Local Risk-Minimization for Defaultable Claims with Recovery Process

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Abstract

We study the local risk-minimization approach for defaultable claims with random *recovery at default time*, seen as payment streams on the random interval $[\![0, \tau \wedge T]\!]$, where T denotes the fixed time-horizon. We find the pseudo-locally risk-minimizing strategy in the case when the agent information takes into account the possibility of a default event (local risk-minimization with $\mathbf{G}$-strategies) and we provide an application in the case of a corporate bond. We also discuss the problem of finding a pseudo-locally risk-minimizing strategy if we suppose the agent obtains her information only by observing the non-defaultable assets.

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1 Introduction

The aim of this paper is to discuss the problem of pricing and hedging defaultable claims, i.e. options that can lose partially or totally their value if a default event occurs, by means of local risk-minimization approach applied to payment streams. We consider a simple financial market model with two non-defaultable primary assets (the money market account and a discounted risky asset) and a defaultable claim with random recovery at default time.

Since it is impossible to hedge against the occurrence of default by using a portfolio consisting only of the primary assets, the default-free market extended with

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the defaultable claim is incomplete and then it is reasonable to use the *local risk-minimization*, that has become a popular criterion for pricing and hedging in incomplete markets.

Other quadratic hedging methods such as *mean-variance hedging* have been extensively studied in the context of defaultable markets by [1], [3], [6], [7] and [8].

The local risk-minimization has been already applied to defaultable markets with recovery schemes *at maturity* in [1, 2]. In [1] the authors have investigated the case where the default time and the underlying Brownian motion were independent. In [2], they have considered the more general case where the dynamics of the risky asset may be influenced by the occurring of a default event and also the default time itself may depend on the assets prices behavior.

Here we allow for mutual dependence between default time and asset prices behavior as in [2], but we focus on defaultable claims that deliver a recovery payment *at default time* in case of default, seen as *payment streams* on the random interval $[\![0, \tau \wedge T]\!]$, where T denotes the maturity date of the contract.

First we extend the results of [27] for local risk-minimization for payment streams to the case of payment streams with *random* delivery date. Then we apply these results to the case of defaultable claims with recovery at time of default and compute explicitly the optimal strategy and the optimal cost.

Another important achievement of this paper is also that we are *not* assuming the hypothesis (H) holds, i.e. the Brownian motion W remains a (continuous) martingale (and then a Brownian motion) with respect to the enlarged filtration $\mathbf{G}$. This is a consequence of the fact that we assume that hedging stops after default.

More precisely, the paper is structured as follows. Section 2 introduces the general setup and Section 3 lays out the local risk-minimization for payment streams adapted to our context. In Section 4, we provide the main result by finding a closed formula for the pseudo-locally risk-minimizing strategy in the case when the agent's information takes into account the possibility of a default event. In particular we compute it explicitly in the case of a corporate bond (see Section 5). Finally in Section 6, we discuss the problem of finding a pseudo-locally risk-minimizing strategy if we suppose the agent obtains her information only by observing the non-defaultable assets.

2 The setting

We consider a simple model of a financial market where we can find a risky asset, the money market account and *defaultable claims*, i.e. contingent agreements traded over-the-counter between default-prone parties. Each side of contract is exposed to the *counterparty risk* of the other party. Hence defaultable claims are derivatives that could fail or lose their own value.

In [1, 2] we have already applied the local risk minimization approach to the case of defaultable markets. However in this paper we study for the first time the problem of finding a pseudo-locally risk-minimizing strategy when the defaultable claim admits a recovery *at default time* τ, seen as a *payment stream* on the interval $[\![0, \tau \wedge T]\!]$ for a fixed time horizon $T \in (0, \infty)$. Since in practice hedging a defaultable claim after default time is usually of minor interest and in our model we have only a single default time, we follow the approach of [9] and assume that hedging stops after default. Hence it makes sense to hedge by using the stopped discounted price process X^τ instead of X.

The random time of default is represented by a nonnegative random variable $\tau : \Omega \to [0, T] \cup \{+\infty\}$, defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$, for each $t \in [0, T]$. The last condition means that the default may not happen during the interval $[0, T]$. For a given default time τ, we introduce the associated *default process* H given by $H_t = \mathbb{I}_{\{\tau \leq t\}}$, for $t \in [0, T]$ and denote by $\mathbf{H} := (\mathcal{H}_t)_{0 \leq t \leq T}$ the filtration generated by the process H, i.e. $\mathcal{H}_t = \sigma(H_u : u \leq t)$ for any $t \in [0, T]$.

Let W be a standard Brownian motion on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and $\mathbf{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ the natural filtration of W. Let $\mathbf{G} := (\mathcal{G}_t)_{0 \leq t \leq T}$ be the filtration given by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, for every $t \in [0, T]$. We put $\mathcal{G} = \mathcal{G}_T$. We remark that all the filtrations are assumed to satisfy the usual hypotheses of completeness and right-continuity.

- Let
$$F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t) \tag{2.1}$$
be the conditional distribution function of the default time τ and assume $F_t < 1$ for $t \in [0, T]$. Then the *hazard process* Γ of τ under $\mathbb{P}$:
$$\Gamma_t = -\ln(1 - F_t)$$
is well-defined for every $t \in [0, T]$. In particular we assume that the hazard process Γ admits the following representation:
$$\Gamma_t = \int_0^t \lambda_s \mathrm{d}s, \quad \forall t \in [0, T],$$
where λ is an $\mathbf{F}$-adapted, non-negative process, with integrable sample paths called *intensity* or *hazard rate*, and that $e^{\Gamma_T} \in L^2(\Omega, \mathcal{G}, \mathbb{P})$. By Proposition 5.1.3 of [10] we obtain that the compensated process M given by
$$M_t := H_t - \int_0^{t \wedge \tau} \lambda_u \mathrm{d}u = H_t - \int_0^t \tilde{\lambda}_u \mathrm{d}u, \ \forall t \in [0, T] \tag{2.2}$$
follows a $\mathbf{G}$-martingale. Notice that for the sake of brevity we have put $\tilde{\lambda}_t := \mathbb{I}_{\{\tau \geq t\}} \lambda_t$. In particular we obtain that the existence of the intensity implies that τ is a *totally inaccessible* $\mathbf{G}$-stopping time ([15], VI.78).

- Since Γ in an increasing process, by Lemma 5.1.6 of [10] we have that W^τ is a $\mathbf{G}$-martingale. By Lévy's Theorem we obtain that W^τ is a Brownian motion on $[\![0, T \wedge \tau]\!]$. Note that if θ is a (sufficiently integrable) $\mathbf{G}$-predictable process, then the stochastic integral $\int \theta_s \mathrm{d}W_s^\tau$ is still well-defined and a $\mathbf{G}$-(local) martingale.

- We denote the money market account by $B_t = \exp\left(\int_0^t r_s \mathrm{d}s\right)$, $t \in [0,T]$, where r is a non-negative $\mathbf{F}$-predictable process. Then we represent the risky asset price by a stochastic process S on $(\Omega, \mathcal{G}, \mathbb{P})$, whose dynamics is given by

$$\begin{cases} \mathrm{d}S_t = \mu_t S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t \\ S_0 = s_0, \quad s_0 \in \mathbb{R}_+, \end{cases} \tag{2.3}$$

where $\sigma_t > 0$ a.s. for every $t \in [\![0, T \wedge \tau]\!]$ and μ, σ, are $\mathbf{F}$-adapted processes such that the discounted asset price process $X_t := \dfrac{S_t}{B_t}$ belongs to $L^2(\mathbb{P})$, $\forall t \in [\![0, T \wedge \tau]\!]$. In the definition of the asset price dynamics we can assume without loss of generality that the dynamics of S are driven by $\mathbf{F}$-adapted coefficients since the price process is stopped at τ. Namely by lemma 4.4 of Chapter IV.2 of [23], any bounded $\mathbf{G}$-predictable process can be decomposed as

$$\mu_t = \mu_t^1 \mathbb{I}_{\{t<\tau\}} + \mu^2(t,\tau)\mathbb{I}_{\{\tau \le t\}},$$

where μ^1 is a bounded $\mathbf{F}$-predictable process and $\mu^2 : \Omega \times [0,T] \times [0,T] \to \mathbb{R}$ is bounded and $\mathcal{B}([0,T]) \otimes \mathcal{P}_{\mathbf{F}}$-measurable, where $\mathcal{P}_{\mathbf{F}}$ denotes the set of $\mathbf{F}$-predictable processes. This decomposition shows how the influence of the default time determines a sudden change in the drift (respectively, in the volatility).

Remark 2.1. *Note that* (2.3) *also provides the semimartingale decomposition of X as $\mathbf{G}$-semimartingale. By [21] we obtain that the $\mathbf{G}$-martingale part of the stopped Brownian motion W^τ is given by*

$$W_t^\tau - \int_0^{\tau\wedge t} \frac{\mathrm{d}\langle W, G\rangle_s}{G_s} = W_t^\tau,$$

since $G_t := 1 - F_t$ is continuous and of finite variation.

