Explicit Representation of Strong Solutions of SDE's driven by Infinite Dimensional Lévy Processes

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Abstract

We develop a white noise framework for Lévy processes on Hilbert spaces. As the main result of this paper, we then employ these white noise techniques to explicitly represent strong solutions of stochastic differential equations driven by a Hilbert-space-valued Lévy process.

Key words and phrases: Lévy processes, infinite dimensional stochastic differential equations, white noise analysis

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1 Introduction

Stochastic differential equations (SDE's) in infinite dimensional spaces have become an indispensible tool to the study of a variety of random phenomena in natural and mathematical sciences. Since the works of [W], [I], [R], [DP] this subject has attracted much interest. SDE's with Gaussian noise on infinite dimensional state spaces were studied by many authors. See e.g. for references in the book of [DP], which provides an overview of analysis and applications of such equations. Infinite dimensional SDE's with jumps typically arise in the modelling of critical phenomena. For example SDE's with non-Gaussian additive noise on Hilbert or conuclear spaces have been applied to neurophysiology to describe fluctuations of membrane potentials of neurones. See e.g. [HKRX] and the references therein. Other applications include subjects like environmental pollution [KX1], infinite interacting particle systems [KR] and zero coupon bond markets in financial mathematics [BDKR].

In this paper, adopting ideas in [LP], [M-BP] we give a new approach to the study of strong solutions of SDE's driven by a Lévy process Z_t on a separable Hilbert space H. More specifically we focus our attention to the investigation of global strong solutions of SDE's of the type

$$dX_t = \gamma(X_{t-})dZ_t,\ X_0 = x,\ 0 \leq t \leq T, \tag{1.1}$$

where $\gamma : H \longrightarrow \mathbb{R}$ is a Borel measurable function.

Given a strong solution of equation (1.1), the main result of this paper is an explicit representation of this solution under certain conditions (see Theorem 3.2). In deriving this result we develop a white noise framework for Lévy processes on Hilbert spaces. We use this tool to construct an explicit distributional object which then is verified to be the strong solution.

The analysis of strong solutions is an important issue in SDE theory, since many applications require solutions, which are functions of the driving process. See e.g. [KR], where applications to the statistical mechanics of infinite particle systems are discussed. Various other applications result from stochastic control theory [K].

In this framework, beside being of interest in itself, we think that our result can be a fruitful starting point for the analysis and understanding of strong solutions in various directions. Interesting aspects for future research using our explicit representation are the study of path properties and existence and uniqueness results of strong solutions of SDE's with irregular coefficients driven by Hilbert-space-valued Lévy processes. Employing the corresponding white noise techniques in the 1-dimensional setting, the authors in [M-BP] derive moment conditions on the Doleans-Dade exponential of a certain process to guarantee existence and uniqueness of strong solutions of Itô diffusions. We also mention that by using different ideas [V1,2,3], [Zv], [ZvK], [GK] obtain results for SDE's with irregular coefficients in the case of Euclidean Gaussian noise. See also [KR], [GM] and [FZ] for a recent development. However, for infinite dimensions very little is known, and it would thus be of high interest to extend the ideas from [M-BP] to infinite dimensions.

Finally let us point out that the techniques presented here potentially carry over to inquire into the case of other types of SDE's like Backward stochastic differential equations or anticipative SDE's. Other topics comprise long time behaviour, Markov or flow property of solutions of (infinite dimensional) SDE's.

The structure of the remaining parts of the paper is as follows. In Section 2 we introduce a white noise framework for Lévy processes on a separable Hilbert space. Then we apply this theory in Section 3 to deduce the explicit representation of strong solutions of SDE (1.1).

2 Framework

In this Section we elaborate a white noise framework for infinite dimensional Lévy processes. We will employ this theory in the forthcoming Sections to study strong solutions of SDE's with irregular coefficients, whose driving noise is given by such Lévy processes. A comprehensive and nice account of Gaussian white noise theory can be found in the books of [HKPS], [Ku] and [O]. For works pertaining to non-Gaussian white noise analysis we refer e.g. to [IKu], [KDS]. See also [LP], [LØP].

