Quadratic Hedging Methods for Defaultable Claims

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Abstract

We apply the local risk-minimization approach to defaultable claims and we compare it with intensity-based evaluation formulas and the mean-variance hedging. We solve analytically the problem of finding respectively the hedging strategy and the associated portfolio for the three methods in the case of a default put option with random *recovery at maturity*.

Key words: Defaultable markets, intensity-based approach, local risk-minimization, minimal martingale measure, mean-variance hedging .

1 Introduction

In this paper we discuss the problem of pricing and hedging defaultable claims, i.e. options that can lose partly or totally their value if a default

event occurs.
We consider a simple market model with two not-defaultable primitive assets (the money market account B_t and the discounted risky asset X_t) and a (discounted) defaultable claim H and we assume that there exists a unique martingale measure $\mathbb{P}^*$ for X_t with square integrable density.
In this context by following the approach of [9], [10] and [11], we first consider the so-called "intensity-based approach", where a defaultable claim is priced by using the risk-neutral valuation formula as the market would be complete. However the market model extended with the defaultable claim is incomplete since the default process is not a traded asset. Hence it is impossible to hedge against the occurrence of a default by using a portfolio consisting only of the (not defaultable) primitive assets. Then this method can only provide pricing formulas for the discounted defaultable payoff H, since it is impossible to find a replicating portfolio for H consisting only of the risky asset and the bond, and it makes sense to apply some of the methods used for pricing and hedging derivatives in incomplete markets.
In particular we focus here on quadratic hedging approaches, i.e. *local risk-minimization* and *mean-variance hedging*[1]. The mean-variance hedging method has been already extensively studied in the context of defaultable markets by [6], [7], [8] and [9]. Here we extend some of their results to the case of *stochastic* drift μ_t and volatility σ_t in the dynamics (4) of the risky asset price, and *random* recovery rate. In fact empirical analysis of recovery rates shows that they may depend on several factors, among which default delays (see for example [12]). For the sake of simplicity here we assume that the recovery rate depends only on the random time of default.
The main contribution of this paper is that we apply for the first time the local risk-minimization method to the pricing and hedging of defaultable derivatives. We focus on the particular case of a default put option with random recovery rate and solve explicitly the problem of finding the pseudo-local risk-minimizing strategy and the portfolio with minimal cost.
For the local risk minimization approach for a general defaultable claim, we refer to [1] and [2].

2 General setting

We make the following assumptions:

- We consider a simple market model, given by:
  - the money market account B_t (riskless asset);

[1]For an extensive survey of the subject, we refer to [25]

- a risky asset S_t, represented by a continuous semimartingale such that it admits an equivalent martingale measure for the discounted price process $X_t = \dfrac{S_t}{B_t}$ (there are no arbitrage opportunities). The price process S_t and the risk-free bond B_t are both defined on the probability space $(\tilde{\Omega}, \mathcal{F}, \mathbb{P})$, endowed with the filtration $(\mathcal{F}_t)_{t\geq 0}$;
- the default time τ, given by a stopping time on the probability space $(\hat{\Omega}, \mathcal{H}, \nu)$. We assume that τ is independent of $\mathcal{F}_t$, for every $t \geq 0$.

- For a given default time τ, we introduce the associated jump process H by setting $H_t = \mathbb{I}_{\{\tau \leq t\}}$ for $t \in \mathbb{R}^+$. H is called the *default process*.
- Let $(\mathcal{H}_t)_{t\geq 0}$ be the filtration generated by the process H, i.e. $\mathcal{H}_t = \sigma(H_u : u \leq t)$, and $\mathcal{H} := \vee_{t\geq 0}\mathcal{H}_t$.

