A GEL'FAND TRIPLE APPROACH TO THE SMALL NOISE PROBLEM FOR DISCONTINUOUS ODE'S

OLIVIER MENOUKEU-PAMEN, THILO MEYER-BRANDIS, AND FRANK PROSKE

This Version : May 3, 2011

Abstract. In this paper, we develop a variational approach to study perturbation problems of ordinary differential equations (ODE's) with discontinuous coefficients. We propose a mathematical framework which can be used to construct stable (and regular) solution processes of discontinuous ODE's.

Key words and phrases: Malliavin calculus, local time, small random perturbations, strong solutions of SDE's.
MSC2010: 60H05, 60G44, 60G48.

1. Introduction

In this paper, we aim at analyzing the small noise problem of discontinuous ODE's. More precisely, we want to provide conditions under which the solutions X_t^n, $n \in \mathbb{N}$, of the stochastic differential equations (SDE's)

$$dX_t^n = b(t, X_t^n)dt + \frac{1}{n} dB_t,\ 0 \leq t \leq 1,\ \ X_0^n = x \in \mathbb{R}^d\,, \tag{1.1}$$

for $n \to \infty$ converge to a solution (process) X_t of the ODE

$$dX_t = b(t, X_t)dt,\ 0 \leq t \leq 1,\ \ X_0 = x \in \mathbb{R}^d\,, \tag{1.2}$$

where the drift term $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is allowed to be a discontinuous function. Here $\{B_t\}_{0 \leq t \leq 1}$ is a d-dimensional $\mathcal{F}_t$-Brownian motion on a probability space $(\Omega, \mathcal{F}, \mu)$, where $\{\mathcal{F}_t\}_{0 \leq t \leq 1}$ is a μ-augmented filtration generated by B..

In the case of continuous drift coefficients b the small noise problem (1.1), (1.2) has been studied by various authors in the literature. See e.g [2, 3, 4, 8, 13, 25] and [26]. The author in [25] introduces the large deviation principle to study the convergence rate of solutions of (1.1) to (1.2) with (Lipschitz-) continuous coefficients. We mention that the authors in [2, 3] and [4] employ the Skorohod embedding in combination with certain boundary value problems to establish criteria for the convergence to solutions processes of (1.2). See also [26]. The work [4] deals with a selection principle based on viscosity solutions to construct Feller solutions of ill-posed degenerate diffusion processes. See also the interesting paper of [13] in the context of (stochastic) superposition solutions of ODE's (SDE's). We shall also refer the reader to [1] and the references therein.

The perturbation problem (1.1), (1.2) for discontinuous or even merely measurable drift terms b is in general challenging and sparsely covered by the current literature. See [7, 9, 15, 16]. In the interesting work [7] the authors use the Skorohod embedding technique

Date: May 3, 2011.

to derive (under fairly general conditions on b) generalized solutions to (1.2) in the sense of Filippov. Further, the papers [15, 16] are concerned with the convergence rate of the probability densities of X^n for some (concrete) non-Lipschitzian drift terms b. The method used in the latter papers are based on large deviation techniques and viscosity solutions of Hamilton-Jacobi equations. We also emphasize the work [9], where the authors develop large deviations techniques to treat ODE's for certain discontinuous coefficients b. Other techniques for the construction of solutions of discontinuous ODE's can be e.g. found in [6, 24].

Our approach to problem (1.1), (1.2) is different from the above mentioned authors' ones and is based on the use of Gel'fand triples

$$\mathbb{D}_{1,2} \hookrightarrow L^2(\mu) \hookrightarrow \mathbb{D}_{-1,2} \tag{1.3}$$

and

$$(\mathcal{S}) \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})^*. \tag{1.4}$$

$\mathbb{D}_{1,2}$ denotes the stochastic Sobolev space of Malliavin differentiable square integrable Brownian functionals and $\mathbb{D}_{-1,2}$ is its topological dual. Further, $(\mathcal{S})$ is the Hida test function space and $(\mathcal{S})^*$ the Hida distribution space. Here the symbol $\hookrightarrow$ stands for continuous inclusions of spaces. We mention that

$$(\mathcal{S}) \hookrightarrow \mathbb{D}_{1,2} \hookrightarrow L^2(\mu) \hookrightarrow \mathbb{D}_{-1,2} \hookrightarrow (\mathcal{S})^*. \tag{1.5}$$

For more information about Malliavin calculus the reader may consult [11, 18] or [21]. As for the construction of the triple (1.4) and its applications in white noise analysis, we recommend the books of [17] or [22].

To be more precise, our method to tackle the perturbation problem (1.1), (1.2) relies on a compactness criterion in $L^2(\mu)$ based on Malliavin calculus (see [10]), a "variational calculus" technique with respect to local time [12], and a compactness criterion for continuous functions with values in $(\mathcal{S})^*$. Using these tools, we are able to show (under certain stochastic conditions on b) that X^n in (1.1) converges in $L^2(\mu)$ (or even in $\mathbb{D}_{1,2}$) for a subsequence to a (possibly Malliavin differentiable) cluster point X_t, which solves the ODE, almost surely (or on a set with positive probability).

We point out that we obtain solutions of discontinuous ODE's which are stable under random perturbations. This approach also provides a natural selection procedure for solutions of discontinuous ODE's which, as one knows, have no unique solutions in general. See e.g [13] for a general discussion of this topic.

2. Main results

In this section, we want to introduce a new technique to study the behavior of the solutions $X^n_\cdot$ of SDE's (1.1) when $n \to \infty$. Before we proceed, we shall send ahead some notions and definitions which we will make use of later on in this paper.

In the following, let $S([0,1]) \subseteq L^2([0,1])$ be the Schwartz space on $[0,1]$ as e.g., constructed in [22]. Using the theorem of Bochner-Minlos, we shall denote by π the unique probability measure on the Borel sets $\mathcal{B}(S'([0,1]))$ of $S'([0,1])$ (topological dual of $S([0,1])$) such that

$$\int_{S'([0,1])} e^{i\langle \omega, \phi \rangle} \pi(d\omega) = e^{-\frac{1}{2}\|\phi\|^2_{L^2([0,1])}}$$

for all $\phi \in S([0,1])$, where $\langle \omega, \phi \rangle$ is the action of $\omega \in S'([0,1])$ on $\phi \in S([0,1])$.

From now on, we assume that the Brownian motion $B_t \in \mathbb{R}^d$ in (1.1) is defined on the probability space

$$(\Omega, \mathcal{F}, \mu) := \left(\prod_{i=1}^{d} \Omega_i,\ \otimes_{i=1}^{d} \mathcal{F}_i,\ \otimes_{i=1}^{d} \mu_i \right), \tag{2.1}$$

where $\Omega_i = S'([0,1]),\ \mathcal{F}_i = \mathcal{B}(S'([0,1])),\ \mu_i = \pi$ for $i = 1, \ldots, d$.