Let

$$\theta_t = \frac{\mu_t - r_t}{\sigma_t}, \quad \forall t \in [\![0, T \wedge \tau]\!] \tag{2.4}$$

be the *market price of risk*. We also assume that μ, σ and r are such that there exists an equivalent martingale measure for the discounted price process X whose density $\mathcal{E}\left(-\int \theta \mathrm{d}W\right)_{T\wedge\tau}$ is square-integrable. If we denote by $\mathcal{P}_e^2(X)$ the set of all equivalent martingale measures $\mathbb{Q}$ for X with $\dfrac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \in L^2(\mathbb{P})$, we have that the convex set $\mathcal{P}_e^2(X)$ is nonempty and the market model is in addition arbitrage-free.

- We assume that the information at time t available to the agent is given by

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t, \quad \forall t \in [0,T].$$

As mentioned above, in this market model there exist defaultable claims, which are formally represented by a triplet $(\bar{X}, Z, \tau)$, where:

- the *promised contingent claim* $\bar{X}$ represents the payoff received by the owner of the claim at time T, if there was no default prior to or at time T. In particular we assume it is represented by a $\mathcal{F}_T$-measurable random variable $\bar{X} \in L^2(\mathbb{P})$;
- the *recovery process* Z represents the recovery payoff at time of default, if default occurs prior to or at the maturity date T. The process Z is supposed to be $\mathbf{F}$-predictable and bounded.

We denote by $N = (N_t)_{0 \leq t \leq \tau \wedge T}$ the process that models all the cash flows received by the owner of the contract, i.e. the total payments on $[\![0, \tau \wedge T]\!]$ arising from the defaultable claim. Indeed, we consider a stream of payments N that delivers only a (random) amount at time τ, whose discounted value is given by:

$$\bar{N}_t = \int_0^t \frac{1}{B_s} \mathrm{d}N_s = \int_{]0,t]} \frac{Z_s}{B_s} \mathrm{d}H_s = \frac{Z_\tau}{B_\tau} \mathbb{I}_{\{\tau \leq t\}}, \quad \text{for } 0 \leq t < T \tag{2.5}$$

and

$$\bar{N}_T = \frac{\bar{X}}{B_T} \mathbb{I}_{\{\tau > T\}}, \quad \text{for } t = T. \tag{2.6}$$

In particular we obtain that $\bar{N}_t \in L^2(\mathbb{P})$, for every $t \in [0, T]$. In this setting we study the problem of a trader wishing to price and hedge a defaultable claim $(\bar{X}, Z, \tau)$ which pays a positive recovery in case of default *at default time* τ. We note that our market model is incomplete even if we assume to trade with $\mathbf{G}$-adapted strategies because M does not represent the value of any tradable asset. In fact it is impossible to hedge against the occurrence of default by using a portfolio consisting only of the (non-defaultable) primary assets. Hence it makes sense to apply some methods used for pricing and hedging in incomplete markets to the case of defaultable options. In particular we focus here on the local risk-minimization approach. This method focuses on the idea of finding a replicating strategy for a given claim. Since the market model is incomplete, this strategy will be in general not self-financing, but it will have a cost. The aim is then to find the replicating strategy with *minimal cost* in a sense that we discuss in Section 3.

3 Local risk-minimization for payment streams with random delivery date

We extend in this section the results of [27] to the case of payment streams with *random delivery date*. Under the hypotheses of Section 2, we introduce first the basic framework and give some definitions. We recall that the asset price dynamics is given by (2.3) and that for every $t \in [0, T]$

$$X_t = \frac{S_t}{B_t}$$

denotes the discounted risky asset price.

- We remark that in our model X belongs to the space $\mathcal{S}^2_{loc}(\mathbb{P})$ of $\mathbf{G}$-semimartingales decomposable as the sum of a locally square-integrable $\mathbf{G}$-local martingale and of a $\mathbf{G}$-predictable process of finite variation null at 0. Indeed by Proposition 4.1 it can be decomposed as follows:

$$X^\tau_t = X_0 + \int_0^t (\mu_s - r_s) X^\tau_s \mathrm{d}s + \int_0^t \sigma_s X^\tau_s \mathrm{d}W^\tau_s, \quad t \in [\![0, T \wedge \tau]\!], \tag{3.1}$$

  where $\int_0^\cdot \sigma_s X^\tau_s \mathrm{d}W^\tau_s$ is a locally square-integrable $\mathbf{G}$-local martingale null at 0 and $\int_0^\cdot (\mu_s - r_s) X^\tau_s \mathrm{d}s$ is a $\mathbf{G}$-predictable process of finite variation null at 0. Moreover, in our case X^τ is a continuous process.

- In our model we have that the so-called **Structure Condition (SC)** is satisfied, that is, X is a special $\mathbf{G}$-semimartingale with canonical decomposition given by (3.1) and the process $\widehat{K}$ given by

$$\widehat{K}_t(\omega) := \int_0^t \theta^2_s(\omega) \mathrm{d}s, \quad t \in [\![0, T \wedge \tau]\!], \tag{3.2}$$

  is finite $\mathbb{P}$-a.s. for each $t \in [0, T]$, where θ is the market price of risk defined in (2.4). Indeed, since X is continuous and the set $\mathcal{P}^2_e(X) \neq \emptyset$ by hypothesis, see Section 2, Theorem 1 of [25] guarantees that (SC) is automatically satisfied. For more general results in this direction, see [11]. Additional results on the relation between (SC) and properties of absence of arbitrage for the process X can be found in [12].

In what follows, we assume that $\widehat{K}$ is uniformly bounded in t and ω, i.e. there exists a constant K such that

$$\widehat{K}_t(\omega) \leq K \text{ a.s}, \quad \forall t \in [\![0, T \wedge \tau]\!]. \tag{3.3}$$

Remark 3.1. *This assumption guarantees the existence of the minimal martingale measure for X (see Definition 3.8). It is possible to choose different hypotheses. However assumption* (3.3) *is the simplest condition that can be assumed. For a complete survey and a discussion of the others, we refer to [26].*

We denote by $\Theta^{\mathbf{F},\tau}_S$ the space of $\mathbf{F}$-predictable processes ξ on Ω such that

$$\mathbb{E}\left[\int_0^{\tau \wedge T} (\xi_s \sigma_s X_s)^2 \mathrm{d}s\right] + \mathbb{E}\left[\left(\int_0^{\tau \wedge T} |\xi_s (\mu_s - r_s) X_s| \mathrm{d}s\right)^2\right] < \infty. \tag{3.4}$$

Definition 3.2. *A pair $\varphi = (\xi, \eta)$ of stochastic processes is said to be an L^2-strategy with random delivery date $\tau \wedge T$ if*

1. $\xi \in \Theta^{\mathbf{F},\tau}_S$;

2. *η is a real-valued* $\mathbf{G}$*-adapted process such that the discounted value process $V(\varphi) = \xi X^\tau + \eta$ is right-continuous and square-integrable, i.e. $V_t(\varphi) \in L^2(\mathbb{P})$, for each $t \in [\![0, \tau \wedge T]\!]$.*

Remark 3.3. *We underline that in Definition 3.2 the assumption of η* $\mathbf{G}$*-adapted plays a crucial role. Indeed, there are no predictable φ such that $V_{\tau\wedge T}(\varphi) = 0$. If $\tau < T$, then the process η will have a jump that will be taken into account in the cost. For further details on this issue, we also refer to the discussion contained in Section 6.*

Here we have supposed that the agent invests in the risky asset according to the information provided by the asset behavior before default and adjusts the portfolio value (by adding or spending money, i.e. modifying the cost), depending on the occurrence or not of the default. In Section 6 we will further comment on other possible choices for L^2-strategies.
The *cost process* of an L^2-strategy $\varphi = (\xi, \eta)$ is given by:

$$C_t^{\bar{N}}(\varphi) := \bar{N}_t + V_t^{\bar{N}}(\varphi) - \int_0^t \xi_s \mathrm{d}X_s^\tau, \quad t \in [\![0, \tau \wedge T]\!]. \tag{3.5}$$

We look now for an L^2-strategy φ for $\bar{N}$ with minimal cost C and such that φ is 0-*achieving*, i.e. the discounted value process satisfies

$$V_{\tau\wedge T}^{\bar{N}}(\varphi) = 0, \quad \mathbb{P} - \text{a.s.} \tag{3.6}$$

The definition of 0-achieving strategies has been introduced in [27] since it is better suited for an extension of the local risk minimization method to the case of payment streams. However the total cost $C_T^{\bar{N}}$ is the same as in the approach of [18], where one uses strategies with terminal value equal to the option payoff.
In which sense is the cost minimal? Although L^2-strategies φ with $V_{\tau\wedge T}^{\bar{N}}(\varphi) = 0$ will in general not be self-financing, it turns out that good L^2-strategies are still self-financing on average in the following sense.