Hence we consider the following product probability space

$$(\Omega, \mathcal{G}, \mathbb{Q}) = (\tilde{\Omega} \times \hat{\Omega}, \mathcal{F}_\infty \otimes \mathcal{H}, \mathbb{P} \otimes \nu)$$

endowed with the filtration

$$\mathcal{G}_t = \mathcal{H}_t \otimes \mathcal{F}_t.$$

All the filtrations are assumed to satisfy the usual hypotheses of completeness and right-continuity.
We introduce the *hazard process* under $\mathbb{Q}$:

$$\Gamma_t := -\ln(1 - F_t), \quad \forall t \in \mathbb{R}^*,$$

where

$$F_t = \mathbb{Q}(\tau \leq t) = \nu(\tau \leq t) \tag{1}$$

is the cumulative distribution function of the default time τ. We assume that the hazard process Γ_t admits the following representation:

$$\Gamma_t = \int_0^t \lambda_s \mathrm{d}s, \quad \forall t \in \mathbb{R}^*,$$

where λ_t is a non-negative, integrable function. The function λ is called *intensity* or *hazard rate*. If F_t is absolutely continuous with respect to Lebesgue measure, that is when $F_t = \int_0^t f_u \mathrm{d}u$ for an integrable positive function f, then we have

$$F_t = 1 - \exp\left(-\Gamma_t\right) = 1 - \exp\left(-\int_0^t \lambda_s \mathrm{d}s\right),$$

where in this case $\lambda_t = \dfrac{f_t}{1 - F_t}$. By Proposition 5.1.3 of [11], we obtain that the compensated process $\hat{M}$ given by the formula

$$\hat{M}_t := H_t - \int_0^{t\wedge\tau} \lambda_u \mathrm{d}u = H_t - \int_0^t \tilde{\lambda}_u \mathrm{d}u, \ \forall t \in \mathbb{R}^+ \tag{2}$$

is a $\mathbb{Q}$-martingale with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$. Notice that for the sake of brevity we have denoted $\tilde{\lambda}_t := \mathbb{I}_{\{\tau\geq t\}}\lambda_t$.
We fix a maturity date $T > 0$. In this framework we can introduce the *defaultable claim* which is represented by a quintuple (X_1, A, X_2, Z, τ), where:

- the *promised contingent claim* X_1 represents the payoff received by the owner of the claim at time T, if there was no default prior to or at time T;
- the process A represents the *promised dividends* - that is, the stream of cash flows received by the owner of the claim prior to default;
- the *recovery process* Z represents the recovery payoff at the time of default, if default occurs prior to or at the maturity date T;
- the *recovery claim* X_2 represents the recovery payoff at time T, if default occurs prior to or at the maturity date T.

For the sake of simplicity, we can assume $A \equiv 0$, i.e. the claim does not pay any dividends prior to default, so in the sequel we will use the simpler notation (X_1, X_2, Z, τ). Furthermore the discounted value of a defaultable claim (X_1, X_2, Z, τ) is given by:

$$H = \frac{X_1}{B_T}\mathbb{I}_{\{\tau>T\}} + \frac{X_2}{B_T}\mathbb{I}_{\{\tau\leq T\}} + \frac{Z_\tau}{B_\tau}\mathbb{I}_{\{\tau\leq T\}}. \tag{3}$$

3 Case of a default-put

We focus in particular on the following model for the risky asset price X_t and consider the case of a default put.

Model:

- Let W_t be a standard Brownian motion on $(\tilde{\Omega}, \mathcal{F}, \mathcal{F}_t^W, \mathbb{P})$ endowed with the natural filtration $\mathcal{F}_t^W$ of W_t. The risky asset price is represented by a stochastic process on $(\tilde{\Omega}, \mathcal{F}, \mathcal{F}_t^W, \mathbb{P})$ whose dynamics are described by the following equations:

$$\begin{cases} \mathrm{d}S_t(\tilde{\omega}) = \mu_t(\tilde{\omega})S_t(\tilde{\omega})\mathrm{d}t + \sigma_t(\tilde{\omega})S_t(\tilde{\omega})\mathrm{d}W_t \\ \mathrm{d}B_t = r(t)B_t\mathrm{d}t, \end{cases} \tag{4}$$