Further, we briefly recall the definition of the S-transform, which can be used to characterize elements of the Hida test function and distribution spaces. See [17]. The S-transform of a $\Phi \in (\mathcal{S})^*$, denoted by $S(\Phi)$ is defined as

$$S(\Phi)(\phi) = \langle \Phi, \widetilde{e}(\phi, \cdot) \rangle \tag{2.2}$$

for $\phi \in S_{\mathbb{C}}([0,1])^d$, where $S_{\mathbb{C}}([0,1])$ is the complexification of $S([0,1]))$ and $\widetilde{e}(\phi, \cdot) \in (\mathcal{S})$ is the exponential functional

$$\widetilde{e}(\phi, \omega) := \exp\left\{ \langle \omega, \phi \rangle - \frac{1}{2} \|\phi\|^2_{L^2([0,1];\mathbb{R}^d)} \right\}$$

for $\omega = (\omega_1, \ldots, \omega_d) \in \Omega,\ \ \Phi = (\Phi^{(1)}, \ldots, \Phi^{(d)}) \in (S([0,1]))^d$, and $\langle \omega, \phi \rangle = \sum_{i=1}^{d} \langle \omega_i, \phi_i \rangle$

In what follows, we shall denote by $D.$ the Malliavin derivative on $(\Omega, \mathcal{F}, \mu)$, which is a linear operator from $\mathbb{D}_{1,2}$ to $L^2(\lambda \otimes \mu)$ (λ Lebesgue measure). See e.g [11] or [21] for the definition of $D.$. We Mention that $\mathbb{D}_{1,2}$ in (1.3) is a Hilbert space with a norm $\|\cdot\|_{1,2}$ given by

$$\|F\|^2_{1,2} := \|F\|^2_{L^2(\mu)} + \|D.F\|^2_{L^2([0,1]\times\Omega, \lambda\otimes\mu)} \tag{2.3}$$

(for $d = 1$). We shall also use the notation δ for the adjoint operator of $D.$, which is referred to as divergence operator.

In this section, we also want to introduce the crucial concept of stochastic integration

$$\int_0^t \int_{\mathbb{R}} f(s, x) L(ds, dx) \tag{2.4}$$

over the plane with respect to Brownian local time $L(t, x)$ for integrands $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ in the Banach space $(\mathcal{H}, \|\cdot\|)$ with the norm

$$\begin{aligned} \|f\| := 2 \left(\int_0^1 \int_{\mathbb{R}} (f(s,x))^2 \exp(-\frac{x^2}{2s}) \frac{ds\, dx}{\sqrt{2\pi s}} \right)^{\frac{1}{2}} \\ + \int_0^1 \int_{\mathbb{R}} |x f(s,x)| \exp(-\frac{x^2}{2s}) \frac{ds\, dx}{s\sqrt{2\pi s}}. \end{aligned} \tag{2.5}$$

See [12]. We need the following auxiliary result ([12, Theorem 3.1, Corollary 3.2])

Lemma 2.1. *Let $f \in \mathcal{H}$. Suppose that for all $t \in [0,1]$ $f(t, \cdot)$, the derivative $f'(t, \cdot)$ (in the generalized sense with respect to the Lebesgue measure) exists and that*

$$\int_0^1 \int_{-A}^{A} \left| f'(s,x) \right| \frac{ds}{\sqrt{s}} dx < \infty$$

for all $A \geq 0$. Then

$$\int_0^t \int_{\mathbb{R}} f(s,x) L(ds, dx) = -\int_0^t f'(s, B_s) ds. \tag{2.6}$$

Later on in this paper, we shall also use the following decomposition of local time space integral (see the proof of Theorem 3.1 in [12])

$$\int_0^t \int_{\mathbb{R}} f_i(s,x) L_i(ds,dx) = \int_0^t f_i(s, B_s^{(i)}) dB_s^{(i)} + \int_{1-t}^1 f_i(1-s, \widehat{B}_s^{(i)}) d\widetilde{W}_s^{(i)} + \int_{1-t}^1 f_i(1-s, \widehat{B}_s^{(i)}) \frac{\widehat{B}_s^{(i)}}{1-s} ds, \tag{2.7}$$

$0 \le t \le 1$, a.e., for $f_i \in \mathcal{H}$, $i = 1, \dots, d$, where $L_i(ds,dx)$ denotes the local time-space with respect to $B^{(i)}$ (the i-th component of B) on (Ω_i, μ_i), $i = 1, \dots, d$. Here $\widehat{B}^{(i)}$ is the i-th component of the time-reversed Brownian motion, that is of

$$\widehat{B}_t := \left(\widehat{B}_t^{(1)}, \dots, \widehat{B}_t^{(d)}\right) := B_{1-t}, \tag{2.8}$$

$0 \le t \le 1$. Further $\widetilde{W}_t^{(i)}$, $0 \le t \le 1$, are independent μ_i-Brownian motions (see (2.1)) with respect to the filtration $\mathcal{F}_t^{\widehat{B}^{(i)}}$ generated by $\widehat{B}_t^{(i)}$, $i = 1, \dots, d$.

Now consider the SDE's (1.1)with Borel measurable drift $b : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$. For our main result (Theorem 2.2) we will need the existence of a sequence $b_p : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$, $p \in \mathbb{N}$, of approximating drift coefficients which fulfill the following five conditions. For notational convenience we set $b_0 := b$.

(C1): The coefficients b_p, $p \in \mathbb{N}$, are continuous with compact support such that $b_p(t, \cdot)$ is continuously differentiable, $0 \le t \le 1$, with bounded derivative on $[0,1] \times \mathbb{R}^d$. It is well known that bounded coefficients admit unique strong solutions $X_t^{n,p}$, $n \in \mathbb{N}, p \in \mathbb{N}$, of the SDE's

$$dX_t^{n,p} = b(t, X_t^{n,p}) dt + \frac{1}{n} dB_t, \ 0 \le t \le 1, \ \ X_0^{n,p} = x \in \mathbb{R}^d. \tag{2.9}$$

(C2): Let $\mathcal{M} \subset \mathbb{R}^{d \times d}$ denote the class of continuous matrix valued functions $\mathcal{M}(t) : [0,1] \to \mathbb{R}^{d \times d}$ such that $\mathcal{M}(t)$ commutes with $\int_s^t \mathcal{M}(u)\, du$ for all $0 \le s \le t \le 1$. Suppose that $b_p'(\cdot, X_\cdot^{n,p}) \in \mathcal{M}$ for all $n \in \mathbb{N}$, $p \in \mathbb{N}$, where the symbol $'$ stands for the derivative with respect to the space variable.

