sgn}({}_iU'(x-\tilde{x}))_iU'(\mathrm{B}(x,\mu) - \mathrm{B}(\tilde{x},\tilde{\mu})) \leq \eta\rho(|x-\tilde{x}|_U) + \lambda^{(i)}\varrho(\vartheta(\mu,\tilde{\mu}))$$

for all $x, \tilde{x} \in \mathbb{R}^m$ and any $\mu, \tilde{\mu} \in \mathscr{P}$ a.e. on $[t_0, \infty)$ a.s. for any $i \in \{1, \dots, m\}$.

(C.5) B is independent of $\mu \in \mathscr{P}$ and for any $n \in \mathbb{N}$ there are $\eta_n \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$ and a concave function $\rho_n \in \mathrm{R}_c$ satisfying

$$\mathrm{sgn}({}_iU'(x-\tilde{x}))_iU'(\hat{\mathrm{B}}(x) - \hat{\mathrm{B}}(\tilde{x})) \leq \eta_n\rho_n(|x-\tilde{x}|_U)$$

for all $x, \tilde{x} \in \mathbb{R}^m$ with $|x| \vee |\tilde{x}| \leq n$ a.e. on $[t_0, \infty)$ a.s. for all $i \in \{1, \dots, m\}$, where $\hat{\mathrm{B}} := \mathrm{B}(\cdot, \mu_0)$ for some $\mu_0 \in \mathscr{P}$.

Example 3.8. Let F be a real-valued admissible function on $[t_0, \infty) \times \Omega \times \mathbb{R} \times \mathscr{P}$ such that B admits the representation

$$\mathrm{B}(x,\mu) = {}_iUF({}_iU'x, \mu)$$

for any $(x, \mu) \in \mathbb{R}^m \times \mathscr{P}$ and fixed $i \in \{1, \dots, m\}$. Then the following two assertions hold:

(1) If there are $\eta \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$, $\lambda \in \mathscr{S}^1_{loc}(\mathbb{R}_+)$ and increasing functions $\rho, \varrho \in \mathrm{R}_c$ such that ρ is concave and

$$\mathrm{sgn}(v-\tilde{v})(F(v,\mu) - F(\tilde{v},\tilde{\mu})) \leq \eta\rho(|v-\tilde{v}|) + \lambda\varrho(\vartheta(\mu,\tilde{\mu}))$$

for any $v, \tilde{v} \in \mathbb{R}$ and all $\mu, \tilde{\mu} \in \mathscr{P}$, then (C.4) is satisfied.

(2) Let F be independent of $\mu \in \mathscr{P}$ and suppose for each $n \in \mathbb{N}$ there are $\eta_n \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$ and a concave increasing function $\rho_n \in \mathrm{R}_c$ such that

$$\mathrm{sgn}(v-\tilde{v})(\hat{F}(v) - \hat{F}(\tilde{v})) \leq \eta_n\rho_n(|v-\tilde{v}|)$$

for all $v, \tilde{v} \in [-n, n]$, where $\hat{F} := F(\cdot, \mu_0)$ for some $\mu_0 \in \mathscr{P}$. Then (C.5) is valid.

For instance, let $\kappa, \tilde{\eta} \in \mathscr{S}(\mathbb{R})$, $\eta \in \mathscr{L}^1_{loc}(\mathbb{R})$, $\lambda \in \mathscr{S}^1_{loc}(\mathbb{R})$, $f : \mathbb{R} \to \mathbb{R}$ be decreasing, $g : \mathbb{R}_+ \to \mathbb{R}$ be Lipschitz continuous and $\mu_0 \in \mathscr{P}$ be such that

$$\tilde{\eta} \geq 0 \quad \text{and} \quad F(v,\mu) = \kappa + \eta v + \tilde{\eta}f(v) + \lambda g(\vartheta(\mu,\mu_0))$$

for any $(v, \mu) \in \mathbb{R} \times \mathscr{P}$. If ϑ satisfies the triangle inequality, then the condition in (1) holds when $\rho(v) = \varrho(v) = v$ for all $v \in \mathbb{R}_+$. If $\lambda = 0$, then the assumption in (2) follows.

Given (C.1), (C.2) and (C.4) hold, pathwise uniqueness can be shown relative to the $\mathbb{R}_+$-valued Borel measurable functional Θ on $[t_0, \infty) \times \mathscr{P}^2 \times \mathscr{P}^1(\mathbb{R}^m)$ defined by

$$\Theta(s,\mu,\tilde{\mu},\nu) := E[\lambda^{(0)}_s]\varrho(\vartheta(\mu,\tilde{\mu})) + \int_{\mathbb{R}^m} \eta(s)\rho(|y|) + c_0\zeta_0(s)|y|\,\nu(dy).$$

If in fact $\Phi_{\rho_0}(\infty)$ is infinite for the modulus of continuity ρ_0 in (3.6) for $\alpha = 1$, then we may disregard the integral expression and simply define $\Theta : [t_0, \infty) \times \mathscr{P}^2 \to \mathbb{R}_+$ via $\Theta(\cdot, \mu, \tilde{\mu}) := E[\lambda^{(0)}]\varrho(\vartheta(\mu, \tilde{\mu}))$, which leads to a relaxed local integrability condition.

Corollary 3.9. *Suppose that* (C.1) *and* (C.2) *are satisfied.*

(i) *Let* (C.4) *be valid and* $\int_0^1((c_0 v)\vee\rho(v)\vee\varrho(c_{\mathscr{P}}v))^{-1}\,dv = \infty$. *Then pathwise uniqueness holds for* (1.2) *relative to* Θ.

(ii) *If* (C.5) *is valid and* $\int_0^1((c_0 v)\vee\rho_n(y))^{-1}\,dv = \infty$ *for each* $n\in\mathbb{N}$, *then pathwise uniqueness for the SDE* (1.2) *follows.*

Remark 3.10. Let B and Σ be deterministic, that is, (2.6) holds. Then the corollary yields pathwise uniqueness for (2.7) in the standard sense if the conditions are specified as follows:

(1) The orthonormal process U is deterministic, in which case (C.1) holds, as shown in Example 3.2, and the Osgood condition (C.2) on compact sets is formulated when ${}_n\hat{\eta}$ is deterministic and belongs to $\mathscr{L}^2_{loc}(\mathbb{R}_+)$ for all $n\in\mathbb{N}$.

(2) The partial uniform continuity condition (C.4) is stated when λ is independent of $\omega\in\Omega$ and lies in $\mathscr{L}^1_{loc}(\mathbb{R}_+)$ and the required estimate (2.8) for ϑ in the L^1-norm is replaced by the domination condition (2.9).

Let us consider a *class of drift maps*, which includes Example 3.8, to which these uniqueness results apply and recall that (C.1) holds for $c_0 = 0$ if U is independent of time.

Example 3.11. Let $\kappa\in\mathscr{S}(\mathbb{R}^m)$ and F, G be two $\mathbb{R}^m$-valued admissible mappings on $[t_0,\infty)\times\Omega\times\mathbb{R}\times\mathscr{P}$ and $[t_0,\infty)\times\Omega\times\mathbb{R}^m\times\mathscr{P}$, respectively, such that

$$\mathrm{B}(x,\mu) = \kappa + U\begin{pmatrix} F^{(1)}(U_1'x,\mu)\\ \vdots \\ F^{(m)}(U_m'x,\mu)\end{pmatrix} + G(x,\mu)$$

for all $(x,\mu)\in\mathbb{R}^m\times\mathscr{P}$. If the Osgood condition (C.2) on compact sets is imposed on Σ, then Corollary 3.9 entails two assertions:

(1) Assume that there are $\eta\in\mathscr{L}^1_{loc}(\mathbb{R}_+)$, $\lambda,\hat{\lambda}\in\mathscr{S}^1_{loc}(\mathbb{R}^m_+)$ and an increasing function $\rho\in\mathrm{R}_c$ such that $\int_0^1((c_1 v)\vee\rho(v))^{-1}\,dv = \infty$ for $c_1 := c_0\vee 1$,

$$\begin{aligned}\mathrm{sgn}(v-\tilde{v})\big(F^{(i)}(v,\mu) - F^{(i)}(\tilde{v},\tilde{\mu})\big) &\leq \eta|v-\tilde{v}| + \lambda^{(i)}\rho(\vartheta(\mu,\tilde{\mu}))\\ \text{and}\quad |{}_iU'(G(x,\mu)-G(\tilde{x},\tilde{\mu}))| &\leq \eta|x-\tilde{x}| + \hat{\lambda}^{(i)}\rho(\vartheta(\mu,\tilde{\mu}))\end{aligned}$$

for all $v,\tilde{v}\in\mathbb{R}$, $x,\tilde{x}\in\mathbb{R}^m$, $\mu,\tilde{\mu}\in\mathscr{P}$ and $i\in\{1,\dots,m\}$. Then, relative to the functional Θ on $[t_0,\infty)\times\mathscr{P}^2$ given by

$$\Theta(s,\mu,\tilde{\mu}) := E[\lambda_s^{(0)} + \hat{\lambda}_s^{(0)}]\rho(\vartheta(\mu,\tilde{\mu})),$$

pathwise uniqueness for (1.2) holds. In particular, $\rho(v) = v$ for all $v > 0$ and $\rho(v) = \alpha v(|\log(v)|+1)$ for any $v>0$ with $\alpha>0$ are feasible choices.

(2) Let F and G be independent of $\mu\in\mathscr{P}$ and suppose that for any $n\in\mathbb{N}$ there are $\eta_n\in\mathscr{L}^1_{loc}(\mathbb{R}_+)$ and a concave increasing function $\rho_n\in\mathrm{R}_c$ with $\int_0^1((c_0 v)\vee\rho_n(v))^{-1}\,dv = \infty$ such that

$$\mathrm{sgn}(v-\tilde{v})\big(\hat{F}^{(i)}(v) - \hat{F}^{(i)}(\tilde{v})\big) \leq \eta\rho_n(|v-\tilde{v}|),\quad |\hat{G}(x)-\hat{G}(\tilde{x})| \leq \eta\rho_n(|x-\tilde{x}|)$$

for all $v,\tilde{v}\in[-n,n]$, any $x,\tilde{x}\in\mathbb{R}^m$ with $|x|\vee|\tilde{x}|\leq n$ and all $i\in\{1,\dots,m\}$, where $\hat{F} := F(\cdot,\mu_0)$ and $\hat{G} := G(\cdot,\mu_0)$ for fixed $\mu_0\in\mathscr{P}$. Then we have pathwise uniqueness for the SDE (1.2).

3.2 An explicit moment estimate and stability in first moment

Now we provide a comparison bound, from which stability results in first moment can be inferred. In this regard, we require a *partial uniform error and mixed Hölder continuity condition* on the random coordinates of B and $\tilde{\mathrm{B}}$ that allows to measure their dependence on each coordinate relative to each Hölder exponent:

(C.6) There are $l \in \mathbb{N}$, $\alpha, \beta \in (0,1]^l$, $\varepsilon \in \mathscr{S}^1_{loc}(\mathbb{R}^m_+)$ and for each $k \in \{1,\dots,l\}$ there are are

$$_k\eta \in \mathscr{S}^{q_{\alpha_k}}_{loc}(\mathbb{R}^{m\times m}) \quad \text{and} \quad _k\lambda \in \mathscr{S}^1_{loc}(\mathbb{R}^m_+)$$

with $_k\eta^{(i,j)} \geq 0$ for any $i,j \in \{1,\dots,m\}$ with $i \neq j$ such that

$$\begin{aligned}\operatorname{sgn}({}_iU'(x-\tilde{x}))\,{}_iU'(\mathrm{B}(x,\mu) - \tilde{\mathrm{B}}(\tilde{x},\tilde{\mu})) &\leq \varepsilon^{(i)} \\ &\quad + \sum_{k=1}^l \bigg(\sum_{j=1}^m {}_k\eta^{(i,j)}|{}_jU'(x-\tilde{x})|^{\alpha_k}\bigg) + {}_k\lambda^{(i)}\vartheta(\mu,\tilde{\mu})^{\beta_k}\end{aligned}$$

for all $x, \tilde{x} \in \mathbb{R}^m$ and any $\mu, \tilde{\mu} \in \mathscr{P}$ a.e. on $[t_0,\infty)$ a.s. for any $i \in \{1,\dots,m\}$.

Remark 3.12. If (C.6) is satisfied and $\alpha_1 < \cdots < \alpha_l$, then (C.3) follows for any $\hat{\alpha} \in [\alpha_l, 1]$ instead of α when $\rho, \varrho \in \mathrm{R}_c$ are given by

$$\rho(v) := \max_{k\in\{1,\dots,l\}} v^{\alpha_k} \quad \text{and} \quad \varrho(v) := \max_{k\in\{1,\dots,l\}} v^{\beta_k},$$

since $\rho(v) = v^{\alpha_1}$ for $v \in [0,1]$ and $\rho(v) = v^{\alpha_l}$ for $v \in [1,\infty)$. Conversely, let (C.3) hold and suppose that there are $l \in \mathbb{N}$, $\hat{\alpha} \in (0,\alpha]^l$ and $\hat{\beta} \in (0,1]^l$ such that

$$\rho(v) \leq \sum_{k=1}^l v^{\hat{\alpha}_k} \quad \text{and} \quad \varrho(v) \leq \sum_{k=1}^l v^{\hat{\beta}_k}$$

for all $v \in \mathbb{R}_+$. Then (C.6) is implied for $(\hat{\alpha}, \hat{\beta})$ instead of (α, β).

We recall the functional $[\cdot]_p$ given by (2.1) for $p \in [1,\infty]$. Then, under conditions (C.1) and (C.6), the functions $\gamma_{\mathscr{P}} \in \mathscr{L}^1_{loc}(\mathbb{R})$ and $\delta_{\mathscr{P}} \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$ defined via

$$\gamma_{\mathscr{P}} := c_0\zeta_0 + \max_{j\in\{1,\dots,m\}} \sum_{k=1}^l \alpha_k \bigg[\sum_{i=1}^m {}_k\eta^{(i,j)}\bigg]_{q_{\alpha_k}} + \beta_k c_{\mathscr{P}}^{\beta_k} E[{}_k\lambda^{(0)}] \tag{3.7}$$

and

$$\delta_{\mathscr{P}} := \sum_{k=1}^l (1-\alpha_k)\bigg(\sum_{j=1}^m \bigg[\sum_{i=1}^m {}_k\eta^{(i,j)}\bigg]_{q_{\alpha_k}}\bigg) + (1-\beta_k)c_{\mathscr{P}}^{\beta_k} E[{}_k\lambda^{(0)}] \tag{3.8}$$

solely depend on the regularity of the random coordinates of B and $\tilde{\mathrm{B}}$. By means of these coefficients we get an *explicit L^1-comparison estimate* relative to the $\mathbb{R}_+$-valued Borel measurable functional Θ on $[t_0,\infty) \times \mathscr{P}^2$ given by

$$\Theta(s,\mu,\tilde{\mu}) := \sum_{k=1}^l E[{}_k\lambda_s^{(0)}]\vartheta(\mu,\tilde{\mu})^{\beta_k}. \tag{3.9}$$

Proposition 3.13. *Let* (C.1), (C.2) *and* (C.6) *be valid, X be a solution to* (1.2) *and $\tilde{X}$ solve* (1.2) *with* $\tilde{\mathrm{B}}$ *instead of* B *such that $E[|Y_{t_0}|] < \infty$ for $Y := X - \tilde{X}$ and $\Theta(\cdot, P_X, P_{\tilde{X}})$ is locally integrable. Then*

$$E[|Y_t|_{U_t}] \leq e^{\int_{t_0}^t \gamma_{\mathscr{P}}(s)\,ds} E[|Y_{t_0}|_{U_{t_0}}] + \int_{t_0}^t e^{\int_s^t \gamma_{\mathscr{P}}(\tilde{s})\,d\tilde{s}}(E[\varepsilon_s^{(0)}] + \delta_{\mathscr{P}}(s))\,ds \tag{3.10}$$

for any $t \in [t_0, \infty)$. In particular, if $\gamma_{\mathscr{P}}^+$, $E[\varepsilon^{(0)}]$ and $\delta_{\mathscr{P}}$ are integrable, then $E[|Y|]$ is bounded. If additionally $\int_{t_0}^\infty \gamma_{\mathscr{P}}^-(s)\,ds = \infty$, then

$$\lim_{t\uparrow\infty} E\big[|Y_t|\big] = 0.$$

Remark 3.14. While ε serves as estimate for $\mathrm{B} - \tilde{\mathrm{B}}$, the coefficient $\delta_{\mathscr{P}}$ arises from all the partial Hölder exponents in $(0,1)$ that appear in (C.6). Namely, $\delta_{\mathscr{P}}(s)$ vanishes for given $s \in [t_0, \infty)$ if and only if each $k \in \{1,\dots,l\}$ satisfies $\sum_{i=1}^m {}_k\eta_s^{(i,j)} \leq 0$ for all $j \in \{1,\dots,m\}$, if $\alpha_k < 1$, and ${}_k\lambda_s = 0$ a.s., if $\beta_k < 1$.

Although the bound in Proposition 3.5 applies to different types of moduli of continuity that are specified in (C.3), the estimate (3.10) is generally sharper, as Example 3.7 and Remark 3.12 show, bearing in mind that $\gamma_{\mathscr{P}}$ may take negative values.

Based on the *partial mixed Hölder continuity condition* (C.6) for B, assuming that $\mathrm{B} = \tilde{\mathrm{B}}$ and $\varepsilon = 0$ there, we get (asymptotic) stability in moment as direct consequence.

Corollary 3.15. *Let* (C.1)*,* (C.2) *and* (C.6) *be satisfied for* $\mathrm{B} = \tilde{\mathrm{B}}$ *and* $\varepsilon = 0$*. Then* (1.2) *is (asymptotically) stable in moment relative to* Θ *defined by* (3.9) *if* $\gamma_{\mathscr{P}}^+$ *and* $\delta_{\mathscr{P}}$ *are integrable (and* $\int_{t_0}^\infty \gamma_{\mathscr{P}}^-(s)\,ds = \infty$*).*

For a *description of the L^1-boundedness and the rate of L^1-convergence for solutions* in Corollary 3.16 below, let us restrict (C.6) to a *partial Lipschitz continuity condition*:

(C.7) There are $\eta \in \mathscr{L}_{loc}^1(\mathbb{R}^{m\times m})$ and $\lambda \in \mathscr{S}_{loc}^1(\mathbb{R}_+^m)$ with $\eta_{i,j} \geq 0$ for all $i, j \in \{1,\dots,m\}$ with $i \neq j$ such that

$$\mathrm{sgn}({}_iU'(x-\tilde{x}))\,{}_iU'\big(\mathrm{B}(x,\mu) - \mathrm{B}(\tilde{x},\tilde{\mu})\big) \leq \sum_{j=1}^m \eta_{i,j}|{}_jU'(x-\tilde{x})| + \lambda^{(i)}\vartheta(\mu,\tilde{\mu})$$

for all $x, \tilde{x} \in \mathbb{R}^m$ and every $\mu, \tilde{\mu} \in \mathscr{P}$ a.e. on $[t_0, \infty)$ a.s. for all $i \in \{1,\dots,m\}$.

Given (C.1) and the preceding condition hold, the coefficient $\delta_{\mathscr{P}}$ in (3.8) vanishes and the *stability coefficient* $\gamma_{\mathscr{P}}$ in (3.7) becomes

$$\gamma_{\mathscr{P}} = c_0\zeta_0 + \max_{j\in\{1,\dots,m\}} \sum_{i=1}^m \eta_{i,j} + c_{\mathscr{P}} E\big[\lambda^{(0)}\big]. \tag{3.11}$$

Remarkably, $\gamma_{\mathscr{P}}$ is merely influenced by the regularity of B relative to the orthonormal process U and satisfies $\gamma_{\mathscr{P}} = \eta + c_{\mathscr{P}}E[\lambda]$ for $m = 1$, since $U \in \{-1, 1\}$ and $\dot{U} = 0$ a.e. in one dimension. Further, under (C.7) the functional in (3.9) is of the form

$$\Theta(\cdot,\mu,\tilde{\mu}) = E\big[\lambda^{(0)}\big]\vartheta(\mu,\tilde{\mu}) \tag{3.12}$$

for any $\mu, \tilde{\mu} \in \mathscr{P}$. To deduce exponential moment stability, we impose an *upper bound on $\gamma_{\mathscr{P}}$ that involves sums of power functions* and entails $\gamma_{\mathscr{P}} \leq 0$ a.e. on $[\hat{t}_1, \infty)$ for some $\hat{t}_1 \in [t_0, \infty)$.