Definition 3.4. *An L^2-strategy φ is called* mean-self-financing *if its cost process $C^{\bar{N}}(\varphi)$ is a* $\mathbf{G}$*-martingale under $\mathbb{P}$ (which is then square-integrable).*

By using Theorem 1.6 of [27] we can give the following definition of $\mathbf{F}$-pseudo-locally risk-minimizing strategy [1] for the payment stream $\bar{N}$.

Definition 3.5. *Let $\bar{N}$ be the payment stream given in* (2.5)-(2.6) *associated to the defaultable claim $(\bar{X}, Z, \tau)$. We say that an L^2-strategy φ is a* $\mathbf{F}$-pseudo-locally

[1] The original definition of *locally risk-minimizing* strategy is given in [26] and formalizes the intuitive idea that changing an optimal strategy over a small time interval increases the risk, at least asymptotically. Since it is a rather technical definition, it has been introduced the concept of *pseudo-locally risk-minimizing* strategy that is both easier to find and to characterize, as Proposition 3.7 will show in the following. Moreover, in the one-dimensional case and if X is sufficiently well-behaved, pseudo-optimal and locally risk-minimizing strategies are the same.

risk-minimizing *(in short* **F**-plrm*) strategy for* $\bar{N}$ *if and only if* φ *is* 0*-achieving and mean-self-financing, and the cost process* $C^{\bar{N}}(\varphi)$ *is a* **G***-martingale strongly orthogonal to the martingale part of* X^τ*.*

Let $\mathcal{M}_0^2(\mathbf{G}, \mathbb{P})$ be the space of all $\mathbb{P}$-square-integrable **G**-martingales null at 0.

Definition 3.6. *A random variable* $N \in L^2(\mathcal{G}_T, \mathbb{P})$ *admits a (stopped) Föllmer-Schweizer decomposition if it can be written as*

$$N = N_0 + \int_0^T \xi_s^N \mathrm{d}X_s^\tau + L_{\tau\wedge T}^N, \quad \mathbb{P} - a.s \tag{3.7}$$

where $N_0 \in \mathbb{R}$*,* $\xi^N \in \Theta_S^{\mathbf{F},\tau}$ *and* $L \in \mathcal{M}_0^2(\mathbf{G}, \mathbb{P})$ *is strongly orthogonal to the martingale part of* X^τ*.*

Proposition 3.7. *Let* $\bar{N}$ *be a payment stream in* L^2 *with random delivery date* $\tau \wedge T$*. Then the payment stream* $\bar{N}$ *admits a* **G***-plrm* L^2*-strategy* φ *if and only if* $\bar{N}_{\tau\wedge T}$ *admits a (stopped) Föllmer-Schweizer decomposition. In that case,* $\varphi = (\xi, \eta)$ *is given by*

$$\xi = \xi^{\bar{N}_{\tau\wedge T}}, \quad \eta = V^{\bar{N}_{\tau\wedge T}} - \xi^{\bar{N}_{\tau\wedge T}} X^\tau \tag{3.8}$$

with

$$V_t^{\bar{N}_{\tau\wedge T}} := \bar{N}_0 + \int_0^t \xi_s^{\bar{N}_{\tau\wedge T}} \mathrm{d}X_s^\tau + L_t^{\bar{N}_{\tau\wedge T}} - \bar{N}_t, \quad t \in [\![0, \tau\wedge T]\!], \tag{3.9}$$

and then the minimal cost is

$$C_t^{\bar{N}}(\varphi) = \bar{N}_0 + L_t^{\bar{N}_{\tau\wedge T}}, \quad t \in [\![0, \tau\wedge T]\!]. \tag{3.10}$$

Proof. It follows from Proposition 5.2 of [27], since the discounted price process X satisfies the (SC) and the process $\widehat{K}$ given in (3.2) is continuous.
More precisely, if $\bar{N}_{\tau\wedge T}$ has a Föllmer-Schweizer decomposition (3.7), then (3.8) and (3.9) define an L^2-strategy φ, see Definition 3.2, whose cost process is given by (3.10). Hence φ is mean-self-financing, see Definition 3.4, and also 0-achieving by (3.7). Thus φ is a **F**-plrm strategy for $\bar{N}$ according to Definition 3.5.
Conversely, if $\varphi = (\xi, \eta)$ is a **F**-plrm strategy for $\bar{N}$, by using (3.5) we can write the condition $V_{\tau\wedge T}^{\bar{N}}(\varphi) = 0$ as follows:

$$\bar{N}_{\tau\wedge T} = C_{\tau\wedge T}^{\bar{N}}(\varphi) + \int_0^T \xi_s \mathrm{d}X_s^\tau = C_0^{\bar{N}}(\varphi) + \int_0^T \xi_s \mathrm{d}X_s^\tau + \left(C_{\tau\wedge T}^{\bar{N}}(\varphi) - C_0^{\bar{N}}(\varphi)\right),$$

and so we have (3.7) for $\bar{N}_{\tau\wedge T}$ with

$$\bar{N}_0 := C_0^{\bar{N}}(\varphi), \quad \xi^{\bar{N}} := \xi, \quad L^{\bar{N}} := C^{\bar{N}}(\varphi) - C_0^{\bar{N}}(\varphi);$$

note that $L^{\bar{N}}$ is a $\mathbb{P}$-square-integrable **G**-martingale strongly orthogonal to the martingale part of X^τ. □

Hence the problem of computing a **F**-plrm strategy for a payment stream boils down to compute the Föllmer-Schweizer decomposition (3.9). In our setting we can find a **F**-plrm strategy for the payment stream $\bar{N}$ by choosing a good equivalent martingale measure for X.

Definition 3.8 (The Minimal Martingale Measure). *A martingale measure $\widehat{\mathbb{P}}$ equivalent to $\mathbb{P}$ with square-integrable density is called* **minimal** *if any square-integrable* **G***-martingale which is strongly orthogonal to the martingale part of X^τ under $\mathbb{P}$ remains a* **G***-martingale under $\widehat{\mathbb{P}}$.*

The minimal measure is the equivalent martingale measure that modifies the martingale structure as little as possible. Under assumption (3.3) we know that the minimal martingale measure $\widehat{\mathbb{P}}$ exists and it is unique. How to use $\widehat{\mathbb{P}}$ to find out the Föllmer-Schweizer decomposition is shown in this well-known Theorem.

Theorem 3.9. *Let $\bar{N}$ be a payment stream in L^2 with random delivery date $\tau \wedge T$. Define the process $\widehat{V}^{\bar{N}}$ as follows*

$$\widehat{V}_t^{\bar{N}} := \widehat{\mathbb{E}}\left[\bar{N}_{\tau\wedge T}\middle|\, \mathcal{G}_t\right], \quad t \in [\![0, \tau \wedge T]\!],$$

where $\widehat{\mathbb{E}}\left[\cdot|\, \mathcal{G}_t\right]$ denotes the **G***-conditional expectation under $\widehat{\mathbb{P}}$. Let*

$$\widehat{V}_t^{\bar{N}} = \widehat{V}_0^{\bar{N}} + \int_0^t \widehat{\xi}_s^{\bar{N}} \mathrm{d}X_s^\tau + \widehat{L}_t^{\bar{N}} \tag{3.11}$$

be the Galtchouk-Kunita-Watanabe (in short GKW) decomposition[2] *of $\widehat{V}^{\bar{N}}$ with respect to X^τ under $\widehat{\mathbb{P}}$. If either $\bar{N}$ admits a Föllmer-Schweizer decomposition or $\widehat{\xi}^{\bar{N}} \in \Theta_S^{\mathbf{F},\tau}$ and $\widehat{L}^{\bar{N}} \in \mathcal{M}_0^2(\mathbb{P})$, then (3.11) for $t = \tau \wedge T$ gives the Föllmer-Schweizer decomposition of $\bar{N}$ stopped at $\tau \wedge T$ with respect to X.*

Proof. Since X is continuous and satisfies (SC), and hypothesis (3.3) guarantees existence of $\widehat{\mathbb{P}}$ and of a Föllmer-Schweizer decomposition for $\bar{N}_{\tau\wedge T}$, then the result follows by Theorem 3.5 of [26]. $\square$

4 Local risk-minimization for defaultable claims with recovery process

Under the hypotheses of Section 2, we apply the results of Section 3 to the class of defaultable claims with recovery at default time.