(C3): For each $n \in \mathbb{N}$

$$\sup_{p \ge 0} \left\| \exp\left\{ 512 \int_0^1 n^2 \left\| b_p(s, \frac{B_s}{n} + x) \right\|^2 ds \right\} \right\|_{L^1(\mu)} < \infty$$

and the sequence of coefficients b_p, $p \in \mathbb{N}$, approximates b in the sense that for each $n \in \mathbb{N}$

$$E[J_{n,p}] \underset{p \to \infty}{\longrightarrow} 0\,,$$

where

$$J_{n,p} = \sum_{j=1}^d \left(2 \int_0^1 \left(n\, b_p^{(j)}(s, \frac{B_s}{n} + x) - n\, b^{(j)}(s, \frac{B_s}{n} + x) \right)^2 ds + \left(\int_0^1 \left| (n\, b_p^{(j)}(s, \frac{B_s}{n} + x))^2 - (n\, b^{(j)}(s, \frac{B_s}{n} + x))^2 \right| ds \right)^2 \right). \tag{2.10}$$

(C4): Using the notation $(\cdot)_{0\le i,j\le d}$ for $\mathbb{R}^{d\times d}$-matrices, we require

$$\sup_{n,p\ge 1}\sup_{0\le t<t'\le 1}\left\|\prod_{i=1}^{4}A_i(n,p,t,t')\right\|_{L^1(\mu)}<\infty, \tag{2.11}$$

where

$$A_1(n,p,t,t')=\exp\left\{\int_0^1 n\,b_p(s,\frac{B_s}{n}+x)dB_s-\frac{1}{2}\int_0^1 n^2\left\|b_p(s,\frac{B_s}{n}+x)\right\|^2 ds\right\}. \tag{2.12}$$

$$\begin{aligned}A_2(n,p,t,t')=\Bigg\|\exp\Bigg\{\Bigg(&-\int_{t'}^{1} n\,b_p^{(j)}(s,\frac{B_s}{n}+x)dB_s^{(i)}-\int_0^{1-t'} n\,b_p^{(j)}(1-s,\frac{\widehat{B}_s}{n}+x)d\widetilde{W}_s^{(i)}\\ &+\int_0^{1-t'} n\,b_p(1-s,\frac{\widehat{B}_s}{n}+x)\frac{\widehat{B}_s^{(i)}}{1-s}ds\Bigg)_{0\le i,j\le d}\Bigg\}\Bigg\|^2.\end{aligned} \tag{2.13}$$

$$\begin{aligned}A_3(n,p,t,t')=\sup_{0\le\lambda\le 1}\Bigg\|\exp\Bigg\{\Bigg(&-\lambda\int_t^{t'} n\,b_p^{(j)}(s,\frac{B_s}{n}+x)dB_s^{(i)}-\lambda\int_{1-t'}^{1-t} n\,b_p^{(j)}(1-s,\frac{\widehat{B}_s}{n}+x)d\widetilde{W}_s^{(i)}\\ &+\lambda\int_{1-t'}^{1-t} n\,b_p^{(j)}(1-s,\frac{\widehat{B}_s}{n}+x)\frac{\widehat{B}_s^{(i)}}{1-s}ds\Bigg)_{0\le i,j\le d}\Bigg\}\Bigg\|^2,\end{aligned} \tag{2.14}$$

$$A_4(n,p,t,t')=\frac{1}{n^2}\frac{\|I_4(n,p,t,t')\|^2}{|t-t'|^{\alpha}},\ \ t\neq t' \tag{2.15}$$

for some $\alpha>\frac{1}{2}$ with

$$\begin{aligned}I_4(n,p,t,t')=\Bigg(&\int_t^{t'} n\,b_p^{(j)}(s,\frac{B_s}{n}+x)dB_s^{(i)}-\int_{1-t'}^{1-t} n\,b_p^{(j)}(1-s,\frac{\widehat{B}_s}{n}+x)d\widetilde{W}_s^{(i)}\\ &+\int_{1-t'}^{1-t} n\,b_p^{(j)}(1-s,\frac{\widehat{B}_s}{n}+x)\frac{\widehat{B}_s^{(i)}}{1-s}ds\Bigg)_{0\le i,j\le d},\end{aligned} \tag{2.16}$$

(C5):

$$\sup_{n,p\ge 1}\sup_{0\le t<t'\le 1}\left\|A_5(n,p,t,t')A_1(n,p,t,t')\right\|_{L^1(\mu)}<\infty, \tag{2.17}$$

where

$$A_5(n,p,t,t')=\frac{\left\|\int_t^{t'} b_p(s,\frac{B_s}{n}+x)ds\right\|^2}{|t-t'|^{\beta}},\ \ t\neq t' \tag{2.18}$$

for some $\beta>\frac{1}{2}$.

Theorem 2.2. *Consider the family of SDE's in* (1.1) *with Borel measurable drift coefficient* $b:[0,1]\times\mathbb{R}^d\to\mathbb{R}^d$. *Suppose there exists a sequence of approximating coefficients* $(b_p)_{p\ge 1}$ *such that* $\{b,(b_p)_{p\ge 1}\}$ *fulfill conditions (C1)-(C5). Then for all* $0\le t\le 1$ *the set of solutions* $(X_t^{n,p})_{n\ge 1,\,p\ge 1}$ *of* (2.9) *is relatively compact in* $L^2(\mu;\mathbb{R}^d)$. *Further, for all* $n\in\mathbb{N}$ *there exist* *a unique strong solution X_t^n of* (1.1) *and the sequence of solutions X_t^n to (1.1) is relatively compact in $L^2(\mu;\mathbb{R}^d)$, $0 \le t \le 1$, and there exists a cluster point $(X_t)_{0\le t\le 1}$ of $(X_t^n)_{0\le t\le 1}$, that is one finds a subsequence $(n_m)_{m\ge 1}$ such that*

$$\lim_{m\to\infty} X_t^{n_m} = X_t \text{ in } L^2(\mu;\mathbb{R}^d) \tag{2.19}$$

for all $0 \le t \le 1$. In particular, if $\|b(t,X_t^n)\|_{L^2(\mu)} \le M < \infty$, $n \ge 1$, t-a.e for some constant M, then

$$X_t = x + \int_0^t \lim_{m\to\infty} b(s, X_s^{n_m})ds \tag{2.20}$$

in $L^2(\mu)$.

