Explicit Solution of a Non-linear Filtering Problem for Lévy Processes with Application to Finance

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Abstract

In this paper we explicitly solve a non-linear filtering problem with mixed observations, modelled by a Brownian motion and a generalized Cox process, whose jump intensity is given in terms of a Lévy measure. Motivated by empirical observations of R. Cont and P. Tankov we propose a model for financial assets, which captures the phenomenon of time-inhomogeneity of the jump size density. We apply the explicit formula to obtain the optimal filter for the corresponding filtering problem.

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1. Introduction

In the present paper we derive an explicit solution for a non-linear filtering problem for Lévy processes. We are interested in the following non-linear filtering model: We consider a process X_t, which follows the dynamic of the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t^X, \tag{1.1}$$

where B_t^X is a standard Wiener process. Assume that X_t is partially observed via the process Y_t, which is described by the the equation

$$dY_t = h(t, X_t)dt + dB_t^Y + \int_{\mathbb{R}_0} \varsigma N_\lambda(dt, d\varsigma), \tag{1.2}$$

where B_t^Y is a standard Wiener process and N_λ an integer valued random measure with predictable compensator

$$\hat{\mu}(dt, d\varsigma, \omega) = \lambda(t, X_t, \varsigma)dt\nu(d\varsigma) \tag{1.3}$$

for a Lévy measure ν and a function $\lambda(t, x, \varsigma)$. It is assumed that (B_t^X, B_t^Y) is a Wiener process independent of N_λ. The process X_t resp. Y_t is called *signal process*

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resp. *observation process.* The jump intensity of the random measure N_λ exhibits a dependence on a hidden state variable, related to the signal process.

The non-linear filtering problem for our model is to find the least square estimate to the (possibly transformed) signal process at time t, given the history of the observation process up to time t, that is to determine the conditional expectation

$$E[f(X_t)\,|\mathcal{F}_t^Y]\,,$$

where f is a given Borel function and where $\mathcal{F}_t^Y$ is the $\sigma-$algebra, generated by $\{Y_s, 0 \leq s \leq t\}$.

The object of this paper is to demonstrate how white noise methods for Lévy processes, developed in [LØP] and [P], can be used to provide an explicit solution of this problem. For this purpose we solve a Zakai equation for the unnormalized conditional density explicitly. This equation is a linear stochastic partial differential equation (SPDE), driven by a Lévy process. We show that the Zakai equation has a unique strong L^p-solution, which can be explicitly represented. As an application we illustrate how this formula can be used in finance.

The theory of non-linear filtering is vastly treated in the literature. Linear filtering theory was initiated by Kalman and Bucy in their pioneering works (see [KB] and the references therein). The generalization of the latter authors' ideas to the non-linear setting was carried out in the sixties and early seventies. See e.g. Lipster and Shiryaev [LS], Kallianpur [Ka], Fleming and Rishel [FRi] for an account of these ideas. See also [D]. Using the innovation approach, Fujisaki, Kallianpur and Kunita derived a SDE for the conditional density of the filter process (see [Ka]). However, this type of equation has the drawback to be difficult to solve. To redress this deficiency Duncan, Mortensen and Zakai (see [Z] and the references therein) derived an equation for the evolution of the unnormalized conditional density, from which the optimal filter can be constructed. This equation is a linear SPDE and reveals a more simple analytical tractability than the above mentioned one.

Existence and uniqueness results for the Zakai equation were obtained by many authors. See e.g. Gyöngy, Krylov [GK1,2] and Grigelionis [Gr]. In the Gaussian case explicit solutions of related Zakai equations to the filter problem (1.1), (1.2) were determined by several authors: Pardoux [Pa1,2] e.g. found an explicit solution, employing Sobolev space techniques. Another contribution is due to Kunita [K], who gave an explicit solution, using the theory of stochastic flows. Further [B] and [BDPV] applied concepts of Gaussian white noise theory and derived a similar formula to the latter ones.

A generalization of the Gaussian setting to the filtering problem (1.1), (1.2) in the case of a doubly stochastic Poisson process, i.e. in the case, when the Lévy measure ν is a Dirac measure, was investigated by [DR]. In this work the authors discuss approximations of the filtering problem, by invoking finite-state Markov chains. Our model contains [DR] as a special case and we are able to give an explicit solution to this problem. Furthermore, our solution formula comprises both the Gaussian case (i.e. the filtering problem without N_λ) and the pure jump case (i.e. without the observation function h and the Brownian motion). Thus our result can be regarded as a generalization of the above solutions in the Gaussian case. Let us mention that a related model to [DR], which captures the description of high frequency data, was studied in [FR]. However, as far as we can see, our methods do not apply here to solve the corresponding non-linear filtering problem explicitly.

Our main tool for solving the non-linear filtering problem are white noise concepts for Lévy processes, introduced in [LØP]. Our work is inspired by [B] and [BDPV], where the Gaussian case is treated.

Section 2 passes in review some basic elements of a white noise theory for Lévy processes. In Section 3 the non-linear filtering problem with mixed observations is set up. Further it is sketched how to derive the Zakai equation for the unnormalized conditional density. Finally, an explicit solution of this equation is given. Section 4 deals with the application of the closed form solution to finance. It has been observed (e.g. [CT]) that time homogeneity of the Lévy measure in exponential Lévy models for financial assets is not appropriate. More precisely, their results indicate that while the intensity (mass of the Lévy measure) stays quite stable over time, the shape of the density changes. Taking into consideration these observations, we set up a model that is able to capture this phenomenon and we apply the results of the previous sections to solve the corresponding filtering problem.

2. White Noise Framework

This section gives a brief outline of some concepts of a white noise theory for Lévy processes, developed in [LØP] and [P]. In Section 3.2 we will apply this approach to solve the Zakai equation (3.1.5), explicitly. For general background information about white noise theory the reader is encouraged to resort to the books of [HKPS], [Ku] and [O].