Let us recollect the notion of a Lévy process on a separable Banach space $(B, \|\cdot\|)$. A $B-$valued adapted stochastic process $(Z_t)_{t\geq 0}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is called a *Lévy process*, if it is stochastically continuous and if it has independent and stationary increments starting at zero, that is if $Z_0 = 0$ a.e. and the increments $Z_t - Z_s$ are independent of $\mathcal{F}_s$ and have the same distribution as $Z_{t-s}, 0 \leq s < t$. Such processes can be represented by the Lévy-Itô decomposition on Banach spaces and are uniquely characterized by the triplet

$$(a, \rho, \nu), \tag{2.1}$$

where $a \in B$, ρ is a centered Gaussian measure on B and where ν is a *Lévy measure* on $B_0 := B - \{0\}$. See e.g. [De], [AG]. The Lévy measure ν gives information about the frequency of jumps of a certain size of the Lévy process and is defined as a σ-finite positive measure on the Borel sets of B_0, which satisfies

$$\widehat{\lambda}(f) = \exp\left\{\int_{B_0} \exp\left(if(x) - 1 - if(x)\chi_{\{\|x\|\leq 1\}}\right)\nu(dx)\right\} \tag{2.2}$$

for all $f \in B^{\prime}$ (dual of B), where $\widehat{\lambda}$ is the characteristic functional of a probability measure λ on B. See [AG], [Li]. If B is a Banach space of cotype 2, it is known that the Lévy measure ν integrates the function $1 \wedge \|x\|^2$. In the following we confine ourselves to separable Hilbert spaces $B = H$, which are spaces both of type 2 and cotype 2.

We carry on to establish a white noise framework for *pure jump Lévy processes*, i.e. for driftless Lévy processes on H without Gaussian part. So we focus on Lévy processes with characteristic tiplet $(0, 0, \nu)$. The extension of this setting to the general case of Lévy processes can be performed as e.g. in [P].

We give the construction of our white noise space. For this purpose define the space $U = \mathbb{R} \times H_0$, $H_0 := H - \{0\}$, and the diffusive Borel measure $\pi = \lambda \times \nu$ on U, where λ stands for the Lebesgue measure on $\mathbb{R}$. Further we consider a dense subspace Φ of $L^2(U, \pi)$ equipped with a consistent sequence of Hilbert norms $\|\cdot\|_p$, $p \geq 0$ such that $\|\cdot\|_0 = \|\cdot\|_{L^2(\pi)}$ as well as

$$\theta \|\phi\|_{p+1} \geq \|\phi\|_p$$

for all $\phi \in \Phi$, $p \geq 0$ with fixed $\theta \in (0,1)$. We make the following three assumptions on the structure of the function space Φ (see e.g. [IKu], [LØP]):

(i) Denoting by $\Phi_p, p \geq 0$ the completions w.r.t the norms $\|\cdot\|_p$ and by Φ_{-p} their duals with corresponding norms $\|\cdot\|_{-p}$ it is assumed that the canonical injection $\Phi_1 \hookrightarrow \Phi_0$ is traceable, that is the mapping $\delta : U \longrightarrow \Phi_{-1}$; $u \longmapsto \delta_u$ is continuous, where δ_u is the evaluation map $\phi \longmapsto \phi(u)$. In addition it is required that

$$\int_U \|\delta_u\|_{-1}^2 \, \pi(du) < \infty.$$

(ii) The space Φ is an algebra w.r.t. to multiplication of functions and the following inequality holds: For all $p \geq 1$ there exists a constant $M_p > 0$ such that

$$\|\varphi\phi\|_p \leq M_p \|\varphi\|_p \|\phi\|_p$$

for all $\varphi, \phi \in \Phi$.

(iii) The chain of inclusions

$$\Phi \hookrightarrow L^2(U, \pi) \hookrightarrow \Phi^{\shortmid}$$

constitutes a Gel'fand triplet ($\Phi^{\shortmid}$ dual of Φ) and the evaluation map δ fulfills the condition

$$\|\delta\|_\infty := \int_U \|\delta_u\|_{-1}\, \pi(du) + \sup_{u \in U} \|\delta_u\|_{-1} < \infty.$$

Let us mention that e.g. the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ satisfies the above assumptions. As a consequence of (i), (ii), (iii) we find that all ϕ in Φ_1 are continuous and bounded and that

$$\|\phi\|_{L^1(\pi)} + \sup_{u \in U} |\phi(y)| \le \|\delta\|_\infty \|\phi\|_{-1} \tag{2.3}$$

holds. Moreover, since assumption (i) ensures the applicability of the famous Bochner-Minlos theorem we conclude the existence of a unique probability measure μ on the Borel sets of $\Phi^{\shortmid}$such that

$$\int_{\Phi^{\shortmid}} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = \exp\left(\int_U (e^{i\phi(u)} - 1)\pi(dy) \right) \tag{2.4}$$

for all $\phi \in \Phi$, where $\langle \omega, \phi \rangle := \omega(\phi)$ denotes the action of $\omega \in \Phi^{\shortmid}$ on $\phi \in \Phi$. We call the measure μ on $\Omega = \Phi^{\shortmid}$ *(pure jump) Lévy white noise probability measure*. We note that in virtue of the assumptions (i), (ii), (iii) μ enjoys the property of the first condition of analyticity in the sense of [KDS].