Remark 2.3. *Note that in case of a bounded drift coefficient b there obviously exists a sequence of approximating coefficients $(b_p)_{p\ge 1}$ that fulfill conditions (C1), (C3), and (C5). In that case, the crucial conditions to check are (C2) and (C4).*

Remark 2.4. *In the case of dimension $d = 1$, the commutativity requirement (C2) is obviously always fulfilled. In the case $d = 2$, condition (C2) can be verified, if e.g.,*

$$b(t,x) = \begin{pmatrix} f(x_1 + x_2) \\ f(x_1 + x_2) \end{pmatrix},$$

where $f : \mathbb{R} \to \mathbb{R}$ is a bounded Borel measurable function. See [19] *for other examples and more general criteria.*

We postpone the proof of Theorem 2.2 to a later time point. In the sequel, we discuss some consequences of the previous result:

Corollary 2.5. *Retain the conditions in Theorem 2.2 and assume additionally that the drift coefficient b in (1.1) is continuous. Then there exists a Malliavin differentiable process X_t such that*

$$X_t = x + \int_0^t b(s, X_s)ds. \tag{2.21}$$

Proof. Equation (2.21) follows from (2.20) and the continuity of b. The Malliavin differentiability of X_t follows from a weak compactness argument. See the proof of Theorem 2.2. $\square$

The next two result treats the case of discontinuous ODE's:

Theorem 2.6. *Keep the conditions in Theorem 2.2 and assume additionally that the drift coefficient b in (1.1) is bounded. Further require that the process X_t in (2.19) doesn't hit the set of points of discontinuity of $b(t,\cdot)$ μ-a.e. for almost all (fixed) t. Then X_t solves the ODE*

$$X_t = x + \int_0^t b(s, X_s)ds. \tag{2.22}$$

Theorem 2.7. *Retain the conditions in Theorem 2.2 and require additionally that the drift coefficient b in (1.1) is bounded and time-homogeneous. Then*

$$X_t^{(i)} \in \mathbb{D}_{1,2}$$

for all $i = 1, \ldots, d$, $0 \le t \le 1$. Moreover, if the Malliavin matrix $\sigma_{X_t} = (\sigma_{X_t}^{i,j})_{1\le i,j\le d}$ with

$$\sigma_{X_t}^{i,j} = (D.X_t^{(i)}, D.X_t^{(j)})_{L^2([0,1])}$$

is invertible a.e for each t, then X_t is a solution of (2.22).

The proofs of these two theorems are also put off to a later time point.
The following result will be needed in the proof of Theorem 2.2.

Lemma 2.8. *Suppose that the conditions of Theorem 2.2 hold. Then the double sequence $(t \longmapsto X_t^{n,p},\ n,p \geq 1)$ is relatively compact in $C([0,1],(\mathcal{S})^*)$.*

Proof. Let ζ belong to the Hida test function space $(\mathcal{S})$. Denote by $\langle F, \rho\rangle$ the dual pairing for $F \in (\mathcal{S})^*,\ \rho \in (\mathcal{S})$. Using the Cauchy-Schwartz inequality, Girsanov's theorem and (C3), and **(C5)** we get that

$$\begin{aligned}\left|\langle X_{t_1}^{n,p} - X_{t_2}^{n,p}, \zeta\rangle\right| &= E\left[\left(X_{t_1}^{n,p} - X_{t_2}^{n,p}\right)\zeta\right] \leq E\left[\left\|X_{t_1}^{n,p} - X_{t_2}^{n,p}\right\|^2\right]^{\frac{1}{2}} E\left[|\zeta|^2\right]^{\frac{1}{2}} \\ &\leq C\,|t_2 - t_1|^{\beta}\, E\left[|\zeta|^2\right]^{\frac{1}{2}}\end{aligned}$$

for some $\beta > \frac{1}{2}$. On the other hand, we directly see that

$$\sup_{0 \leq t \leq T} \|X_t^{n,p}\|_{L^2(\mu)} \leq M$$

for all $n, p \geq 1$. The desired result then follows from Mitoma's theorem (see [20]) applied to the conuclear space $(\mathcal{S})^*$ and Arzelá-Ascoli's theorem with respect to $C([0,1])$. □

Proof. (Theorem 2.2).
We first want to employ a compactness criterion based on Malliavin calculus [10, Theorem 1] to show that $(X_t^{n,p})_{p\geq 0, n \geq 1}$ is relatively compact in $L^2(\mu;\mathbb{R}^d)$ for all $t \geq 0$. To this end we assume without loss of generality that $t = 1$. Our assumptions and the chain rule of the Malliavin derivative D_t (see e.g., [21]) imply that

$$D_t X_1^{n,p} = \frac{1}{n}\exp\left\{\int_t^1 b_p'(s, \frac{X_s^{n,p}}{n})ds\right\} \in \mathbb{R}^{d\times d},\ \ 0 \leq t \leq 1,\ \ n,p \geq 1. \tag{2.23}$$

Fix $0 \leq t < t' \leq 1$. Then using Girsanov's theorem we find that

$$\begin{aligned}&E\left[\|D_t X_1^{n,p} - D_{t'} X_1^{n,p}\|^2\right] \\ &= \frac{1}{n^2} E\left[\left\|\exp\left\{\int_t^1 b_p'(s, \frac{B_s}{n} + x)ds\right\} - \exp\left\{\int_{t'}^1 b_p'(s, \frac{B_s}{n} + x)ds\right\}\right\|^2 A_1\right],\end{aligned}$$

where $A_1 = \exp\left\{\int_0^1 n\, b_p(s, \frac{B_s}{n} + x)dB_s - \frac{1}{2}\int_0^1 n^2 \left\|b_p(s, \frac{B_s}{n} + x)\right\|^2 ds\right\}$.

Applying the properties of evolution operators for linear systems of ODE's and the mean value theorem, we have

$$
\begin{aligned}
&E\left[\|D_t X_1^{n,p} - D_{t'} X_1^{n,p}\|^2\right] \\
&= \frac{1}{n^2} E\left[\left\|\exp\left\{\int_{t'}^1 b_p'(s, \frac{B_s}{n} + x) ds\right\}\right\|^2 \left\|\exp\left\{\int_t^{t'} b_p'(s, \frac{B_s}{n} + x) ds\right\} - 1\right\|^2 A_1\right] \\
&\leq \frac{1}{n^2} C E\left[\left\|\exp\left\{\int_{t'}^1 b_p'(s, \frac{B_s}{n} + x) ds\right\}\right\|^2 \left\|\left\{\int_t^{t'} b_p'(s, \frac{B_s}{n} + x) ds\right\}\right\|^2 . \right. \\
&\qquad \left. \sup_{0\leq\lambda\leq 1} \left\|\exp\left\{\lambda \int_t^{t'} b_p'(s, \frac{B_s}{n} + x) ds\right\}\right\|^2 A_1\right]
\end{aligned}
$$