Let us recapitulate that a *Lévy process* can be defined to be a stochastic process $\eta(t)$ on $\mathbb{R}_+$, which has independent and stationary increments starting at zero, i.e. $\eta(0) = 0$. Such processes form a prototype of semimartingales with the characteristic triplet

$$(B_t, C_t, \hat{\mu}) = (a \cdot t, \sigma \cdot t, dt\nu(dz)), \tag{2.1}$$

where a, σ are constants and where ν denotes a compensating measure on $\mathbb{R}_0 := \mathbb{R} - \{0\}$, called *Lévy measure*, that integrates the function $1 \wedge z^2$. See e.g. [B], [Sa] or [JS] for details.

In order to build a white noise theory for combinations of Gaussian and Poisson (random measure) noise, we first confine ourselves to pure jump Lévy processes, that is we assume $a = \sigma = 0$ in (2.1).

We recall the construction of the white noise space $\widetilde{\mathcal{S}}(X)$ in [LØP] in the case of the space-time dimension $d = 1$. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space on $\mathbb{R}^d$ and denote by $\mathcal{S}'(\mathbb{R}^d)$ its dual, i.e. the space of tempered distribution. It is well-known that the topology of $\mathcal{S}(\mathbb{R}^d)$ can be induced by increasing, compatible pre-Hilbertian norms

$\|\cdot\|_p$, $p \in \mathbb{N}$. We set $X = \mathbb{R} \times \mathbb{R}_0$ and we define a closed nuclear subalgebra of $\mathcal{S}(\mathbb{R}^2)$ (w.r.t. the restrictions of the norms $\|\cdot\|_p$) by

$$\mathcal{S}(X) := \left\{ \varphi \in \mathcal{S}(\mathbb{R}^2) : \varphi(t,0) = (\frac{\partial}{\partial z}\varphi)(t,0) = 0 \right\} \tag{2.2}$$

The space $\widetilde{\mathcal{S}}(X)$ is defined as the quotient algebra

$$\widetilde{\mathcal{S}}(X) = \mathcal{S}(X)/\mathcal{N}_\pi, \tag{2.3}$$

where $\mathcal{N}_\pi$ is the closed ideal in $\mathcal{S}(X)$, given by

$$\mathcal{N}_\pi := \{\phi \in \mathcal{S}(X) : \|\phi\|_{L^2(\pi)} = 0\} \tag{2.4}$$

with $\pi = \lambda \times \nu$ for λ the Lebesgue measure. The space $\widetilde{\mathcal{S}}(X)$ forms a (countably Hilbertian) nuclear algebra and we denote by $\widetilde{\mathcal{S}}'(X)$ its dual.

Then the Bochner-Minlos theorem implies the existence of a unique probability measure μ on the Borel sets of $\widetilde{\mathcal{S}}'(X)$, satisfying

$$\int_{\widetilde{\mathcal{S}}'(X)} e^{i\langle\omega,\phi\rangle} d\mu(\omega) = \exp\left(\int_X (e^{i\phi} - 1) d\pi\right) \tag{2.5}$$

for all $\phi \in \widetilde{\mathcal{S}}(X)$, where $\langle\omega,\phi\rangle = \omega(\phi)$ is the action of $\omega \in \widetilde{\mathcal{S}}'(X)$ on $\phi \in \widetilde{\mathcal{S}}(X)$. The measure μ on $\Omega = \widetilde{\mathcal{S}}'(X)$ is referred to as *(pure jump) Lévy white noise probability measure*. This measure turns out to fulfill the *first condition of analyticity* and to be *non-degenerate* (see [LØP]). Let us point out that these properties ensure the existence of *generalized Charlier polynomials* $C_n(\omega)$ (see [KDS] for the definition), which have the following generating property: Define the function α via $\alpha(\phi) = \log(1+\varphi) \bmod \mathcal{N}_\pi$, if $\phi = \varphi \bmod \mathcal{N}_\pi$ for $\varphi(x) > -1$. Then α is holomorphic at zero and invertible and the exponential $\widetilde{e}(\phi,\omega) := \frac{\exp\langle\omega,\alpha(\phi)\rangle}{E_\mu[e^{\langle\omega,\alpha(\phi)\rangle}]}$ can be expanded into a power series around zero in $\widetilde{\mathcal{S}}(X)$, i.e.

$$\widetilde{e}(\phi,\omega) = \sum_{n\geq 0} \frac{1}{n!} \left\langle C_n(\omega), \phi^{\otimes n} \right\rangle, \tag{2.6}$$

where $\phi^{\otimes n} \in \widetilde{\mathcal{S}}(X)^{\widehat{\otimes} n}$ denotes the n-th completed symmetric tensor product of $\widetilde{\mathcal{S}}(X)$ with itself.

Our approach to solve the Zakai equation necessitates the definition of the Lévy-Hida test function and distribution space. These spaces are constructed by means of a certain orthogonal basis of $L^2(\mu)$, given in terms of the generalized Charlier polynomials in (2.6). We persue a short review of this concepts. In the following we shall denote by $\mathcal{J}$ the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, ...)$ with finitely many non-zero entries $\alpha_i \in \mathbb{N}_0$. Let $Index(\alpha) = \max\{i : \alpha_i \neq 0\}$ and $|\alpha| = \sum_i \alpha_i$ for