In the following we assume that the compensated Poisson random measure

$$\widetilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt$$

associated with our Lévy process Z_t is defined on the *white noise probability space*

$$(\Omega, \mathcal{F}, P) = \left(\Phi^{\shortmid}, \mathcal{B}(\Phi^{\shortmid}), \mu \right).$$

The theory of holomorphic functions on infinite dimensional spaces (see e.g. [Di]) guarantees the existence of symmetric polynomials, called *generalized Charlier polynomials* $C_n(\omega) \in \left(\Phi^{\widehat{\otimes} n} \right)^{\shortmid}$ (dual of the n-th completed symmetric tensor product of Φ with itself), from which an orthogonal basis $\{\mathcal{K}_\alpha(\omega)\}_{\alpha \in \mathcal{J}}$ of $L^2(\mu)$ can be constructed as

$$\mathcal{K}_\alpha(\omega) = \left\langle C_{|\alpha|}(\omega), \delta^{\widehat{\otimes} \alpha} \right\rangle. \tag{2.5}$$

The symbol $\mathcal{J}$ denotes the collection of all multiindices $\alpha = (\alpha_1, \alpha_2, ...)$, whose entries $\alpha_i \in \mathbb{N}_0$ are finitely often non-zero. The expression $\delta^{\widehat{\otimes}\alpha}$ stands for the symmetrization of $\delta_1^{\otimes\alpha_1} \otimes ... \otimes \delta_j^{\otimes\alpha_j}$, where $\{\delta_j\}_{j\geq 1}$ constitutes an orthonormal basis of $L^2(\pi)$. We choose $\{\delta_j\}_{j\geq 1}$ to be contained in Φ.

Thus every $f \in L^2(\mu)$ has the chaos representation

$$f = \sum_{\alpha\in\mathcal{J}} c_\alpha \mathcal{K}_\alpha$$

with unique Fourier coefficients $c_\alpha \in \mathbb{R}$, for which the isometry

$$\|f\|^2_{L^2(\mu)} = \sum_{\alpha\in\mathcal{J}} \alpha! c_\alpha^2 \tag{2.6}$$

holds, where $\alpha! := \alpha_1!\ \alpha_1!...$ for $\alpha \in \mathcal{J}$. We continue to introduce the stochastic test function space $(\mathcal{S})$ and stochastic distribution space $(\mathcal{S})^*$. The procedure to define these spaces corresponds to a second quantization argument in Gaussian white noise analysis. Define the weights

$$\Gamma^{\otimes k\alpha} = \prod_{j=1}^{Index(\alpha)} (2j(1 + \|\delta_j\|^2_\infty))^{k\alpha_j} \tag{2.7}$$

for $k \in \mathbb{Z}, \alpha \in \mathcal{J}$, where $Index(\alpha) := \max\{i : \alpha_i \neq 0\}$ and where δ_j is the $L^2(\mu)-$basis. Note that $\|\delta_j\|_\infty = \sup_x |\delta_j(x)| < \infty$ for all j because of (2.3). The *Lévy-Hida test function space* $(\mathcal{S})$ can be characterized as the space of all $f \in L^2(\mu)$ with chaos representation $f = \sum_{\alpha\in\mathcal{J}} c_\alpha \mathcal{K}_\alpha$ such that the growth condition

$$\|f\|^2_{0,k} := \sum_{\alpha\in\mathcal{J}} \alpha! c_\alpha^2 \Gamma^{\otimes k\alpha} < \infty \tag{2.8}$$

is satisfied for all $k \in \mathbb{N}_0$. The space $(\mathcal{S})$ is endowed with the projective topology based on the norms $(\|\cdot\|_{0,k})_{k\in\mathbb{N}_0}$ in (2.8). We define the *Lévy-Hida distribution space*, denoted by $(\mathcal{S})^*$ as the topological dual of $(\mathcal{S})$. By construction we observe that

$$(\mathcal{S}) \hookrightarrow L^2(\mu) \hookrightarrow (\mathcal{S})^* \tag{2.9}$$

is Gel'fand triplet. We enrich the structure of $(\mathcal{S})^*$ by defining a multiplication $\diamond$ of distributions. This non-linear operation, which makes $(\mathcal{S})^*$ a topological algebra, is called *Wick product* and is defined as

$$(\mathcal{K}_\alpha \diamond \mathcal{K}_\beta)(\omega) = (\mathcal{K}_{\alpha+\beta})(\omega),\ \ \alpha, \beta \in \mathcal{J} \tag{2.10}$$

Linear extension gives an operation on the whole space $(\mathcal{S})^* \times (\mathcal{S})^*$. As an example we shall mention that

$$\langle C_n(\omega), f_n\rangle \diamond \langle C_m(\omega), g_m\rangle = \left\langle C_{n+m}(\omega), f_n \widehat{\otimes} g_m \right\rangle \tag{2.11}$$

for $f_n \in \Phi^{\widehat{\otimes} n}$and $g_m \in \Phi^{\widehat{\otimes} m}$. See e.g. [LØP].