Consider the local time-space $L_i(ds, dx)$ with respect to $B^{(i)}$ (the i-th component of B) on (Ω_i, μ_i), $i = 1, \dots, d$. Using Lemma 2.1 and the decomposition (2.7), we get

$$
\begin{aligned}
&E\left[\|D_t X_1^{n,p} - D_{t'} X_1^{n,p}\|^2\right] \\
&\leq C E\left[\left\|\exp\left\{\left(-\int_{t'}^1 \int_{\mathbb{R}} n\, b_p^{(j)}(s, \frac{x}{n}) L_i(ds, dx)\right)_{1\leq i,j\leq d}\right\}\right\|^2 \right. \\
&\left\|\left\{\left(-\int_t^{t'} \int_{\mathbb{R}} n\, b_p^{(j)}(s, \frac{x}{n}) L_i(ds, dx)\right)_{1\leq i,j\leq d}\right\}\right\|^2 \\
&\left. \sup_{0\leq\lambda\leq 1} \left\|\exp\left\{\lambda\left(-\int_t^{t'} \int_{\mathbb{R}} n\, b_p^{(j)}(s, \frac{x}{n}) L_i(ds, dx)\right)_{1\leq i,j\leq d}\right\}\right\|^2 A_1\right] \\
&\leq C \left|t' - t\right|^{\alpha} \left(\sup_{n,p\geq 1} \sup_{0\leq t<t'\leq 1} \left\|\prod_{i=1}^4 A_i(n, p, t, t')\right\|_{L^1(\mu)}\right)
\end{aligned}
$$

for some constant C. In particular, we see that the family $(X_1^{n,p})_{p\geq 0, n\geq 1}$ is bounded in $\mathbb{D}_{1,2}$. Then the relative compactness of $(X_1^{n,p})_{p\geq 0, n\geq 1}$ follows from [[10], Lemma 1] in connection with [10, Theorem 1].

In the next step of the proof we aim at constructing a solution process X_t to the ODE's (1.2) based on the double sequence $(X_t^{n,p})_{p\geq 1, n\geq 1}$. Using the condition **(C3)** in connection with Theorem 4 in [19], we obtain that for all $n \geq 1$ there exists a subsequence $(p_{k,n})$ (independent of t) such that

$$
X_t^n = \lim_{k\to\infty} X_t^{n, p_{k,n}} \in L^2(\mu; \mathbb{R}^d)
$$

satisfies the SDE's (1.1). In particular, $(X_t^n)_{n\geq 1}$ is relatively compact in $L^2(\mu; \mathbb{R}^d)$ for each t. We also mention that X_t^n is Malliavin differentiable for all n, t by a weak compactness argument (see [19, Lemma 1,2,3]).

On the other hand, it follows from Lemma 2.8 that there exists a subsequence (n_k) such that

$$X_t^{n_k} \underset{k\to\infty}{\longrightarrow} X_t \text{ in } (\mathcal{S})^*$$

uniformly in t. The latter and the uniqueness of chaos decompositions in $(\mathcal{S})^*$ entail that

$$X_t^{n_k} \underset{k\to\infty}{\longrightarrow} X_t \text{ in } L^2(\mu;\mathbb{R}^d)$$

for all t.

Finally, if the drift coefficient is bounded, we can apply dominated convergence for functions from $[0,1]$ to $L^2(\mu;\mathbb{R}^d)$ and obtain (2.20). □

Proof. (Theorem 2.6).
We shall argue by contradiction. Assume that $b(t, X_t^n)$ does not converge to $b(t, X_t)$ in $L^2(\mu)$ for some t for which the points of discontinuity cannot be reached. Then there exists a $\epsilon > 0$ and a subsequence (n_k) such that

$$\|b(t, X_t^{n_k}) - b(t, X_t)\|_{L^2(\mu)} > \epsilon. \tag{2.24}$$

We know that

$$X_t^{n_{\tilde{n}_l(t)}} \longrightarrow X_t \text{ a.e.}$$

for some subsequence $(\tilde{n}_l(t))$. Using the fact that X_t doesn't hit the points of discontinuity of $b(t,\cdot)$ a.e., we see that

$$b(t, X_t^{n_{\tilde{n}_l(t)}}) \longrightarrow b(t, X_t) \text{ a.e.}$$

Since b is bounded, it follows from the dominated convergence theorem that

$$\left\| b(t, X_t^{n_{\tilde{n}_l(t)}}) - b(t, X_t)\right\|_{L^2(\mu)} \underset{l\to\infty}{\longrightarrow} 0.$$

For $k = \tilde{n}_l(t)$, this leads to a contradiction to (2.24). Therefore

$$\lim_{n\to\infty} b(t, X_t^n) = b(t, X_t) \text{ in } L^2(\mu), \ \ t\text{-a.e.}$$

□

Proof. (Theorem 2.7).
We recall that each X_s^n is Malliavin differentiable (see [19]). We want to justify that we may set $b_p = b$ for all $p \geq 1$ in the proof of Theorem 2.2. To this end we shall derive a certain representation for $D_t X_s^n$ by employing the S-transform (see (2.2)). Without loss of generality, we assume that $s = 1$ and $d = 1$ (one-dimensional case). Let us evaluate

$$S(D_t X_1^{n,p})(\phi), \ \ \phi \in S_{\mathbb{C}}(\mathbb{R}), \ \ n \geq 1.$$

Then, using Girsanov's theorem and the local time-space decomposition (2.7), we find that

$$\begin{aligned}
&S(D_t X_1^{n,p})(\phi) \\
&= E\Bigg[\frac{1}{n}\exp\Bigg\{\int_t^1 n\, b_p(\frac{1}{n}B_s + x)dB_s - \int_0^{1-t} n\, b_p(\frac{1}{n}B_s + x)d\widetilde{W}_s \\
&\quad + \int_0^{1-t} n\, b_p(\frac{1}{n}\widehat{B}_s + x)\frac{\widehat{B}_s}{1-s}ds\Bigg\} \\
&\exp\left\{\int_0^1 \left(n\, b_p(\frac{1}{n}B_s + x) + \phi(x)\right) dB_s - \frac{1}{2}\int_0^1 \left(n\, b_p(\frac{1}{n}B_s + x) + \phi(x)\right)^2 ds\right\}\Bigg]
\end{aligned} \tag{2.25}$$

for all $\phi \in S(\mathbb{R})$. By analyticity, we see that relation (2.25) also holds for all $\phi \in S_{\mathbb{C}}(\mathbb{R})$.