$\alpha \in \mathcal{J}$. Then let us choose an orthonormal basis $\{\delta_k(x,z)\}_{k\geq 1} \subset \mathcal{S}(X)$ of $L^2(X)$. In assuming that $Index(\alpha) = j$ and $|\alpha| = m$ for $\alpha \in \mathcal{J}$ we define the function $\delta^{\otimes\alpha}$ by

$$\delta^{\otimes\alpha}((x_1,z_1),...,(x_m,z_m)) = \tag{2.7}$$

$$\delta_1^{\otimes\alpha_1} \otimes ... \otimes \delta_j^{\otimes\alpha_j}((x_1,z_1),...,(x_m,z_m)) = \delta_1(x_1,z_1)\cdot ... \cdot \delta_1(x_{\alpha_1},z_{\alpha_1})$$

$$\cdot ... \cdot \delta_j(x_{\alpha_1+...+\alpha_{j-1}+1}, z_{\alpha_1+...+\alpha_{j-1}+1}) \cdot ... \cdot \delta_j(x_m,z_m),$$

where $\delta_i^{\otimes 0} := 1$. The *symmetrized tensor product* of the δ_k's, indicated by $\delta^{\widehat{\otimes}\alpha}$, is the symmetrization of $\delta^{\otimes\alpha}$ with respect to the variables $(x_1,z_1),...,(x_m,z_m)$. Then the family $\{K_\alpha(\omega)\}_{\alpha\in\mathcal{J}}$ of random variables, given by

$$K_\alpha(\omega) = \left\langle C_{|\alpha|}(\omega), \delta^{\widehat{\otimes}\alpha} \right\rangle, \tag{2.8}$$

constitutes an orthogonal basis of $L^2(\mu)$. So every $F \in L^2(\mu)$ can be written as

$$F = \sum_{\alpha\in\mathcal{J}} c_\alpha K_\alpha \tag{2.9}$$

for a unique sequence of real numbers $(c_\alpha)_{\alpha\in\mathcal{J}}$, where

$$\|F\|^2_{L^2(\mu)} = \sum_{\alpha\in\mathcal{J}} \alpha! c_\alpha^2, \tag{2.10}$$

with $\alpha! := \alpha_1!\alpha_2!...$, if $\alpha = (\alpha_1, \alpha_2, ...) \in \mathcal{J}$. The *Lévy-Hida test function space* $(\mathcal{S})$ is characterized as the space of all $f = \sum_{\alpha\in\mathcal{J}} c_\alpha K_\alpha \in L^2(\mu)$ such that the growth condition

$$\|f\|^2_{0,k} := \sum_{\gamma\in\mathcal{J}^m} \alpha! c_\alpha^2 (2\mathbb{N})^{k\alpha} < \infty \tag{2.11}$$

holds for all $k \in \mathbb{N}_0$ with weight $(2\mathbb{N})^{k\alpha} = (2\cdot 1)^{k\alpha_1}(2\cdot 2)^{k\alpha_2}...(2\cdot l)^{k\alpha_l}$, if $Index(\alpha) = l$. The space $(\mathcal{S})$ is equipped with projective topology, based on the family of norms $(\|\cdot\|_{0,k})_{k\in\mathbb{N}_0}$ in (2.11). The *Lévy-Hida distribution space*, denoted by $(\mathcal{S})^*$ is defined as the topological dual of $(\mathcal{S})$. Thus we just constructed the following Gel'fand triple

$$(\mathcal{S}) \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})^*. \tag{2.12}$$

We introduce on $(\mathcal{S})^*$ a multiplication of distributions by means of the *Wick product* $\diamond$, given by

$$(K_\alpha \diamond K_\beta)(\omega) = (K_{\alpha+\beta})(\omega), \ \alpha, \beta \in \mathcal{J} \tag{2.13}$$

The product is linearly extended to the whole space. It can be shown e.g. that

$$\langle C_n(\omega), f_n\rangle \diamond \langle C_m(\omega), g_m\rangle = \left\langle C_{n+m}(\omega), f_n \widehat{\otimes} g_m \right\rangle \tag{2.14}$$

for symmetric functions $f_n \in L^2(\pi^{\times n})$ and $g_m \in L^2(\pi^{\times m})$ (see [LØP]). Since $(\mathcal{S})^*$ forms a topological algebra with respect to the Wick product, it is possible e.g. to introduce the Wick version of the exponential function exp by

$$\exp^\diamond X := \sum_{n\geq 0} \frac{1}{n!} X^{\diamond n} \tag{2.15}$$

for $X \in (\mathcal{S})^*$, where the Wick powers in (2.15) are defined as

$$X^{\diamond n} = X \diamond X \diamond ... \diamond X \qquad \text{(n times)}.$$

The Wick product elicits an interesting relation to Itô integration: Let

$$\widetilde{N}(dt, dx) = N(dt, dz) - dt\nu(dz)$$

denote the compensated Poisson random measure associated with the Lévy process $\eta(t)$ and let $\overset{\bullet}{\widetilde{N}}(t,z)$ be the white noise of $\widetilde{N}(dt, dx)$, which takes values in $(\mathcal{S})^*$ $\lambda \times \nu-$a.e. If $Y(t, z, \omega)$ is a predictable process, satisfying the condition

$$E \int_0^T \int_{\mathbb{R}_0} Y^2(t, z, \omega) dt\nu(dz) < \infty,$$

then $Y(t, z, \omega) \diamond overset\bullet\widetilde{N}(t,z)$ is $\lambda \times \nu$-Bochner ntegrable in $(\mathcal{S})^*$ and

$$\int_0^T \int_{\mathbb{R}_0} Y(t, z, \omega)\ \widetilde{N}(dt, dx) = \int_0^T \int_{\mathbb{R}_0} Y(t, z, \omega) \diamond \overset{\bullet}{\widetilde{N}}(t, z) dt\nu(dz). \tag{2.16}$$