In the forthcoming sections we invoke the *Lévy Hermite transform* $\mathcal{H}$ as a crucial tool to scrutinize SDE's driven by infinite dimensional Lévy processes. As an algebra monomorphism from $(\mathcal{S})^*$ into the algebra of power series in infinitely many (complex) variables the Lévy Hermite transform provides a characterization of distributions (see characterization theorem 2.3.8 in [LØP]). By exploiting the chaos expansion of distributions along the basis $\{\mathcal{K}_\alpha(\omega)\}_{\alpha\in\mathcal{J}}$ we define just as in the Gaussian case the *Lévy Hermite transform* of $X(\omega) = \sum_\alpha c_\alpha \mathcal{K}_\alpha(\omega) \in (\mathcal{S})^*$, denoted by $\mathcal{H}X$, as

$$\mathcal{H}X(z) = \sum_\alpha c_\alpha z^\alpha \in \mathbb{C} \ , \tag{2.12}$$

where $z = (z_1, z_2, ...) \in \mathbb{C}^{\mathbb{N}}$, i.e. in the space of complex-valued sequences, and where $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} ...$ It can be e.g. shown that $\mathcal{H}X(z)$ in (2.12) is absolutely convergent on the infinite dimensional neighbourhood

$$\mathbb{K}_q(R) := \left\{ (z_1, z_2, ...) \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha\neq 0} |z^\alpha|^2 \, \Gamma^{\otimes k\alpha} < R^2 \right\} \tag{2.13}$$

for some $0 < q \leq R < \infty$. By the mapping properties of $\mathcal{H}$ we see that

$$\mathcal{H}(X \diamond Y)(z) = \mathcal{H}(X)(z) \cdot \mathcal{H}(Y)(z)$$

holds for $X, Y \in (\mathcal{S})^*$ on some $\mathbb{K}_q(R)$. The latter commutation relation gives rise to a generalization to Wick versions of complex analytical functions $g : \mathbb{C} \longrightarrow \mathbb{C}$, whose Taylor expansion around $\xi_0 = \mathcal{H}(X)(0)$ has real valued coefficients: Using the proof of Theorem 2.3.8 in [LØP] one can find a unique distribution $Y \in (\mathcal{S})^*$ such that

$$\mathcal{H}(Y)(z) = g\left(\mathcal{H}(X)(z)\right) \tag{2.14}$$

on $\mathbb{K}_q(R)$ for some $0 < q \leq R < \infty$. We shall write $g^\diamond(X)$ for Y to indicate the Wick version of g.

For example, the Wick version of the exponential function exp can be calculated as

$$\exp^\diamond X = \sum_{n\geq 0} \frac{1}{n!} X^{\diamond n}. \tag{2.15}$$

Another important concept which we make use of in the next sections is the *(singular) white noise* $\overset{\bullet}{\widetilde{N}}(t,x)$ of the Poisson random measure $\widetilde{N}(dt,dx)$. It can be regarded as a formal Radon-Nikodym derivative of $\widetilde{N}(dt,dx)$ and is defined as

$$\overset{\bullet}{\widetilde{N}}(t,x) = \sum_{k\geq 1} \delta_k(t,x)\mathcal{K}_{\varepsilon_k}(\omega),$$

where

$$\varepsilon_k(j) := \begin{cases} 1, & j=k \\ 0, & \text{else} \end{cases} .$$

Denoting by $\|\cdot\|_{-0,-2}$ the dual norm of $\|\cdot\|_{0,2}$ it can be verified that

$$\sup_{(t,x)\in U} \left\| \overset{\bullet}{\widetilde{N}}(t,x) \right\|_{-0,-2} < \infty. \tag{2.16}$$

Hence $\overset{\bullet}{\widetilde{N}}(t,x)$ is contained in $(\mathcal{S})^*$ for all t,x. We directly ascertain that

$$\mathcal{H}(\overset{\bullet}{\widetilde{N}}(t,x))(z) = \sum_{k\geq 1} \delta_k(t,x) z_k. \tag{2.17}$$

We conclude this section by pointing out an interesting property of the Wick product. The Wick product reveals a relation to stochastic integrals w.r.t. to $\widetilde{N}(dt,dx)$: Let $Y(t,x,\omega)$ be a predictable process such that $E\int_0^T \int_{\mathbb{R}_0} Y^2(t,x,\omega)dt\nu(dz) < \infty$. Then $Y(t,x,\omega) \diamond \overset{\bullet}{\widetilde{N}}(t,x)$ is $\lambda \times \nu$-Bochner integrable in $(\mathcal{S})^*$ and

$$\int_0^T \int_{H_0} Y(t,x,\omega)\ \widetilde{N}(dt,dx) = \int_0^T \int_{H_0} Y(t,x,\omega) \diamond \overset{\bullet}{\widetilde{N}}(t,x) dt\nu(dx). \tag{2.18}$$

See [LØP] or [ØP] for similar proofs.