[2] We recall for reader's convenience the definition of Galtchouk-Kunita-Watanabe (GKW) decomposition: if X is a $\mathbb{P}$-local martingale, any $H \in L^2(\mathcal{G}_T, \mathbb{P})$ admits a GKW decomposition with respect to X, i.e. it can be uniquely written as

$$H = \mathbb{E}[H] + \int_0^T \xi_s^H \mathrm{d}X_s + L_T^H, \quad \mathbb{P}-\text{a.s.},$$

for some **G**-predictable process ξ^H that satisfies $\mathbb{E}\left[\int_0^T (\xi_s^H)^2 \sigma_s^2 X_s^2 \mathrm{d}s\right] < \infty$, and some $L^H \in \mathcal{M}_0^2(\mathbb{P})$ which is strongly orthogonal to X.

We now show that every **G**-martingale stopped at τ can be represented in terms of a stochastic integral with respect to (W^τ, M).

Proposition 4.1. *Let $(Z_t)_{0\le t\le T}$ be a **G**-martingale. Then the stopped martingale $(Z_t^\tau)_{0\le t\le T}$ admits the martingale decomposition*

$$Z_t^\tau = Z_0 + \underbrace{\int_0^t e^{\Gamma_s}\xi_s^{\hat m}\mathrm{d}W_s^\tau}_{:=L^{W,Z}} + \underbrace{\int_{]0,t]}(\hat m_u - D_u)\mathrm{d}M_u}_{:=L^{M,Z}} \tag{4.1}$$

where $L^{W,Z}$ and $L^{M,Z}$ are strongly orthogonal martingales and

$$\hat m_t = \mathbb{E}\left[\int_0^T \hat Z_u \mathrm{d}F_u \middle| \mathcal{F}_t\right] = \hat m_0 + \int_0^t \xi_s^{\hat m}\mathrm{d}W_s,$$

*for some **F**-predictable process $\xi^{\hat m}$ and*

$$D_t = e^{\Gamma_t}\mathbb{E}\left[\int_t^T \hat Z_u \mathrm{d}F_u \middle| \mathcal{F}_t\right],$$

*and $\hat Z$ is an **F**-predictable process such that*

$$\hat Z_\tau = \mathbb{E}\left[Z_{\tau\wedge T}\middle| \mathcal{G}_{\tau-}\right] = \mathbb{E}\left[Z_{\tau\wedge T}\middle| \mathcal{F}_{\tau-}\right].$$

Proof. Consider the stopped **G**-martingale

$$Z_{t\wedge\tau} = \mathbb{E}\left[Z_{\tau\wedge T}\middle| \mathcal{G}_t\right]$$

and let $\hat Z$ be an **F**-predictable process such that

$$\hat Z_\tau = \mathbb{E}\left[Z_{\tau\wedge T}\middle| \mathcal{G}_{\tau-}\right].$$

Existence of such process $\hat Z$ is ensured by [14], (68.1), page 126. Then we have

$$\begin{aligned}\mathbb{E}\left[Z_{T\wedge\tau}\middle| \mathcal{G}_t\right] &= \mathbb{E}\left[Z_{T\wedge\tau}\left(\mathbb{I}_{\{\tau\le t\}} + \mathbb{I}_{\{\tau>t\}}\right)\middle| \mathcal{G}_t\right]\\ &= \mathbb{E}\left[Z_\tau \mathbb{I}_{\{\tau\le t\}}\middle| \mathcal{G}_t\right] + \mathbb{E}\left[Z_{\tau\wedge T}\mathbb{I}_{\{\tau>t\}}\middle| \mathcal{G}_t\right].\end{aligned} \tag{4.2}$$

Since Γ is a continuous process, by Corollary 5.1.3 of [10], we have

$$\mathbb{E}\left[Z_{T\wedge\tau}\mathbb{I}_{\{\tau>t\}}\middle| \mathcal{G}_t\right] = \mathbb{I}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}\left[\int_t^T \hat Z_u \mathrm{d}F_u\middle| \mathcal{F}_t\right].$$

Hence we can rewrite (4.2) as

$$Z_{t\wedge\tau} = \mathbb{E}\left[Z_{T\wedge\tau}\middle| \mathcal{G}_t\right] = \mathbb{I}_{\{\tau\le t\}}Z_{T\wedge\tau} + \mathbb{I}_{\{\tau>t\}}e^{\Gamma_t}\mathbb{E}\left[\int_t^T \hat Z_u \mathrm{d}F_u\middle| \mathcal{F}_t\right]. \tag{4.3}$$

By Lemma 4.3 of [2] and by Lemma 5.1.5 of [10] we have that

$$\begin{aligned} H_t\hat{Z}_\tau &= H_t\mathbb{E}\left[Z_{T\wedge\tau}|\mathcal{F}_t \vee \mathcal{H}_T\right] \\ &= H_t\mathbb{E}\left[Z_{T\wedge\tau}|\,\mathcal{G}_t\right] \\ &= H_t Z_{T\wedge\tau} = H_t Z_\tau. \end{aligned}$$

Hence we can rewrite (4.3) as

$$Z_{t\wedge\tau} = H_t\hat{Z}_\tau + \mathbb{I}_{\{\tau>t\}} e^{\Gamma_t}\mathbb{E}\left[\int_t^T \hat{Z}_u \mathrm{d}F_u \middle| \mathcal{F}_t\right].$$

Consider now the $\mathbf{F}$-martingale

$$\hat{m}_t = \mathbb{E}\left[\int_0^T \hat{Z}_u \mathrm{d}F_u \middle| \mathcal{F}_t\right].$$

Then $\hat{m}^\tau$ stopped at τ is also a $\mathbf{G}$-martingale by Lemma 5.1.6 of [10]. By doing the same steps as in Proposition 5.2.1 of [10] we can show that

$$Z_{t\wedge\tau} = Z_0 + \int_0^t e^{\Gamma_s}\mathrm{d}\hat{m}_s^\tau + \int_{]0,t]}\left(\hat{Z}_u - D_u\right)\mathrm{d}M_u, \tag{4.4}$$

where

$$D_t = e^{\Gamma_t}\mathbb{E}\left[\int_t^T \hat{Z}_u \mathrm{d}F_u \middle| \mathcal{F}_t\right].$$

Since τ is a $\mathbf{G}$-stopping time, by using the properties of the stochastic integral we obtain

$$\begin{aligned} \int_0^t e^{\Gamma_s}\mathrm{d}\hat{m}_s^\tau = \int_{]0,t\wedge\tau]} e^{\Gamma_s}\mathrm{d}\hat{m}_s &= \int_{]0,t\wedge\tau]} e^{\Gamma_s}\xi_s^m \mathrm{d}W_s \\ &= \int_0^t e^{\Gamma_s}\xi_s^m \mathrm{d}W_s^\tau, \end{aligned} \tag{4.5}$$

where

$$\hat{m}_t = \hat{m}_0 + \int_0^t \xi_s^{\hat{m}}\mathrm{d}W_s, \quad t\in[0,T],$$

is the Brownian representation of $\hat{m}$ with respect to $\mathbf{F}$, where $\xi^{\hat{m}}$ is an $\mathbf{F}$-predictable process. Here we have used also the fact that $\hat{m}$ is a $\mathbf{G}$-semimartingale, see [19] or [21]. By substituting (4.5) in (4.4) we obtain decomposition (4.1). Note that $L^{W,Z}$ and $L^{M,Z}$ are strongly orthogonal since

$$[W^\tau, M]_t = [W,M]_t^\tau = 0, \quad \forall t \in [\![0, T\wedge\tau]\!].$$

$\square$

Consequently, by hypothesis (3.3), Girsanov Theorem and Proposition 4.1 we have that the minimal martingale measure exists and its density is given by

$$\frac{\mathrm{d}\widehat{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(-\int \theta \mathrm{d}W\right)_{T\wedge\tau}. \tag{4.6}$$

By (4.6) we have that $\widehat{W}_t^\tau = W_t^\tau + \int_0^{t\wedge\tau} \theta_s \mathrm{d}s$, $t \in [\![0, T\wedge\tau]\!]$, is a **G**-Brownian motion. Furthermore the pair $(\widehat{W}^\tau, M)$ has the predictable representation property also under $\widehat{\mathbb{P}}$ by Theorem 2.1 of [20] (see also [24] for the case when the hypothesis (H) holds). In fact we recall that M is not affected by the change of measure from $\mathbb{P}$ to $\widehat{\mathbb{P}}$ by the definition of minimal measure, and that $\widehat{W}^\tau$ is again a **G**-Brownian motion strongly orthogonal to M:

$$[\widehat{W}^\tau, M]_t = [\widehat{W}, M^\tau]_t = [\widehat{W}, M]_t = 0, \quad t \in [\![0, \tau\wedge T]\!],$$

since $M = M^\tau$. We now compute the GKW decomposition of $\bar{N}_{\tau\wedge T}$ under $\widehat{\mathbb{P}}$, i.e. the Föllmer-Schweizer decomposition for $\bar{N}_{\tau\wedge T}$.
Under the equivalent martingale measure $\widehat{\mathbb{P}}$, the discounted value process $\widehat{V}^{\bar{N}}$ of $\bar{N}_{\tau\wedge T}$ at time $t \in [\![0, \tau\wedge T]\!]$ is given by:

$$\begin{aligned}\widehat{V}_t^{\bar{N}} &= \widehat{\mathbb{E}}\left[\bar{N}_{\tau\wedge T}\middle|\, \mathcal{G}_t\right]\\ &= \widehat{\mathbb{E}}\left[\frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau>T\}} + \frac{Z_\tau}{B_\tau}\mathbb{I}_{\{\tau\le T\}}\middle|\, \mathcal{G}_t\right].\end{aligned}$$