See [LØP] or [ØP] for definitions. We are coming to the *Lévy Hermite transform* $\mathcal{H}$ as an important tool for the study of SPDE's. Just as in the Gaussian case, the construction of $\mathcal{H}$ employs the use of the expansion along the basis $\{K_\alpha(\omega)\}_{\alpha\in\mathcal{J}}$ in (2.8). The *Lévy Hermite transform* of $X(\omega) = \sum_\alpha c_\alpha K_\alpha(\omega) \in (\mathcal{S})^*$, indicated by $\mathcal{H}X$ or $\widetilde{X}$, is defined by

$$\mathcal{H}X(z) = \widetilde{X}(z) = \sum_\alpha c_\alpha z^\alpha \in \mathbb{C} \quad \text{(when convergent)}, \tag{2.17}$$

where $z = (z_1, z_2, ...) \in \mathbb{C}^\mathbb{N}$, i.e. in the space of $\mathbb{C}-$valued sequences, and where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}...$ Let us mention that $\mathcal{H}X(z)$ in (2.17) converges in the space of sequences with compact support, $\mathbb{C}_c^\mathbb{N}$. Since the Hermite transform maps the algebra $(\mathcal{S})^*$ into the algebra of power series in infinitely many complex variables, homomorphically, we find above all that

$$\mathcal{H}(X \diamond Y)(z) = \mathcal{H}(X)(z) \cdot \mathcal{H}(Y)(z) \tag{2.18}$$

holds. Finally we remark that any distribution in $(\mathcal{S})^*$ is uniquely characterized by its $\mathcal{H}$-transform (see characterization theorem 2.3.8 in [LØP]).

We conclude this Section with a short description of how the methods, elaborated above, can be generalized to cover the case of Lévy processes with Brownian motion and pure jump part (see [P]). Denote by μ_G the Gaussian white noise measure on the measurable space

$$(\Omega_G, \mathcal{F}_G) = (\mathcal{S}'(\mathbb{R}), \mathcal{B}(\mathcal{S}'(\mathbb{R}))).$$

Further recollect the construction of the orthogonal $L^2(\mu_G)$ basis $\{H_\alpha(\omega)\}_{\alpha\in J}$, given by

$$H_\alpha(\omega) = \prod_{j\geq 1} h_{\alpha_j}(\langle \omega, \xi_j \rangle),$$

where $\langle \omega, \cdot \rangle = \omega(\cdot)$ and where ξ_j resp. $h_j, j = 1, 2, ...$ are the Hermite functions resp. Hermite polynomials. Using μ_L to denote the pure jump white noise measure on $(\Omega_L, \mathcal{F}_L) = (\widetilde{\mathcal{S}}'(X), \mathcal{B}(\widetilde{\mathcal{S}}'(X)))$, we can define the *Lévy white noise measure* μ as the product measure $\mu_G \times \mu_L$ on

$$(\Omega, \mathcal{F}) = (\Omega_G \times \Omega_L, \mathcal{F}_G \otimes \mathcal{F}_L). \tag{2.19}$$

Define

$$L_\gamma(\omega) = L_\gamma(\omega_1, \omega_2) = H_\alpha(\omega_1) K_\beta(\omega_2), \tag{2.20}$$

if $\gamma = (\alpha, \beta) \in \mathcal{I} := \mathcal{J}^2$. Then $(L_\gamma(\omega)_{\gamma\in\mathcal{I}}$ forms an $L^2(\mu)-$basis with norm expression

$$\|L_\gamma\|^2_{L^2(\mu)} = \gamma!,$$

where $\gamma! := \alpha!\beta!$ for $\gamma = (\alpha, \beta) \in \mathcal{I}$.

Just as in the pure jump case, we can use the basis $(L_\gamma(\omega)_{\gamma\in\mathcal{I}}$ to extend the concepts of Hida space, Wick product or Hermite transform to the mixture of Gaussian and pure jump Lévy noise.

3. Explicit Solution of the Zakai Equation

In this Section we specify the set-up of our non-linear filtering problem, based on a Lévy process as driving noise. Further we outline how to derive the corresponding Zakai equation, yielding a (linear) SPDE for the unnormalized conditional density of the filter process. Then we demonstrate how the framework in Section 2 can be applied to provide an explicit strong L^p-solution of the Zakai equation.

3.1. **Non-linear filtering with respect to Lèvy processes, Zakai equation.** We wish to investigate a nonlinear filtering problem of the following type: Assume a partially observable process $(X_t, Y_t) \in \mathbb{R}^{n+m}$, $0 \leq t \leq T$, defined on a probability space $(\Omega, \mathcal{F}, \pi)$. X_t stands for the unobservable component of the process, referred to as the *signal process*, whereas Y_t is the observable part, called *observation process*. Suppose that the dynamics of the process is described by the following SDE:

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dB^X_t \\ dY_t &= h(t, X_t)dt + dB^Y_t + \int_{\mathbb{R}_0} \varsigma N_\lambda(dt, d\varsigma), \end{aligned} \tag{3.1}$$

where (B_t^X, B_t^Y) is a Wiener process, being independent of the integer valued random measure N_λ, which has the predictable compensator

$$\hat{\mu}(dt, d\varsigma, \omega) = \lambda(t, X_t, \varsigma) dt \nu(d\varsigma)$$

for a Lévy measure ν and a function $\lambda(t, x, \varsigma)$. The initial condition X_0 is assumed to be a random variable independent of B_t^X, B_t^Y. In particular, the case $h(t, x) \equiv 0$, when B_t^Y is omitted, corresponds to the pure jump observation process

$$dY_t = \int_{\mathbb{R}_0} \varsigma N_\lambda(dt, d\varsigma).$$

In the following we restrict the Lévy measure ν to be finite. Our main theorem (Theorem 3.3) will be proven under this assumption. However, we stress that a corresponding result for general Lévy measures can be obtained by using similar arguments and conditions.

For the sake of argument we limit ourselves to the case $n = m = 1$. To guarantee a unique (strong) solution of system (3.1) we require that the coefficients b and σ are Lipschitz continuous and satisfy the linear growth condition. Further conditions on b, σ, the observation function h and the intensity rate λ are listed in C1-C8 in Section 3.2.