3 Explicit representability of strong solutions of a pure jump Lévy diffusion

In this Section we assume that the Poisson random measure $N(dt,dx)$ associated with the Lévy process Z_t in (3.1) is constructed on the white noise probability space $(\Omega, \mathcal{F}, \mu)$. Our objects of study are Lévy noise driven infinite dimensional SDE's of the type

$$\begin{aligned} dX_t &= \gamma(X_{t-})dZ_t \\ &= \int_{H_0} \gamma(X_{t-})xN(dt,dx),\ X_0 = y,\ 0 \le t \le T\,, \end{aligned} \tag{3.1}$$

where Z_t is the uncompensated pure jump Lévy process given by

$$Z_t = \int_0^t \int_{H_0} xN(dt,dx)\,,$$

and where γ is a Borel measurable function from H to $\mathbb{R}$. The stochastic integral in (3.1), which is of the form

$$\int_0^t \int_{H_0} \Phi(t,x,\omega)N(dt,dx),$$

is defined for predictable integrands $\Phi(t,x,\omega)$ in the sense of [IW] in the Hilbert space setting. For information about stochastic integration with respect to (compensated) Poisson random measures on Banach spaces or conuclear spaces the reader is referred to [KX3], [Ü], [Rü], [De]. We impose on γ to be (locally) Lipschitz continuous and of linear growth. It is e.g. shown in [KX3], [Z] that these conditions entail the existence of a unique càdlàg adapted process $X_t \in L^2(\mu; H)$, which globally solves (3.1) in the strong sense. Further, we require non-degeneracy of the diffusion coefficient γ in the sense that

$$|\gamma(y)| > 0 \tag{3.2}$$

for all $y \in H$.

Remark 3.1 The degeneracy condition (3.2) guarantees the equivalence of the jump measure of X_t and the jump measure of Z_t. In the proof of Theorem 3.4 this fact is needed to transform the compensating measure of X_t into the Lévy measure ν under a change of probability.

In the sequel we make the restriction that the dilation measure ν^λ , given by

$$\nu^\lambda(\Gamma) := \nu(\lambda \cdot \Gamma), \Gamma\ \in \mathcal{B}(H), \tag{3.3}$$

is absolutely continuous w.r.t. to the Lévy measure ν for all $\lambda \in \mathbb{R}_0$. We choose the Radon-Nikodym density of ν^λ, denoted by $\vartheta(\lambda, x)$, to be strictly positiv.

Example 3.2 Consider a Lévy measure ν on H, whose generalized exponent is a $p-$stable measure ($0 < p \leq 2$). Then ν can be represented as

$$\nu(\Gamma) = \int_{\mathbb{R}_+} \int_{\partial B} \chi_\Gamma(tx)\sigma(dx)t^{-p-1}dt, \ \Gamma \ \in \mathcal{B}(H)$$

for a finite measure σ on $\partial B := \{x \in H : \|x\| = 1\}$. See e.g. [Li]. Thus the Radon-Nikodym density of the dilation ν^λ w.r.t. ν is given by

$$\vartheta(\lambda, x) = \lambda^{-p}.$$

We are coming to the main result of this Section.

Theorem 3.3 Assume that γ in (3.1) is locally Lipschitz continuous and of linear growth. Retain the conditions (3.2) and (3.3). Further suppose that either the Lévy measure ν is finite or the integrability condition

$$\begin{aligned} & E_\mu \left[\exp \left\{ 2 \int_0^T \int_{H_0} \big(-\log(\vartheta(\gamma(X_{s-})^{-1}, x)) \right. \right. \\ & \quad \left. \left. +2\vartheta(\gamma(X_{s-})^{-1}, x) - 1 \big) \, \nu(dx)ds \right\} \right] \\ < & \quad \infty \end{aligned} \tag{3.4}$$

holds, where X_t is the global strong solution of (3.1). Then X_t takes the explicit form

$$X_t = \sum_{i \geq 1} \alpha_i(t) e_i, \tag{3.5}$$

where

$$\alpha_i(t) = E_{\widetilde{\mu}} \left[\left\langle \widetilde{Z}_t, e_i \right\rangle J_T^\diamond \right]$$

and where $J_T^\diamond$ is defined as

$$\begin{aligned} & J_T^\diamond = \\ & \exp^\diamond \{ \int_0^T \int_{H_0} \log^\diamond \left((1 + \overset{\bullet}{\widetilde{N}}(\omega, s, \gamma(\widetilde{Z}_{s-})^{-1}x)) \vartheta(\gamma(\widetilde{Z}_{s-})^{-1}, x) \right) N(\widetilde{\omega}, ds, dx) \} \\ & \diamond \exp^\diamond \{ \int_0^T \int_{H_0} \left(1 - (1 + \overset{\bullet}{\widetilde{N}}(\omega, s, \gamma(\widetilde{Z}_{s-})^{-1}x)) \vartheta(\gamma(\widetilde{Z}_{s-})^{-1}, x) \right) \nu(dx)ds \}. \end{aligned}$$