(C.8) Both (C.1) and (C.7) are valid and there are $l \in \mathbb{N}$, $\alpha \in (0,\infty)^l$ and $\hat{\lambda}, s \in \mathbb{R}^l$ with $\alpha_1 < \cdots < \alpha_l$ and $\hat{\lambda}_l < 0$ such that

$$\gamma_{\mathscr{P}}(s) \leq \sum_{k=1}^l \hat{\lambda}_k\alpha_k(s-s_k)^{\alpha_k - 1} \quad \text{for a.e. } s \in [t_1, \infty),$$

where $t_1 \in [t_0, \infty)$ satisfies $\max_{k\in\{1,\dots,l\}} s_k \leq t_1$.

By using the negativity of the constant $\hat{\lambda}_l$ associated to the greatest power α_l in the preceding condition, the announced exponential moment stability follows.

Corollary 3.16. *Under* (C.2) *the following two assertions hold:*

(i) If (C.1) *and* (C.7) *are valid and* $\gamma_{\mathscr{P}}^+$ *is integrable, then the difference* Y *of any two solutions* X *and* $\tilde{X}$ *to* (1.2) *satisfies*

$$\sup_{t\in[t_0,\infty)} e^{\int_{t_0}^t \gamma_{\mathscr{P}}^-(s)\,ds} E[|Y_t|] < \infty,$$

provided $E[|Y_{t_0}|] < \infty$ *and* $\Theta(\cdot, P_X, P_{\tilde{X}})$ *is locally integrable. If in addition* $\gamma_{\mathscr{P}}^-$ *fails to be integrable, then* $\lim_{t\uparrow\infty} \exp(\alpha \int_{t_0}^t \gamma_{\mathscr{P}}^-(s)\,ds) E[|Y_t|] = 0$ *for any* $\alpha \in [0,1)$.

(ii) Let (C.8) *be valid. Then* (1.2) *is* α_l*-exponentially stable in moment relative to* Θ *with any moment* α_l*-Lyapunov exponent in* $(\hat{\lambda}_l, 0)$*. Further,* $\hat{\lambda}_l$ *serves as Lyapunov exponent as soon as*

$$\max_{k\in\{1,\dots,l\}} \hat{\lambda}_k \leq 0 \quad \text{and} \quad s_l \leq t_0.$$

Let us conclude with a specification of Example 3.11, which for $m = 1$ plays a major role in the volatility modelling in [9] and includes *one-dimensional square-root diffusions.* For this purpose, let $U = u$ for some orthonormal matrix $u \in \mathbb{R}^{m\times m}$.

Example 3.17. Let $\kappa \in \mathscr{S}(\mathbb{R}^m)$, $\eta \in \mathscr{L}^1_{loc}(\mathbb{R}^m)$ and $l, N \in \mathbb{N}$ as well as $\hat{m} \in \mathbb{N}^l$. Suppose that for any $k \in \{1,\dots,l\}$ there are

$$_{k,1}\eta, \dots, {}_{k,N}\eta \in \mathscr{S}(\mathbb{R}^m_+),$$

$\mathbb{R}^m$-valued measurable mappings $f_{k,1},\dots,f_{k,N}$ on $\mathbb{R}$, some $_k\lambda \in \mathscr{S}(\mathbb{R}^{m\times\hat{m}_k})$ and a Borel measurable map $g_k : \mathscr{P} \to \mathbb{R}^{\hat{m}_k}$ such that

$$\mathrm{B}(x,\mu) = \kappa + u\mathrm{diag}(\eta)u'x + \sum_{k=1}^l {}_k\lambda g_k(\mu) + u\sum_{n=1}^N \mathrm{diag}({}_{k,n}\eta)\begin{pmatrix} f_{k,n,1}(u_1'x) \\ \vdots \\ f_{k,n,m}(u_m'x)\end{pmatrix} \tag{3.13}$$

for any $(x,\mu) \in \mathbb{R}^m \times \mathscr{P}$. For instance, in the one-dimensional setting $m = 1$ and the constant choice $u = 1$ this representation reduces to

$$\mathrm{B}(x,\mu) = \kappa + \eta x + \sum_{k=1}^l {}_k\lambda g_k(\mu) + \sum_{n=1}^N {}_{k,n}\eta f_{k,n}(x)$$

for all $(x,\mu) \in \mathbb{R}\times\mathscr{P}$. In the general case we set $|A|_u := \sum_{i=1}^m |u_i'A|$ for any $\tilde{m} \in \mathbb{N}$ and each $A \in \mathbb{R}^{m\times\tilde{m}}$. Then (C.6) follows for $\alpha, \beta \in (0,1]^l$ from the following two conditions:

(1) For each $k \in \{1,\dots,l\}$ there are $\lambda_{k,1},\dots,\lambda_{k,N}$ in $\mathbb{R}^m$ such that

$$\mathrm{sgn}(v-\tilde{v})\big(f_{k,n,i}(v) - f_{k,n,i}(\tilde{v})\big) \leq \lambda_{k,n,i}|v-\tilde{v}|^{\alpha_k}$$

for all $v, \tilde{v} \in \mathbb{R}$ and any $(n,i) \in \{1,\dots,N\}\times\{1,\dots,m\}$. Further, $_k\tilde{\eta} \in \mathscr{S}(\mathbb{R}^m)$ given by $_k\tilde{\eta}^{(i)} := \sum_{n=1}^N {}_{k,n}\eta^{(i)}\lambda_{k,n,i}$ belongs to $\mathscr{S}^{q_{\alpha_k}}(\mathbb{R}^m)$.

(2) $E[|_k\lambda|_u]$ is locally integrable and $|g_k(\mu) - g_k(\tilde{\mu})| \leq \vartheta(\mu,\tilde{\mu})^{\beta_k}$ for each $k \in \{1,\dots,l\}$ and any $\mu, \tilde{\mu} \in \mathscr{P}$.

If in addition (C.2) is met by Σ, then Proposition 3.13 and Corollaries 3.15 and 3.16 yield the following statements:

(3) The difference Y of any two solutions X and $\tilde{X}$ to (1.2) for which $E[|Y_{t_0}|] < \infty$ and $\sum_{k=1}^{l} E[|_k\lambda|_u]\vartheta(P_X, P_{\tilde{X}})^{\beta_k}$ is locally integrable satisfies the estimate (3.10) with

$$\gamma_{\mathscr{P}} = \max_{j\in\{1,\dots,m\}} \eta_j + \sum_{k=1}^{l} \alpha_k \left[{}_k\tilde{\eta}^{(j)}\right]_{q_{\alpha_k}} + \beta_k c_{\mathscr{P}}^{\beta_k} E[|_k\lambda|_u],$$

$\delta_{\mathscr{P}} = \sum_{k=1}^{l}(1-\alpha_k)(\sum_{j=1}^{m}[{}_k\tilde{\eta}^{(j)}]_{q_{\alpha_k}}) + (1-\beta_k)c_{\mathscr{P}}^{\beta_k}E[|_k\lambda|_u]$ and $\varepsilon = 0$. Thereby, we recall for any $k \in \{1,\dots,l\}$ that $[\cdot]_{q_{\alpha_k}}$ may take negative values only if $\alpha_k = 1$.

(4) Define $\underline{\eta} \in \mathscr{L}^1_{loc}(\mathbb{R}^m)$ and $\lambda \in \mathscr{S}^1_{loc}(\mathbb{R}_+)$ by

$$\underline{\eta}_j := \eta_j + \sum_{k=1,\,\alpha_k=1}^{l} [{}_k\tilde{\eta}^{(j)}]_\infty \quad \text{and} \quad \lambda := \sum_{k=1}^{l} |_k\lambda|_u. \tag{3.14}$$

If $\underline{\eta}_j^+$, $\sum_{k=1,\,\alpha_k<1}^{l}[{}_k\tilde{\eta}^{(j)}]_{q_{\alpha_k}}$ and $E[\lambda]$ are integrable for all $j \in \{1,\dots,m\}$, then $E[|Y|]$ is bounded. In this case, $\lim_{t\uparrow\infty} E[|Y_t|] = 0$ follows from

$$\int_{t_0}^{\infty} \min_{j\in\{1,\dots,m\}} \underline{\eta}_j^-(s)\,ds = \infty. \tag{3.15}$$

(5) Assume that each $k \in \{1,\dots,l\}$ satisfies ${}_k\tilde{\eta}^{(j)} \le 0$ for all $j \in \{1,\dots,m\}$, if $\alpha_k < 1$, and also $\beta_k = 1$. This ensures the validity of (C.7) and we define an $\mathbb{R}_+$-valued Borel measurable functional Θ on $[t_0,\infty) \times \mathscr{P}^2$ by

$$\Theta(s,\mu,\tilde{\mu}) := E[\lambda_s]\vartheta(\mu,\tilde{\mu}). \tag{3.16}$$

   (i) If $\underline{\eta}_1^+,\dots,\underline{\eta}_m^+$ are integrable (and (3.15) holds), then (1.2) is (asymptotically) stable in moment with respect to Θ.

   (ii) If there are $\hat{\lambda} < 0$ and $\hat{\alpha} > 0$ such that

$$\max_{j\in\{1,\dots,m\}} \underline{\eta}_j(s) + c_{\mathscr{P}}E[\lambda_s] \le \hat{\lambda}\hat{\alpha}(s-t_0)^{\hat{\alpha}-1} \quad \text{for a.e. } s \in [t_0,\infty), \tag{3.17}$$

   then we have $\hat{\alpha}$-exponential stability in moment relative to Θ with Lyapunov exponent $\hat{\lambda}$.

3.3 Pathwise stability and moment growth estimates

In this section, our first aim is to derive pathwise exponential stability for (1.2). For this purpose, we replace the Osgood condition (C.2) on compact sets by the following stronger 1/2-*Hölder continuity condition* for the diffusion:

(C.9) There is $\hat{\eta} \in \mathscr{L}^2_{loc}(\mathbb{R}^m_+)$ such that $|_iU'(\Sigma(x) - \Sigma(\tilde{x}))| \le \hat{\eta}_i|_iU'(x-\tilde{x})|^{1/2}$ for any $x,\tilde{x} \in \mathbb{R}^m$ a.e. on $[t_0,\infty)$ a.s. for all $i \in \{1,\dots,m\}$.

Moreover, we require that the partial Lipschitz condition (C.7) for the drift B holds, and the regularity coefficients λ and $\hat{\eta}$ satisfy a suitable growth condition, which follows if $E[\lambda^{(0)}]$ and $\hat{\eta}_0$ are locally bounded, for instance.

(C.10) Conditions (C.7) and (C.9) are satisfied and there is $\hat{\delta} > 0$ such that

$$\sup_{t\in[t_0\infty)} \left(\int_t^{t+\hat{\delta}} E\big[\lambda_s^{(0)}\big]\,ds\right) \vee \left(\int_t^{t+\hat{\delta}} \hat{\eta}_0(s)^2\,ds\right) < \infty.$$

Under the following abstract condition on the stability coefficient $\gamma_{\mathscr{P}}$, we obtain a *general pathwise stability estimate* as special case of Theorem 4.13, a pathwise result for random Itô processes based on the Borel-Cantelli Lemma.

(C.11) Condition (C.10) holds and there are $\hat{\varepsilon} \in (0,1)$ and a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $[t_0,\infty)$ such that $\gamma_{\mathscr{P}} \leq 0$ a.e. on $[t_1,\infty)$,

$$\sup_{n\in\mathbb{N}}(t_{n+1}-t_n) < \hat{\delta}, \quad \lim_{n\uparrow\infty} t_n = \infty$$

and $\sum_{n=1}^{\infty} \exp((\varepsilon/2)\int_{t_1}^{t_n}\gamma_{\mathscr{P}}(s)\,ds) < \infty$ for all $\varepsilon \in (0,\hat{\varepsilon})$.

Proposition 3.18. *Assume* (C.1) *and* (C.11) *and let* X, $\tilde{X}$ *be solutions to* (1.2) *so that* $E[\lambda^{(0)}]\vartheta(P_X,P_{\tilde{X}})$ *is locally integrable. Then* $Y := X - \tilde{X}$ *satisfies*

$$\limsup_{t\uparrow\infty}\frac{1}{\varphi(t)}\log(|Y_t|_{U_t}) \leq \frac{1}{2}\limsup_{n\uparrow\infty}\frac{1}{\varphi(t_n)}\int_{t_1}^{t_n}\gamma_{\mathscr{P}}(s)\,ds \quad a.s.$$

for every increasing function $\varphi : [t_1,\infty) \to \mathbb{R}_+$ *that is positive on* (t_1,∞)*, provided* $E[|Y_{t_0}|] < \infty$ *or* $\lambda = 0$.

Now we employ the upper bound (C.8) on $\gamma_{\mathscr{P}}$ to derive pathwise exponential stability, since (C.11) already follows from (C.10) in the case, as will be shown.

Corollary 3.19. *Let* (C.1), (C.10), (C.8) *hold and define* $\Theta : [t_0,\infty)\times\mathscr{P}^2 \to \mathbb{R}$ *by* (3.12).

(i) Then (1.2) *is pathwise* α_l*-exponentially stable with Lyapunov exponent* $\hat{\lambda}_l/2$ *relative to an initial moment and* Θ.

(ii) If in fact B *is independent of* $\mu \in \mathscr{P}$*, then the SDE* (1.2) *is pathwise* α_l*-exponentially stable with Lyapunov exponent* $\hat{\lambda}_l/2$*, unrestrictedly.*

Under the hypothesis that there is a an orthonormal matrix $u \in \mathbb{R}^{m\times m}$ such that $U = u$, we may directly apply the preceding corollary to the class of drift coefficients in Example 3.17.

Example 3.20. Suppose that B is of the form (3.13) and let (C.9) hold for Σ when $\hat{\eta}$ is locally bounded. Then conditions (1) and (2) in Example 3.17 together with the following condition imply (C.10):

(3) $E[|_k\lambda|_u]$ is locally bounded, $|g_k(\mu) - g_k(\tilde{\mu})| \leq \vartheta(\mu,\tilde{\mu})$ for each $k \in \{1,\ldots,l\}$ and all $\mu,\tilde{\mu} \in \mathscr{P}$ and ${}_k\tilde{\eta}^{(j)} \leq 0$ for all $(k,j) \in \{1,\ldots,l\}\times\{1,\ldots,m\}$ with $\alpha_k < 1$.

For example, if $f_{k,n,j}$ is decreasing for fixed $(k,j) \in \{1,\ldots,l\}\times\{1,\ldots,m\}$ and every $n \in \{1,\ldots,N\}$, then ${}_k\tilde{\eta}^{(j)} \leq 0$. Further, Corollary 3.19 entails two assertions if $\underline{\eta}$ and λ given by (3.14) satisfy the bound (3.17) for some $\hat{\lambda} < 0$ and $\hat{\alpha} > 0$:

(4) Equation (1.2) is pathwise $\hat{\alpha}$-exponentially stable with Lyapunov exponent $\hat{\lambda}/2$ with respect to an initial moment and $\Theta : [t_0,\infty)\times\mathscr{P}^2 \to \mathbb{R}_+$ defined by (3.16).

(5) If ${}_1\lambda = \cdots = {}_l\lambda = 0$, in which case B is independent of $\mu \in \mathscr{P}$, then the SDE (1.2) is pathwise $\hat{\alpha}$-exponentially stable with Lyapunov exponent $\hat{\lambda}/2$.

Next, we give two first moment bounds for solutions to (1.2), each showing that their moment functions are locally bounded. Hence, local integrability relative to the functional Θ in each of the Corollaries 3.9, 3.15, 3.16 and 3.19 holds automatically in these cases.

First, we require an *Osgood growth condition on compact sets* for the random coordinate processes of Σ, by using the cone R_c, introduced before Example 2.10.

(C.12) For any $n \in \mathbb{N}$ there are ${}_n\hat{v} \in \mathscr{S}(\mathbb{R}_+)$ and an increasing function $\hat{\phi} \in \mathrm{R}_c$ such that $\int_{t_0}^t {}_n\hat{v}_s^2\,ds < \infty$ for all $t \in [t_0,\infty)$, $\int_0^1 \hat{\phi}_n(v)^{-2}\,dv = \infty$ and

$$|{}_iU'\Sigma(x)| \leq {}_n\hat{v}\hat{\phi}(|{}_iU'x|)$$

for any $x \in \mathbb{R}^m$ with $|x| \leq n$ a.e. on $[t_0,\infty)$ a.s. for every $i \in \{1,\dots,m\}$.

Remark 3.21. If Σ satisfies the Osgood condition (C.2) on compact sets and $\Sigma(0) = 0$ a.e. on $[t_0,\infty)$ a.s., then (C.12) follows. In particular, this implication readily applies to Example 3.4 when ${}_0\hat{\eta} = 0$ there.

Let us also consider two *partial growth conditions* on B. While the first allows for various kinds of growth behaviour, the second is of affine type and explicitly measures the growth components:

(C.13) There are $\alpha \in (0,1]$, $\kappa, \chi \in \mathscr{S}_{loc}^1(\mathbb{R}_+^m)$, $v \in \mathscr{S}_{loc}^{q_\alpha}(\mathbb{R}_+^m)$ and $\phi, \varphi \in \mathrm{R}_c$ such that $\phi^{1/\alpha}$ is concave, φ is increasing and

$$\mathrm{sgn}({}_iU'x){}_iU'\mathrm{B}(x,\mu) \leq \kappa^{(i)} + v^{(i)}\phi(|x|_U) + \chi^{(i)}\varphi(\vartheta(\mu,\delta_0))$$

for any $(x,\mu) \in \mathbb{R}^m \times \mathscr{P}$ a.e. on $[t_0,\infty)$ a.s. for each $i \in \{1,\dots,m\}$.

(C.14) There are $l \in \mathbb{N}$, $\alpha, \beta \in (0,1]^l$, $\kappa \in \mathscr{S}_{loc}^1(\mathbb{R}_+^m)$ and for each $k \in \{1,\dots,l\}$ there are

$${}_kv \in \mathscr{S}_{loc}^{q_{\alpha_k}}(\mathbb{R}^{m\times m}) \quad \text{and} \quad {}_k\chi \in \mathscr{S}_{loc}^1(\mathbb{R}_+^m)$$

with ${}_kv^{(i,j)} \geq 0$ for any $i,j \in \{1,\dots,m\}$ with $i \neq j$ such that

$$\mathrm{sgn}({}_iU'x){}_iU'\mathrm{B}(x,\mu) \leq \kappa^{(i)} + \sum_{k=1}^l \bigg(\sum_{j=1}^m {}_kv^{(i,j)}|{}_jU'x|^{\alpha_k}\bigg) + {}_k\chi^{(i)}\vartheta(\mu,\delta_0)^{\beta_k}$$

for all $(x,\mu) \in \mathbb{R}^m \times \mathscr{P}$ a.e. on $[t_0,\infty)$ a.s. for any $i \in \{1,\dots,m\}$.

If (C.1) and (C.13) hold, for given $\beta \in (0,1]$ we introduce $f,g \in \mathscr{L}_{loc}^1(\mathbb{R}_+)$ by

$$f := c_0\zeta_0 + \alpha\big[v^{(0)}\big]_{q_\alpha} + \beta E\big[\chi^{(0)}\big] \quad \text{and} \quad g := (1-\alpha)\big[v^{(0)}\big]_{q_\alpha} + (1-\beta)E\big[\chi^{(0)}\big]$$

and infer a *quantitative first moment estimate* from Theorem 4.5, which reduces to an explicit bound in the framework of Example 3.7.