Given a Borel measurable function f, the non-linear filtering problem comes down to determine the least square estimate of $f(X_t)$, given the observations up to time t. In other words the problem consists in evaluating the *optimal filter*

$$E[f(X_t) \left| \mathcal{F}_t^Y \right.] ,$$

where $\mathcal{F}_t^Y$ is the $\sigma-$algebra, generated by $\{Y_s, 0 \leq s \leq t\}$.

If the conditional distribution $P[X_t \left| \mathcal{F}_t^Y \right.]$ is absolutely continuous with respect to the Lebesgue measure with conditional density $p(t, x, \omega)$, i.e.

$$P[X_t \in dx \left| \mathcal{F}_t^Y \right.] (\omega) = p(t, x, \omega) dx,$$

it will be conceivable to solve the so-called Fujisaki-Kallianpur-Kunita equation (see e.g. [Ka]). In the purely Gaussian case (i.e. (3.1) with $\lambda \equiv 0$) this equation is a stochastic partial differential equation for $p(t, x, \omega)$, which involves the adjoint of the infinitesimal generator of X_t and the innovation process $d\nu_t = dY_t - E[h(t, X_t) \left| \mathcal{F}_t^Y \right.] dt$ as the driving process. However it is not easy to cope with finding a solution of this equation. In order to overcome this difficulty, M. Zakai (see [Z]) introduced the *unnormalized conditional density*, i.e. a process Φ, which is related to $p(t, x, \omega)$ in the following way:

$$p(t, x, \omega) = \frac{\Phi(t, x, \omega)}{\int_{\mathbb{R}} \Phi(t, x, \omega) dx}. \tag{3.2}$$

Thus the solution of the original problem can be retrieved by using Φ, which fulfills a linear SPDE, called Zakai equation. It turns out that this equation is less difficult to tackle than the above mentioned one.

Following a line of reasoning similar to the paper of Zakai [Z], by exploiting the Markov property of processes and the change of measure method in (3.1), one shows that a corresponding Zakai equation for Φ in (3.2) can be established (see also [Gr] or [DR]). To be more precise, let us first define the equivalent measure μ on $(\Omega, \mathcal{F})$ via $d\pi = \Lambda_T d\mu$ with Radon-Nikodym density

$$\begin{aligned} \Lambda_t &= \exp\{\int_0^t h(s,X_s)dB_s^Y - \frac{1}{2}\int_0^t h^2(s,X_s)ds \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \log\lambda(s,X_s,\varsigma)N_\lambda(ds,d\varsigma) + \int_0^t\int_{\mathbb{R}_0}(1-\lambda(s,X_s,\varsigma))ds\nu(d\varsigma)\}. \end{aligned} \tag{3.3}$$

Using the Girsanov theorem for random measures and the uniqueness of semimartigale characteristics (see e.g. [JS]), one sees that the processes (3.1) get decoupled under the measure μ in the sense that system (3.1) transforms to

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dB_t^X \\ dY_t &= B_t + L_t \end{aligned}$$

where Y_t is a Lévy process independent of X_t under μ with

$$B_t = B_t^Y - \int_0^t h(s,X_s)ds$$

the Brownian motion part and

$$L_t = \int_0^t\int_{\mathbb{R}_0} \varsigma N(ds,d\varsigma)$$

is the jump component, driven by the Poisson random measure $N(ds,d\varsigma) := N_\lambda(ds,d\varsigma)$ with compensation $ds\nu(d\varsigma)$.

The unnormalized conditional density Φ is defined as

$$\Phi(t,x,\omega) = E_\mu[\Lambda_t^{-1}\left|\mathcal{F}_t^Y\right](\omega)p(t,x,\omega). \tag{3.4}$$

Just as in the purely Gaussian setting (see [DM]), it can be proven that Φ, provided it exists and fulfills certain regularity assumptions, necessarily solves the following Zakai equation:

$$\begin{aligned} \Phi(t,x) &= \int_0^t \mathcal{L}^*\Phi(s,x)ds + \\ &\quad \int_0^t h(s,x)\Phi(s,x)dB_s + \int_0^t\int_{\mathbb{R}_0}(\lambda(s,x,\varsigma)-1)\Phi(s,x)\widetilde{N}(ds,d\varsigma) \\ \Phi(0,x) &= p_0(x), \end{aligned} \tag{3.5}$$

where $\mathcal{L}^*$ is the adjoint operator of the generator $\mathcal{L}$ of X_t, $\widetilde{N}(ds,d\varsigma) := N(ds,dr) - ds\nu(d\varsigma)$ and where $p_0(x)$ is the density function of the initial condition X_0. One observes that $\mathcal{L}^*$ can be written as

$$\mathcal{L}^* = \mathcal{A} - c \tag{3.6}$$

for a function $c(t,x)$, where $\mathcal{A}$ is the generator of a diffusion.

3.2. **Explicit solution.** In the following we denote by $C_b^{n,m}(\mathbb{R}_+ \times \mathbb{R})$ the space of continuously differentiable functions (n times in t, m times in x) with all partial derivatives bounded. Further $C^{n+\beta}$ stands for the space of functions whose partial derivatives up to order n are Hölder continuous of order $0 < \beta \leq 1$.

For convenience we sum up all the conditions, which are imposed on our filtering model throughout this Section:

- $C1$: The Lévy measure ν is finite.
- $C2$: The generator $\mathcal{A}$ in (3.6) is uniformly elliptic.
- $C3$: The coefficients b and σ are Hölder continuous and belong to $C_b^{1,3}(\mathbb{R}_+ \times \mathbb{R})$.
- $C4$: The initial condition p_0 in (3.5) is positive and element of $C_b^{2+\beta}(\mathbb{R})$.
- $C5$: The intensity rate λ is strictly positive and $\lambda(\cdot, \varsigma) \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}) \cap C^{2+\beta}(\mathbb{R}_+ \times \mathbb{R})$ uniformly in ς.
- $C6$: The function c in (3.6) is in $C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}) \cap C^{2+\beta}(\mathbb{R}_+ \times \mathbb{R})$.
- $C7$: The observation function h is contained in $C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}) \cap C^{2+\beta}(\mathbb{R}_+ \times \mathbb{R})$.
- $C8$: Λ_t in (3.3) is a martingale.