Using an appropriate sequence of coefficients $b_p,\ p \geq 1$, which approximates the bounded function b (compare e.g the proof of [19, Lemma 12]) and a weak compactness argument in Hilbert spaces, we deduce that

$$\begin{aligned}
&S\left(\int_0^1 D_t X_1^n . h(t)dt\right)(\phi) \\
&= E\Bigg[\int_0^1 \Bigg(\frac{1}{n}\exp\Bigg\{-\int_t^1 n\, b(\frac{B_s}{n}+x)dB_s - \int_0^{1-t} n\, b(\frac{B_s}{n}+x)d\widetilde{W}_s \\
&\quad + \int_0^{1-t} n\, b(\frac{\widehat{B}_s}{n}+x)\frac{\widehat{B}_s}{1-s}ds\Bigg\} \qquad (2.26)\\
&\exp\left\{\int_0^1 \left(n\, b(\frac{B_s}{n}+x)+\phi(s)\right)dB_s - \frac{1}{2}\int_0^1\left(n\, b(\frac{B_s}{n}+x)+\phi(s)\right)^2 ds\right\}\Bigg)h(t)dt\Bigg]
\end{aligned}$$

for all bounded Borel-measurable functions h on $[0,1]$, $\phi \in S_{\mathbb{C}}(\mathbb{R})$ and $n \geq 1$. Repeated use of the local time-space decomposition (2.7), Girsanov's theorem and the Itô-Tanaka formula for continuous semimartingales in [23, p.220] give that

$$S(D_t X_1^n)(\phi) = S(\Psi_t^n)(\phi)$$

for all $\phi \in S_{\mathbb{C}}(\mathbb{R})$, where

$$\Psi_t^n = \frac{1}{n}\exp\left\{\int_t^1 \int_{\mathbb{R}} n\, b(\frac{y}{n}+x)L^{n(X^n-x)}(ds,dy)\right\},$$

where $L^{n(X^n-x)}(s,y)$ denotes the local time at y of $n(X_\cdot^n - x)$. Thus

$$D_\cdot X_1^n = \Psi_\cdot^n \tag{2.27}$$

for all n.

Using this representation and the line of reasoning in the proof of Theorem 2.2 in connection with the weak compactness in $\mathbb{D}_{1,2}$, we conclude that X_t is Malliavin differentiable for all t.

The last statement of Theorem 2.7 is a direct consequence of [21, Theorem 2.1.2] □

Remark 2.9. *Assume $b : \mathbb{R} \to \mathbb{R}$ satisfies the assumptions of Theorem 2.6. Consider the case, when*

$$D_\cdot X_u = 0 \tag{2.28}$$

on a measurable set A such that $(\lambda \otimes \mu)(A) > 0$ for some $0 < u \leq 1$. Then using relation (2.27) in the proof of Theorem 2.6 in connection with Girsanov's theorem shows that there is a subsequence n_k such that

$$-\log n_k + \mathcal{L}_1(n_k,t,u) + \mathcal{L}_2(n_k,u) \underset{k\to\infty}{\longrightarrow} -\infty \tag{2.29}$$

on A (t,ω)-a.e., where

$$\begin{aligned}
\mathcal{L}_1(n,t,u) = &\int_t^u n\, b_p(\frac{1}{n}B_s+x)dB_s - \int_{1-u}^{1-t} n\, b_p(\frac{1}{n}B_s+x)d\widetilde{W}_s \\
&+ \int_{1-u}^{1-t} n\, b_p(\frac{1}{n}\widehat{B}_s+x)\frac{\widehat{B}_s}{1-s}ds
\end{aligned}$$

and

$$\mathcal{L}_2(n,u) = \int_0^u n\, b_p(\frac{1}{n}B_s + x)dB_s - \frac{1}{2}\int_0^u n^2 b_p^2(\frac{1}{n}B_s + x)ds.$$

So (2.29) is a necessary condition for (2.28). In particular, if $(\lambda \otimes \mu)(A) < 1$ there is a set B of positive measure such that the conditional density of X_u with respect to B exists and condition (2.29) is violated.

The next result provides a sufficient condition for the assumptions of Theorem 2.6 in the one dimensional case.

Theorem 2.10. *Let the drift coefficient $b : [0,1] \times \mathbb{R} \to \mathbb{R}$ and its approximating sequence $b_p : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfy the assumptions of Theorem 2.2. Further suppose that*

$$E\left[\left(\int_0^s A_2(n,p,u,s)du\right)^{-4} A_1(n,p,u,s)\right] < \infty \tag{2.30}$$

for all $0 < s \le 1,\ n \ge 1,\ p \ge 1$, and that for all compact sets $K \subseteq \mathbb{R}$ there exists a constant $M < \infty$ such that

$$\begin{aligned}\int_K \Bigg(E\Bigg[\int_0^s m\chi_{(y,y+\frac{1}{m})}(\frac{1}{n}B_s + x)A_2(n,p,u,s)\\ \cdot\left(\int_0^s A_2(n,p,u,s)du\right)^{-1} A_1(n,p,u,s)\, du\Bigg]\Bigg)^2 dy < M\end{aligned} \tag{2.31}$$

for all $m, n \ge 1,\ p \ge 1$. Then there exists a cluster point $X_t,\ 0 \le t \le 1$ of the processes $X_t^n,\ 0 \le t \le 1$ in (1.1) such that $X_\cdot$ solves the ODE's (1.2).

Proof. For convenience we assume that $K = \mathbb{R}$. Using Girsanov's theorem and the local time-space decomposition (2.7) we see that the condition (2.30) is equivalent to

$$E\left[\|D_\cdot X_s^{n,p}\|_{L^2[0,1]}^{-8}\right] < \infty.$$

The latter and our assumptions on $b_p,\ p \ge 1$ imply that $\frac{D_\cdot X_s^{n,p}}{\|D_\cdot X_s^{n,p}\|_{L^2[0,1]}^2}$ is in the domain of the divergence operator δ for all $0 < s \le 1$. See e.g [21].

From this it follows that $X_s^{n,p}$ has a continuous and bounded probability density $\rho_s^{n,p}$ which has the representation

$$\rho_s^{n,p}(y) = E\left[\chi_{(y,\infty)}(X_s^{n,p})\delta\left(\frac{D_\cdot X_s^{n,p}}{\|D_\cdot X_s^{n,p}\|_{L^2[0,1]}^2}\right)\right],\ \ y \in \mathbb{R},\ \ n,p \ge 1. \tag{2.32}$$

See [21, Proposition 2.1] or [11]. Consider now the sequence of Lipschitz continuous functions $0 \le \varrho_m \le \chi_{(x,\infty)}$ with $\varrho_m(z) \to \chi_{(y,\infty)}(z),\ \ z \in \mathbb{R}$ given by

$$\varrho_m(z) = \begin{cases} mz - my & , \quad y < z < y + \frac{1}{m} \\ 0 & , \quad z \le y \\ 1 & , \quad z \ge y + \frac{1}{m}\end{cases}$$