Lemma 3.22. *Let* (C.1), (C.12) *and* (C.13) *be valid and* X *be a solution to* (1.2) *for which* $E[|X_{t_0}|] < \infty$ *and* $E[\chi^0]\vartheta(P_X,\delta_0)$ *is locally integrable. Define* $\phi_0, \varphi_0 \in C(\mathbb{R}_+)$ *by*

$$\phi_0(v) := (c_0v) \vee \phi(v)^{\frac{1}{\alpha}} \quad \textit{and} \quad \varphi_0(v) := \phi_0(v) \vee \varphi(c_{\mathscr{P}}v)^{\frac{1}{\beta}}$$

and suppose that $\Phi_{\varphi_0}(\infty) = \infty$. *Then* $E[|X|]$ *is locally bounded and*

$$\sup_{s\in[t_0,t]} E\big[|X_s|_{U_s}\big] \leq \Psi_{\varphi_0}\bigg(E\big[|X_{t_0}|_{U_{t_0}}\big] + \int_{t_0}^t E\big[\kappa_s^{(0)}\big] + g(s)\,ds, \int_{t_0}^t f(s)\,ds\bigg)$$

for all $t \in [t_0,\infty)$. *In particular, if* $E[\kappa^{(0)}]$, f *and* g *are integrable, then* $E[|X|]$ *is bounded.*

Given (C.14) holds, we define $f_{\mathscr{P}} \in \mathscr{L}_{loc}^1(\mathbb{R})$ and $g_{\mathscr{P}} \in \mathscr{L}_{loc}^1(\mathbb{R}_+)$ by

$$f_{\mathscr{P}} := c_0\zeta_0 + \max_{j\in\{1,\dots,m\}} \sum_{k=1}^l \alpha_k\bigg[\sum_{i=1}^m {}_kv^{(i,j)}\bigg]_{q_{\alpha_k}} + \beta_k c_{\mathscr{P}}^{\beta_k}E\big[{}_k\chi^{(0)}\big] \tag{3.18}$$

and

$$g_{\mathscr{P}} := \sum_{k=1}^{l}(1-\alpha_k)\bigg(\sum_{j=1}^{m}\Big[\sum_{i=1}^{m}{}_k\upsilon^{(i,j)}\Big]_{q_{\alpha_k}}\bigg) + (1-\beta_k)c_{\mathscr{P}}^{\beta_k}E\big[{}_k\chi^{(0)}\big]. \tag{3.19}$$

These formulas are in spirit the same as those for the stability coefficients in (3.7) and (3.8), since the following *explicit* L^1*-growth bound* follows, just as the L^1-comparison estimate in Proposition 3.13, from Theorem 4.6.

Lemma 3.23. *Let* (C.1), (C.12) *and* (C.14) *be satisfied and* X *be a solution to* (1.2) *such that* $E[|X_{t_0}|] < \infty$ *and* $\sum_{k=1}^{l} E[{}_k\chi^{(0)}]\vartheta(P_X,\delta_0)^{\beta_k}$ *is locally integrable. Then*

$$E[|X_t|_{U_t}] \leq e^{\int_{t_0}^{t} f_{\mathscr{P}}(s)\,ds}E[|X_{t_0}|_{U_{t_0}}] + \int_{t_0}^{t} e^{\int_{s}^{t} f_{\mathscr{P}}(\tilde{s})\,ds}\big(E[\kappa_s^{(0)}] + g_{\mathscr{P}}(s)\big)\,ds \tag{3.20}$$

for each $t \in [t_0,\infty)$*. In particular, suppose that* $f_{\mathscr{P}}^{+}$*,* $E[\kappa^{(0)}]$ *and* $g_{\mathscr{P}}$ *are integrable. Then* $E[|X|]$ *is bounded, and* $\lim_{t\uparrow\infty} E[|X_t|] = 0$ *as soon as* $\int_{t_0}^{\infty} f_{\mathscr{P}}^{-}(s)\,ds = \infty$.

To improve our intuition we consider the previous class of drift maps and suppose that there is an orthonormal matrix $u \in \mathbb{R}^{m\times m}$ satisfying $U = u$.

Example 3.24. Let B satisfy the representation (3.13) of Example 3.17. Then (C.14) is implied by the following two conditions

(1) For each $k \in \{1,\dots,l\}$ there exist $c_{k,1},\dots,c_{k,N}$ in $\mathbb{R}^m$ such that $\mathrm{sgn}(v)f_{k,n,i}(v) \leq c_{k,n,i}|v|^{\alpha_k}$ for every $v \in \mathbb{R}$ and all $(n,i) \in \{1,\dots,N\}\times\{1,\dots,m\}$. Further, ${}_k\upsilon \in \mathscr{S}(\mathbb{R}^m)$ given by ${}_k\upsilon^{(i)} := \sum_{n=1}^{N} {}_{k,n}\eta^{(i)}c_{k,n,i}$ lies in $\mathscr{S}^{q_{\alpha_k}}(\mathbb{R}^m)$.

(2) $E[|\kappa|_u]$ and $E[|{}_k\lambda|_u]$ are locally integrable and $|g_k(\mu)| \leq c_k + \hat{c}_k\vartheta(\mu,\delta_0)^{\beta_k}$ for all $k \in \{1,\dots,l\}$, any $\mu \in \mathscr{P}$ and some $c,\hat{c} \in \mathbb{R}_+^l$.

In this case, we define $\hat{\kappa} \in \mathscr{S}_{loc}^1(\mathbb{R}_+^m)$ by $\hat{\kappa}^{(i)} := |u_i'\kappa| + \sum_{k=1}^{l}|u_i'{}_k\lambda|c_k$ and suppose that (C.12) holds for Σ. Then two facts follow from Lemma 3.23:

(3) For any solution X to (1.2) for which $E[|X_{t_0}|] < \infty$ and $\sum_{k=1}^{l}\hat{c}_kE[|{}_k\lambda|_u]\vartheta(P_X,\delta_0)^{\beta_k}$ is locally integrable the bound (3.20) holds with

$$f_{\mathscr{P}} = \max_{j\in\{1,\dots,m\}}\eta_j + \sum_{k=1}^{l}\alpha_k\big[{}_k\upsilon^{(j)}\big]_{q_{\alpha_k}} + \beta_k\hat{c}_kc_{\mathscr{P}}^{\beta_k}E\big[|{}_k\lambda|_u\big],$$

$\delta_{\mathscr{P}} = \sum_{k=1}^{l}(1-\alpha_k)(\sum_{j=1}^{m}[{}_k\upsilon^{(j)}]_{q_{\alpha_k}}) + (1-\beta_k)\hat{c}_kc_{\mathscr{P}}^{\beta_k}E[|{}_k\lambda|_u]$ and κ replaced by $\hat{\kappa}$.

(4) Let $\underline{\upsilon} \in \mathscr{L}_{loc}^1(\mathbb{R}^m)$ be given by

$$\underline{\upsilon}_j := \eta_j + \sum_{k=1,\,\alpha_k=1}^{l}\big[{}_k\upsilon^{(j)}\big]_{\infty}.$$

Then $E[|X|]$ is bounded as soon as $\underline{\upsilon}_j^{+}$, $\sum_{k=1,\,\alpha_k<1}^{l}[{}_k\upsilon^{(j)}]_{q_{\alpha_k}}$ and $\sum_{k=1}^{l}\hat{c}_kE[|{}_k\lambda|_u]$ are integrable for each $j \in \{1,\dots,m\}$. If in addition

$$\int_{t_0}^{\infty}\min_{j\in\{1,\dots,m\}}\underline{\upsilon}_j^{-}(s)\,ds = \infty,\ \text{then}\ \lim_{t\uparrow\infty}E[|X_t|] = 0.$$

3.4 Strong solutions with locally bounded moment functions

In this section, let b and σ be two Borel measurable maps on $[t_0,\infty)\times\mathbb{R}^m\times\mathscr{P}$ and $[t_0,\infty)\times\mathbb{R}^m$ with values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively, and $\xi:\Omega\to\mathbb{R}^m$ be $\mathscr{F}_{t_0}$-measurable. We derive a strong solution X to (2.7) such that $X_{t_0}=\xi$ a.s. and the measurable moment function

$$[t_0,\infty)\to[0,\infty],\quad t\mapsto E[|X_t|]$$

is finite and locally bounded but not necessarily continuous, by combining the preceding results with a fixed-point approach. Thereby, neither $b(s,\cdot,\mu)$ nor $\sigma(s,\cdot)$ need to be locally Lipschitz continuous for any $(s,\mu)\in[t_0,\infty)\times\mathscr{P}$.

For a Borel measurable map $\mu:[t_0,\infty)\to\mathscr{P}$ we define an $\mathbb{R}^m$-valued measurable map b_μ on $[t_0,\infty)\times\mathbb{R}^m$ by $b_\mu(s,x):=b(s,x,\mu(s))$ and show that the induced SDE

$$dX_t=b_\mu(t,X_t)\,dt+\sigma(t,X_t)\,dW_t\quad\text{for } t\in[t_0,\infty)\tag{3.21}$$

admits a solution. To this end, σ should vanish at the origin of $\mathbb{R}^m$ at almost any time and satisfy an Osgood continuity condition on compact sets in terms of an $\mathbb{R}^{m\times m}$-valued orthonormal locally absolutely continuous map u on $[t_0,\infty)$:

(D.1) $\sigma(\cdot,0)=0$ a.e. and for any $n\in\mathbb{N}$ there are $\hat{\eta}_n\in\mathscr{L}^2_{loc}(\mathbb{R}_+)$ and an increasing function $\hat{\rho}_n\in\mathrm{R}_c$ such that $\int_0^1\hat{\rho}_n(v)^{-2}\,dv=\infty$ and

$$|u_i'(\sigma(\cdot,x)-\sigma(\cdot,\tilde{x}))|\le\hat{\eta}_n\hat{\rho}_n(|u_i'(x-\tilde{x})|)$$

for all $x,\tilde{x}\in\mathbb{R}^m$ with $|x|\vee|\tilde{x}|\le n$ a.e. on $[t_0,\infty)$ for all $i\in\{1,\dots,m\}$.

Remark 3.25. This condition still allows for Example 3.4 when $\Sigma=\sigma$, $\varphi(0)=0$, ${}_0\hat{\eta}=0$ and ${}_k\hat{\eta}$ is deterministic and lies in $\mathscr{L}^2_{loc}(\mathbb{R}^d)$ for all $k\in\{1,\dots,l\}$.

We set $c_0:=0$, if u is independent of time, and $c_0:=1$, otherwise, and note that $\zeta\in\mathscr{L}^1_{loc}(\mathbb{R}^m_+)$ defined by $\zeta_j:=\max_{j\in\{1,\dots,m\}}|u_j'\dot{u}_i|$ satisfies (C.1) for $U=u$. Given c_0, we introduce a partial growth condition on b, a continuity and boundedness condition on (b,σ) and a partial Osgood condition on compact sets on b_μ:

(D.2) There are $\kappa,\upsilon,\chi\in\mathscr{L}^1_{loc}(\mathbb{R}^m_+)$ and $\phi,\varphi\in\mathrm{R}_c$ so that ϕ is concave, φ is increasing, $\int_1^\infty((c_0v)\vee\phi(v))^{-1}\,dv=\infty$ and

$$\mathrm{sgn}(u_i'x_i)u_i'b(\cdot,x,\mu)\le\kappa_i+\upsilon_i\phi(|x|_u)+\chi_i\varphi(\vartheta(\mu,\delta_0))$$

for all $(x,\mu)\in\mathbb{R}^m\times\mathscr{P}$ a.e. on $[t_0,\infty)$ for every $i\in\{1,\dots,m\}$.

(D.3) $b(s,\cdot,\mu)$ and $\sigma(s,\cdot)$ are continuous for all $\mu\in\mathscr{P}$ for a.e. $s\in[t_0,\infty)$ and for any $n\in\mathbb{N}$ there is $c_n>0$ such that

$$|b(s,x,\mu)|\vee|\sigma(s,x)|\le c_n$$

for all $s\in[t_0,t_0+n]$, each $x\in\mathbb{R}^m$ with $|x|\le n$ and any $\mu\in\mathscr{P}$ with $\vartheta(\mu,\delta_0)\le n$.

(D.4) For each $n\in\mathbb{N}$ there are $\eta_n\in\mathscr{L}^1_{loc}(\mathbb{R}_+)$ and a concave function $\rho_n\in\mathrm{R}_c$ satisfying $\int_0^1((c_0v)\vee\rho_n(v))^{-1}\,dv=\infty$ and

$$\mathrm{sgn}(u_i'(x-\tilde{x}))u_i'(b(\cdot,x,\mu)-b(\cdot,\tilde{x},\mu))\le\eta_n\rho_n(|x-\tilde{x}|_u)$$

for all $x,\tilde{x}\in\mathbb{R}^m$ with $|x|\vee|\tilde{x}|\le n$ and any $\mu\in\mathscr{P}$ a.e. on $[t_0,\infty)$ for every $i\in\{1,\dots,m\}$.

Let $B_{b,loc}(\mathscr{P})$ denote the set of all Borel measurable maps $\mu : [t_0, \infty) \to \mathscr{P}$ for which the function $[t_0, \infty) \to \mathbb{R}_+$, $t \mapsto \vartheta(\mu(t), \delta_0)$ is locally bounded. By means of a local weak existence result from [25] and Corollary 3.9, we concisely settle questions of uniqueness and existence of solutions to (3.21).

Proposition 3.26. *For $\mu \in B_{b,loc}(\mathscr{P})$ the following three assertions hold:*

(i) *Let* (D.1) *and* (D.4) *be valid. Then pathwise uniqueness for the SDE* (3.21) *follows.*

(ii) *Let* (D.1)-(D.3) *hold. Then* (3.21) *admits a weak solution X with $\mathscr{L}(X_{t_0}) = \mathscr{L}(\xi)$. Further, if ξ is integrable, then the moment function of X is locally bounded.*

(iii) *If* (D.1)-(D.4) *are satisfied, then there is a unique strong solution $X^{\xi,\mu}$ to* (3.21) *such that $X_{t_0}^{\xi,\mu} = \xi$ a.s.*

In the sequel, let $B_{b,loc}(\mathscr{P}_1(\mathbb{R}^m))$ be the convex space of all Borel measurable maps $\mu : [t_0, \infty) \to \mathscr{P}_1(\mathbb{R}^m)$ for which $\vartheta_1(\mu, \delta_0)$ is locally bounded, equipped with the topology of local uniform convergence.

That means, a sequence $(\mu_n)_{n \in \mathbb{N}}$ in this space converges locally uniformly to some $\mu \in B_{b,loc}(\mathscr{P}_1(\mathbb{R}^m))$ if $\lim_{n \uparrow \infty} \sup_{s \in [t_0,t]} \vartheta_1(\mu_n, \mu)(s) = 0$ for every $t \in [t_0, \infty)$. Then, as the Wasserstein metric ν_1 is complete, $B_{b,loc}(\mathscr{P}_1(\mathbb{R}^m))$ is completely metrisable.

To construct a solution to (2.7) from a local uniform limit of a Picard iteration in $B_{b,loc}(\mathscr{P}_1(\mathbb{R}^m))$, we replace the partial Osgood condition (D.4) on compact sets on b_μ for given $\mu \in B_{b,loc}(\mathscr{P})$ by a partial Lipschitz condition on b:

(D.5) There are $\eta \in \mathscr{L}^1_{loc}(\mathbb{R}^{m \times m})$ and $\lambda \in \mathscr{L}^1_{loc}(\mathbb{R}^m_+)$ with $\eta_{i,j} \geq 0$ for all $i, j \in \{1, \ldots, m\}$ with $i \neq j$ such that

$$\operatorname{sgn}(u_i'(x - \tilde{x}))u_i'(b(\cdot, x, \mu) - b(\cdot, \tilde{x}, \tilde{\mu})) \leq \sum_{j=1}^m \eta_{i,j}|u_j'(x - \tilde{x})| + \lambda_i \vartheta(\mu, \tilde{\mu})$$

for any $x, \tilde{x} \in \mathbb{R}^m$ and every $\mu, \tilde{\mu} \in \mathscr{P}$ a.e. on $[t_0, \infty)$ for all $i \in \{1, \ldots, m\}$.

If (D.5) is satisfied, which implies (C.7) for $\mathrm{B} = b$, then we use the formula in (3.11) when $\lambda = 0$ for the definition of $\gamma_{\mathscr{P},0} \in \mathscr{L}^1_{loc}(\mathbb{R})$. Namely, we set

$$\gamma_{\mathscr{P},0} := c_0\zeta_0 + \max_{j \in \{1,\ldots,m\}} \sum_{i=1}^m \eta_{i,j}.$$

Further, if the following partial affine growth condition for b holds, which is stronger than (D.2), then we can deduce an estimate in $B_{b,loc}(\mathscr{P}_1(\mathbb{R}^m))$ on the Picard iteration.

(D.6) There are $l \in \mathbb{N}$, $\alpha, \beta \in (0, 1]^l$, $\kappa \in \mathscr{L}^1_{loc}(\mathbb{R}^m_+)$ and for every $k \in \{1, \ldots, l\}$ there are

$${}_k\upsilon \in \mathscr{L}^1_{loc}(\mathbb{R}^{m \times m}) \quad \text{and} \quad {}_k\chi \in \mathscr{L}^1_{loc}(\mathbb{R}^m_+)$$

with ${}_k\upsilon_{i,j} \geq 0$ for all $i, j \in \{1, \ldots, m\}$ with $i \neq j$ such that

$$\operatorname{sgn}(u_i'x)u_i'b(\cdot, x, \mu) \leq \kappa_i + \sum_{k=1}^l \left(\sum_{j=1}^m {}_k\upsilon_{i,j}|u_j'x|^{\alpha_k} \right) + {}_k\chi_i \vartheta(\mu, \delta_0)^{\beta_k}$$

for any $(x, \mu) \in \mathbb{R}^m \times \mathscr{P}$ a.e. on $[t_0, \infty)$ for all $i \in \{1, \ldots, m\}$.

Since (D.6) entails (C.14) for $\mathrm{B} = b$, we may use the formulas (3.18) and (3.19) for $f_{\mathscr{P}} \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R})$ and $g_{\mathscr{P}} \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}_+)$ when all appearing coefficients are deterministic. Based on these considerations, we get a *strong existence result with explicit error and growth estimates.*

Theorem 3.27. *Let* (D.1)-(D.3) *and* (D.5) *hold,* $\mathscr{P}_1(\mathbb{R}^m) \subset \mathscr{P}$, $\mu_0 \in B_{b,loc}(\mathscr{P})$ *and* $E[|\xi|] < \infty$. *Further, define* $\Theta : [t_0, \infty) \times \mathscr{P}^2 \to \mathbb{R}_+$ *by* $\Theta(s, \mu, \tilde{\mu}) := \lambda_0(s)\vartheta(\mu, \tilde{\mu})$.

(i) Pathwise uniqueness for (2.7) *relative to* Θ *holds and there exists a unique strong solution* X^ξ *to* (2.7) *such that* $X^\xi_{t_0} = \xi$ *a.s. and* $E[|X^\xi|]$ *is locally bounded.*

(ii) The map $[t_0, \infty) \to \mathscr{P}_1(\mathbb{R}^m)$, $t \mapsto \mathscr{L}(X^\xi_t)$ *is the local uniform limit of the sequence* $(\mu_n)_{n\in\mathbb{N}}$ *in* $B_{b,loc}(\mathscr{P}_1(\mathbb{R}^m))$ *recursively given by* $\mu_n := \mathscr{L}(X^{\xi,\mu_{n-1}})$ *and*

$$\sup_{s\in[t_0,t]} \vartheta_1(\mu_n(s), \mathscr{L}(X^\xi_s)) \leq \Delta(t) \sum_{i=n}^{\infty} \frac{c^i_{\mathscr{P}}}{i!} \bigg(\int_{t_0}^t e^{\int_s^t \gamma_{\mathscr{P},0}(\tilde{s})\,d\tilde{s}} \lambda_0(s)\,ds \bigg)^i \tag{3.22}$$

for all $t \in [t_0, \infty)$ *with* $\Delta(t) := \sup_{s\in[t_0,t]} c^{-1}_{\mathscr{P}} \vartheta(\mathscr{L}(X^{\xi,\mu_0}), \mu_0)(s)$.