Since $\bar{X} \in L^1(\mathcal{F}_T, \widehat{\mathbb{P}})$, then we get the following classic evaluation:

$$\widehat{\mathbb{E}}\left[\frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau>T\}}\middle|\, \mathcal{G}_t\right] = \mathbb{I}_{\{t<\tau\}}\widehat{\mathbb{E}}\left[e^{-\int_t^T \lambda_s \mathrm{d}s}\frac{\bar{X}}{B_T}\middle|\, \mathcal{F}_t\right],$$

see, e.g., Chapter 5 in [10] or [22]. Moreover, since $\mathcal{G}_{\tau-} = \mathcal{F}_{\tau-}$ we note that

$$\widehat{\mathbb{E}}\left[\frac{Z_\tau}{B_\tau}\mathbb{I}_{\{\tau\le T\}}\middle|\, \mathcal{G}_t\right] = \frac{Z_\tau}{B_\tau}H_t + \widehat{\mathbb{E}}\left[\frac{Z_\tau}{B_\tau}\mathbb{I}_{\{t<\tau\le T\}}\middle|\, \mathcal{G}_t\right].$$

In addition, since $\dfrac{Z}{B}$ is an **F**-predictable bounded process, then

$$\widehat{\mathbb{E}}\left[\frac{Z_\tau}{B_\tau}\mathbb{I}_{\{t<\tau\le T\}}\middle|\, \mathcal{G}_t\right] = \mathbb{I}_{\{t<\tau\}}e^{\int_0^t \lambda_s \mathrm{d}s}\widehat{\mathbb{E}}\left[\int_t^T \frac{Z_s}{B_s}e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s\middle|\, \mathcal{F}_t\right],$$

see, e.g., Proposition 5.1.1. in [10]. We thus obtain, for any $t \in [\![0, \tau\wedge T]\!]$,

$$\widehat{V}_t^{\bar{N}} = \frac{Z_\tau}{B_\tau}H_t + \mathbb{I}_{\{t<\tau\}}e^{\int_0^t \lambda_s \mathrm{d}s}\widehat{\mathbb{E}}\left[e^{-\int_0^T \lambda_s \mathrm{d}s}\frac{\bar{X}}{B_T} + \int_t^T \frac{Z_s}{B_s}e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s\middle|\, \mathcal{F}_t\right]. \tag{4.7}$$

Now, we introduce the continuous **F**-martingale m by setting for each $t \in [0, T]$

$$m_t = \widehat{\mathbb{E}}\left[e^{-\int_0^T \lambda_s \mathrm{d}s}\frac{\bar{X}}{B_T} + \int_0^T \frac{Z_s}{B_s}e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s\middle|\, \mathcal{F}_t\right]. \tag{4.8}$$

Similarly to the proof of Proposition 2.2 in [4], where however hypothesis (H) is assumed, in view of (4.7) the process $\widehat{V}^{\bar{N}}$ can be represented as follows, for $t \in [0,T]$:

$$\widehat{V}_t^{\bar{N}} = \frac{Z_\tau}{B_\tau} H_t + \mathbb{I}_{\{t<\tau\}} e^{\int_0^t \lambda_s \mathrm{d}s} U_t, \tag{4.9}$$

where the auxiliary process U equals

$$U_t = m_t^\tau - \int_0^{t\wedge\tau} \frac{Z_s}{B_s} e^{-\int_0^s \lambda_u \mathrm{d}u} \lambda_s \mathrm{d}s. \tag{4.10}$$

An application of Itô's formula leads to

$$U_t e^{\int_0^t \lambda_s \mathrm{d}s} = m_0 + \int_0^t e^{\int_0^s \lambda_u \mathrm{d}u} \mathrm{d}m_s^\tau + \int_0^{t\wedge\tau} \left(U_s e^{\int_0^s \lambda_u \mathrm{d}u} - \frac{Z_s}{B_s} \right) \lambda_s \mathrm{d}s, \quad t \in [\![0, T\wedge\tau]\!].$$

Furthermore, since $U e^{\int_0^\cdot \lambda_s \mathrm{d}s}$ is a continuous process, by using the integration by parts formula and the same steps as in (4.5) we can rewrite the process $V^{\bar{N}}$ given in (4.9) for all $t \in [\![0, \tau\wedge T]\!]$ as:

$$\begin{aligned}
\widehat{V}_t^{\bar{N}} &= \frac{Z_\tau}{B_\tau} H_t + m_0 + \int_0^t e^{\int_0^s \lambda_u \mathrm{d}u} \mathrm{d}m_s^\tau - \int_{]0,t]} U_s e^{\int_0^s \lambda_u \mathrm{d}u} \mathrm{d}M_s - \int_{]0,t]} \frac{Z_s}{B_s} \lambda_s \mathrm{d}s \\
&= m_0 + \int_0^t \xi_s^m e^{\int_0^s \lambda_u \mathrm{d}u} \mathrm{d}\widehat{W}_s^\tau + \int_0^t \left(\frac{Z_s}{B_s} - U_s e^{\int_0^s \lambda_u \mathrm{d}u} \right) \mathrm{d}M_s,
\end{aligned}$$

where in particular ξ^m is the $\mathbf{F}$-predictable process satisfying $\int_0^t (\xi_s^m)^2 \mathrm{d}s < \infty$, for all $t \in [0,T]$, that appears in the Brownian representation of the $\mathbf{F}$-martingale m, i.e.

$$m_t = m_0 + \int_0^t \xi_s^m \mathrm{d}\widehat{W}_s, \quad t \in [0,T]. \tag{4.11}$$

Hence the Föllmer-Schweizer decomposition of $\bar{N}_{\tau\wedge T}$ defined in (2.5)-(2.6) with respect to X stopped at time $\tau \wedge T$, is given by:

$$\bar{N}_{\tau\wedge T} = m_0 + \int_0^T \frac{1}{\sigma_s X_s} \xi_s^m e^{\int_0^s \lambda_u \mathrm{d}u} \mathrm{d}X_s^\tau + \int_0^T \left(\frac{Z_s}{B_s} - U_s e^{\int_0^s \lambda_u \mathrm{d}u} \right) \mathrm{d}M_s. \tag{4.12}$$

Proposition 4.2. *In the market model outlined in Section 2, the payment stream $\bar{N}$ given in* (2.5)-(2.6) *admits a $\mathbf{F}$-plrm strategy $\varphi = (\xi, \eta)$, that is given by:*

$$\begin{aligned}
\xi_t &= \frac{1}{\sigma_t X_t} \xi_t^m e^{\int_0^t \lambda_s \mathrm{d}s}, \\
\eta_t &= V_t^{\bar{N}_{\tau\wedge T}} - \frac{1}{\sigma_t} \xi_t^m e^{\int_0^t \lambda_s \mathrm{d}s}
\end{aligned}$$

for $t \in [\![0, \tau\wedge T]\!]$, with

$$V_t^{\bar{N}_{\tau\wedge T}} := m_0 + \int_0^t \frac{1}{\sigma_s X_s} \xi_s^m e^{\int_0^s \lambda_u \mathrm{d}u} \mathrm{d}X_s^\tau + \int_0^t \left(\frac{Z_s}{B_s} - U_s e^{\int_0^s \lambda_u \mathrm{d}u} \right) \mathrm{d}M_s - \bar{N}_t, \tag{4.13}$$

and cost process

$$C_t^{\bar{N}}(\varphi) = m_0 + \int_0^t \left(\frac{Z_s}{B_s} - U_s e^{\int_0^s \lambda_u \mathrm{d}u}\right) \mathrm{d}M_s,$$

for every $t \in [\![0, \tau \wedge T]\!]$, where the processes M, m, U, ξ^m are introduced respectively in (2.2), (4.8), (4.10) *and* (4.11).