Then the functions $\rho_s^{m,n,p}$ defined as

$$\rho_s^{m,n,p}(y) = E\left[\varrho_m(X_s^{n,p})\delta\left(\frac{D_\cdot X_s^{n,p}}{\|D_\cdot X_s^{n,p}\|_{L^2[0,1]}^2}\right)\right]$$

converge to $\rho_s^{n,p}$, pointwisely for all s, n, p. On the other hand one infers from the duality relation and the chain rule of the Malliavin derivative (see e.g [21, 11]) that

$$\rho_s^{m,n,p}(y) = E\left[\int_0^s \chi_{(y,y+\frac{1}{m})}(X_s^{n,p}) \frac{(D_u X_s^{n,p})^2}{\|D_{\cdot} X_s^{n,p}\|_{L^2[0,1]}^2} du\right].$$

Then we obtain from (2.31) in connection with the Girsanov's theorem and the decomposition (2.7) that

$$\|\rho_s^{m,n,p}\|_{L^2(\mathbb{R})}^2 \le M < \infty \quad \text{for all} \quad m, n, p.$$

Using weak compactness of $\rho_s^{m,n,p}$, m, n, p in $L^2(\mathbb{R})$, pointwise convergence of $\rho_s^{m,n,p}$ with respect to m and the fact that $X_s^{n,p}$ converges to X_s^n in $L^2(\mu)$ (for a subsequence), we observe that X_s^n has a probability density ρ_s^n and that ρ_s^n is weakly compact in $L^2(\mathbb{R})$. Repeated use of weak compactness and $L^2(\mu)$-convergence shows that the cluster point X_s in Theorem 2.6 has a density ρ_s, $0 < s \le 1$. So the result follows. □

Finally, we give an application of Theorem 2.6 in the case of a discontinuous ODE.

Example 2.11. *Consider the ODE (1.2) with initial value x and the drift coefficient b given by the sign function, that is the special case of a step function*

$$b(t,y) = \operatorname{sign}(y) = \begin{cases} 1 & , \quad y \ge 0 \\ -1 & , \quad y < 0. \end{cases}$$

We want to show that there exists a subsequence (n_k) such that the solutions X_s^n converge in $\mathbb{D}_{1,2}$ to a deterministic process X_s, $0 \le s \le 1$ (for certain $x \ne 0$).

Without loss of generality, let $s = 1$. Since the sign function is bounded, we know from the proof of Theorem 2.6 that

$$D_t X_1^n = \frac{1}{n} \exp\left\{-\int_t^1 \int_{\mathbb{R}} n \operatorname{sign}\left(\frac{1}{n} y + x\right) L^{n(X^n - x)}(ds, dy)\right\},$$

*where $L^{n(X^n-x)}(s,y)$ is the local time at y of $n(X^n - x)$. Using the latter representation, we may replace the coefficient b_p, $p \ge 1$ in Theorem 2.2 by the sign function itself. In order to verify condition **(C4)** we apply Girsanov's theorem and Hölder's inequality and find that it is sufficient to show that*

$$I_1(n,t,t') \cdot I_2(n,t,t') \le C \cdot \left|t - t'\right|^{\alpha}, \quad 0 \le t \le t' \le 1 \tag{2.33}$$

for some $\alpha > \frac{1}{2}$ and a constant C (independent of n), where

$$I_1(n,t,t') := \frac{1}{n^2} E\left[\left(-\int_t^{t'} \int_{\mathbb{R}} n \operatorname{sign}\left(\frac{1}{n} y + x\right) L(ds, dy)\right)^4\right]^{\frac{1}{2}} \tag{2.34}$$

and

$$\begin{aligned} I_2(n,t,t') := E\Bigg[&\exp\left\{-4\int_t^{t'} \int_{\mathbb{R}} n \operatorname{sign}\left(\frac{1}{n} y + x\right) L(ds, dy)\right\} \\ &. \exp\left\{\int_0^1 n \operatorname{sign}\left(\frac{1}{n} B_s + x\right) dB_s - \frac{1}{2}\int_0^1 n^2 ds\right\}\Bigg]^{\frac{1}{2}}. \end{aligned} \tag{2.35}$$

Using the Itô-Tanaka formula and Burkholder's inequality we find that

$$\begin{aligned}
&I_1(n,t,t') \\
&= \frac{1}{n^2} E\Bigg[\bigg(\int_t^{t'} n^2 (\operatorname{sign}(X_u^n))^2 du + \int_t^{t'} n \operatorname{sign}(X_u^n) dB_u - (|n\, X_{t'}^n - nx| - |n\, X_t^n - nx)\bigg)^4\Bigg]^{\frac{1}{2}} \\
&\leq C\, n^4 \left|t - t'\right| \qquad (2.36)
\end{aligned}$$

for some constant C.

On the other hand, by applying [12, Corollary 3.2] *we get that*

$$\begin{aligned}
-\int_t^1 \int_{\mathbb{R}} n \operatorname{sign}\left(\frac{1}{n} y + x\right) L(ds, dy) &= \int_{\mathbb{R}} \left(\int_t^1 2n\, d_s L_s^y\right) \delta_{\{-nx\}}(dy) \\
&= 2n \left(L(1, -nx) - L(t, -nx)\right),
\end{aligned}$$

where $\delta_{\{-nx\}}$ is the Dirac measure in $-nx$.

Repeated use of Girsanov's theorem and the formula of Itô-Tanaka gives that

$$\begin{aligned}
I_2(n,t,t') = E\,&[\exp\left\{8n\left(L(1,-nx) - L(t,-nx)\right)\right\} \\
&\exp\left\{n\left(|B_1 + nx| - n\,|x| - 2L(1,-nx) - \frac{1}{2}n\right)\right\}\bigg]^{\frac{1}{2}} \\
\leq E&\left[\exp\left\{6nL(1,-nx) + n\,|B_1 + nx| - n^2(|x| + \frac{1}{2})\right\}\right]^{\frac{1}{2}}.
\end{aligned}$$

Then using the probability density of $(L(s,y), B_s)$ for a Brownian motion starting in a (see e.g. [5, p.155]*) that is*

$$\begin{aligned}
&P_a\left(L(t,x) \in dy, B_t \in dz\right) \\
&= \frac{1}{t\sqrt{2\pi t}} \left(y + |z - x| + |x - a|\right) \exp\left\{-\frac{(y + |z - x| + |x - r|)^2}{2t}\right\} dz\, dy,
\end{aligned}$$

we obtain that

$$\begin{aligned}
I_2^2(n,t,t') \leq \int_0^\infty \int_{\mathbb{R}} &\exp\left\{6ny + n\,|z + nx| - n^2(|x| + \frac{1}{2})\right\} \\
&\frac{1}{\sqrt{2\pi}} \left(y + |z + nx| + |nx|\right) \exp\left\{-\frac{(y + |z + nx| + |nx|)^2}{2}\right\} dz\, dy.
\end{aligned}$$

Using substitution and the fact that

$$\frac{2}{\sqrt{\pi}} \int_r^\infty e^{-v^2} dv \cong \frac{1}{\sqrt{\pi} r} e^{-r^2}$$

for $r \to \infty$ (see e.g. [5]*). We conclude that*

$$\begin{aligned}
&I_2^2(n,t,t') \\
&\leq \frac{2}{(n(|x| - 1) - 1)\,(n(|x| - 11) - 1)} \exp\left\{72n^2 - 7n^2\,|x| - (n(|x| - 11) - 1)^2\right\} \qquad (2.37)
\end{aligned}$$

for $n \geq n_0$ and $|x| > 11$.