(iii) If in fact (D.6) *holds, then* $(\mu_n)_{n\in\mathbb{N}}$ *is a sequence in the closed and convex set* M *of all* $\mu \in B_{b,loc}(\mathscr{P}_1(\mathbb{R}^m))$ *satisfying*

$$\vartheta_1(\mu(t), \delta_0) \leq e^{\int_{t_0}^t f_{\mathscr{P}}(s)\,ds} E\big[|\xi|_{u(t_0)}\big] + \int_{t_0}^t e^{\int_s^t f_{\mathscr{P}}(\tilde{s})\,d\tilde{s}} (\kappa_0 + g_{\mathscr{P}})(s)\,ds \tag{3.23}$$

for each $t \in [t_0, \infty)$ *as soon as* $\mu_0 \in M$.

Remark 3.28. For $\mu_0 = \delta_0$ we may use that $\Delta(t) \leq \sup_{s\in[t_0,t]} E\big[|X^{\xi,\delta_0}_s|\big]$ in (3.22) for any $t \in [t_0, \infty)$. If instead $\mu_0 = \mathscr{L}(X^\xi)$, then $\Delta = 0$ and $\mu_n = \mu_0$ for all $n \in \mathbb{N}$.

Let us conclude with a variation of Example 3.17 and suppose that u is independent in time, in which case we have set $c_0 = 0$.

Example 3.29. Let $g : [t_0, \infty) \times \mathbb{R}^m \times \mathscr{P} \to \mathbb{R}^m$ be Borel measurable and for $N \in \mathbb{N}$ let $\eta_1, \ldots, \eta_N \in \mathscr{L}^1_{loc}(\mathbb{R}^m_+)$ and $f_1, \ldots, f_N$ be $\mathbb{R}^m$-valued continuous maps on $\mathbb{R}$ such that

$$b(s, x, \mu) = g(s, x, \mu) + u \sum_{n=1}^N \operatorname{diag}(\eta_n(s)) \begin{pmatrix} f_{n,1}(u_1' x) \\ \vdots \\ f_{n,m}(u_m' x) \end{pmatrix}$$

for any $(s, x, \mu) \in [t_0, \infty) \times \mathbb{R}^m \times \mathscr{P}$. Then the following three assertions hold:

(1) Let $\lambda_1, \ldots, \lambda_N \in \mathbb{R}^m$ and $\hat{\eta}, \lambda \in \mathscr{L}^1_{loc}(\mathbb{R}^m_+)$ be such that $\operatorname{sgn}(v - \tilde{v})(f_{n,i}(v) - f_{n,i}(\tilde{v})) \leq \lambda_{n,i}|v - \tilde{v}|$ for all $v, \tilde{v} \in \mathbb{R}$ and every $n \in \{1, \ldots, N\}$ and

$$\text{and} \quad |u_i'(g(\cdot, x, \mu) - g(\cdot, \tilde{x}, \tilde{\mu}))| \leq \hat{\eta}_i |x - \tilde{x}|_u + \lambda_i \vartheta(\mu, \tilde{\mu})$$

for any $x, \tilde{x} \in \mathbb{R}^m$ and all $\mu, \tilde{\mu} \in \mathscr{P}$ a.e. for each $i \in \{1, \ldots, m\}$. Then (D.5) is valid.

(2) Suppose that $c_1, \ldots, c_N \in \mathbb{R}^m$ and $\kappa, \hat{v}, \chi \in \mathscr{L}^1_{loc}(\mathbb{R}^m_+)$ satisfy $\operatorname{sgn}(v) f_{n,i}(v) \leq c_{n,i}|v|$ for every $v \in \mathbb{R}$ and all $n \in \{1, \ldots, N\}$ and

$$|u_i' g(\cdot, x, \mu)| \leq \kappa_i + \hat{v}_i |x|_u + \chi_i \vartheta(\mu, \delta_0)$$

for any $(x, \mu) \in \mathbb{R}^m \times \mathscr{P}$ a.e. for all $i \in \{1, \ldots, m\}$. Then (D.6) follows.

(3) (D.3) is satisfied if the conditions in (1) and (2) hold such that $\eta_1, \ldots, \eta_N$ and $\kappa, \hat{v}, \chi$ as well as $\hat{\eta}, \lambda$ are locally bounded.

For example, if $f_{n,i}$ is decreasing, $f_{n,i} \geq 0$ on $(-\infty, 0)$ and $f_{n,i} \leq 0$ on $(0,\infty)$ for fixed $n \in \{1,\dots,N\}$ and all $i \in \{1,\dots,m\}$, then $c_n = \lambda_n = 0$ is feasible in (1) and (2). More specifically, we may take

$$f_n(x) = -(x_1^{l_{n,1}}, \dots, x_m^{l_{n,m}}) \quad \text{for all } x \in \mathbb{R}^m$$

and some $l_n \in \mathbb{N}^m$ with odd coordinates. In general, if σ satisfies (D.1) and $\mathscr{P}_1(\mathbb{R}^m) \subset \mathscr{P}$, then all assertions of Theorem 3.27 apply and the coefficients reduce to

$$\gamma_{\mathscr{P},0} = \hat{\eta}_0 + \max_{j \in \{1,\dots,m\}} \eta_j, \quad f_{\mathscr{P}} = \hat{\upsilon}_0 + \max_{j \in \{1,\dots,m\}} \upsilon_j + c_{\mathscr{P}}\chi_0 \quad \text{and} \quad g_{\mathscr{P}} = 0,$$

where $\eta, \upsilon \in \mathscr{L}^1_{loc}(\mathbb{R})$ are given by $\eta_i := \sum_{n=1}^N \eta_{n,i}\lambda_{n,i}$ and $\upsilon_i := \sum_{n=1}^N \eta_{n,i}c_{n,i}$.

4 Moment and pathwise asymptotic estimations for random Itô processes

4.1 Auxiliary moment bounds

In the sequel, let $\hat{\mathrm{B}} \in \mathscr{S}(\mathbb{R}^m)$ and $\hat{\Sigma} \in \mathscr{S}(\mathbb{R}^{m\times d})$ be such that $\int_{t_0}^t |\hat{\mathrm{B}}_s| + |\hat{\Sigma}_s|^2\,ds < \infty$ for each $t \in [t_0,\infty)$. For an $\mathbb{R}^m$-valued adapted continuous process Y satisfying

$$Y_t = Y_{t_0} + \int_{t_0}^t \hat{\mathrm{B}}_s\,ds + \int_{t_0}^t \hat{\Sigma}_s\,dW_s \quad \text{for all } t \in [t_0,\infty) \text{ a.s.},$$

which we call a *random Itô process* with drift $\hat{\mathrm{B}}$ and diffusion $\hat{\Sigma}$, we will derive quantitative L^1-estimates. In this regard, given $\hat{m} \in \mathbb{N}$ and an $\mathbb{R}^{m\times\hat{m}}$-valued adapted locally absolutely continuous process V, we allow for the seminorm $|\cdot|_V$, defined in (3.1).

First, we recall an approximation of the identity function on $\mathbb{R}_+$, used by Yamada and Watanabe [37] to prove pathwise uniqueness for SDEs. For any $i \in \{1,\dots,\hat{m}\}$ and each increasing function $\hat{\rho}_i \in \mathrm{R}_c$ with $\int_0^1 \hat{\rho}_i(v)^{-2}\,dv = \infty$, there are a strictly decreasing sequence $(a_{i,n})_{n\in\mathbb{N}_0}$ in $(0,1]$ converging to zero and an increasing sequence $(\psi_{i,n})_{n\in\mathbb{N}}$ of non-negative functions in $C^2(\mathbb{R}_+)$ such that

$$\psi_{i,n}' \in [0,1], \quad \psi_{i,n}' = \psi_{i,n}'\mathbb{1}_{(0,a_{i,n-1})} + \mathbb{1}_{[a_{i,n-1},\infty)} \quad \text{and} \quad 0 \leq \psi_{i,n}'' \leq \frac{2}{n}\hat{\rho}_i^{-2}\mathbb{1}_{(a_{i,n},a_{i,n-1})}$$

for any $n \in \mathbb{N}$ and hence, $\psi_{i,n}(0) = \psi_{i,n}'(0) = \psi_{i,n}''(0) = 0$. These conditions ensure that $\sup_{n\in\mathbb{N}} \psi_{i,n}(x) = x$ and $\lim_{n\uparrow\infty} \psi_{i,n}'(x) = 1$ for all $x > 0$, which we combine with an application of Itô's formula.

Lemma 4.1. *Let $\psi \in C^2(\mathbb{R}_+)$ satisfy $\psi'(0) = \psi''(0) = 0$ and U be an $\mathbb{R}^m$-valued adapted locally absolutely continuous process. Then*

$$\begin{aligned} u(t)\psi(|U_t'Y_t|) &= u(t_0)\psi(|U_{t_0}'Y_{t_0}|) + \int_{t_0}^t \psi(|U_s'Y_s|)\,du(s) \\ &+ \int_{t_0}^t u(s)\Big(\psi'(|U_s'Y_s|)\mathrm{sgn}(U_s'Y_s)(\dot{U}_s'Y_s + U_s'\hat{\mathrm{B}}_s) + \frac{1}{2}\psi''(|U_s'Y_s|)|U_s\hat{\Sigma}_s|^2\Big)\,ds \\ &+ \int_{t_0}^t u(s)\psi'(|U_s'Y_s|)\mathrm{sgn}(U_s'Y_s)U_s'\hat{\Sigma}_s\,dW_s \end{aligned}$$

for all $t \in [t_0,\infty)$ a.s. and any $u \in C([t_0,\infty))$ that is locally of bounded variation.

Proof. We define $\varphi \in C^{2,2}(\mathbb{R}^m \times \mathbb{R}^m)$ by $\varphi(x,y) := \psi(|x'y|)$ with first- and second-order derivatives with respect to the first coordinate

$$D_x\varphi(x,y) = \psi'(|x'y|)\mathrm{sgn}(x'y)y' \quad \text{and} \quad D_x^2\varphi(x,y) = \psi''(|x'y|)yy'$$

for any $x, y \in \mathbb{R}^m$. Its first- and second-order derivatives relative to the second coordinate satisfy $D_y\varphi(x,y) = D_x\varphi(y,x)$ and $D_y^2\varphi(x,y) = D_x^2\varphi(y,x)$, as $\varphi(x,y) = \varphi(y,x)$. Thus,

$$\begin{aligned}\varphi(U_t, Y_t) = \varphi(U_{t_0}, Y_{t_0}) &+ \int_{t_0}^t u(s)D_x\varphi(Y_s, U_s)\hat{\Sigma}_s\, dW_s \\ &+ \int_{t_0}^t D_x\varphi(U_s, Y_s)\dot{U}_s + D_x\varphi(Y_s, U_s)\hat{\mathrm{B}}_s + \frac{1}{2}\mathrm{tr}\big(D_x^2\varphi(Y_s, U_s)\hat{\Sigma}_s\hat{\Sigma}_s'\big)\, ds\end{aligned}$$

for any $t \in [t_0, \infty)$ a.s., by Itô's formula. As u is locally of bounded variation, the asserted identity follows from Itô's product rule. ☐

For any $i \in \{1, \dots, \hat{m}\}$ let ${}_iV$ denote the i-th column of the process V. Then the representation $V = ({}_1V, \dots, {}_{\hat{m}}V)$ holds and we can state an auxiliary moment estimate.

Proposition 4.2. *Let $E[|Y_{t_0}|_{V_{t_0}}] < \infty$ and τ be a stopping time. Suppose that there are $Z \in \mathscr{S}(\mathbb{R}^{\hat{m}})$ and $\hat{\eta} \in \mathscr{S}(\mathbb{R}_+)$ satisfying $\int_{t_0}^t |Z_s| + \hat{\eta}_s^2\, ds < \infty$ for all $t \in [t_0, \infty)$ and*

$$\mathrm{sgn}({}_iV_s'Y_s)\big({}_i\dot{V}_s'Y_s + {}_iV_s'\hat{\mathrm{B}}_s\big) \leq Z_s^{(i)} \quad \text{on } \{{}_iV_s'Y_s \neq 0\} \quad \text{and} \quad |{}_iV_s'\hat{\Sigma}_s| \leq \hat{\eta}_s\hat{\rho}_i(|{}_iV_s'Y_s|)$$

for a.e. $s \in [t_0, \infty)$ with $s < \tau$ a.s. for any $i \in \{1, \dots, \hat{m}\}$. If $u : [t_0, \infty) \to \mathbb{R}_+$ is locally absolutely continuous and $E[\int_{t_0}^{t\wedge\tau} |\dot{u}(s)|Y_s|_{V_s} + u(s)\sum_{i=1}^{\hat{m}} Z_s^{(i)}\mathbb{1}_{\{{}_iV_s'Y_s \neq 0\}}|\, ds] < \infty$, then

$$\begin{aligned}E\big[u(t\wedge\tau)|Y_t^\tau|_{V_t^\tau}\big] &\leq u(t_0)E\big[|Y_{t_0}|_{V_{t_0}}\big] \\ &+ E\bigg[\int_{t_0}^{t\wedge\tau} \dot{u}(s)|Y_s|_{V_s} + u(s)\sum_{i=1}^{\hat{m}} Z_s^{(i)}\mathbb{1}_{\{{}_iV_s'Y_s \neq 0\}}\, ds\bigg]\end{aligned} \tag{4.1}$$

for any $t \in [t_0, \infty)$.

Proof. For fixed $n \in \mathbb{N}$ we infer from the preceding lemma that the $\mathbb{R}_+$-valued adapted continuous process ${}_nX := \sum_{i=1}^{\hat{m}} \psi_{i,n}(|{}_iV'Y|)$ is a semimartingale satisfying

$$\begin{aligned}u(t\wedge\tau)_nX_t^\tau \leq u(t_0)_nX_{t_0} &+ \int_{t_0}^{t\wedge\tau} \dot{u}(s)_nX_s + u(s)\bigg(\sum_{i=1}^{\hat{m}} \psi_{i,n}'(|{}_iV_s'Y_s|)Z_s^{(i)} + \frac{\hat{m}}{n}\hat{\eta}_s^2\bigg)\, ds \\ &+ \int_{t_0}^{t\wedge\tau} u(s)\sum_{i=1}^{\hat{m}} \psi_{i,n}'(|{}_iV_s'Y_s|)\mathrm{sgn}({}_iV_s'Y_s){}_iV_s'\hat{\Sigma}_s\, dW_s\end{aligned}$$

for any $t \in [t_0, \infty)$ a.s. In this context, we readily notice that $E[{}_nX_{t_0}] \leq E[|Y_{t_0}|_{V_{t_0}}]$ and

$$\sum_{i=1}^{\hat{m}} |\psi_{i,n}'(|{}_iV_s'Y_s|)\mathrm{sgn}({}_iV_s'Y_s){}_iV_s'\hat{\Sigma}_s| \leq \hat{\eta}_s\sum_{i=1}^{\hat{m}} \hat{\rho}_i(|Y_s|_{V_s})$$

for a.e. $s \in [t_0, \infty)$ with $s < \tau$ a.s. Thus, for given $k \in \mathbb{N}$ we define a stopping time by $\tau_k := \inf\{t \in [t_0, \infty) \,|\, |Y_t|_{V_t} \geq k \text{ or } \int_{t_0}^t |Z_s| + \hat{\eta}_s^2\, ds \geq k\} \wedge \tau$ to get that

$$\begin{aligned}E\big[u(t\wedge\tau_k)_nX_t^{\tau_k}\big] &\leq u(t_0)E[{}_nX_{t_0}] \\ &+ E\bigg[\int_{t_0}^{t\wedge\tau_k} \dot{u}(s)_nX_s + u(s)\bigg(\sum_{i=1}^{\hat{m}} \psi_{i,n}'(|{}_iV_s'Y_s|)Z_s^{(i)} + \frac{\hat{m}}{n}\hat{\eta}_s^2\bigg)\, ds\bigg]\end{aligned}$$

for fixed $t \in [t_0, \infty)$. By monotone and dominated convergence, we may take the limit $n \uparrow \infty$ to deduce (4.1) when τ is replaced by τ_k, as $({}_nX)_{n\in\mathbb{N}}$ is an increasing sequence converging pointwise to $|Y|_V$.

Finally, Fatou's lemma, another application of the dominated convergence theorem and the fact that $\sup_{k\in\mathbb{N}} \tau_k = \tau$ give the claimed bound. $\square$

Remark 4.3. If there are $\alpha \in \mathscr{S}(\mathbb{R})$ and a measurable function $\rho : \mathbb{R}_+ \to \mathbb{R}$ with $\rho(0) = 0$ such that $Z^{(i)} = \alpha + \rho(|{}_iV'Y|)$ for some $i \in \{1, \dots, \hat{m}\}$, then

$$Z_s^{(i)} \mathbb{1}_{\{{}_iV_s'Y_s \neq 0\}} \leq \alpha_s^+ + \rho(|{}_iV_s'Y_s|) \quad \text{for any } s \in [t_0, \infty).$$

We recall a Burkholder-Davis-Gundy inequality for stochastic integrals driven by W from [33][Theorem 7.3]. Namely, for $p \in [2, \infty)$ let $\overline{w}_p := (p^{p+1}/(2(p-1)^{p-1}))^{p/2}$, if $p > 2$, and $\overline{w}_p := 4$, if $p = 2$. Then

$$E\bigg[\sup_{\tilde{s}\in[t_0,t]}\bigg|\int_{t_0}^{\tilde{s}} X_s\,dW_s\bigg|^p\bigg] \leq \overline{w}_p E\bigg[\bigg(\int_{t_0}^t |X_s|^2\,ds\bigg)^{\frac{p}{2}}\bigg] \tag{4.2}$$

for each $X \in \mathscr{S}(\mathbb{R}^{m\times d})$ and any $t \in [t_0, \infty)$ for which $\int_{t_0}^t |X_s|^2\,ds < \infty$. Now we conclude with an auxiliary moment estimate in a supremum seminorm.

Proposition 4.4. *Let $p \in [1, \infty)$, τ be a stopping time, $Z \in \mathscr{S}(\mathbb{R}^{\hat{m}})$ and $\hat{\eta} \in \mathscr{L}^2_{loc}(\mathbb{R}^{\hat{m}}_+)$ be such that $\int_{t_0}^t |Z_s|\,ds < \infty$ for all $t \in [t_0, \infty)$ and*

$$\mathrm{sgn}({}_iV_s'Y_s)({}_i\dot{V}_s'Y_s + {}_iV_s'\hat{\mathrm{B}}_s) \leq Z_s^{(i)} \quad \text{on } \{{}_iV_s'Y_s \neq 0\} \quad \text{and} \quad |{}_iV_s'\hat{\Sigma}_s| \leq \hat{\eta}_i(s)\hat{\rho}_i(|{}_iV_s'Y_s|)$$

for a.e. $s \in [t_0, \infty)$ with $s < \tau$ a.s. for all $i \in \{1, \dots, \hat{m}\}$. Then any locally absolutely continuous function $u : [t_0, \infty) \to \mathbb{R}_+$ satisfies

$$\begin{aligned}
&E\bigg[\bigg(\sup_{s\in[t_0,t]} u(s\wedge\tau)|Y_s^\tau|_{V_s^\tau} - u(t_0)|Y_{t_0}|_{V_{t_0}}\bigg)^p\bigg]^{\frac{1}{p}} \\
&\leq E\bigg[\bigg(\int_{t_0}^{t\wedge\tau}\bigg(\dot{u}(s)|Y_s|_{V_s} + u(s)\sum_{i=1}^{\hat{m}} Z_s^{(i)}\mathbb{1}_{\{{}_iV_s'Y_s\neq 0\}}\bigg)^+ ds\bigg)^p\bigg]^{\frac{1}{p}} \\
&\quad + \bigg(\int_{t_0}^t \hat{\eta}_0(s)^2\,ds\bigg)^{\frac{1}{2}-\frac{1}{p_0}}\bigg(\overline{w}_{p_0}\int_{t_0}^t \hat{\eta}_0(s)^2 u(s)^{p_0} E[\hat{\rho}(|Y_s|_{V_s})^{p_0}\mathbb{1}_{\{\tau>s\}}]\,ds\bigg)^{\frac{1}{p_0}}
\end{aligned} \tag{4.3}$$

for all $t \in [t_0, \infty)$ with $p_0 := p \vee 2$ and $\hat{\rho} := \max_{i\in\{1,\dots,\hat{m}\}} \hat{\rho}_i$.