with $S_0(\tilde{\omega}) = s_0 \in \mathbb{R}^+$, where $r(t)$ is *deterministic*, $\sigma_t(\tilde{\omega}) > 0$ for every $t \in [0, T]$ and $\mu_t(\tilde{\omega})$, $\sigma_t(\tilde{\omega})$ are $\mathcal{F}^W$-adapted processes such that the discounted price process $X_t := \dfrac{S_t}{B_t}$ belongs to $L^2(\mathbb{P})$, $\forall t \in [0, T]$. In addition we assume that $\mu_t(\tilde{\omega})$ is adapted to the filtration $\mathcal{F}_t^S$ generated by S_t. We remark that if $\sigma_t(\tilde{\omega})$ has a right-continuous version, then it is $\mathcal{F}^S$-adapted (see [15]) since

$$\int_0^t \sigma_s^2 S_s^2 \mathrm{d}s = \lim_{\sup_i |t_{i+1}-t_i| \to 0} \sum_i^n |S_{t_{i+1}} - S_{t_i}|^2,$$

where $0 = t_0 \leq t_1 \leq \cdots t_n = t$ is a partition of $[0, t]$. Hence we obtain that $\mathcal{F}_t^S = \mathcal{F}_t^W$ for any $t \in [0, T]$ and from now on we assume it as the *reference filtration* $\mathcal{F}_t := \mathcal{F}_t^S = \mathcal{F}_t^W$ on $(\tilde{\Omega}, \mathcal{F}, \mathbb{P})$.

- We denote by

$$\theta_t = \frac{\mu_t - r(t)}{\sigma_t} \tag{5}$$

  the *market price of risk*. We also assume that μ, σ and r are such that there exists a unique equivalent martingale measure for the discounted price process X_t whose density $\dfrac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} := \mathcal{E}\left(-\int \theta \mathrm{d}W\right)_T$ is square-integrable. If we denote by $\mathcal{P}_e^2(X)$ the set of equivalent martingale measures for X_t with square-integrable density, we have then that the no-arbitrage hypothesis $\mathcal{P}_e^2(X) \neq \emptyset$ is satisfied.

- There exists a stopping time τ with diffuse law on $\mathbb{R}^+$ that represents the random time of default. By [14] IV.107, this implies that τ is a *totally inaccessible* stopping time.

Note that by construction W_t is a Brownian motion also with respect to $\mathcal{G}_t$.

Definition 3.1. *The buyer of a* default put *has to pay a premium to the seller who undertakes the default risk linked to the underlying asset. If a* credit event *occurs before the maturity date T of the option, the seller has to pay to the put's owner an amount* (default payment)*, which can be fixed or variable.*

If we restrict our attention to the simple case of

$$Z \equiv 0,$$

the default put is given by a triplet (X_1, X_2, τ), where

1. the promised claim is given by the payoff of a standard put option with *strike price* and *exercise date* T:
$$X_1 = (K - S_T)^+; \tag{6}$$
2. the recovery payoff at time T is given by
$$X_2 = \delta(K - S_T)^+, \tag{7}$$
where $\delta = \delta(\omega)$ is supposed to be a ***random*** recovery rate.

In particular we assume that $\delta(\omega) = \delta(\tilde{\omega}, \hat{\omega}) = \delta(\hat{\omega})$ is represented by a $\mathcal{H}_T$-measurable random variable in $L^2(\hat{\Omega}, \mathcal{H}_T, \nu)$, i.e.

$$\delta(\omega) = h(\tau(\omega) \wedge T) \tag{8}$$

for some square-integrable Borel function $h : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $0 \leq h \leq 1$. Here we differ from the approach of [11], since we assume that X_2 is $\mathcal{G}_T$-measurable and not necessarily $\mathcal{F}_T$-measurable. This is due to the fact that in our model we allow the recovery rate δ to depend on the default time τ. This represents a generalization of the models presented in [9] and [11].