Proof. It follows by hypothesis (3.3), Theorem 3.9 and from the mutual orthogonality of M and W^τ. More precisely, let $\bar{N}$ the payment stream given in (2.5)-(2.6). By Proposition 3.7, we know that $\bar{N}$ admits a $\mathbf{F}$-plrm L^2-strategy if and only if $\bar{N}_{\tau\wedge T}$ admits a (stopped) Föllmer-Schweizer decomposition. We note that since $\frac{\mathrm{d}\widehat{\mathbb{P}}}{\mathrm{d}\mathbb{P}} \in L^2(\mathbb{P})$, then $L^2(\mathbb{P}) \subset L^1(\widehat{\mathbb{P}})$. Consequently $\bar{N}_{\tau\wedge T} \in L^1(\widehat{\mathbb{P}})$ since $\bar{N}_{\tau\wedge T} \in L^2(\mathbb{P})$ in our setting, and by Proposition 4.1 we obtain decomposition (4.12).
The $\mathbf{G}$-martingale $\int_0^\cdot \left(\frac{Z_s}{B_s} - U_s e^{\int_0^s \lambda_u \mathrm{d}u}\right) \mathrm{d}M_s$ is strongly orthogonal to the martingale part of X, hence (4.12) gives the GKW decomposition of $\bar{N}_{\tau\wedge T}$ under $\widehat{\mathbb{P}}$. Since by hypothesis $\frac{\mathrm{d}\widehat{\mathbb{P}}}{\mathrm{d}\mathbb{P}} \in L^2(\mathbb{P})$ and X is continuous, then by Theorem 3.5. of [18] the associated density process

$$\tilde{Z}_t = \widehat{\mathbb{E}}\left[\frac{\mathrm{d}\widehat{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\middle| \mathcal{G}_t\right], \quad t \in [0, T],$$

is a square-integrable $\mathbf{G}$-martingale. Since hypothesis (3.3) is in force, we can apply Theorem 3.9 and conclude that (4.12) is the Föllmer-Schweizer decomposition of $\bar{N}_{\tau\wedge T}$. □

Remark 4.3. *We note that formula* (4.13) *has also the following interpretation. By* (2.5), (3.5), (3.6), (4.13) *we have that*

$$\Delta N_\tau + \Delta V_\tau^{\bar{N}} = \frac{Z_\tau}{B_\tau} - U_\tau e^{\int_0^\tau \lambda_u \mathrm{d}u},$$

i.e.

$$-V_{\tau-}^{\bar{N}} = \frac{Z_\tau}{B_\tau} - U_\tau e^{\int_0^\tau \lambda_u \mathrm{d}u} - \frac{Z_\tau}{B_\tau}\mathbb{I}_{\{\tau<T\}}$$

if $\tau < T$. By Theorem 3.3.2 of [5], we have that

$$V_{t-}^{\bar{N}} = U_t e^{\int_0^t \lambda_u \mathrm{d}u}$$

for all $t < T$.

5 Example: case of a corporate bond

In this example, we wish to find a $\mathbf{F}$-plrm strategy for a corporate bond with recovery at default that we hedge by using a Treasury bond. To simplify the

computations, *we work out the example directly under* $\widehat{\mathbb{P}}$.
We fix $T>0$ and assume that the discounted price process X is $\mathbf{F}$-adapted. Here we assume that the process X represents the discounted price of a Treasury bond that expires at time T, with the following representation

$$X_t = \widehat{\mathbb{E}}\left[e^{-\int_0^T r_s \mathrm{d}s} \middle| \mathcal{F}_t \right], \quad t \in [0,T],$$

and that the discounted recovery process $\frac{Z}{B}$ is given by

$$\frac{Z_t}{B_t} = \delta X_t, \quad t \in [0,T],$$

where δ is a constant belonging to the interval $]0,1[$. As we said, we put $\bar{X}=1$ so that the discounted value $\bar{N}$ of the payment stream on $[\![0,\tau\wedge T]\!]$ associated to the corporate bond, is given by:

$$\bar{N}_t = \delta \int_{]0,t]} X_s \mathrm{d}H_s, \quad \text{for } 0 \le t < T$$

and

$$\bar{N}_T = \frac{1}{B_T}\mathbb{I}_{\{\tau > T\}}, \quad \text{for } t = T.$$

We make also the following hypotheses:

- r is an affine process, in particular it satisfies the following equation under $\widehat{\mathbb{P}}$:

$$\begin{cases} \mathrm{d}r_t = (b + \beta r_t)\mathrm{d}t + \alpha\sqrt{r_t}\mathrm{d}\widehat{W}_t \\ r_0 = 0, \end{cases} \tag{5.1}$$

  where $b,\ \alpha \in \mathbb{R}_+$ and β is arbitrary. This is the Cox-Ingersoll-Ross model and we know it has a unique strong solution $r \ge 0$ for every $r_0 \ge 0$. See [16] for further details.

- The $\mathbf{F}$-intensity λ is supposed to be a positive *deterministic* function of time. We remark that in this case τ is independent of $\mathbf{F}$.

This last assumption allows us to compute explicitly the processes D, m and the Föllmer-Schweizer decomposition for $N_{\tau\wedge T}$ in the sequel.
We consider the discounted value of the payment stream N at time $\tau \wedge T$:

$$\bar{N}_{\tau\wedge T} = \frac{1}{B_T}(1 - H_T) + \delta X_\tau H_T. \tag{5.2}$$

We compute now the terms appearing in decomposition (4.13) for this case. First we focus on the process $e^{\int_0^\cdot \lambda(s)\mathrm{d}s}U$, that in this setting has the following form:

$$\begin{aligned} e^{\int_0^t \lambda(s)\mathrm{d}s}U_t &= e^{-\int_t^T \lambda(s)\mathrm{d}s}\widehat{\mathbb{E}}\left[\frac{1}{B_T}\middle| \mathcal{F}_t\right] + e^{\int_0^t \lambda(s)\mathrm{d}s}\widehat{\mathbb{E}}\left[\int_t^T \delta X_s e^{-\int_0^s \lambda(u)\mathrm{d}u}\lambda_s \mathrm{d}s \middle| \mathcal{F}_t\right] \\ &= e^{-\int_t^T \lambda(s)\mathrm{d}s} X_t + \widehat{\mathbb{E}}\left[\int_t^T \delta X_s e^{-\int_t^s \lambda(u)\mathrm{d}u}\lambda_s \mathrm{d}s \middle| \mathcal{F}_t\right]. \end{aligned} \tag{5.3}$$

Then, we consider $\bar{\xi}$, the $\mathbf{F}$-predictable process such that $\int_0^t (\bar{\xi}_s)^2 \mathrm{d}s < \infty$, for every $t \in [0,T]$, that appears in the integral representation of the $\mathbf{F}$-martingale $\widehat{\mathbb{E}}\left[\frac{1}{B_T}\middle|\mathcal{F}_t\right]$ with respect to the Brownian motion $\widehat{W}$:

$$\widehat{\mathbb{E}}\left[\frac{1}{B_T}\middle|\mathcal{F}_t\right] = \widehat{\mathbb{E}}\left[\frac{1}{B_T}\right] + \int_0^t \bar{\xi}_s \mathrm{d}\widehat{W}_s, \quad t \in [\![0, \tau \wedge T]\!].$$

By following Section 5 of [2], since r is an affine process whose dynamics is given in (5.1), we have

$$\begin{aligned}\widehat{\mathbb{E}}\left[\frac{1}{B_T}\middle|\mathcal{F}_t\right] &= e^{-\int_0^t r_s \mathrm{d}s} e^{-A(t,T)-B(t,T)r_t} \\ &= e^{-A(0,T)} - \int_0^t e^{-A(s,T)-B(s,T)r_s} \frac{B(s,T)}{B_s}\sqrt{r_s}\mathrm{d}\widehat{W}_s, \end{aligned} \tag{5.4}$$

where the functions $A(t,T)$, $B(t,T)$ satisfy the following equations:

$$\begin{aligned}\partial_t B(t,T) &= \frac{\alpha^2}{2} B^2(t,T) - \beta B(t,T) - 1, \quad B(T,T) = 0 \\ \partial_t A(t,T) &= -bB(t,T), \quad A(T,T) = 0,\end{aligned}$$

that admit explicit solutions (see for instance [17]). Thus, we can rewrite (5.3) as follows:

$$\begin{aligned} e^{\int_0^t \lambda(s)\mathrm{d}s} U_t = e^{-\int_t^T \lambda(s)\mathrm{d}s} &\left(e^{-A(0,T)} - \int_0^t e^{-A(s,T)-B(s,T)r_s} \frac{B(s,T)}{B_s}\sqrt{r_s}\mathrm{d}\widehat{W}_s \right) \\ &+ \widehat{\mathbb{E}}\left[\int_t^T \delta X_s e^{-\int_t^s \lambda(u)\mathrm{d}u} \lambda_s \mathrm{d}s \middle| \mathcal{F}_t \right]. \end{aligned} \tag{5.5}$$