Proof. For given $k, n \in \mathbb{N}$ we set $\tau_k := \inf\{t \in [t_0, \infty)\,|\,|Y_t|_{V_t} \geq k \text{ or } \int_{t_0}^t |Z_s|\,ds \geq k\} \wedge \tau$ and ${}_nX := \sum_{i=1}^{\hat{m}} \psi_{i,n}(|{}_iV'Y|)$. Then Lemma 4.1 and Minkowski's inequality show that

$$\begin{aligned}
&E\bigg[\bigg(\sup_{s\in[t_0,t]} u(s\wedge\tau_k){}_nX_s^{\tau_k} - u(t_0){}_nX_{t_0}\bigg)^p\bigg]^{\frac{1}{p}} \\
&\qquad \leq E\bigg[\bigg(\int_{t_0}^{t\wedge\tau_k}\bigg(\dot{u}(s){}_nX_s + u(s)\sum_{i=1}^{\hat{m}}\psi_{i,n}'(|{}_iV_s'Y_s|)Z_s^{(i)}\bigg)^+ ds\bigg)^p\bigg]^{\frac{1}{p}} \\
&\qquad\quad + \frac{1}{n}\int_{t_0}^t u(s)\sum_{i=1}^{\hat{m}}\hat{\eta}_i(s)^2\,ds + E\bigg[\bigg(\sup_{s\in[t_0,t]} {}_nI_s^{\tau_k}\bigg)^p\bigg]^{\frac{1}{p}}
\end{aligned} \tag{4.4}$$

for fixed $t \in [t_0, \infty)$, where ${}_nI$ denotes a continuous local martingale with ${}_nI_{t_0} = 0$ that is indistinguishable from the stochastic integral

$$\int_{t_0}^{\cdot} u(s)\sum_{i=1}^{\hat{m}}\psi_{i,n}'(|{}_iV_s'Y_s|)\mathrm{sgn}({}_iV_s'Y_s){}_iV_s'\hat{\Sigma}_s\,dW_s.$$

Moreover, as $\sum_{i=1}^{\hat{m}} |\psi_{i,n}'(|_i V_s' Y_s|)_i V_s' \hat{\Sigma}_s| \leq \hat{\eta}_0(s)\hat{\rho}(|Y_s|_{V_s})$ for a.e. $s \in [t_0, t]$ with $s < \tau_k$ a.s., it follows from Hölder's inequality, (4.2) and Jensen's inequality that

$$\begin{aligned} \overline{w}_{p_0}^{-1} E\Big[\sup_{s\in[t_0,t]} |{}_n I_s^{\tau_k}|^p\Big]^{\frac{p_0}{p}} &\leq E\Big[\Big(\int_{t_0}^{t\wedge\tau_k} \hat{\eta}_0(s)^2 u(s)^2 \hat{\rho}(|Y_s|_{V_s})\,ds\Big)^{\frac{p_0}{2}}\Big] \\ &\leq \Big(\int_{t_0}^t \hat{\eta}_0(s)^2\,ds\Big)^{\frac{p_0}{2}-1} \int_{t_0}^t \hat{\eta}_0(s)^2 u(s)^{p_0} E\big[\hat{\rho}(|Y_s|_{V_s})^{p_0} \mathbb{1}_{\{\tau_k > s\}}\big]\,ds. \end{aligned} \tag{4.5}$$

Now recall that any sequence $(x_n)_{n\in\mathbb{N}}$ of real-valued functions on $[t_0, t]$ and each function $x : [t_0, t] \to \mathbb{R}$ such that $x(s) \leq \liminf_{n\uparrow\infty} x_n(s)$ for all $s \in [t_0, t]$ satisfies

$$\sup_{s\in[t_0,t]} x(s) \leq \liminf_{n\uparrow\infty} \sup_{s\in[t_0,t]} x_n(s).$$

In combination with Fatou's lemma, this shows that (4.3) follows when τ is replaced by τ_k from (4.4), (4.5) and dominated convergence. As $\sup_{k\in\mathbb{N}} \tau_k = \tau$, monotone convergence yields the asserted estimate. □

4.2 Quantitative first moment estimates

To deduce an L^1-estimate based on Bihari's inequality from the results of the preceding section, we fix $l \in \mathbb{N}$ and $\alpha, \beta \in (0, 1]^l$ and introduce two assumptions on the random Itô process Y:

(A.1) For any $n \in \mathbb{N}$ there are ${}_n\hat{\eta} \in \mathscr{S}(\mathbb{R}_+)$ and increasing functions $\hat{\rho}_{1,n}, \ldots, \hat{\rho}_{\hat{m},n} \in \mathrm{R}_c$ with $\int_{t_0}^t {}_n\hat{\eta}_s^2\,ds < \infty$ for all $t \in [t_0, \infty)$ such that $\int_0^1 \hat{\rho}_{i,n}(v)^{-2}\,dv = \infty$ and

$$|{}_i V_s' \hat{\Sigma}_s| \leq {}_n\hat{\eta}_s \hat{\rho}_{i,n}(|{}_i V_s' Y_s|)$$

for a.e. $s \in [t_0, \infty)$ with $|Y_s|_{V_s} \leq n$ a.s. for every $i \in \{1, \ldots, \hat{m}\}$.

(A.2) There exist $\kappa \in \mathscr{S}_{loc}^1(\mathbb{R}_+^{\hat{m}})$, a measurable map $\theta : [t_0, \infty) \to \mathbb{R}_+^l$ and for every $k \in \{1, \ldots, l\}$ there are

$$\rho_k, \varrho_k \in \mathrm{R}_c, \quad {}_k\eta \in \mathscr{S}_{loc}^{q_{\alpha_k}}(\mathbb{R}_+^{\hat{m}}) \quad \text{and} \quad {}_k\lambda \in \mathscr{S}_{loc}^1(\mathbb{R}_+^{\hat{m}})$$

such that ρ_k^{1/α_k} is concave, ϱ_k is increasing and

$$\mathrm{sgn}({}_i V' Y)({}_i \dot{V}' Y + {}_i V' \hat{\mathrm{B}}) \leq \kappa^{(i)} + \sum_{k=1}^l {}_k\eta^{(i)} \rho_k(|Y|_V) + {}_k\lambda^{(i)} \varrho_k \circ \theta_k$$

a.e. on $[t_0, \infty)$ a.s. for each $i \in \{1, \ldots, \hat{m}\}$.

For given ${}_1\lambda, \ldots, {}_l\lambda \in \mathscr{S}_{loc}^1(\mathbb{R}_+^{\hat{m}})$ and a measurable map $\theta : [t_0, \infty) \to \mathbb{R}_+^l$, as in (A.2), we rely on a domination condition:

(A.3) There is $c \in \mathbb{R}_+^l$ satisfying $\theta_k(s) \leq c_k E[|Y_s|_{V_s}]$ for a.e. $s \in [t_0, \infty)$ with $E[{}_k\lambda_s^{(0)}] > 0$ for any $k \in \{1, \ldots, l\}$.

If (A.2) is satisfied, then, by using the functional $[\cdot]_p$ given by (2.1) for $p \in [1, \infty]$, we define two functions $\gamma, \delta \in \mathscr{L}_{loc}^1(\mathbb{R}_+)$ via

$$\gamma := \sum_{k=1}^l \alpha_k \big[{}_k\eta^{(0)}\big]_{q_{\alpha_k}} + \beta_k E\big[{}_k\lambda^{(0)}\big], \quad \delta := \sum_{k=1}^l (1 - \alpha_k)\big[{}_k\eta^{(0)}\big]_{q_{\alpha_k}} + (1 - \beta_k) E\big[{}_k\lambda^{(0)}\big].$$

We also recall the definitions of the functions Φ_ρ and Ψ_ρ in (3.4) and (3.5) for $\rho \in \mathrm{R}_c$ that lead to the following general estimation result.

Theorem 4.5. *Let* (A.1)-(A.3) *hold,* $E[|Y_{t_0}|_{V_{t_0}}] < \infty$*,* $\sum_{k=1}^l E[{}_k\lambda^{(0)}]\varrho_k \circ \theta_k$ *be locally integrable and* $\rho_0, \varrho_0 \in C(\mathbb{R}_+)$ *be defined by*

$$\rho_0(v) := \max_{k\in\{1,\dots,l\}} \rho_k(v)^{\frac{1}{\alpha_k}} \quad \textit{and} \quad \varrho_0(v) := \rho_0(v) \vee \max_{k\in\{1,\dots,l\}} \varrho_k(c_k v)^{\frac{1}{\beta_k}}.$$

If $\Phi_{\rho_0}(\infty) = \infty$ *or* $\sum_{k=1}^l E[{}_k\eta^{(0)}\rho_k(|Y|_V)]$ *is locally integrable, then* $E[|Y|_V]$ *is locally bounded and*

$$\sup_{s\in[t_0,t]} E\big[|Y_s|_{V_s}\big] \le \Psi_{\varrho_0}\bigg(E\big[|Y_{t_0}|_{V_{t_0}}\big] + \int_{t_0}^t E[\kappa_s^{(0)}] + \delta(s)\,ds, \int_{t_0}^t \gamma(s)\,ds\bigg)$$

for any $t \in [t_0, t_0^+)$*, where* t_0^+ *denotes the supremum over all* $t \in [t_0,\infty)$ *for which*

$$\bigg(E\big[|Y_{t_0}|_{V_{t_0}}\big] + \int_{t_0}^t E[\kappa_s^{(0)}] + \delta(s)\,ds, \int_{t_0}^t \gamma(s)\,ds\bigg) \in D_{\varrho_0}.$$

Proof. We introduce the stopping time $\tau_n := \inf\{t \in [t_0,\infty)\,|\,|Y_t|_{V_t} \ge n\}$ for fixed $n \in \mathbb{N}$ and set $\hat\kappa := E[\kappa^{(0)}] + \sum_{k=1}^l E[{}_k\lambda^{(0)}]\varrho_k \circ \theta_k$. Then Proposition 4.2 and Remark 4.3 yield

$$E\big[|Y_t^{\tau_n}|_{V_t^{\tau_n}}\big] \le E\big[|Y_{t_0}|_{V_{t_0}}\big] + \int_{t_0}^t \hat\kappa(s) + \sum_{k=1}^l E\big[{}_k\eta_s^{(0)}\rho_k(|Y_s|_{V_s})\mathbb{1}_{\{\tau_n > s\}}\big]\,ds \tag{4.6}$$

for given $t \in [t_0,\infty)$. Thereby, we notice that the local integrability of the measurable function $[t_0,\infty) \to \mathbb{R}_+$, $s \mapsto E[{}_k\eta_s^{(0)}\rho_k(|Y_s|_{V_s})\mathbb{1}_{\{\tau_n>s\}}]$ follows from (2.2), which yields

$$E\big[{}_k\eta_s^{(0)}\rho_k(|Y_s|_{V_s})\mathbb{1}_{\{\tau>s\}}\big] \le \big[{}_k\eta_s^{(0)}\big]_{q_{\alpha_k}}\big(1 - \alpha_k + \alpha_k\rho_k\big(E\big[|Y_s^\tau|_{V_s^\tau}\big]\big)^{\frac{1}{\alpha_k}}\big) \tag{4.7}$$

for all $s \in [t_0,t]$, each $k \in \{1,\dots,l\}$ and every stopping time τ for which $E[|Y^\tau|_{V^\tau}]$ is finite, because ρ_k^{1/α_k} is concave, by assumption.

Thus, let us set $\hat\delta := \sum_{k=1}^l (1-\alpha_k)[{}_k\eta^{(0)}]_{q_{\alpha_k}}$. If $\Phi_{\rho_0}(\infty) = \infty$ holds, then we apply Bihari's inequality to (4.6) and infer from Fatou's lemma that

$$\begin{aligned} E\big[|Y_t|_{V_t}\big] &\le \liminf_{n\uparrow\infty} E\big[|Y_t^{\tau_n}|_{V_t^{\tau_n}}\big] \\ &\le \Psi_{\rho_0}\bigg(E\big[|Y_{t_0}|_{V_{t_0}}\big] + \int_{t_0}^t (\hat\kappa + \hat\delta)(s)\,ds, \int_{t_0}^t \sum_{k=1}^l \alpha_k\big[{}_k\eta_s^{(0)}\big]_{q_{\alpha_k}}\,ds\bigg), \end{aligned}$$

as the domain of Ψ_{ρ_0} satisfies $D_{\rho_0} = \mathbb{R}_+^2$. For this reason, $E[|Y|_V]$ is locally bounded in this case. By choosing $\tau = \infty$ in (4.7), we see that it suffices to consider the case when $\sum_{k=1}^l E[{}_k\eta^{(0)}\rho_k(|Y|_V)]$ is locally integrable. Then

$$E\big[|Y_t|_{V_t}\big] \le E\big[|Y_{t_0}|_{V_{t_0}}\big] + \int_{t_0}^t (\hat\kappa + \hat\delta)(s) + \sum_{k=1}^l \alpha_k\big[{}_k\eta_s^{(0)}\big]_{q_{\alpha_k}}\rho_k\big(E\big[|Y_s|_{V_s}\big]\big)^{\frac{1}{\alpha_k}}\big)\,ds$$

follows from (4.6), Fatou's lemma and (4.7). Thereby, we readily checked that $E[|Y|_V]$ is actually locally bounded. Finally, Young's inequality gives us that

$$E\big[{}_k\lambda^{(0)}\big]\varrho_k \circ \theta_k \le E\big[{}_k\lambda^{(0)}\big]\big(1 - \beta_k + \beta_k\varrho_k\big(c_k E\big[|Y|_V\big]\big)^{\frac{1}{\beta_k}}\big)$$

a.e. on $[t_0,t]$ for all $k \in \{1,\dots,l\}$ and we conclude the proof with another application of Bihari's inequality, since $\hat\delta + \sum_{k=1}^l (1-\beta_k)E[{}_k\lambda^{(0)}] = \delta$. □

For a stability analysis we consider another condition, which explicitly measures the dependence on each coordinate and implies (A.2):

(A.4) There are $\kappa \in \mathscr{S}^1_{loc}(\mathbb{R}^{\hat{m}}_+)$, a measurable mapping $\theta : [t_0, \infty) \to \mathbb{R}^l_+$ and for each $k \in \{1, \dots, l\}$ there are

$$
{}_k\eta \in \mathscr{S}^{q_{\alpha_k}}_{loc}(\mathbb{R}^{\hat{m}\times\hat{m}}) \quad \text{and} \quad {}_k\lambda \in \mathscr{S}^1_{loc}(\mathbb{R}^{\hat{m}}_+)
$$

with ${}_k\eta^{(i,j)} \geq 0$ for all $i, j \in \{1, \dots, \hat{m}\}$ with $i \neq j$ such that

$$
\operatorname{sgn}({}_iV'Y)\big({}_i\dot{V}'Y + {}_iV'\hat{\mathrm{B}}\big) \leq \kappa^{(i)} + \sum_{k=1}^{l} \bigg(\sum_{j=1}^{\hat{m}} {}_k\eta^{(i,j)} |{}_jV'Y|^{\alpha_k} \bigg) + {}_k\lambda^{(i)} \theta_k^{\beta_k}
$$

a.e. on $[t_0, \infty)$ a.s. for any $i \in \{1, \dots, \hat{m}\}$.

If (A.4) and (A.3) hold, then we may use $\gamma_c \in \mathscr{L}^1_{loc}(\mathbb{R})$ and $\delta_c \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$ given by

$$
\gamma_c := \max_{j\in\{1,\dots,\hat{m}\}} \sum_{k=1}^{l} \alpha_k \bigg[\sum_{i=1}^{\hat{m}} {}_k\eta^{(i,j)} \bigg]_{q_{\alpha_k}} + \beta_k c_k^{\beta_k} E\big[{}_k\lambda^{(0)}\big] \tag{4.8}
$$

and

$$
\delta_c := \sum_{k=1}^{l} (1 - \alpha_k) \bigg(\sum_{j=1}^{\hat{m}} \bigg[\sum_{i=1}^{\hat{m}} {}_k\eta^{(i,j)} \bigg]_{q_{\alpha_k}} \bigg) + (1 - \beta_k) c_k^{\beta_k} E\big[{}_k\lambda^{(0)}\big] \tag{4.9}
$$

to get an explicit moment estimate, which yields sufficient conditions for boundedness and convergence in first moment.

Theorem 4.6. *Let* (A.1), (A.4) *and* (A.3) *be valid,* $E[|Y_{t_0}|_{V_{t_0}}] < \infty$ *and* $\sum_{k=1}^{l} E[{}_k\lambda^{(0)}]\theta_k^{\beta_k}$ *be locally integrable. Then*

$$
E\big[|Y_t|_{V_t}\big] \leq e^{\int_{t_0}^{t} \gamma_c(s)\,ds} E\big[|Y_{t_0}|_{V_{t_0}}\big] + \int_{t_0}^{t} e^{\int_s^t \gamma_c(\tilde{s})\,d\tilde{s}} \big(E\big[\kappa_s^{(0)}\big] + \delta_c(s)\big)\,ds \tag{4.10}
$$

for all $t \in [t_0, \infty)$. *In particular if* γ_c^+, $E[\kappa^{(0)}]$ *and* δ_c *are integrable, then* $E[|Y|_V]$ *is bounded. If in addition* $\int_{t_0}^{\infty} \gamma_c^-(s)\,ds = \infty$, *then* $\lim_{t\uparrow\infty} E\big[|Y_t|_{V_t}\big] = 0$.

Proof. First, we observe that (A.2) holds when ${}_k\eta$ is replaced by ${}_k\tilde{\eta} \in \mathscr{S}^{q_{\alpha_k}}_{loc}(\mathbb{R}^{\hat{m}}_+)$ defined coordinatewise by ${}_k\tilde{\eta}^{(i)} := \sum_{j=1}^{\hat{m}} ({}_k\eta^{(i,j)})^+$ and it holds that

$$
\rho_k(v) = v^{\alpha_k} \quad \text{and} \quad \varrho_k(v) = v^{\beta_k} \quad \text{for all } v \in \mathbb{R}_+
$$

and each $k \in \{1, \dots, l\}$. As $\rho_0 \in C(\mathbb{R}_+)$ given by $\rho_0(v) = v$ satisfies $\Phi_{\rho_0}(\infty) = \infty$, Theorem 4.5 shows us that $E[|Y|_V]$ is locally bounded.