In order to derive an explicit solution of the Zakai equation (3.5), we will invoke the Feynman-Kac representation formula. For the sake of completeness we recall here a version of this formula, which is suitable for our purposes (see e.g. [F]).

Theorem 3.1. *Suppose that $k : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and bounded and that $p_0 : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and positive. Let $u : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ be of class $C^{1,2}([0,T) \times \mathbb{R})$ solving the Cauchy-problem*

$$\begin{aligned} \frac{\partial u}{\partial t} + ku &= \mathcal{A}u \\ u(0,x) &= p_0(x), \end{aligned}$$

where $\mathcal{A}$ is the infinitesimal generator of a unique weak solution X_s of an Itô-diffusion with coefficients of linear growth. Further assume that u satisfies the polynomial growth condition

$$\max_{0 \leq t \leq T} |u(t,x)| \leq M(1 + |x|^{2\mu})$$

for some $M > 0$, $\mu \geq 1$. *Then* u *can be represented as*

$$u(t,x) = E^{T-t,x}\left[p_0(X_T)\exp\{-\int_{T-t}^{T} k(T-t,X_s)ds\}\right]$$

on $[0,T]\times\mathbb{R}$. *Therefore such a solution is unique.*

Remark 3.2. Sufficient conditions for the validity of this formula are: The generator $\mathcal{A}$ is uniformly elliptic. The coefficients b, σ and k are Hölder continuous and bounded. The function p_0 is of polynomial growth. See e.g. [F].

Assume that an Itô integrable solution $\Phi(t,x)$ in (3.5) exists. Then by (2.16) and a similar relation for the Brownian motion, the Zakai equation can be equivalently reformulated as

$$\begin{aligned}\Phi(t,x) &= \int_0^t \mathcal{A}\Phi(s,x) - c(s,x)\Phi(s,x)ds + \int_0^t h(s,x)\Phi(s,x)\diamond W_s ds \\ &\quad + \int_0^t\int_{\mathbb{R}_0}(\lambda(s,x,\varsigma)-1)\Phi(s,x)\diamond \overset{\bullet}{\widetilde{N}}(s,\varsigma)ds\nu(d\varsigma) \\ \Phi(0,x) &= p_0(x),\end{aligned}$$

where W_t resp. $\overset{\bullet}{\widetilde{N}}(t,\varsigma)$ is the white noise of B_t resp. $\widetilde{N}(t,\varsigma)$ in (3.5).

By applying (formally) the Hermite transform to (3.7) we deduce that

$$\begin{aligned}\frac{\partial}{\partial t}\widetilde{\Phi}(t,x,z) &= \mathcal{A}\widetilde{\Phi}(t,x,z) - k(t,x,z)\widetilde{\Phi}(t,x,z) \\ \widetilde{\Phi}(0,x,z) &= p_0(x),\end{aligned} \tag{3.7}$$

where

$$k(t,x,z) = (c(t,x) - h(t,x)\widetilde{W}_t(z) - \int_{\mathbb{R}_0}(\lambda(t,x,\varsigma)-1)\mathcal{H}(\overset{\bullet}{\widetilde{N}}(t,\varsigma))(z))\nu(d\varsigma)$$

and where $\widetilde{\Phi}(t,x,z) = \mathcal{H}(\Phi(t,x))(z)$ and $\widetilde{W}_t(z) = \mathcal{H}(W_t)(z)$ for $z \in \mathbb{C}_c^{\mathbb{N}}$ (see (2.17)).

Using the inverse Hermite transform we obtain by means of the Feynman-Kac formula a solution candidate for the Zakai equation (3.5).

Theorem 3.3. *Suppose that conditions* $C1 - C8$ *hold. Then equation (3.7) has a unique strong solution in* $(\mathcal{S})^*$. *Moreover its solution takes the explicit form*

$$\begin{aligned}&\Phi(t,x) \\ &= E^x[p_0(X_t(\theta)\exp(-\int_0^t c(s,X_{t-s}(\theta)ds) \\ &\quad \exp^{\diamond}\left\{\int_0^t h(s,X_{t-s}(\theta))dB_s(\omega) + \int_0^t\int_{\mathbb{R}_0}(\lambda(s,X_{t-s}(\theta),\varsigma)-1)\widetilde{N}(ds,d\varsigma,\omega)\right\}],\end{aligned} \tag{3.8}$$

where $X_s(\theta) = X_s^x(\theta)$ *is a diffusion process associated with* $\mathcal{A}$, *which starts at time zero in* x *and which is defined on an auxialiary probability space* $(\Theta,\mathcal{G},\vartheta)$. E^x *is* *the expectation with respect to the measure* ϑ *of* $X_s = X_s^x$ *and the Wick eponential* $\exp^\diamond$ *is defined as in (2.15).*

Proof. By assumption we can apply the Feynman-Kac formula to the Cauchy problem (3.8) to obtain the stochastic representation of the solution as

$$\widetilde{\Phi}(t,x,z) = E^x[p_0(X_t(\theta))\exp(-\int_0^t k(s, X_{t-s}(\theta), z)ds)].$$

By taking (2.15) and (2.17) into account, i.e.