We now compute the second term on the right-hand side of (5.5). By applying Fubini-Tonelli Theorem, we have

$$\begin{aligned}\widehat{\mathbb{E}}\left[\int_t^T \delta X_s e^{-\int_t^s \lambda(u)\mathrm{d}u} \lambda_s \mathrm{d}s \middle| \mathcal{F}_t \right] &= \int_t^T e^{-\int_t^s \lambda(u)\mathrm{d}u} \lambda(s) \delta \widehat{\mathbb{E}}\left[X_s \middle| \mathcal{F}_t\right] \mathrm{d}s \\ &= \int_t^T e^{-\int_t^s \lambda(u)\mathrm{d}u} \lambda(s) \delta X_t \mathrm{d}s \\ &= \delta X_t \left(1 - e^{-\int_t^T \lambda(s)\mathrm{d}s}\right),\end{aligned}$$

since λ is a deterministic function and finally

$$\begin{aligned} e^{\int_0^t \lambda(s)\mathrm{d}s} U_t &= e^{-\int_t^T \lambda(s)\mathrm{d}s} X_t + \delta X_t \left(1 - e^{-\int_t^T \lambda(s)\mathrm{d}s}\right) \\ &= e^{-\int_t^T \lambda(s)\mathrm{d}s} X_t \left(\delta e^{\int_t^T \lambda(s)\mathrm{d}s} - \delta + 1\right). \end{aligned}$$

It only remains to compute the $\mathbf{F}$-martingale m introduced in (4.8) and in particular its integral representation with respect to the Brownian motion $\widehat{W}$. In virtue of (4.10), we can rewrite m in terms of U:

$$\begin{aligned}
& m_t \\
&= U_t + \int_0^t \delta X_s e^{-\int_0^s \lambda(v)\mathrm{d}v}\lambda(s)\mathrm{d}s \\
&= e^{-\int_0^T \lambda(s)\mathrm{d}s}\Bigg(e^{-A(0,T)} - \int_0^t \underbrace{e^{-A(s,T)-B(s,T)r_s}\frac{B(s,T)}{B_s}\sqrt{r_s}}_{\zeta_s}\,\mathrm{d}\widehat{W}_s\Bigg) \\
&\qquad + \delta X_t\left(e^{-\int_0^t \lambda(s)\mathrm{d}s} - e^{-\int_0^T \lambda(s)\mathrm{d}s}\right) + \int_0^t \delta X_s e^{-\int_0^s \lambda(v)\mathrm{d}v}\lambda(s)\mathrm{d}s \qquad (5.6)\\
&= e^{-\int_0^T \lambda(s)\mathrm{d}s}\left(e^{-A(0,T)} - \int_0^t \zeta_s \mathrm{d}\widehat{W}_s\right) \\
&+ \delta\left[X_t\left(e^{-\int_0^t \lambda(s)\mathrm{d}s} - e^{-\int_0^T \lambda(s)\mathrm{d}s}\right) + \int_0^t \left(e^{-A(0,T)} - \int_0^s \zeta_u \mathrm{d}\widehat{W}_u\right)e^{-\int_0^s \lambda(v)\mathrm{d}v}\lambda(s)\mathrm{d}s\right] \\
&= e^{-\int_0^T \lambda(s)\mathrm{d}s}\left(e^{-A(0,T)} - \int_0^t \zeta_s \mathrm{d}\widehat{W}_s\right) + \delta\left[e^{-A(0,T)}\left(1 - e^{-\int_0^T \lambda(s)\mathrm{d}s}\right)\right. \\
&\quad \left. - \left(e^{-\int_0^t \lambda(s)\mathrm{d}s} - e^{-\int_0^T \lambda(s)\mathrm{d}s}\right)\int_0^t \zeta_s \mathrm{d}\widehat{W}_s - \int_0^t\int_0^s \zeta_u e^{-\int_0^s \lambda(v)\mathrm{d}v}\lambda(s)\mathrm{d}\widehat{W}_s\mathrm{d}s\right],
\end{aligned} \tag{5.7}$$

where ζ is an $\mathbf{F}$-predictable process such that

$$\widehat{\mathbb{E}}\left[\int_0^T \zeta_s^2 \mathrm{d}s\right] = \widehat{\mathbb{E}}\left[\int_0^T \left(e^{-A(s,T)-B(s,T)r_s}\frac{B(s,T)}{B_s}\sqrt{r_s}\right)^2 \mathrm{d}s\right] < \infty.$$

Moreover we note that

$$\widehat{\mathbb{E}}\left[\int_0^T\int_0^T \zeta_u^2 e^{-\int_0^s \lambda(v)\mathrm{d}v}\lambda(s)\mathrm{d}u\mathrm{d}s\right] = \left(1 - e^{-\int_0^T \lambda(s)\mathrm{d}s}\right)\widehat{\mathbb{E}}\left[\int_0^T \zeta_u^2 \mathrm{d}u\right]\mathrm{d}s < \infty,$$

since by (5.4)

$$\widehat{\mathbb{E}}\left[\int_0^T \zeta_u^2 \mathrm{d}u\right] = \widehat{\mathbb{E}}\left[\left(\int_0^T \zeta_u \mathrm{d}\widehat{W}_u\right)^2\right] = \widehat{\mathbb{E}}\left[(e^{-A(0,T)} - X_T)^2\right] \le (e^{-A(0,T)} + 1)^2,$$

because X takes values in $(0,1)$. Since all the integrability conditions are satisfied, by applying the Fubini's Theorem for stochastic integrals, we have

$$\begin{aligned}
-\int_0^t\int_0^t e^{-\int_0^s \lambda(v)\mathrm{d}v}\lambda(s)\zeta_u \mathbb{I}_{\{u\le s\}}\mathrm{d}\widehat{W}_u\mathrm{d}s &= \int_0^t \left(-\int_u^t e^{-\int_0^s \lambda(v)\mathrm{d}v}\lambda(s)\mathrm{d}s\right)\zeta_u \mathrm{d}\widehat{W}_u \\
&= \int_0^t \left(e^{-\int_0^t \lambda(v)\mathrm{d}v} - e^{-\int_0^u \lambda(v)\mathrm{d}v}\right)\zeta_u \mathrm{d}\widehat{W}_u.
\end{aligned} \tag{5.8}$$

Hence by (5.6), (5.7) and (5.8), we obtain

$$m_0 = e^{-A(0,T)-\int_0^T \lambda(s)\mathrm{d}s}\left[\delta\left(e^{\int_0^T \lambda(s)\mathrm{d}s} - 1\right) + 1\right]$$

and

$$m_t = e^{-A(0,T)-\int_0^T \lambda(s)\mathrm{d}s}\left[\delta\left(e^{\int_0^T \lambda(s)\mathrm{d}s} - 1\right) + 1\right] - \delta\int_0^t e^{-\int_0^u \lambda(v)\mathrm{d}v}\zeta_u \mathrm{d}\widehat{W}_u.$$

Hence the Föllmer-Schweizer decomposition of $\bar{N}_{\tau\wedge T}$ defined in (5.2) stopped at time $\tau \wedge T$, is given by:

$$\begin{aligned}
&\bar{N}_{\tau\wedge T}\\
&= e^{-A(0,T)-\int_0^T \lambda(s)\mathrm{d}s}\left[\delta\left(e^{\int_0^T \lambda(s)\mathrm{d}s} - 1\right) + 1\right] - \delta\int_0^{\tau\wedge T} \frac{\zeta_s}{\sigma_s X_s}\mathrm{d}X_s\\
&\quad + \int_0^{\tau\wedge T} e^{-\int_s^T \lambda(u)\mathrm{d}u} X_s(\delta - 1)\mathrm{d}M_s.
\end{aligned}$$

In particular, the cost process is given by

$$C_t = e^{-A(0,T)-\int_0^T \lambda(s)\mathrm{d}s}\left[\delta\left(e^{\int_0^T \lambda(s)\mathrm{d}s} - 1\right) + 1\right] + \int_0^t e^{-\int_s^T \lambda(u)\mathrm{d}u} X_s(\delta - 1)\mathrm{d}M_s,$$

for every $t \in [\![0, \tau\wedge T]\!]$.

6 Local risk-minimization with G-strategies

We now comment on our choice of L^2-strategies given by Definition 3.2. The following Lemma shows that this is equivalent to local risk-minimization by using **F**-strategies.

Lemma 6.1. *For any* **G***-predictable process* φ *there exists a unique* **F***-predictable process* $\tilde{\varphi}$ *such that*

$$\mathbb{I}_{\{\tau\geq t\}}\tilde{\varphi}_t = \mathbb{I}_{\{\tau\geq t\}}\varphi_t, \quad t \in [0, T].$$

Proof. It follows from [13], since **G** is the filtration given by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, for each $t \in [0, T]$ and the process F defined in (2.1) is such that the inequality $F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t) < 1$ holds for every $t \in [0, T]$. □

By Lemma 6.1 we obtain that there exists a unique **F**-predictable process $\tilde{X}$ such that

$$\mathbb{I}_{\{\tau\geq t\}}\tilde{X}_t = \mathbb{I}_{\{\tau\geq t\}}X_t, \quad t \in [0, T].$$

Following [8] and [9] we refer to $\tilde{X}$ as the *pre-default value* of X. In practice, since hedging stops after default, the agent observes the pre-default (discounted) value $\tilde{X}$ and hedges by using $\tilde{X}$ until the default happens.