The Wick product $\diamond$ refers to ω and $\widetilde{Z}_t$ is a Lévy process, which has the same characteristics as Z_t on a copy $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mu})$ of the initial white noise space

$(\Omega, \mathcal{F}, \mu)$. The integrals in (3.6) define (stochastic) Bochner integrals on the Lévy-Hida space.

Remark 3.4 As stated in Theorem 3.3, the explicit representation is always valid in case the Lévy measure ν is finite. If the Lévy process has infinite activity, a class of SDE's satisfying the integrability condition (3.4) is is given by those Lévy measures with singularity of order one around zero. For example, we could define

$$\nu(\Gamma) := \int_{\mathbb{R}_+} \int_{\partial B} \chi_\Gamma(tx)\sigma(dx)\varphi(t)dt,$$

where

$$\varphi(t) = \begin{cases} t^{-1}, & 0 < t < 1 \\ e^{-t+1}, & t \geq 1 \end{cases},$$

and B is as in Example 3.2. Then

$$\vartheta(\lambda, x) = \frac{\lambda\varphi(\lambda \|x\|)}{\varphi(\|x\|)}.$$

Here we assume that $\epsilon < \gamma(y) \leq M < \infty$, $0 < \epsilon$, for all y. Other prominent examples belonging to this class in the case of finite dimensional Hilbert spaces are the Gamma or the variance Gamma Lévy processes.

Proof *(Theorem 3.3)* One finds that the Hermite transform of X_t can be written as

$$\mathcal{H}(X_t)(z) = E_\mu\left[X_t\mathcal{E}(\int_0^T \int_{H_0} \phi_z(s,x)\widetilde{N}(ds,dx))\right],$$

on some neighbourhood $\mathbb{K}_q(R)$, where $\phi_z(s,x) = \sum_k z_k\delta_k(s,x)$, $z \in \mathbb{C}_c^{\mathbb{N}}$ and $\widetilde{N}(ds,dx) = N(ds,dx) - \nu(dx)ds$. We can express the Doleans-Dade exponential in the above equation by

$$\begin{aligned} & \mathcal{E}(\int_0^T \int_{H_0} \phi_z(s,x)\widetilde{N}(ds,dx)) \\ = \; & \exp\{\int_0^T \int_{H_0} \log(1+\phi_z(s,x))N(ds,dx) - \int_0^T \int_{H_0} \phi_z(s,x)\nu(dx)ds\}. \end{aligned}$$

on $\mathbb{K}_q(R)$ for some q, R. By applying the Girsanov theorem for random measures (see [JS]), we obtain that

$$\mathcal{H}(X_t)(z) = E_{\mu^*}[X_t]$$

where μ^* is the equivalent probability measure with density process

$$\frac{d\mu^*}{d\mu} = \mathcal{E}(\int_0^T \int_{H_0} \phi_z(s,x)\widetilde{N}(ds,dx)).$$

The predictable compensation of the jump measure N_X of X_t w.r.t. μ^* can be evaluated as

$$\nu^*(B\times\Gamma) = \int_{[0,T)\times H_0} \chi_{B\times\Gamma}(s,\gamma(X_{s-})x)\,(1+\phi_z(s,x))\,\nu(dx)ds$$

for $B\times\Gamma \in \mathcal{B}([0,T))\otimes\mathcal{B}(H_0)$. By assumption on the dilation measure of ν we get that

$$\nu^*(ds,dx) = \left(1+\phi_z(s,\gamma(X_{s-})^{-1}x)\right)\vartheta(\gamma(X_{s-})^{-1},x)\nu(dx)ds.$$