Thus, we define ${}_k\hat{\eta} \in \mathscr{S}^{q_{\alpha_k}}_{loc}(\mathbb{R}^{\hat{m}})$ by ${}_k\hat{\eta}^{(j)} := \sum_{i=1}^{\hat{m}} {}_k\eta^{(i,j)}$ for all $k \in \{1, \dots, l\}$. Then Proposition 4.2 and Remark 4.3 imply that

$$
\begin{aligned}
u(t)E\big[|Y_t|_{V_t}\big] &\leq u(t_0)E\big[|Y_{t_0}|_{V_{t_0}}\big] + \int_{t_0}^{t} u(s)E\big[\kappa_s^{(0)}\big]\,ds \\
&+ \int_{t_0}^{t} \dot{u}(s)E\big[|Y_s|_{V_s}\big] + u(s)\bigg(\sum_{k=1}^{l} \bigg(\sum_{j=1}^{\hat{m}} E\big[{}_k\hat{\eta}_s^{(j)} |{}_jV_s'Y_s|^{\alpha_k}\big] \bigg) + E\big[{}_k\lambda_s^{(0)}\big]\theta_k(s)^{\beta_k} \bigg)\,ds
\end{aligned} \tag{4.11}
$$

for fixed $t \in [t_0, \infty)$ and any $\mathbb{R}_+$-valued locally absolutely continuous function u on $[t_0, \infty)$. From (2.2) we directly obtain that

$$
E\big[{}_k\hat{\eta}^{(j)} |{}_jV'Y|^{\alpha_k}\big] \leq \big[{}_k\hat{\eta}^{(j)}\big]_{q_{\alpha_k}} \big(1 - \alpha_k + \alpha_k E\big[|{}_jV'Y|\big]\big)
$$

and $E[{}_k\lambda^{(0)}]\theta_k^{\beta_k} \leq c_k^{\beta_k} E[{}_k\lambda^{(0)}](1 - \beta_k + \beta_k E[|Y|_V])$ a.e. on $[t_0, t]$ for any $j \in \{1, \dots, \hat{m}\}$ and each $k \in \{1, \dots, l\}$. As a consequence,

$$\sum_{k=1}^{l} \bigg(\sum_{j=1}^{\hat{m}} E\big[{}_k\hat{\eta}^{(j)} |_j V'Y|^{\alpha_k}\big] \bigg) + E\big[{}_k\lambda^{(0)}\big]\theta_k^{\beta_k} \leq \delta_c + \gamma_c E\big[|Y|_V\big]$$

a.e. on $[t_0, t]$. Thus, by choosing $u(\tilde{s}) = \exp(-\int_{t_0}^{\tilde{s}} \gamma_c(s)\, ds)$ for any $\tilde{s} \in [t_0, \infty)$ in (4.11), we get the asserted estimate after dividing by $u(t)$.

Next, let us assume that γ_c^+, $E[\kappa^{(0)}]$ and δ_c are integrable. Then the second assertion follows from the bound

$$\sup_{t \in [t_0, \infty)} E\big[|Y_t|_{V_t}\big] \leq e^{\int_{t_0}^{\infty} \gamma_c^+(s)\, ds} \bigg(E\big[|Y_{t_0}|_{V_{t_0}}\big] + \int_{t_0}^{\infty} E[\kappa_s^{(0)}] + \delta_c(s)\, ds \bigg).$$

For the last claim, let additionally $\int_{t_0}^{\infty} \gamma_c^-(s)\, ds = \infty$. Then $\lim_{t \uparrow \infty} \exp(\int_s^t \gamma_c(\tilde{s})\, d\tilde{s}) = 0$ for every $s \in [t_0, \infty)$, as any measurable function $f : [s, \infty) \to \mathbb{R}_+$ satisfies $\lim_{t \uparrow \infty} \int_s^t f(\tilde{s})\, d\tilde{s} = \int_s^{\infty} f(\tilde{s})\, d\tilde{s}$, by monotone convergence. Thus,

$$\lim_{t \uparrow \infty} \int_{t_0}^{t} e^{\int_s^t \gamma_c(\tilde{s})\, d\tilde{s}} \big(E[\kappa_s^{(0)}] + \delta_c(s) \big)\, ds = 0$$

follows from dominated convergence, which completes the proof. □

Remark 4.7. Suppose that $\kappa = 0$ a.s. and let δ_c vanish a.e., which holds if $\alpha_k = \beta_k = 1$ for any $k \in \{1, \dots, l\}$. If γ_c^+ is integrable, then

$$\sup_{t \in [t_0, \infty)} e^{\int_{t_0}^{t} \gamma_c^-(s)\, ds} E\big[|Y_t|_{V_t}\big] \leq \sup_{t \in [t_0, \infty)} e^{\int_{t_0}^{t} \gamma_c^+(s)\, ds} E\big[|Y_{t_0}|_{V_{t_0}}\big] < \infty.$$

If additionally $\int_{t_0}^{\infty} \gamma_c^-(s)\, ds = \infty$, then from $\alpha\gamma_c^- + \gamma_c = \gamma_c^+ - (1 - \alpha)\gamma_c^-$ we infer that

$$\lim_{t \uparrow \infty} e^{\alpha \int_{t_0}^{t} \gamma_c^-(s)\, ds} E\big[|Y_t|_{V_t}\big] = 0 \quad \text{for all } \alpha \in [0, 1).$$

These two facts give more insight into the rate of convergence.

4.3 Moment estimates in a supremum seminorm

In this section we deduce L^p-moment estimates in a supremum seminorm for $p \in [1, 2]$. To this end, we require a Hölder condition instead of the weaker Osgood condition (A.1) on compact sets and restrict (A.2) and (A.4) as follows:

(A.5) There exists $\hat{\eta} \in \mathscr{L}_{loc}^2(\mathbb{R}_+^{\hat{m}})$ such that $|{}_iV'\Sigma| \leq \hat{\eta}_i |{}_iV'Y|^{1/2}$ a.e. on $[t_0, \infty)$ a.s. for all $i \in \{1, \dots, \hat{m}\}$.

(A.6) Assumption (A.2) holds when ${}_k\eta = {}_k\overline{\eta}$ for some ${}_k\overline{\eta} \in \mathscr{L}_{loc}^1(\mathbb{R}_+^{\hat{m}})$ and $\rho_k(v) = v^{\alpha_k}$ for all $v \in \mathbb{R}_+$ and each $k \in \{1, \dots, l\}$.

(A.7) Assumption (A.4) is valid and for each $k \in \{1, \dots, l\}$ there is ${}_k\overline{\eta} \in \mathscr{L}_{loc}^1(\mathbb{R}^{\hat{m} \times \hat{m}})$ such that ${}_k\eta = {}_k\overline{\eta}$.

Under (A.5) and (A.6), we define two $[0, \infty]$-valued measurable functions f_p and g_p on the set of all $(t_1, t) \in [t_0, \infty)^2$ with $t_1 \leq t$ by

$$f_p(t_1, t) := E\bigg[\bigg(\int_{t_1}^{t} \kappa_s^{(0)} + \sum_{k=1}^{l} {}_k\lambda_s^{(0)} \varrho_k(\theta_k(s))\, ds \bigg)^p\bigg]^{\frac{1}{p}} + 2\bigg(\int_{t_1}^{t} \hat{\eta}_0(s)^2 E[|Y_s|_{V_s}]\, ds \bigg)^{\frac{1}{2}}$$

and

$$g_p(t_1,t) := f_p(t_1,t) + \sum_{k=1}^{l} E\big[|Y_{t_1}|_{V_{t_1}}^{\alpha_k p}\big]^{\frac{1}{p}} \int_{t_1}^{t} {}_k\overline{\eta}_0(s)\,ds,$$

which are finite if (A.3) holds, $E[|Y|_V]$ is locally bounded and $E[|Y|_V^p]$ is finite. In addition, let us set $\underline{\alpha} := \min_{k\in\{1,\dots,l\}} \alpha_k$ and $\overline{\alpha} := \max_{k\in\{1,\dots,l\}} \alpha_k$.

Proposition 4.8. *Let* (A.5), (A.6) *and* (A.3) *be valid,* $\sum_{k=1}^{l} E[{}_k\lambda^{(0)}]\varrho_k \circ \theta_k$ *be locally integrable and* $\rho_0 \in C(\mathbb{R}_+)$ *be given by* $\rho_0(v) := v^{\underline{\alpha}}\mathbb{1}_{(0,1]}(v) + v^{\overline{\alpha}}\mathbb{1}_{(1,\infty)}(v)$. *If*

$$E\big[|Y_{t_0}|_{V_{t_0}}^p\big], \quad E\bigg[\bigg(\int_{t_0}^{t} \kappa_s^{(0)}\,ds\bigg)^p\bigg] \quad \text{and} \quad E\bigg[\bigg(\int_{t_0}^{t}\sum_{k=1}^{l} {}_k\lambda_s^{(0)}\,ds\bigg)^p\bigg] \tag{4.12}$$

are finite, then $\sup_{s\in[t_0,t]} |Y_s|_{V_s}$ *is p-fold integrable and*

$$E\bigg[\bigg(\sup_{s\in[t_1,t]} |Y_s|_{V_s} - |Y_{t_1}|_{V_{t_1}}\bigg)^p\bigg] \leq \Psi_{\rho_0}\bigg((l+1)^{p-1}g_p(t_1,t)^p, (l+1)^{p-1}\sum_{k=1}^{l}\bigg(\int_{t_1}^{t} {}_k\overline{\eta}_0(s)\,ds\bigg)^p\bigg)$$

for any $t_1, t \in [t_0,\infty)$ *with* $t_1 \leq t$. *In particular,* $E[|Y|_V^p]$ *is continuous.*

Proof. If the integrability assertion is true, then $\lim_{n\uparrow\infty} E[|Y_{t_n}|_{V_{t_n}}^p] = E[|Y_t|_{V_t}^p]$ for any sequence $(t_n)_{n\in\mathbb{N}}$ that converges to some $t \in [t_0,\infty)$, by dominated convergence. For this reason, it suffices to show the first two claims.

We set $\tau_n := \inf\{t \in [t_1,\infty) \,|\, |Y_t|_{V_t} \geq n\}$ for given $t_1 \in [t_0,\infty)$ and $n \in \mathbb{N}$. Then Proposition 4.4 and the inequalities of Minkowski and Jensen give

$$\begin{aligned}
E\bigg[\bigg(\sup_{s\in[t_1,t]} |Y_s^{\tau_n}|_{V_s^{\tau_n}} - |Y_{t_1}|_{V_{t_1}}\bigg)^p\bigg]^{\frac{1}{p}} &\leq 2\bigg(\int_{t_1}^{t} \hat{\eta}_0(s)^2 E\big[|Y_s^{\tau_n}|_{V_s^{\tau_n}}\big]\,ds\bigg)^{\frac{1}{2}} \\
&\quad + E\bigg[\bigg(\int_{t_1}^{t\wedge\tau_n} \kappa_s^{(0)} + \sum_{k=1}^{l} {}_k\overline{\eta}_0(s)|Y_s|_{V_s}^{\alpha_k} + {}_k\lambda_s^{(0)}\varrho_k(\theta_k(s))\,ds\bigg)^p\bigg]^{\frac{1}{p}} \\
&\leq f_p(t_1,t) + \sum_{k=1}^{l}\bigg(\int_{t_1}^{t} {}_k\overline{\eta}_0(s)\bigg)^{1-\frac{1}{p}}\bigg(\int_{t_1}^{t} {}_k\overline{\eta}_0(s) E\big[|Y_s^{\tau_n}|_{V_s^{\tau_n}}^{\alpha_k p}\big]\,ds\bigg)^{\frac{1}{p}}
\end{aligned}$$

for fixed $t \in [t_1,\infty)$. We recall that $\Phi_{\rho_0}(\infty) = \infty$ and $D_{\rho_0} = \mathbb{R}_+^2$. Thus, if $E[|Y_{t_1}|_{V_{t_1}}^p] < \infty$, then another application of Minkowski's inequality together with Bihari's inequality yield the asserted bound for

$$E\bigg[\bigg(\sup_{s\in[t_1,t]} |Y_s^{\tau_n}|_{V_s^{\tau_n}} - |Y_{t_1}|_{V_{t_1}}\bigg)^p\bigg]$$

and from Fatou's lemma we readily infer the claimed result. In this context, we may use the fact that $E[|Y|_V]$ is locally bounded, by Theorem 4.5. This ensures that $g(t_1,t)$ is finite in this case.

Further, by choosing $t_1 = t_0$ the p-fold integrability of $\sup_{s\in[t_0,t]} |Y_s|_{V_s}$ follows from that of $|Y_{t_0}|_{V_{t_0}}$ and the finiteness of $g_p(t_0,t)$, which completes the proof. □

If (A.7) holds, then the definitions (4.8) and (4.9) for the choice ${}_1\lambda = \cdots = {}_l\lambda = 0$ lead us to two functions in $\mathscr{L}_{loc}^1(\mathbb{R}_+)$ given by

$$\gamma_{c,0} := \max_{j\in\{1,\dots,\hat{m}\}} \sum_{k=1,\,\alpha_k<1}^{l} \alpha_k\bigg(\sum_{i=1}^{\hat{m}} {}_k\overline{\eta}_{i,j}\bigg)^+ + \sum_{k=1,\,\alpha_k=1}^{l}\sum_{i=1}^{\hat{m}} {}_k\overline{\eta}_{i,j}$$

and

$$\delta_{c,0} := \sum_{k=1}^{l}\sum_{j=1}^{\hat{m}}(1-\alpha_k)\Big(\sum_{i=1}^{\hat{m}} {}_k\overline{\eta}_{i,j}\Big)^{+}.$$

For given $\gamma \in \mathscr{L}^1_{loc}(\mathbb{R})$ we also introduce an $[0,\infty]$-valued measurable function $h_{\gamma,p}$ on the set of all $(t_1,t) \in [t_0,\infty)^2$ with $t_1 \leq t$ by

$$h_{\gamma,p}(t_1,t) := E\bigg[\bigg(\int_{t_1}^{t} e^{-\int_{t_1}^{s}\gamma(s_0)\,ds_0}\Big(\kappa_s^{(0)} + \delta_{c,0}(s) + \sum_{k=1}^{l} {}_k\lambda_s^{(0)}\theta_k(s)^{\beta_k}\Big)\,ds\bigg)^{p}\bigg]^{\frac{1}{p}}$$
$$+ 2\bigg(\int_{t_1}^{t}\hat{\eta}_0(s)^2 e^{-2\int_{t_1}^{s}\gamma(s_0)\,ds_0}E\big[|Y_s|_{V_s}\big]\,ds\bigg)^{\frac{1}{2}}$$

and state the analogue of Proposition 4.8 when (A.7) instead of (A.6) holds.

Lemma 4.9. *Let* (A.5), (A.7) *and* (A.3) *be satisfied and* $\sum_{k=1}^{l} E[{}_k\lambda^{(0)}]\theta_k^{\beta_k}$ *be locally integrable. If the expectations in* (4.12) *are finite, then*

$$E\bigg[\bigg(\sup_{s\in[t_1,t]} e^{-\int_{t_1}^{s}\gamma(s_0)\,ds_0}|Y_s|_{V_s} - |Y_{t_1}|_{V_{t_1}}\bigg)^{p}\bigg]^{\frac{1}{p}} \leq h_{\gamma,p}(t_1,t)$$
$$+ \bigg(\int_{t_1}^{t}(\gamma_{c,0}-\gamma)^{+}(s)\,ds\bigg)^{1-\frac{1}{p}}\bigg(\int_{t_1}^{t}(\gamma_{c,0}-\gamma)^{+}(s)e^{-p\int_{t_1}^{s}\gamma(s_0)\,ds_0}E\big[|Y_s|_{V_s}^{p}\big]\,ds\bigg)^{\frac{1}{p}}$$

for each $\gamma \in \mathscr{L}^1_{loc}(\mathbb{R})$ *and all* $t_1, t \in [t_0,\infty)$ *with* $t_1 \leq t$.

Proof. As (A.7) is a special case of (A.6), Proposition 4.8 entails that $\sup_{s\in[t_0,t]}|Y_s|_{V_s}$ is p-fold integrable. Moreover, we readily see that

$$\sum_{k=1}^{l}\sum_{i,j=1}^{\hat{m}} {}_k\overline{\eta}_{i,j}|{}_jV'Y|^{\alpha_k} \leq \delta_{c,0} + \gamma_{c,0}|Y|_V,$$

by Young's inequality. Hence, the claim follows immediately from Proposition 4.4 and the inequalities of Minkowski and Jensen. □

We consider a last restriction that still allows for the mixed Hölder condition in (A.4) on a finite time interval:

(A.8) Assumptions (A.5) and (A.7) hold and there are $t_1 \in [t_0,\infty)$, $\hat{\delta} > 0$ and $\overline{c}_0 \in \mathbb{R}_+$ with $\kappa = (1-\beta_k){}_k\lambda = 0$ and $(1-\alpha_k)\sum_{i=1}^{\hat{m}} {}_k\overline{\eta}_{i,j} \leq 0$ on $[t_1,\infty)$ for any $j \in \{1,\ldots,\hat{m}\}$ and each $k \in \{1,\ldots,l\}$ and

$$\sup_{t\in[t_1,\infty)}\bigg(\int_{t}^{t+\hat{\delta}} E\big[{}_k\lambda_s^{(0)}\big]\,ds\bigg) \vee \bigg(\int_{t}^{t+\hat{\delta}}\hat{\eta}_0(s)^2\,ds\bigg) \leq \overline{c}_0.$$

for every $k \in \{1,\ldots,l\}$ with $\beta_k = 1$.

Then the following first moment estimate in a supremum seminorm will contribute to the pathwise asymptotic analysis of Y in the next section.

Proposition 4.10. *Let* (A.8) *and* (A.3) *hold,* $E[|Y_{t_0}|_{V_{t_0}}] < \infty$ *and* $\sum_{k=1}^{l} E[{}_k\lambda^{(0)}]\theta_k^{\beta_k}$ *be locally integrable. Further, suppose that there are* $\gamma \in \mathscr{L}^1_{loc}(\mathbb{R})$ *and* $\overline{c}_{\gamma,-1},\ldots,\overline{c}_{\gamma,3} \in \mathbb{R}_+$ *such that*

$$\int_{t_2}^{t}(\gamma_{c,0}-\gamma)^{+}(s)\,ds \leq \overline{c}_{\gamma,-1},\quad \int_{t_2}^{t}(\gamma_{c,0}-q\gamma)(s)\,ds \leq \overline{c}_{\gamma,q},\quad \int_{t_2}^{t}\gamma(s)\,ds \leq \overline{c}_{\gamma,3}$$

for all $t_2, t \in [t_1, \infty)$ with $t_2 \leq t < \hat{\delta}$ and any $q \in \{0, 1, 2\}$. Then there is $\overline{c} > 0$ such that

$$E\bigg[\sup_{s \in [t_2, t]} |Y_s|_{V_s}\bigg] \leq \overline{c}\varphi\bigg(E[|Y_{t_0}|_{V_{t_0}}] + \int_{t_0}^{t_1} E[\kappa_s^{(0)}] + \delta_c(s)\, ds\bigg) e^{\frac{1}{2}\int_{t_1}^{t_2} \gamma_c(s)\, ds}$$

for any $t_2, t \in [t_1, \infty)$ with $t_2 \leq t < \hat{\delta}$ and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\varphi(x) := x + \sqrt{x}$.