$$\mathcal{H}(\exp^\diamond(F)) = \exp(\mathcal{H}(F)),\ F \in (\mathcal{S})^*,$$

we can extract the $\mathcal{H}$ transform from the left hand side of the last relation and get

$$\widetilde{\Phi}(t,x,z) = \widetilde{U}(t,x,z),$$

where U denotes the expression on the right hand side of (3.9). Then the characterization theorem in [LØP] entails that

$$\Phi(t,x) = U(t,x).$$

In order to confirm that U actually solves (3.7), one has to show that the $\mathcal{H}$-transform and the derivatives in (3.7) can be interchanged. The general theory of parabolic partial differential equations applied to (3.8) (see Theorem 2.78 in [ES]) shows that for every open set $G = (0,T) \times D$ relatively compact in $\mathbb{R}_+ \times \mathbb{R}$ there exists a constant C such that

$$\left\|\widetilde{\Phi}\right\|_{C^{1,2+\gamma}(G)} \le C(\left\|L\widetilde{\Phi}\right\|_{C^{1,2+\gamma}(G)} + \|p_0\|_{C^{2+\gamma}(\partial G)},$$

where L is the partial differential operator defined by

$$Lu(t,x) = \frac{\partial u}{\partial t} - \mathcal{A}u + k(t,x,z)u.$$

The last inequality in connection with a Lévy version of Theorem 4.1.1 in [HØUZ] implies that this commutation of operators can be performed. Compare e.g. with [B] or [P]. ∎

We can verify that the solution $\Phi(t,x)$ in Theorem 3.3 takes values in a considerably smaller subspace of $(\mathcal{S})^*$.

Lemma 3.4. *The solution* $\Phi(t,x)$ *in (3.9) belongs to* $L^p(\mu)$, $p \ge 1$.

Proof. Since B_t and $\widetilde{N}(t,\varsigma)$ are independent we give the proof w.l.o.g. for the jump case, that is we consider the case $h = 0$ without B_t^Y in (3.9). We conclude from (2.14) that

$$\left\langle C_n(\omega), \varphi^{\otimes n}\right\rangle = \langle C_1(\omega), \varphi\rangle^{\diamond n},$$

where

$$\langle C_1(\omega), \varphi\rangle = \int_0^t \int_{\mathbb{R}_0} \varphi \widetilde{N}(ds, d\varsigma, \omega).$$

Therefore, in virtue of (2.6) and (2.15) the relation

$$(3.9)\qquad \exp^{\diamond}\{\int_0^t \int_{\mathbb{R}_0} \varphi \widetilde{N}(ds, d\varsigma, \omega)\}$$
$$= \exp\{\int_0^t \int_{\mathbb{R}_0} \log(1+\varphi)\widetilde{N}(ds, d\varsigma, \omega) + \int_0^t \int_{\mathbb{R}_0} (\log(1+\varphi) - \varphi) ds\nu(d\varsigma)\}$$

holds. So using Tonelli we find the estimate

$$E_\mu[|\Phi(t,x)|^p]$$
$$\leq \|p_0\|_\infty^p e^{pt\|c\|_\infty t} E^x E_\mu[\exp\{p\int_0^t \int_{\mathbb{R}_0} \log(\lambda(s, X_{t-s}(\theta), \varsigma)) N(ds, d\varsigma, \omega)$$
$$-p\int_0^t \int_{\mathbb{R}_0} (\lambda(s, X_{t-s}(\theta), \varsigma) - 1) ds\nu(d\varsigma)\}]$$
$$\leq \|p_0\|_\infty^p e^{pt\|c\|_\infty t} \exp\{(p+1)\nu(\mathbb{R}_0)t(\|\lambda\|_\infty + 1)^p\} < \infty.$$

Thus the result follows. ∎

We are coming to our main result.

Theorem 3.5. *Under assumptions $C1 - C8$ there exists a unique strong solution $\Phi(t,x)$ in $L^p(\mu), p \geq 1$, of the Zakai equation, which is twice continuously differentiable in x. The solution is explicitly given by*

$$(3.10)\qquad \Phi(t,x)$$
$$= E^x[p_0(X_t(\theta)\exp(-\int_0^t c(s, X_{t-s}(\theta)ds)$$
$$\exp\{\int_0^t h(s, X_{t-s}(\theta))dB_s(\omega) - \frac{1}{2}\int_0^t (h(s, X_{t-s}(\theta)))^2 ds$$
$$+\int_0^t \int_{\mathbb{R}_0} \log(\lambda(s, X_{t-s}(\theta), \varsigma))\widetilde{N}(ds, d\varsigma, \omega)$$
$$+\int_0^t \int_{\mathbb{R}_0} (\log(\lambda(s, X_{t-s}(\theta), \varsigma)) - (\lambda(s, X_{t-s}(\theta), \varsigma) - 1)) ds\nu(d\varsigma)\}].$$

Proof. One checks that $\Phi(\cdot, x)$ is a càdlàg adapted process. By the independence of B and $\widetilde{N}$ and by (3.10) and a similar relation for B we get the representation in (3.11). Following the same arguments as in [BDPV], based on Kolmogorov's continuity theorem for random fields (see also [K] for the use of stochastic flow arguments), one verifies with the help of Lemma 3.4 that Φ is twice continuously differentiable w.r.t. x. ∎

Remark 3.6. Theorem 3.5 can be extended e.g. to signal processes, which are modelled by a Lévy-Itô diffusion, i.e. by diffusions involving a Poisson random measure as additional term. In a more sophisticated way, using flow property arguments, the above proofs carry over to such processes under modified conditions.

4. Application to Finance: Calibration of Jump diffusion Models

In financial markets, one important issue in pricing and risk analysis is to estimate the dynamics of the underlying assets. Traditionally, financial assets have been described by diffusions, the Black-Scholes model being the classical one. However, diffusion models have been shown to be insufficient to explain certain empirical properties of asset returns and option prices. This has led to a development of a variety of jump diffusion models, of which one widely studied class is that of exponential Lévy processes. In this natural generalization of the Black-Scholes model one assumes the asset to be of the form

$$S_t = \exp(Y_t),$$

where Y_t is a Lévy process defined by its characteristics (a, σ, ν). Then, if one assumes the Lévy measure $\nu = v_\theta$ to be parametrized by θ, parameter estimation methods can be employed to optimally fit (a, σ, ν) to market data. Information about σ is quite rich in the market (e.g implied volatility), and there also exist methods based on quadratic variation to calculate σ. The evaluation of the drift a and the Lévy measure v_θ, however, is more delicate. A possible estimation procedure could be filtering.