Next define

$$\varphi_z(\omega,s,x) = \frac{1}{(1+\phi_z(s,\gamma(X_{s-})^{-1}x))\,\vartheta(\gamma(X_{s-})^{-1},x)} - 1$$

on $\mathbb{K}_q(R)$ for some q, R. Then Lemma 3.6 below and the Girsanov theorem imply

$$\begin{aligned}
& E_{\mu^*}[X_t] \\
= \; & E_{\mu^*}[X_t\mathcal{E}^{-1}(\int_0^T \int_{H_0} \varphi_z(\widetilde{\omega},s,x)(N_X-\nu^*)(ds,dx)) \\
& \mathcal{E}(\int_0^T \int_{H_0} \varphi_z(\widetilde{\omega},s,x)(N_X-\nu^*)(ds,dx))] \\
= \; & E_{\widetilde{\mu}}\Big[\widetilde{Z}_t \exp\{\int_0^T \int_{H_0} \log\left(\left(1+\phi_z(s,\gamma(\widetilde{Z}_{s-})^{-1}x)\right)\vartheta(\gamma(\widetilde{Z}_{s-})^{-1},x)\right) N(\widetilde{\omega},ds,dx) \\
& \cdot \exp\{\int_0^T \int_{H_0} \left(1-\left(1+\phi_z(s,\gamma(\widetilde{Z}_{s-})^{-1}x)\right)\vartheta(\gamma(\widetilde{Z}_{s-})^{-1},x)\right)\nu(dx)ds\Big], \qquad (3.7)
\end{aligned}$$

for $z\in\mathbb{K}_q(R)$ with some $0<q,R<\infty$, where $\widetilde{Z}_t\in H$ is a Lévy process with the same characteristic triplet as Z_t on a copy $\left(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mu}\right)$ of the white noise space $(\Omega,\mathcal{F},\mu)$.

Because of a Lévy version of Theorem 2.6.12 in [HØUZ] and Lemma 3.5 we can apply the inverse Hermite transform on both sides of (3.7) to obtain the result. ∎

Lemma 3.5 The processes $Y_t^{(i)} := \left\langle \widetilde{Z}_t, e_i \right\rangle J_T^{\diamond}$, $i \geq 1$ with $\widetilde{Z}_t$ and $J_T^{\diamond}$ as in Theorem 3.2 are Bochner integrable w.r.t. $\widetilde{\mu}$ on $(\mathcal{S})^*$.

Proof It suffices to show that there exist q, R such that

$$E_{\widetilde{\mu}}\left[\sup_{z\in\mathbb{K}_q(R)}\left|\mathcal{H}(\left\langle \widetilde{Z}_t, e_i \right\rangle J_T^{\diamond})(z)\right|\right] < \infty \tag{3.8}$$

for all $i \geq 1$. It can be verified that

$$\sup_{z\in\mathbb{K}_q(R)} |\phi_z(s,x)| \leq R \left\| \overset{\bullet}{\widetilde{N}}(s,x) \right\|_{-0,-2} \quad \text{for all } s, x, \tag{3.9}$$

where $\phi_z(s,x)$ denotes the Hermite transform of $\overset{\bullet}{\widetilde{N}}(s,x)$ and where $\|\cdot\|_{-0,-2}$ is the dual norm w.r.t. the completion of $\left((\mathcal{S}), \|\cdot\|_{0,2}\right)$ (see Section 2). Since we have that

$$\begin{aligned}
&\int_0^T \int_{H_0} \left(R \left\| \overset{\bullet}{\widetilde{N}}(s, \gamma(\widetilde{Z}_{s-})^{-1}x) \right\|_{-0,-2} \vartheta(\gamma(\widetilde{Z}_{s-})^{-1}, x) \right) \nu(dx)ds \\
= &\int_0^T \int_{H_0} \left(R \left\| \overset{\bullet}{\widetilde{N}}(s, \gamma(\widetilde{Z}_{s-})^{-1}x) \right\|_{-0,-2} \right) \nu^{\gamma(\widetilde{Z}_{s-})^{-1}}(dx)ds \\
= &\int_0^T \int_{H_0} \left(R \left\| \overset{\bullet}{\widetilde{N}}(s,x) \right\|_{-0,-2} \right) \nu(dx)ds \qquad (3.10) \\
\leq &\ const. \int_0^T \int_{H_0} \left(\left\| \overset{\bullet}{\widetilde{N}}(s,x) \right\|_{-0,-2} \right)^2 \nu(dx)ds < \infty
\end{aligned}$$