Proof. Since $E[\kappa^{(0)}] = \delta_{c,0} = (1 - \beta_k)E[{}_k\lambda^{(0)}] = 0$ a.e. on $[t_1, \infty)$ for any $k \in \{1, \ldots, l\}$, from Lemma 4.9 we directly get that

$$\begin{aligned} E\bigg[\sup_{s \in [t_2, t]} e^{-\int_{t_2}^{s} \gamma(s_0)\, ds_0} |Y_s|_{V_s}\bigg] &\leq E[|Y_{t_2}|_{V_{t_2}}] + \int_{t_2}^{t} \hat{\gamma}_c(s) e^{-\int_{t_2}^{s} \gamma(s_0)\, ds_0} E[|Y_s|_{V_s}]\, ds \\ &\quad + 2\bigg(\int_{t_2}^{t} \hat{\eta}_0(s)^2 e^{-2\int_{t_2}^{s} \gamma(s_0)\, ds_0} E[|Y_s|_{V_s}]\, ds\bigg)^{\frac{1}{2}} \end{aligned} \tag{4.13}$$

for $\hat{\gamma}_c := (\gamma_{c,0} - \gamma)^+ + \sum_{k=1,\, \beta_k = 1}^{l} E[{}_k\lambda^{(0)}]c_k$. Further, as $\delta_c = \delta_{c,0} + \sum_{k=1}^{l}(1 - \beta_k)c_k^{\beta_k} E[{}_k\lambda^{(0)}]$, we infer from the moment stability estimate of Theorem 4.6 that

$$\begin{aligned} e^{-q\int_{t_2}^{s} \gamma(s_0)\, ds_0 - c_{\gamma,q}} E[|Y_s|_{V_s}] &\leq e^{\int_{t_0}^{t_2} \gamma_c(s_0)\, ds_0} E[|Y_{t_0}|_{V_{t_0}}] \\ &\quad + \int_{t_0}^{t_1} e^{\int_{s_0}^{t_2} \gamma_c(s_1)\, ds_1} (E[\kappa_{s_0}^{(0)}] + \delta_c(s_0))\, ds_0 \end{aligned}$$

for any $s \in [t_2, t]$ and each $q \in \{0, 1, 2\}$, where $c_{\gamma,q} := \overline{c}_{\gamma,q} + \overline{c}_0 \sum_{k=1,\, \beta_k = 1}^{l} c_k$. Hence, the first two terms on the right-hand side in (4.13) do not exceed

$$\overline{c}_1\bigg(E[|Y_{t_0}|_{V_{t_0}}] + \int_{t_0}^{t_1} E[\kappa_s^{(0)}] + \delta_c(s)\, ds\bigg) e^{\frac{1}{2}\int_{t_1}^{t_2} \gamma_c(s)\, ds}$$

with $\overline{c}_1 := \exp(\int_{t_0}^{t_1} \gamma_c^+(s)\, ds + c_{\gamma,0}/2)(1 + e^{c_{\gamma,1}}(\overline{c}_{\gamma,-1} + \overline{c}_0 \sum_{k=1,\, \beta_k = 1}^{l} c_k))$. Moreover, the third expression in (4.13) is bounded by

$$\overline{c}_2\bigg(E[|Y_{t_0}|_{V_{t_0}}] + \int_{t_0}^{t_1} E[\kappa_s^{(0)}] + \delta_c(s)\, ds\bigg)^{\frac{1}{2}} e^{\frac{1}{2}\int_{t_1}^{t_2} \gamma_c(s)\, ds}$$

for $\overline{c}_2 := 2\overline{c}_0^{1/2} \exp((1/2)(\int_{t_0}^{t_1} \gamma_c^+(s)\, ds + c_{\gamma,2}))$. Since $\exp(-\int_{t_2}^{s} \gamma(s_0)\, ds_0) \geq \exp(-\overline{c}_{\gamma,3})$ for all $s \in [t_2, t]$, the assertion follows for $\overline{c} := \exp(\overline{c}_{\gamma,3})(\overline{c}_1 \vee \overline{c}_2)$. ☐

4.4 Pathwise asymptotic behaviour

To purpose of this section is to deduce the limiting behaviour of Y under the random seminorm $|\cdot|_V$ from the moment estimate of Proposition 4.10, by using the following application of the Borel-Cantelli Lemma.

Lemma 4.11. *Let $A \in \mathscr{F}$ and X be an $\mathbb{R}_+$-valued right-continuous process for which there are a strictly increasing sequence $(t_n)_{n \in \mathbb{N}}$ in $[t_0, \infty)$ with $\lim_{n \uparrow \infty} t_n = \infty$ and a sequence $(c_n)_{n \in \mathbb{N}}$ in $(0, \infty)$ such that*

$$\sum_{n=1}^{\infty} P\bigg(\sup_{s \in (t_n, t_{n+1}]} X_s \mathbb{1}_A > c_n\bigg) < \infty. \tag{4.14}$$

Then for any lower semicontinuous function $\varphi : (t_1, \infty) \to (0, \infty)$ it holds that

$$\limsup_{t \uparrow \infty} \frac{\log(X_t)}{\varphi(t)} \leq \limsup_{n \uparrow \infty} \frac{\log(c_n)}{\inf_{s \in (t_n, t_{n+1}]} \varphi(s)} \quad \text{a.s. on } A. \tag{4.15}$$

Proof. By the Borel-Cantelli Lemma, there is a null set $N \in \mathscr{F}$ such that for any fixed $\omega \in N^c \cap A$ there is $n_0 \in \mathbb{N}$ so that $\sup_{s\in(t_n,t_{n+1}]} X_s(\omega) \leq c_n$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Hence,

$$\sup_{n\in\mathbb{N}:\, n\geq n_1} \sup_{s\in(t_n,t_{n+1}]} \frac{\log(X_s(\omega))}{\varphi(t)} \leq \sup_{n\in\mathbb{N}:\, n\geq n_1} \frac{\log(c_n)}{\inf_{s\in(t_n,t_{n+1}]} \varphi(s)}$$

for every $n_1 \in \mathbb{N}$ with $n_1 \geq n_0$. This in turn shows us that

$$\limsup_{t\uparrow\infty} \frac{\log(X_t(\omega))}{\varphi(t)} = \inf_{n\in\mathbb{N}:\, n\geq n_0} \sup_{s\in(t_n,\infty)} \frac{\log(X_s(\omega))}{\varphi(s)} \leq \limsup_{n\uparrow\infty} \frac{\log(c_n)}{\inf_{s\in(t_n,t_{n+1}]} \varphi(s)},$$

as claimed. □

Remark 4.12. Suppose that instead of (4.14) there are $\hat{c} > 0$ and $\hat{\varepsilon} \in (0,1)$ such that $E[\sup_{s\in(t_n,t_{n+1}]} X_s \mathbb{1}_A] \leq \hat{c} c_n$ for every $n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} c_n^{\varepsilon} < \infty \quad \text{for all } \varepsilon \in (0,\hat{\varepsilon}). \tag{4.16}$$

Then (4.15) follows as well. Indeed, for $\varepsilon \in (0,\hat{\varepsilon})$ Chebyshev's inequality and Lemma 4.11 yield a null set $N_\varepsilon \in \mathscr{F}$ such that

$$\limsup_{t\uparrow\infty} \frac{\log(X_t(\omega))}{\varphi(t)} \leq (1-\varepsilon) \limsup_{n\uparrow\infty} \frac{\log(c_n)}{\inf_{s\in(t_n,t_{n+1}]} \varphi(s)} \tag{4.17}$$

for all $\omega \in N_\varepsilon^c \cap A$. So, any $\omega \in A$ that lies in the complement of $N := \bigcup_{\varepsilon\in\mathbb{Q}\cap(0,\hat{\varepsilon})} N_\varepsilon$ satisfies (4.17) for $\varepsilon = 0$, which is the sharpest bound, as $\lim_{n\uparrow\infty} c_n = 0$.

More specifically, one may derive the conditions in Remark 4.12 in the case that there are $n_0 \in \mathbb{N}$ and a decreasing function $\psi : [n_0,\infty) \to \mathbb{R}_+$ such that $c_n = \psi(n)$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Then (4.16) holds for some $\hat{\varepsilon} \in (0,1)$ if and only if

$$\int_{n_0}^{\infty} \psi(v)^{\varepsilon}\, dv < \infty \quad \text{for all } \varepsilon \in (0,\hat{\varepsilon}), \tag{4.18}$$

as the integral test for the convergence of series shows. We conclude with a pathwise estimate and stress the fact that the fraction 1/2 comes from the Hölder condition (A.5) on the diffusion $\hat{\Sigma}$.

Theorem 4.13. *Let* (A.8) *and* (A.3) *be valid and* $\sum_{k=1}^{l} E[{}_k\lambda^{(0)}]\theta_k^{\beta_k}$ *be locally integrable. Suppose that* $\gamma_c \leq 0$ *a.e. on* $[t_1,\infty)$ *and there is a strictly increasing sequence* $(t_n)_{n\in\mathbb{N}\setminus\{1\}}$ *in* $[t_1,\infty)$ *such that*

$$\sup_{n\in\mathbb{N}} (t_{n+1}-t_n) < \hat{\delta}, \quad \lim_{n\uparrow\infty} t_n = \infty$$

and $\sum_{n=1}^{\infty} \exp((\varepsilon/2)\int_{t_1}^{t_n} \gamma_c(s)\, ds) < \infty$ *for every* $\varepsilon \in (0,\hat{\varepsilon})$ *and some* $\hat{\varepsilon} \in (0,1)$. *If* $E[|Y_{t_0}|_{V_{t_0}}] < \infty$ *or* ${}_1\lambda = \cdots = {}_l\lambda = 0$, *then*

$$\limsup_{t\uparrow\infty} \frac{1}{\varphi(t)} \log(|Y_t|_{V_t}) \leq \frac{1}{2} \limsup_{n\uparrow\infty} \frac{1}{\varphi(t_n)} \int_{t_1}^{t_n} \gamma_c(s)\, ds \quad a.s.$$

for any increasing function $\varphi : [t_1,\infty) \to \mathbb{R}_+$ *that is positive on* (t_1,∞).

Proof. As $\gamma_c = \gamma_{c,0} + \sum_{k=1}^{l} \beta_k c_k^{\beta_k} E[{}_k\lambda^{(0)}]$, we have $\gamma_{c,0} \leq 0$ a.e. on $[t_1,\infty)$. Thus, if $|Y_{t_0}|_{V_{t_0}}$ is integrable, then we may choose $\gamma = \hat{\alpha}\gamma_{c,0}$ with $\hat{\alpha} \in [0,1/2]$ in Proposition 4.10 to get $\hat{c} > 0$ such that

$$E\Big[\sup_{s\in[t_n,t_{n+1}]} |Y_s|_{V_s}\Big] \leq \hat{c} e^{\frac{1}{2}\int_{t_1}^{t_n} \gamma_c(s)\, ds}$$

for every $n \in \mathbb{N}$. In the case that ${}_1\lambda = \cdots = {}_l\lambda = 0$ we set $A_k := \{|Y_{t_0}|_{V_{t_0}} \leq k\}$ for fixed $k \in \mathbb{N}$ and note that $Y\mathbb{1}_{A_k}$ is a random Itô process with drift $\hat{\mathrm{B}}\mathbb{1}_{A_k}$ and diffusion $\hat{\Sigma}\mathbb{1}_{A_k}$.

Since (A.4) and (A.5) are satisfied by $(\hat{\mathrm{B}}\mathbb{1}_{A_k}, \hat{\Sigma}\mathbb{1}_{A_k})$ instead of $(\hat{\mathrm{B}}, \hat{\Sigma})$ such that (A.8) holds, Proposition 4.10 gives $\hat{c}_k > 0$ so that

$$E\Big[\sup_{s\in[t_n,t_{n+1}]} |Y_s\mathbb{1}_{A_k}|_{V_s}\Big] \leq \hat{c}_k e^{\frac{1}{2}\int_{t_1}^{t_n}\gamma_c(s)\,ds}$$

for each $n \in \mathbb{N}$. In either case, the asserted implication follows from Lemma 4.11 and Remark 4.12, by noting that $\bigcup_{k\in\mathbb{N}} A_k = \Omega$. □

Remark 4.14. Let $\gamma \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$ satisfy $\gamma > 0$ a.e. on $(t_1, t_1 + \tilde{\delta})$ for some $\tilde{\delta} > 0$. Then the continuous function $[t_1, \infty) \to \mathbb{R}_+$, $t \mapsto \int_{t_0}^t \gamma(s)\,ds$ is positive on (t_1, ∞). Thus,

$$\limsup_{t\uparrow\infty} \frac{\log(|Y_t|_{V_t})}{\int_{t_0}^t \gamma(s)\,ds} \leq -\frac{1}{2}(1 - \hat{\gamma}) \quad \text{a.s.}$$

as soon as $\gamma_c \leq -\gamma$ a.e. on $[t_1, \infty)$ with $\hat{\gamma} := \int_{t_0}^{t_1}\gamma(s)\,ds / \int_{t_0}^{\infty}\gamma(s)\,ds \in [0, 1)$. In this case, the sharpest bound is attained for $\hat{\gamma} = 0$, which occurs if and only if $\gamma = 0$ a.e. on $[t_0, t_1]$ or γ fails to be integrable.

5 Proofs of the main results

5.1 Proofs for admissible spaces of probability measures

Proof of Proposition 2.2. Let S be a metric space, $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{P})$ be a probability space and $X : S \times \tilde{\Omega} \to \mathbb{R}^m$ be a continuous process so that $\tilde{P}_{X_s} \in \mathscr{P}$ for all $s \in S$. By condition (i), the sequence $(F_n)_{n\in\mathbb{N}}$ of $\mathscr{P}$-valued maps on S defined via $F_n(s) := \tilde{P}_{\varphi_n \circ X_s}$ satisfies

$$\lim_{n\uparrow\infty} F_n(s) = \tilde{P}_{X_s} \quad \text{in } \mathscr{P} \quad \text{for any given } s \in S.$$

Since $(X_{s_k})_{k\in\mathbb{N}}$ converges pointwise to X_s for any sequence $(s_k)_{k\in\mathbb{N}}$ in S converging to s, it follows from (ii) that $\lim_{k\uparrow\infty} F_n(s_k) = F_n(s)$ for each $n \in \mathbb{N}$. Thus, the map $S \to \mathscr{P}$, $s \mapsto \tilde{P}_{X_s}$ is Borel measurable, as pointwise limit of a sequence of continuous maps. □

Proof of Corollary 2.4. We show that the conditions of Proposition 2.2 are met by the sequence $(\varphi_n)_{n\in\mathbb{N}}$ of radial retractions, introduced in Example 2.3.

Let $\mu \in \mathscr{P}$ and note that condition (i) directly yields $\mu \circ \varphi_n^{-1} \in \mathscr{P}$ for fixed $n \in \mathbb{N}$. If we define $\phi_n : \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$ by $\phi_n(x) := (\varphi_n(x), x)$, then $\theta_n := \mu \circ \phi_n^{-1}$ belongs to $\mathscr{P}(\mu \circ \varphi_n^{-1}, \mu)$ and

$$\int_{\mathbb{R}^m\times\mathbb{R}^m} \rho(|x - y|)\,\theta_n(dx, dy) = \int_{\mathbb{R}^m} \rho(|\varphi_n(x) - x|)\,\mu(dx),$$

by the measure transformation formula. Since $\rho(|\varphi_n(x) - x|) \leq \rho(|x|)$ for all $x \in \mathbb{R}^m$, the integral on the right-hand side converges to zero as $n \uparrow \infty$, by dominated convergence. Thus, from (ii) we infer that $\lim_{n\uparrow\infty} \mu \circ \varphi_n^{-1} = \mu$ in $\mathscr{P}$.

Now let $(\mu_k)_{k\in\mathbb{N}}$ be a sequence in $\mathscr{P}$ that converges stochastically to some $\mu \in \mathscr{P}$. That is, there is a sequence $(\theta_k)_{k\in\mathbb{N}}$ of Borel measures on $\mathbb{R}^m \times \mathbb{R}^m$ such that $\theta_k \in \mathscr{P}(\mu_k, \mu)$ for any $k \in \mathbb{N}$ and (2.4) holds for every $\delta > 0$.

For the map $\psi_n : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$ given by $\psi_n(x, y) := (\varphi_n(x), \varphi_n(y))$ the measure $\hat{\theta}_k := \theta_k \circ \psi_n^{-1}$ lies in $\mathscr{P}(\mu_k \circ \varphi_n^{-1}, \mu \circ \varphi_n^{-1})$ and

$$\int_{\mathbb{R}^m\times\mathbb{R}^m} \rho(|x - y|)\,\hat{\theta}_k(dx, dy) = \int_{\mathbb{R}^m\times\mathbb{R}^m} \rho(|\varphi_n(x) - \varphi_n(y)|)\,\theta_k(dx, dy)$$

for all $n \in \mathbb{N}$. For given $\varepsilon > 0$ we choose $\delta > 0$ and $k_0 \in \mathbb{N}$ such that $\rho(2\delta) < \varepsilon/2$ and $\theta_k(\{(x,y) \in \mathbb{R}^m \times \mathbb{R}^m \,|\, |x-y| \geq \delta\}) < (\varepsilon/2)(1+\rho(2n))$ for any $k \in \mathbb{N}$ with $k \geq k_0$. Then the integral on the right-hand side cannot exceed ε for every such $k \in \mathbb{N}$. This shows $\lim_{k\uparrow\infty} \mu_k \circ \varphi_n^{-1} = \mu \circ \varphi_n^{-1}$ in $\mathscr{P}$ and the assertion follows. □

5.2 Proofs of the moment estimates, uniqueness and moment stability

Proof of Proposition 3.5. We define $\hat{\mathrm{B}} \in \mathscr{S}(\mathbb{R}^m)$ and $\hat{\Sigma} \in \mathscr{S}(\mathbb{R}^{m\times d})$ by

$$\hat{\mathrm{B}}_s := \mathrm{B}_s(X_s, P_{X_s}) - \tilde{\mathrm{B}}_s(\tilde{X}_s, P_{\tilde{X}_s}) \quad \text{and} \quad \hat{\Sigma}_s := \Sigma_s(X_s) - \Sigma_s(\tilde{X}_s). \tag{5.1}$$

Then the difference Y of X and $\tilde{X}$ is a random Itô process with drift $\hat{\mathrm{B}}$ and diffusion $\hat{\Sigma}$ such that

$$\mathrm{sgn}({}_iU'Y)\big({}_iU'\hat{\mathrm{B}} + {}_i\dot{U}'Y\big) \leq \varepsilon^{(i)} + c_0\zeta_i|Y|_U + \eta^{(i)}\rho(|Y|_U) + \lambda^{(i)}\varrho\circ\theta$$

a.e. on $[t_0,\infty)$ a.s. for all $i \in \{1,\ldots,m\}$ with the measurable function $\theta := \vartheta(P_X, P_{\tilde{X}})$. Hence, the assertion follows from an application of Theorem 4.5. □

Proof of Corollary 3.9. To show uniqueness in both cases, we suppose that X and $\tilde{X}$ are two solutions to (1.2). In case (i) we require the local integrability of $E[\lambda^{(0)}]\varrho(\vartheta(P_X, P_{\tilde{X}}))$ or

$$E[\lambda^{(0)}]\varrho(\vartheta(P_X, P_{\tilde{X}})) + \eta E[\rho(|X-\tilde{X}|_U)] + c_0\zeta_0 E[|X-\tilde{X}|_U],$$

depending on whether $\Phi_{\rho_0}(\infty)$ is infinite or finite, respectively. Note that in the second case the local integrability of $\Theta(\cdot, P_X, P_{\tilde{X}}, P_{X-\tilde{X}})$ is a sufficient condition for this, as Jensen's inequality and the continuity of ρ show.

Moreover, $E[|X_t - \tilde{X}_t|]$ vanishes for all $t \in [t_0,\infty)$ in either case, due to Proposition 3.5. Indeed, $(0,w) \in D_{\varrho_0}$ and $\Psi_{\varrho_0}(0,w) = 0$ for all $w \in \mathbb{R}_+$, as $\Phi_{\varrho_0}(0) = -\infty$. So, $X = \tilde{X}$ a.s., by the continuity of paths.