*Combining this with the estimate in (2.36) we see that **(C4)** is fulfilled for initial values with $|x| > 11$. On the other hand the boundedness of the sign function implies the validity of the conditions **(C3)** and **(C5)** for $|x| > 11$. So it follows from Theorem 2.2 that the solutions*

$X_s^n,\ 0 \leq s \leq 1$ *converge to* $X_s,\ 0 \leq s \leq 1$ *in* $L^2(\mu)$ *for a subsequence if* $|x| > 11$. *Moreover, by weak compactness and the estimates in (2.36) and (2.37) we can even deduce that this convergence is in* $\mathbb{D}_{1,2}$ *and that*

$$D.X_s = 0,\ \ 0 \leq s \leq 1.$$

Hence, $X_s,\ 0 \leq s \leq 1$ *is a deterministic process. On the other hand, since* $|x| > 11$ *we get that*

$$|X_s| \geq ||x| - s| \geq 10 \ \text{for all}\ 0 \leq s \leq 1,\ \text{a.e.},$$

that is X_s *cannot hit the discontinuity point zero.*

So $X.$ *must be a deterministic solution (i.e.,* $x \pm t$*) of the ODE (1.2).*

Remark 2.12. *The arguments in Example 2.11 show that we may also consider drift coefficients* b *given by e.g. step functions of the form*

$$b(x) = \sum_{i=1}^{n} \xi_i \chi_{[0,b_i)},$$

$\xi_i \geq 0,\ \ b_i \in [0,\infty],\ \ i = 1,\ldots,n.$

References

[1] L. Ambrosio: *Transport equation and Cauchy problem for BV vector fields.* Invent. math. (2004). doi: 10.1007/s00222-004-0367-2

[2] R. Bafico: *On the convergence of the weak solutions of stochastic differential equations when the noise intensity goes to zero.* Boll. Unione Mat. Ital. Sez. B **17**, 308–324 (1980)

[3] R. Bafico, P. Baldi: *Small random perturbations of Peano phenomena.* Stochastics **6** (3 & 4), 279–292 (1982)

[4] V.S. Borkar, K. Suresh Kumar: *A new Markov selection procedure for degenerate diffusions.* J Theor. Probab. (2009).

[5] A.N. Borodin, P. Salminen: *Handbook of Brownian Motion & Facts and Formulae.* Second edition. Birkhauser Verlag, (2002)

[6] A. Bressan, W. Sheng: *On discontinuous differential equations.* In: Differential Inclusions and Optimal Control, Lect. Notes Nonlin. Anal., **2**, 73–87 (1988)

[7] R. Buckdahn, Y. Ouknine, M. Quincampoix: *On limiting values of stochastic differential equations with small noise intensity tending to zero.* Bull. Sci. math. **133** (3), 229–237 (2009)

[8] T-S. Chiang, C-H. Hwang: *On the non-uniqueness of the limit points of diffusions with a small parameter.* Stochastics **10** (2), 149–153 (1983)

[9] T-S. Chiang, S-J. Sheu: *Large deviation of diffusion processes with discontinuous drift and their occupation times.* Annals of Probability **28** (1), 140–165 (2000)

[10] G. Da Prato, P. Malliavin, D. Nualart: *Compact families of Wiener functionals.* C. R. Acad. Sci. Paris, Sér. I **315**, 1287–1291 (1992)

[11] G. Di Nunno, B. Øksendal, F. Proske: *Malliavin Calculus for Lévy Processes with Applications to Finance.* Springer (2008).

[12] N. Eisenbaum: *Integration with respect to local time.* Potential Analysis **13**, 303–328 (2000)

[13] F. Flandoli: *Remarks on uniqueness and strong solutions to deterministic and stochastic differential equations.* Metrika **69**, 101–123 (2009)

[14] M. Gradinaru, S. Herrmann, B. Roynette: *A singular large deviations phenomenon.* Ann. Inst. H. Poincaré Probab. Statist. **37** (5), 555–580 (2001)

[15] S. Herrmann, Ph.D. Thesis, Université Henri Poincaré, Nancy, (2001)

[16] S. Herrmann: *Phénomène de Peano et grandes déviations.* C. R. Acad. Sci. Paris, Sér. I **332**, 1019–1024 (2001)

[17] T. Hida, H.-H. Kuo, J. Potthoff, L. Streit: *White Noise: An Infinite Dimensional Calculus.* Kluwer Academic, (1993)

[18] P. Malliavin: *Stochastic Analysis.* Springer (1997)

[19] T. Meyer-Brandis, F. Proske: *Construction of strong solutions of SDE's via Malliavin calculus.* Journal of Funct. Anal. **258**, 3922–3953 (2010)
[20] I. Mitoma: *Tightness of Probabilities on* $C([0,1],S')$ *and* $D([0,1],S')$. Annals of Probability, 989–999 **11** (4) (1983)
[21] D. Nualart: *The Malliavin Calculus and Related Topics.* Springer (1995)
[22] N. Obata: *White Noise Calculus and Fock Space.* LNM **1577**, Springer (1994)
[23] D. Revuz, M. Yor: *Continuous Martingales and Brownian Motion.* Third edition, Springer (2004)
[24] W. Rzymowski: *Existence of Solutions for a Class of Discontinuous Differential Equations in* $\mathbb{R}^n$. J. Math. Anal. Appl. **233**, 634–643 (1999)
[25] S.R.S. Varadhan, *Large deviations and applications.* CBMS-NSF Regional Conference Series in Applied Mathematics, **46**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA (1984)
[26] A.Y. Veretennikov: *Approximation of ordinary differential equations by stochastic ones.* Mat. Zametki **33** (6), 929–932 (1983) (in Russian)

CMA, Department of Mathematics, University of Oslo, Moltke Moes vei 35, P.O. Box 1053 Blindern, 0316 Oslo, Norway.
The research of this author was supported by the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no [228087].
E-mail address: `o.m.pamen@cma.uio.no`

Department of Mathematics,LMU, Theresienstr. 39, D-80333 Munich, Germany
E-mail address: `meyerbr@math.uio.no`

CMA, Department of Mathematics, University of Oslo, Moltke Moes vei 35, P.O. Box 1053 Blindern, 0316 Oslo, Norway.
E-mail address: `proske@math.uio.no`