we get the estimate

$$
\begin{aligned}
& E_{\widetilde{\mu}}\left[\sup_{z\in\mathbb{K}_q(R)}\left|\mathcal{H}(\left\langle \widetilde{Z}_t, e_i\right\rangle J_T^{\diamond})(z)\right|\right] \\
\leq\ & const.E\left[\left\|\widetilde{Z}_t\right\|\mathcal{E}\left(\int_0^T\int_{H_0}\left\{\left(1+R\left\|\overset{\bullet}{\widetilde{N}}(s,\gamma(\widetilde{Z}_{s-})^{-1}x)\right\|_{-0,-2}\right)\right.\right.\right. \\
& \left.\left.\cdot\vartheta(\gamma(\widetilde{Z}_{s-})^{-1},x)-1\right\}\widetilde{N}(\widetilde{\omega},ds,dx)\right) \\
& \cdot\exp\left\{\int_0^T\int_{H_0}\left\{\left(1+R\left\|\overset{\bullet}{\widetilde{N}}(s,\gamma(\widetilde{Z}_{s-})^{-1}x)\right\|_{-0,-2}\right)\right.\right. \\
& \left.\left.\left.\cdot\vartheta(\gamma(\widetilde{Z}_{s-})^{-1},x)-1\right\}\nu(dx)ds\right\}\right] \\
\leq\ & const.E\left[\left\|\widetilde{Z}_t\right\|\mathcal{E}\left(\int_0^T\int_{H_0}\left\{\left(1+R\left\|\overset{\bullet}{\widetilde{N}}(s,\gamma(\widetilde{Z}_{s-})^{-1}x)\right\|_{-0,-2}\right)\right.\right.\right. \\
& \left.\left.\left.\cdot\vartheta(\gamma(\widetilde{Z}_{s-})^{-1},x)-1\right\}\widetilde{N}(\widetilde{\omega},ds,dx)\right)\right]
\end{aligned}
$$

Let

$$
\begin{aligned}
& M_t \\
=\ & \int_0^t\int_{H_0}\left\{\left(1+R\left\|\overset{\bullet}{\widetilde{N}}(s,\gamma(\widetilde{Z}_{s-})^{-1}x)\right\|_{-0,-2}\right)\vartheta(\gamma(\widetilde{Z}_{s-})^{-1},x)-1\right\}\widetilde{N}(\widetilde{\omega},ds,dx)
\end{aligned}
$$

We infer from (3.10) and the Itô isometry that

$$E\left[M_t^2\right]<\infty.$$

The latter implies that the stochastic exponential $\mathcal{E}(M_t)$ is square integrable (see [LM]). In connection with the finiteness of the Lévy measure we conclude (3.8). ∎

Lemma 3.6 Adopt the definitions of φ_z, ν^*, μ^* and N_X in the proof of Theorem 3.2. Suppose that either ν is finite or the integrability condition (3.4) is fulfilled. Then the Doleans-Dade exponential

$$\mathcal{E}(\int_0^t\int_{H_0}\varphi_z(\widetilde{\omega},s,x)(N_X-\nu^*)(ds,dx)) \qquad (3.11)$$

is a martingale on $\mathbb{K}_q(R)$ for some q, R.

Proof Let first the Lévy measure ν be finite and denote by $M := \nu(H_0) = \nu^\lambda(H_0)$ the total mass of ν (and also of ν^λ). Then it is straight forward to see that

$$\int_0^t \int_{H_0} \varphi_z(\widetilde{\omega}, s, x)(N_X - \nu^*)(ds, dx) < c \cdot M$$

for a constant c, and the Doleans-Dade exponential is a martingale. If the integrability condition (3.4) is fulfilled, then using (2.16) and the inequality (3.9) it can be shown that

$$\begin{aligned} & E_{\mu^*}\left[\left\{\exp \int_0^T \int_{H_0} ((1 + \varphi_z(\widetilde{\omega}, s, x)) \log(1 + \varphi_z(\widetilde{\omega}, s, x)) - \varphi_z(\widetilde{\omega}, s, x)) \right.\right. \\ & \left.\left. \nu^*(ds, dx))\right\}\right] \\ \leq \ & const. E_\mu \left[\exp\left\{2 \int_0^T \int_{H_0} \left(-\log(\vartheta(\gamma(X_{s-})^{-1}, x))\right.\right.\right. \\ & \left.\left.\left. +2\vartheta(\gamma(X_{s-})^{-1}, x) - 1\right) \nu(dx) ds\right\}\right]^{1/2} \\ < \ & \infty \end{aligned}$$

on $\mathbb{K}_q(R)$ for some q, R. By invoking a Novikov condition in [LM] the result follows. ∎

Remark 3.7 (i) The condition on the dilation measure ν^λ in (3.3) can be weakened by assuming that there exists a Lévy measure $\widehat{\nu}$ such that

$$\nu^\lambda \ll \widehat{\nu}$$

for all λ in a set $D \subset \mathbb{R}_+$. In this case the proof of Theorem 3.2 carries over to attain a similar result. However, the Lévy process $\widetilde{Z}_t$ in (3.5) must be replaced by one with Lévy measure $\widehat{\nu}$. Further, γ is restricted to take values in D.

(ii) If we consider instead of (3.1) the more general SDE

$$dX_t = \int_{H_0} \gamma(X_{t-}, x) N(dt, dx), \;\; X_0 = y, 0 \leq t \leq T$$

with the diffusion coefficient $\gamma : H \times H \to \mathbb{R}$ it will be conceivable to retrieve analogous results to Theorem 3.2 by studying more general transformations of the Lévy measure than those of dilations.

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