In [CT] a non-parametric calibration method to fit the risk neutral Lévy measure to market data has been developed. Their empirical results show that already a small Lévy intensity (the total mass of the Lévy measure) is sufficient to account for observed implied volatility patterns. However, the assumption of time homogenous Lévy measures seems not to be appropriate. Actually, the Lévy intensity appears constant over time, but the shape of the density (the weight on different jumps) changes over time.

Motivated by these empirical findings we will in the following set up a filter problem in the framework of this paper, that takes this time dependence pattern into account. Here, the power of filtering becomes evident since this method is able to capture the dynamics of the estimate, constantly updated by the observations.

We choose to model an observed asset price process S_t by

$$Y_t = \log(S_t) = at + \sigma B_t + \int_0^t \int_{R_0} \varsigma N_\lambda(ds, d\varsigma), \tag{4.1}$$

where a, σ is supposed to be known, w.l.o.g. we assume $\sigma = 1$, B_t is the Brownian motion and $N_\lambda(ds, d\varsigma)$ is the jump measure with compensating measure $\lambda(t, X_t, \varsigma)dt\nu(d\varsigma)$ in (3.1) of the form

$$\begin{aligned} \lambda(t, X_t, \varsigma) &= \frac{\mathcal{N}_\varsigma(\mu_t, \rho_t)}{\mathcal{N}_\varsigma(m, s)} \\ \nu(d\varsigma) &= \hat{\lambda} \cdot \mathcal{N}_\varsigma(m, s) d\varsigma, \end{aligned} \tag{4.2}$$

where $\mathcal{N}(\mu, \rho)$ is the Gaussian density with mean μ and standard deviation ρ and where X_t is the multidimensional signal process (4.3). Here, $\mathcal{N}(m, s)$ is just an alibi density to induce a Lévy measure ν. In other words, we assume the compensated jump measure at time t to be of Gaussian type with mean μ_t, standard deviation ρ_t and given constant intensity $\hat{\lambda}$. Further we suppose that μ_t is described by an mean-reverting Ornstein-Uhlenbeck process and that ρ_t is constant over time. Then the problem of estimating the dynamic of the financial asset comes down to estimating μ_t and ρ_t. We remark that this model does not reflect reality very well, but we have chosen this setting for notational simplicity. Including the other model parameters in the estimation or choosing a more realistic parametrized compensated measure than of Gaussian type would follow the same principle.

Setting up the corresponding filter problem, results in the following five-dimensional signal process

$$X_t = \begin{cases} d\mu_t = d_t(c_t - \mu_t)dt + b_t dB_t \\ dd_t = 0 \\ dc_t = 0 \\ db_t = 0 \\ d\rho_t = 0 \end{cases} \tag{4.3}$$

and the observation process Y_t, given by (4.1) (with $\sigma = 1$).

We suppose that a prior distribution for $(\mu_0, d_0, c_0, b_0, \rho_0)$ is given by the joint density $p_0(\mu, d, c, b, \rho)$. The dual of the infinitesimal generator of the signal process is here

$$\mathcal{L}^* = \frac{1}{2} b^2 \frac{\partial}{\partial \mu^2} - d(c - \mu)\frac{\partial}{\partial \mu} + d,$$

which can be written as $\mathcal{A} - c$, where $\mathcal{A}$ is the generator of the system

$$\begin{aligned} d\widetilde{\mu}_t &= -\widetilde{d}_t(\widetilde{c}_t - \widetilde{\mu}_t)dt + \widetilde{b}_t dB_t \\ d\widetilde{d}_t &= 0 \\ d\widetilde{c}_t &= 0 \\ d\widetilde{b}_t &= 0 \\ d\widetilde{\rho}_t &= 0. \end{aligned} \tag{4.4}$$

The observation process is given by the log-price

$$dY_t = a dt + dB_t + \int_{\mathbb{R}_0} \varsigma N_\lambda(ds, d\varsigma). \tag{4.5}$$

Theorem 3.5 cannot be directly applied to this problem, since e.g. the drift in (4.4) and the intensity $\lambda(t, X_t, \varsigma)$ appear to be unbounded. However, since (4.4) is a Gausssian driven diffusion of linear type one checks that the proof of the Feynman-Kac formula as recalled in Theorem 3.1 carries over to this case. The linearity of (4.4) together with dominated convergence finally gives the desired regularity of the unnormalized conditional density. Therefore we get as a solution for the unnormalized conditional density of $\mu_t, d_t, c_t, b_t, \rho_t$:

$$
\begin{aligned}
(4.6) \qquad & \Phi(t, \mu, d, c, b, \rho) \\
= \ & E^{(\mu,d,c,b,\rho)}[p_0(\widetilde{\mu}_t^{\mu}(\theta), d, c, b, \rho) \exp(td + taB_t(\omega) - \frac{1}{2}ta^2) \\
& \exp\{\int_0^t \int_{\mathbb{R}_0} \log\left(\frac{\mathcal{N}_\varsigma(\widetilde{\mu}_{t-s}(\theta), \rho_{t-s})}{\mathcal{N}_\varsigma(m,s)}\right) N(ds, d\varsigma, \omega) \\
& + \int_0^t \int_{\mathbb{R}_0} \left(1 - \frac{\mathcal{N}_\varsigma(\widetilde{\mu}_{t-s}(\theta), \rho_{t-s})}{\mathcal{N}_\varsigma(m,s)}\right) ds\nu(d\varsigma)\}],
\end{aligned}
$$

where the above expectation is taken w.r.t. the measure $\vartheta(d\theta)$.

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