In case (ii) we set $\tau_n := \inf\{t \in [t_0,\infty) \,|\, |X_t| \geq n \text{ or } |\tilde{X}_t| \geq n\}$ and let φ_n be the radial retraction of the ball $\{x \in \mathbb{R}^m \,|\, |x| \leq n\}$ for fixed $n \in \mathbb{N}$, as in Example 2.3. Further, we define two admissible maps ${}_n\hat{\mathrm{B}}$ and ${}_n\Sigma$ on $[t_0,\infty)\times\Omega\times\mathbb{R}^m$ with values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively, via

$${}_n\hat{\mathrm{B}}_s(x) := \hat{\mathrm{B}}_s(\varphi_n(x))\mathbb{1}_{\{\tau_n > s\}} \quad \text{and} \quad {}_n\Sigma_s(x) := \Sigma_s(x)\mathbb{1}_{\{\tau_n > s\}}.$$

Then X^{τ_n} and $\tilde{X}^{\tau_n}$ solve (1.2) when (B,Σ) is replaced by $({}_n\hat{\mathrm{B}}, {}_n\Sigma)$. We see that (C.2) holds for ${}_n\Sigma$ instead of Σ and (C.3) is satisfied in the case that $\mathrm{B} = \tilde{\mathrm{B}} = {}_n\hat{\mathrm{B}}$, $\varepsilon = 0$ and $(\eta^{(i)}, \lambda^{(i)}) = (\eta_n, 0)$ for each $i \in \{1,\ldots,m\}$. Thus, Proposition 3.5 gives $X^{\tau_n} = \tilde{X}^{\tau_n}$ a.s. From $\sup_{n\in\mathbb{N}} \tau_n = \infty$ we conclude that $X = \tilde{X}$ a.s. □

Proof of Proposition 3.13. As in the proof of Proposition 3.5, we let $\hat{\mathrm{B}} \in \mathscr{S}(\mathbb{R}^m)$ and $\hat{\Sigma} \in \mathscr{S}(\mathbb{R}^{m\times d})$ be given by (5.1). Then $\mathrm{sgn}({}_iU'Y)\big({}_iV'\hat{\mathrm{B}} + {}_i\dot{U}'Y\big)$ is bounded by

$$\varepsilon^{(i)} + c_0\zeta_i|Y|_U + \sum_{k=1}^{l}\bigg(\sum_{j=1}^{m} {}_k\eta^{(i,j)}|{}_jU'Y|^{\alpha_k}\bigg) + {}_k\lambda^{(i)}\vartheta(P_X, P_{\tilde{X}})^{\beta_k}$$

a.e. on $[t_0,\infty)$ a.s. for any $i \in \{1,\ldots,m\}$. For this reason, all assertions are implied by Theorem 4.6. □

Proof of Corollary 3.15. By Definition 2.11, the stability assertions are directly implied by Proposition 3.13 in the case that $\mathrm{B} = \tilde{\mathrm{B}}$ and $\varepsilon = 0$. □

Proof of Corollary 3.16. (i) Both claims follow from the reasoning in Remark 4.7 when $\gamma_{\mathscr{P}}$ takes the role of γ_c there.

(ii) As before, let X and $\tilde{X}$ be two solutions to (1.2) for which $E[|X_{t_0} - \tilde{X}_{t_0}|]$ is finite and $E[\lambda^{(0)}]\vartheta(P_X, P_{\tilde{X}})$ is locally integrable. Then Proposition 3.13 gives

$$E\big[|X_t - \tilde{X}_t|\big] \leq \sqrt{m} e^{\int_{t_0}^t \gamma_{\mathscr{P}}(s)\,ds} E\big[|X_{t_0} - \tilde{X}_{t_0}|\big]$$

for each $t \in [t_0, \infty)$. Thus, for the first assertion let $P(X_{t_0} \neq \tilde{X}_{t_0}) > 0$, as otherwise $E[|X - \tilde{X}|]$ vanishes. Then from (C.8) we get that

$$\limsup_{t\uparrow\infty} \frac{1}{t^{\alpha_l}} \log\left(E\big[|X_t - \tilde{X}_t|\big]\right) \leq \limsup_{t\uparrow\infty} \sum_{k=1}^{l} \hat{\lambda}_k \frac{(t-s_k)^{\alpha_k} - (t_1 - s_k)^{\alpha_k}}{t^{\alpha_l}} = \hat{\lambda}_l,$$

since $\lim_{t\uparrow\infty}(t - s_k)^{\alpha_k}/t^{\alpha_l} = \mathbb{1}_{\{l\}}(k)$ for any $k \in \{1, \dots, l\}$. Now Remark 2.12 yields the correct result.

For the second assertion it is sufficient to consider the case $l = 1$. First, we set $\hat{c}_0 := \max_{t\in[t_0,t_1]} \exp(\int_{t_0}^t \gamma_{\mathscr{P}}(s)\,ds - \hat{\lambda}_1(t - t_0)^{\alpha_1})$ and get that

$$E\big[|X_t - \tilde{X}_t|\big] \leq \sqrt{m}\hat{c}_0 e^{\hat{\lambda}_1(t-t_0)^{\alpha_1}} E\big[|X_{t_0} - \tilde{X}_{t_0}|\big] \tag{5.2}$$

for each $t \in [t_0, t_1]$. Since $\int_{t_1}^t \gamma_{\mathscr{P}}(s)\,ds \leq \hat{\lambda}_1((t - s_1)^{\alpha_1} - (t_1 - s_1)^{\alpha_1})$ for all $t \in [t_1, \infty)$ and $s_1 \leq t_0$, we see that (5.2) holds for all $t \in [t_0, \infty)$ if we replace $\hat{c}_0$ by the constant

$$\hat{c} := \hat{c}_0 \vee \left(e^{\int_{t_0}^{t_1} \gamma_{\mathscr{P}}(s)\,ds - \hat{\lambda}_1(t_1 - s_1)^{\alpha_1}}\right).$$

□

5.3 Proofs for pathwise stability and moment growth estimates

Proof of Proposition 3.18. As we know, Y is a random Itô process with drift $\hat{\mathrm{B}} \in \mathscr{S}(\mathbb{R}^m)$ and diffusion $\hat{\Sigma} \in \mathscr{S}(\mathbb{R}^{m\times d})$ given by (5.1) when $\tilde{\mathrm{B}} = \mathrm{B}$. Therefore, Theorem 4.13 entails the claim. □

For the proof of Corollary 3.19 we need to check whether the series in (C.11) converges when the upper bound for $\gamma_{\mathscr{P}}$ in (C.8) is used.

Lemma 5.1. *Let $l \in \mathbb{N}$, $\alpha \in (0,\infty)^l$ and $\beta, s \in \mathbb{R}^l$ satisfy $\alpha_1 < \cdots < \alpha_l$, $\beta_l < 0$ and $\max_{k\in\{1,\dots,l\}} s_k \leq t_1$ for some $t_1 \in \mathbb{R}_+$. Then*

$$\int_0^\infty \exp\left(\varepsilon \sum_{k=1}^{l} \beta_k \int_{t_1}^{t_1+\delta t} \alpha_k (s - s_k)^{\alpha_k - 1}\,ds\right) dt$$

is finite for all $\delta, \varepsilon > 0$.

Proof. As β may be replaced by $\varepsilon\beta$, it suffices to consider the case $\varepsilon = 1$. So, we set $\Delta_k := t_1 - s_k$ for all $k \in \{1, \dots, l\}$ and $c := \sum_{k=1}^l \beta_k \Delta_k^{\alpha_k}$. Then

$$\sum_{k=1}^{l} \beta_k \int_{t_1}^{t_1+\delta t} \alpha_k (s - s_k)^{\alpha_k - 1}\,ds = c - \sum_{k=1}^{l} \beta_k (\delta t + \Delta_k)^{\alpha_k} \leq \frac{\beta_l}{2}(\delta t)^{\alpha_l}$$

for all $t \in [t_2, \infty)$ and some $t_2 > 0$, since $\lim_{t\uparrow\infty}(\delta t + \Delta_k)^{\alpha_k}/(\delta t)^{\alpha_l} = \mathbb{1}_{\{l\}}(k)$ for each $k \in \{1, \dots, l\}$. Further, a substitution shows us that

$$\int_0^\infty e^{-\gamma(\delta t)^{\alpha_l}}\,dt = \frac{\Gamma(1/\alpha_l)}{\alpha_l \gamma^{\alpha_l}\delta} \quad \text{for any } \gamma > 0.$$

Hence, the claim follows. □

Proof of Corollary 3.19. To apply Proposition 3.18, we verify (C.11). For this purpose, we choose $\hat{t}_1 \in [t_1,\infty)$ such that $\gamma_{\mathscr{P}} \leq 0$ a.e. on $[\hat{t}_1,\infty)$ and define a sequence $(t_n)_{n\in\mathbb{N}\backslash\{1\}}$ in $[t_0,\infty)$ by $t_n := \hat{t}_1 + \tilde{\delta}(n-1)$ for some $\tilde{\delta} > 0$. The integral test for the convergence of series, recalled in (4.18), shows that

$$\sum_{n=1}^{\infty} e^{\varepsilon\int_{\hat{t}_1}^{t_n}\gamma_{\mathscr{P}}(s)\,ds} < \infty \quad \text{if and only if} \quad \int_0^{\infty}\exp\left(\varepsilon\int_{\hat{t}_1}^{\hat{t}_1+\tilde{\delta}t}\gamma_{\mathscr{P}}(s)\,ds\right)dt < \infty$$

for any given $\varepsilon > 0$ and the latter condition is always satisfied, due to the imposed upper bound on $\gamma_{\mathscr{P}}$ in (C.8) and Lemma 5.1.

Thus, for (C.11) to be valid, it suffices to take $\tilde{\delta} < \hat{\delta}$. Then the difference Y of any two solutions X and $\tilde{X}$ to (1.2) for which $E[\lambda^{(0)}]\vartheta(P_X,P_{\tilde{X}})$ is locally integrable satisfies

$$\limsup_{t\uparrow\infty}\frac{\log\left(|Y_t|\right)}{t^{\alpha_l}} \leq \frac{1}{2}\limsup_{n\uparrow\infty}\sum_{k=1}^{l}\hat{\lambda}_k\frac{(t_n-s_k)^{\alpha_k}-(\hat{t}_1-s_k)^{\alpha_k}}{t_n^{\alpha_l}} = \frac{\hat{\lambda}_l}{2} \quad \text{a.s.,}$$

by Proposition 3.18, since $\lim_{t\uparrow\infty}(t-s_k)^{\alpha_k}/t^{\alpha_l} = \mathbb{1}_{\{l\}}(k)$ for each $k \in \{1,\dots,l\}$, as we have checked previously. □

Proof of Lemma 3.22. Because X is a random Itô process with drift $\hat{\mathrm{B}} \in \mathscr{S}(\mathbb{R}^m)$ and $\hat{\Sigma} \in \mathscr{S}(\mathbb{R}^{m\times d})$ given by $\hat{\mathrm{B}} := \mathrm{B}(X,P_X)$ and $\hat{\Sigma} := \Sigma(X)$, the claim is an immediate consequence of Theorem 4.5. □

Proof of Lemma 3.23. By the same reasoning as in Lemma 3.22, the assertions follow from an application of Theorem 4.6. □

5.4 Derivation of strong solutions with locally bounded law

Proof of Proposition 3.26. (i) As the partial uniform continuity condition (C.5) holds in the case that $\mathrm{B} = b_\mu$, pathwise uniqueness for (3.21) is implied by Corollary 3.9.

(ii) We shall first suppose that ξ is essentially bounded. Then the support of $\mathscr{L}(\xi)$ is compact and it essentially follows from Theorem 2.3 in [25][Chapter IV] that there is a local weak solution $\tilde{X}$ to (3.21).

By using the one-point compactification of $\mathbb{R}^m$, we can view $\tilde{X}$ as an $\mathbb{R}^m\cup\{\infty\}$-valued adapted continuous process on a filtered probability space $(\tilde{\Omega},\tilde{\mathscr{F}},(\tilde{\mathscr{F}}_t)_{t\in\mathbb{R}_+},\tilde{P})$ that allows for an $(\tilde{\mathscr{F}}_t)_{t\in\mathbb{R}_+}$-Brownian motion $\tilde{W}$ such that the usual and the following conditions are satisfied:

(1) If $\tilde{X}_s(\omega) = \infty$ for some $(s,\omega) \in [t_0,\infty)\times\Omega$, then $\tilde{X}_t(\omega) = \infty$ for all $t \in [s,\infty)$.

(2) $\mathscr{L}(\tilde{X}_{t_0}) = \mathscr{L}(\xi)$ and the supremum τ of the sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times given by $\tau_n := \inf\{t \in [t_0,\infty)\,|\,|\tilde{X}_t| \geq n\}$ satisfies $\tau > t_0$ a.s.

(3) For any $n \in \mathbb{N}$ the process $\tilde{X}^{\tau_n}$ is a solution to (1.2) on $(\tilde{\Omega},\tilde{\mathscr{F}},(\tilde{\mathscr{F}}_t)_{t\in\mathbb{R}_+},\tilde{P})$ relative to $\tilde{W}$ when B and Σ are replaced by the admissible maps

$$[t_0,\infty)\times\tilde{\Omega}\times\mathbb{R}^m \to \mathbb{R}^m, \quad (s,\tilde{\omega},x) \mapsto b_\mu(s,x)\mathbb{1}_{\{\tau_n>s\}}(\tilde{\omega})$$

and $[t_0,\infty)\times\tilde{\Omega}\times\mathbb{R}^m \to \mathbb{R}^{m\times d}$, $(s,\tilde{\omega},x) \mapsto \sigma(s,x)\mathbb{1}_{\{\tau_n>s\}}(\tilde{\omega})$, respectively.

We readily observe that (D.2) implies the partial growth condition (C.13) for b_μ instead of B. In fact, we may define ${}_\mu\kappa \in \mathscr{L}^1_{loc}(\mathbb{R}_+)$ by ${}_\mu\kappa_i := \kappa_i + \chi_i\varphi(\vartheta(\mu,\delta_0))$ and get that

$$\mathrm{sgn}(u_i(s)'x)u_i(s)'b_\mu(s,x)\mathbb{1}_{\{\hat{\tau}>s\}} \leq {}_\mu\kappa_i(s) + \upsilon_i(s)\phi(|x|_{u(s)})$$

for any $x \in \mathbb{R}^m$ for a.e. $s \in [t_0, \infty)$ for all $i \in \{1, \dots, m\}$ and each stopping time $\hat{\tau}$. Further, $|u_i(s)'\sigma(s, \cdot)\mathbb{1}_{\{\hat{\tau}>s\}}| \le |u_i(s)'\sigma(s, \cdot)|$ for each $s \in [t_{0,}\infty)$. Hence, we set $f := c_0\zeta_0 + \upsilon_0$ and infer from Fatou's lemma that

$$\tilde{E}[|\tilde{X}_t^\tau|] \le \liminf_{n\uparrow\infty} \tilde{E}[|\tilde{X}_t^{\tau_n}|] \le \Psi_{\phi_0}\bigg(E[|\xi|_{u(t_0)}] + \int_{t_0}^t {}_\mu\kappa_0(s)\,ds, \int_{t_0}^t f(s)\,ds\bigg) \tag{5.3}$$

for all $t \in [t_0, \infty)$ with $\phi_0 \in C(\mathbb{R}_+)$ given by $\phi_0(v) := (c_0 v) \vee \phi(v)$, by the virtue of Lemma 3.22. In particular, $\tau = \infty$ and $\tilde{X} \in \mathbb{R}^m$ $\tilde{P}$-a.s. So, $X : [t_0, \infty) \times \tilde{\Omega} \to \mathbb{R}^m$ given by $X_t(\tilde{\omega}) := \tilde{X}_t(\tilde{\omega})$, if $\tau(\tilde{\omega}) = \infty$, and $X_t(\tilde{\omega}) := 0$, otherwise, is a weak solution to (3.21) and $\tilde{E}[|X|]$ is locally bounded.

Now we remove the boundedness hypothesis on ξ. As we have shown that to each $x \in \mathbb{R}^m$ there is a weak solution to (3.21) with inital value condition x, it follows from Remark 2.1 in [25][Chapter IV] that there exists a weak solution X to (3.21) satisfying $X_{t_0} = \xi$ a.s. Further, its moment function is locally bounded if ξ is integrable, as it cannot exceed the right-hand estimate in (5.3) in this case, due to Lemma 3.22.

(iii) By the first two assertions, we have pathwise uniqueness for (3.21) and there is a weak solution for any $\mathbb{R}^m$-valued $\mathscr{F}_{t_0}$-measurable random vector serving as initial condition. As postulated by Theorem 1.1 in [25][Chapter IV], there is a unique strong solution $X^{\xi,\mu}$ to (3.21) with $X_{t_0}^{\xi,\mu} = \xi$ a.s. □

Proof of Theorem 3.27. (i) and (ii) Pathwise uniqueness is an immediate consequence of Corollary 3.9 and Remark 3.10. In particular, there is at most a unique solution X to (2.7) such that $X_{t_0} = \xi$ a.s. and $E[|X|]$ is locally bounded, since λ_0 is locally integrable.

Regarding existence, we recall that for any $\mu \in B_{b,loc}(\mathscr{P})$ there is a unique strong solution $X^{\xi,\mu}$ to (3.21) such that $X_{t_0}^{\xi,\mu} = \xi$ a.s. and, as ξ is integrable, $E[|X^{\xi,\mu}|]$ is locally bounded, by Proposition 3.26. The definition of b_μ entails that $X^{\xi,\mu}$ solves (2.7) if μ is a fixed-point of the operator

$$\Psi : B_{b,loc}(\mathscr{P}) \to B_{b,loc}(\mathscr{P}_1(\mathbb{R}^m)), \quad \Psi(\nu)(t) := \mathscr{L}(X_t^{\xi,\nu}).$$

In this case, $X^{\xi,\mu}$ must be a strong solution. For any $\mu, \tilde{\mu} \in B_{b,loc}(\mathscr{P})$ we readily see that condition (C.6) holds for $(b_\mu, b_{\tilde{\mu}})$ instead of $(\mathrm{B}, \tilde{\mathrm{B}})$. For this reason, Proposition 3.13 yields that

$$\vartheta_1(\Psi(\mu), \Psi(\tilde{\mu}))(t) \le E[|X_t^{\xi,\mu} - X_t^{\xi,\tilde{\mu}}|] \le \int_{t_0}^t e^{\int_s^t \gamma_{\mathscr{P},0}(\tilde{s})\,d\tilde{s}} \lambda_0(s)\vartheta(\mu, \tilde{\mu})(s)\,ds \tag{5.4}$$

for all $t \in [t_0, \infty)$. We note that the function $[s, \infty) \to \mathbb{R}_+$, $t \mapsto \exp(\int_s^t \gamma_{\mathscr{P},0}(\tilde{s})\,d\tilde{s})\lambda_0(s)$ is increasing for any $s \in [t_0, \infty)$. Consequently, Gronwall's inequality entails that Ψ admits at most a unique fixed-point.

As $B_{b,loc}(\mathscr{P}_1(\mathbb{R}^m))$ is completely metrisable, existence and the error estimate (3.22), which implies the local uniform convergence assertion, follow from an application of the fixed-point theorem for time evolution operators in [29]. In fact,

$$\vartheta_1(\mu_m, \mu_n)(t) \le \Delta(t) \sum_{i=n}^{m-1} \frac{c_{\mathscr{P}}^i}{i} \bigg(\int_{t_0}^t e^{\int_s^t \gamma_{\mathscr{P},0}(\tilde{s})\,d\tilde{s}} \lambda_0(s)\,ds\bigg)^i \tag{5.5}$$

for any $m, n \in \mathbb{N}$ with $m > n$ and all $t \in [t_0, \infty)$, by induction and the triangle inequality. So, $(\mu_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $B_{b,loc}(\mathscr{P}_1(\mathbb{R}^m))$ and, due to (5.4), its limit μ is a fixed-point of Ψ. Hence, (3.22) follows from (5.5) by taking the limit $m \uparrow \infty$.

(iii) Since the bound in (3.23) is independent of $\mu \in B_{b,loc}(\mathscr{P}_1(\mathbb{R}^m))$, the set M is closed and convex. We set $f_{\mathscr{P},1} := \sum_{k=1}^l {}_k c_{\mathscr{P}}^{\beta_k} \beta_k \chi_0$ and $f_{\mathscr{P},0} = f_{\mathscr{P}} - f_{\mathscr{P},1}$. Then

$$\vartheta_1(\Psi(\mu)(t), \delta_0) \le e^{\int_{t_0}^t f_{\mathscr{P},0}(s)\,ds} E[|\xi|_{u(t_0)}] + \int_{t_0}^t e^{\int_s^t f_{\mathscr{P},0}(s)\,ds}(\kappa_0 + g_{\mathscr{P}} + f_{\mathscr{P},1}\vartheta_1(\mu, \delta_0))(s)\,ds$$

for each $\mu \in B_{b,loc}(\mathscr{P})$ and any $t \in [t_0, \infty)$, by Lemma 3.23 and Young's inequality. Then it follows from the Fundamental Theorem of Calculus for Lebesgue-Stieltjes integrals and Fubini's theorem that Ψ maps M into itself. Hence, the claim holds. □

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