We observe that there do not exist **F**-pseudo-locally risk-minimizing strategies, if we use the usual definition. In fact, finding a **F**-plrm strategy $\varphi^{\mathbf{F}} = (\xi, \eta)$ is equivalent to find a pair of processes (ξ, C) such that:

- $\xi \in \Theta_S^{\mathbf{F},\tau}$,

- the cost process C is an $\mathbf{F}$-martingale strongly orthogonal to the martingale part of $\tilde{X}^\tau$, with

$$V_t(\varphi^{\mathbf{F}}) = \int_0^t \xi_s \mathrm{d}X_s^\tau + C_t(\varphi^{\mathbf{F}}) - \bar{N}_t, \quad t \in [\![0, \tau \wedge T]\!]$$

  and $V_{\tau\wedge T}(\varphi^{\mathbf{F}}) = 0$, i.e. $\int_0^{\tau\wedge T} \xi_s \mathrm{d}X_s + C_{\tau\wedge T}(\varphi^{\mathbf{F}}) = \bar{N}_{\tau\wedge T}$.

From this definition follows that $C_t \equiv \text{cost}$, for each $t \in [\![0, \tau \wedge T]\!]$, since C is an $\mathbf{F}$-martingale strongly orthogonal to W and W has the predictable representation property with respect to $\mathbf{F}$. Equivalently, one could see that if a plrm strategy with respect to $\mathbf{F}$ would exist with $V_{\tau\wedge T} = 0$, then $\tau \wedge T$ would be an hitting time of 0 of the $\mathbf{F}$-adapted process V, which is not possible. As already remarked, the agent observes the pre-default (discounted) value $\tilde{X}$ until the default happens. Hence we cannot hedge against the occurring of a default by using only the information contained in the pre-default asset prices. This is one of the differences with respect to the mean-variance hedging, where the optimal $\mathbf{F}$-strategy is given by the replicating strategy for $\widehat{\mathbb{E}}\left[\bar{N}_{\tau\wedge T} \middle| \mathcal{F}_t\right]$, (if it exists). See [3] for further details. Another possible choice would be to consider $\mathbf{G}$-predictable strategies. Let $\Theta_S^{\mathbf{G},\tau}$, the space of $\mathbf{G}$-predictable processes ξ on Ω such that

$$\mathbb{E}\left[\int_0^{\tau\wedge T} (\xi_s \sigma_s X_s)^2 \mathrm{d}s\right] + \mathbb{E}\left[\left(\int_0^{\tau\wedge T} |\xi_s(\mu_s - r_s) X_s| \mathrm{d}s\right)^2\right] < \infty.$$

Definition 6.2. *Let $\bar{N}$ be the payment stream given in* (2.5)-(2.6) *associated to the defaultable claim $(\bar{X}, Z, \tau)$. A pair $\varphi^{\mathbf{G}} = (\xi, C)$ of stochastic processes is said an $\mathbf{G}$-pseudo-locally risk-minimizing (in short $\mathbf{G}$-plrm) strategy for $\bar{N}$, if*

1. *$\xi \in \Theta_S^{\mathbf{G},\tau}$;*

2. *the cost process C is a $\mathbf{G}$-martingale strongly orthogonal to the martingale part of $\tilde{X}^\tau$;*

3. *the discounted value process $V(\varphi^{\mathbf{G}}) = \xi X^\tau + \eta$ is such that*

$$V_t(\varphi^{\mathbf{G}}) = \int_0^t \xi_s \mathrm{d}X_s^\tau + C_t(\varphi^{\mathbf{F}}) - \bar{N}_t, \quad t \in [\![0, \tau \wedge T]\!],$$

   and $V_{\tau\wedge T}(\varphi^{\mathbf{G}}) = 0$.

Clearly the component η invested in the money market account, is given by

$$\eta_t = V_t(\varphi^{\mathbf{G}}) - \xi_t X_t^\tau, \quad t \in [\![0, \tau \wedge T]\!].$$

Remark 6.3. *At a first look it may appear that there is no difference in between the sets of the* $\mathbf{F}$*-plrm strategy (see Definition 6.2) and of the* $\mathbf{G}$*-plrm strategy (see Definition 3.8). However, these sets are not equal, but they are related in the following way. For any* $\varphi^{\mathbf{G}} = (\xi, \eta)$ $\mathbf{G}$*-plrm strategy there exists a unique* $\mathbf{F}$*-plrm strategy* $\varphi^{\mathbf{F}} = (\tilde{\xi}, C)$ *such that*

$$\mathbb{I}_{\{\tau \geq s\}} \xi_s = \mathbb{I}_{\{\tau \geq s\}} \tilde{\xi}_s,$$

i.e., $\tilde{\xi}$ *is the pre-default value of* ξ *according to Lemma 6.1, and*

$$\begin{aligned} \eta_t &= V_t(\varphi^{\mathbf{F}}) - \tilde{\xi}_t X_t^\tau \\ &= \int_0^t \tilde{\xi}_s \mathrm{d}X_s^\tau + C_t(\varphi^{\mathbf{F}}) - \bar{N}_t - \tilde{\xi}_t X_t^\tau. \end{aligned}$$

The two strategies differ only for what concerns the first component. Note that given $H \in L^2(\mathcal{G}_t, \mathbb{P})$*, the (stopped) Föllmer-Schweizer decompositions of it with respect to* $\mathbf{F}$*-plrm strategies or to* $\mathbf{G}$*-plrm strategies coincide by Lemma 6.4. The two different sets have been introduced only to stress the fact that an agent may invest in the risky asset without taking into account the possibility of a default until* τ *happens. At the moment when the default occurs she is then forced to readjust the portfolio by using the cost.*

Lemma 6.4. *Given a* $\mathbf{G}$*-predictable process* φ *such that*

$$\widehat{\mathbb{E}}\left[\int_0^T \varphi_s^2 \mathrm{d}\langle X \rangle_s\right] < \infty, \tag{6.1}$$

let $\tilde{\varphi}$ *be the* $\mathbf{F}$*-predictable process such that* $\mathbb{I}_{\{\tau \geq t\}} \varphi_t = \mathbb{I}_{\{\tau \geq t\}} \tilde{\varphi}_t$*, for each* $t \in [0, T]$*. Then for every* $t \leq T$

$$\int_0^t \tilde{\varphi}_s \mathrm{d}X_s^\tau = \int_0^t \varphi_s \mathrm{d}X_s^\tau, \quad \forall t \in [0, T].$$

Proof. Since X^τ is a continuous local $\mathbf{G}$-martingale under $\widehat{\mathbb{P}}$ and (6.1) holds, we have that for $t \leq T$

$$\begin{aligned} \int_0^t \varphi_s \mathrm{d}X_s^\tau &= \int_0^t \mathbb{I}_{\{\tau \geq s\}} \varphi_s \mathrm{d}X_s^\tau \\ &= \int_0^t \mathbb{I}_{\{\tau \geq s\}} \tilde{\varphi}_s \mathrm{d}X_s^\tau = \int_0^t \tilde{\varphi}_s \mathrm{d}X_s^\tau. \end{aligned}$$

We only need to check that the integral $\int_0^t \tilde{\varphi}_s \mathrm{d}X_s^\tau$ exists for each $t \in [0, T]$ and that is well-defined if the integral $\int_0^t \varphi_s \mathrm{d}X_s^\tau$ exists and it is well-defined for each

$t \in [0,T]$. This is clear since if $\widehat{\mathbb{E}}\left[\int_0^T \varphi_s^2 \mathrm{d}\langle X^\tau\rangle_s\right] < \infty$, we have

$$\begin{aligned}\infty > \widehat{\mathbb{E}}\left[\int_0^T \varphi_s^2 \mathrm{d}\langle X^\tau\rangle_s\right] &= \widehat{\mathbb{E}}\left[\left(\int_0^T \varphi_s \mathrm{d}X_s^\tau\right)^2\right] \\ &= \widehat{\mathbb{E}}\left[\left(\int_0^T \varphi_s \mathbb{I}_{\{\tau \geq s\}} \mathrm{d}X_s\right)^2\right] \\ &= \widehat{\mathbb{E}}\left[\int_0^T \tilde{\varphi}_s^2 \mathrm{d}\langle X^\tau\rangle_s\right],\end{aligned}$$

since $\mathbb{I}_{\{\tau \geq t\}}\varphi_t = \mathbb{I}_{\{\tau \geq t\}}\tilde{\varphi}_t$, for each $t \in [0,T]$, by hypothesis. □

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