Example 3.2. *We remark that in this paper we restrict our attention to the case when the recovery rate depends only on the random time of default. For example $\delta(\omega)$ can be of the form:*

$$\delta(\omega) = \delta_1 \mathbb{I}_{\{\tau \leq T_0\}} + \delta_2 \mathbb{I}_{\{T \geq \tau > T_0\}},$$

when $\delta_1, \delta_2 \in \mathbb{R}_0^+$ and $0 < T_0 < T$. In this example we are considering a case when we obtain a portion of the underlying option according to the fact that the default occurs before or after a certain date. The recovery claim is always handled out at time T of maturity.

In this case the discounted value of the default put can be represented as follows:

$$\begin{aligned} H &= \frac{X_1}{B_T} \mathbb{I}_{\{\tau > T\}} + \frac{X_2}{B_T} \mathbb{I}_{\{\tau \leq T\}} \\ &= \frac{(K - S_T)^+}{B_T} \left(\mathbb{I}_{\{\tau > T\}} + \delta(\omega) \mathbb{I}_{\{\tau \leq T\}} \right) \\ &= \frac{(K - S_T)^+}{B_T} \left(1 + (\delta(\omega) - 1) \mathbb{I}_{\{\tau \leq T\}} \right), \end{aligned} \tag{9}$$

where δ is given in (8). Our aim is then to apply three different hedging methods for H in this setting:

1. Reduced-Form model;
2. Local-Risk Minimization;
3. Mean-Variance Hedging.

4 Reduced-Form model

In this section we present the main results that can be obtained through the intensity-based approach to the valuation of defaultable claims and then we see an application to the case of a default-put. We follow here the approach of [9], [10] and [11].
Under the assumption of Section 3 the no-defaultable market is complete since there exists a unique equivalent martingale measure $\mathbb{P}^*$ for the discounted price process $X_t = \frac{S_t}{B_t}$. See [22] for further details. We put

$$\mathbb{Q}^* = \mathbb{P}^* \otimes \nu$$

in the sequel. Note that by construction, $\mathbb{Q}^*$ is still a martingale measure for X_t with respect to the filtration $\mathcal{G}_t$.
By using no-arbitrage arguments, in Section 8.1.1 of [11] they show that a valuation formula for a defaultable claim can be obtained by the usual risk-neutral valuation formula as follows.
Under the probability measure $\mathbb{Q}^*$, if $A \equiv 0$ and $Z_\tau \equiv 0$, the discounted price process of the default put at time t is given by:

$$\begin{aligned}
\frac{S_t}{B_t} &= E^{\mathbb{Q}^*}\left[\frac{X_1}{B_T}\mathbb{I}_{\{\tau>T\}} + \frac{X_2}{B_T}\mathbb{I}_{\{\tau\leq T\}}\bigg|\mathcal{G}_t\right] \\
&= B_t E^{\mathbb{Q}^*}\left[\frac{(K-S_T)^+}{B_T}\left(1+(\delta(\omega)-1)\mathbb{I}_{\{\tau\leq T\}}\right)\bigg|\mathcal{G}_t\right] \\
&= \underbrace{B_t E^{\mathbb{Q}^*}\left[\frac{(K-S_T)^+}{B_T}\bigg|\mathcal{G}_t\right]}_{a)}\underbrace{E^{\mathbb{Q}^*}\left[(1+(\delta(\omega)-1)H_T)\bigg|\mathcal{G}_t\right]}_{b)},
\end{aligned}$$

where the last equality follows from the fact that S_T and H_T are independent. We compute separately the terms a) and b).

a) This term represents the well-known price P_t of a standard put option:

$$\begin{aligned}
P_t = B_t E^{\mathbb{Q}^*}\left[\frac{(K-S_T)^+}{B_T}\bigg|\mathcal{G}_t\right] &= E^{\mathbb{Q}^*}\left[e^{-\int_t^T r(s)\mathrm{d}s}(K-S_T)^+\bigg|\mathcal{G}_t\right] \\
&= E^{\mathbb{Q}^*}\left[e^{-\int_t^T r(s)\mathrm{d}s}(K-S_T)^+\bigg|\mathcal{F}_t\right] \\
&= Ke^{-\int_t^T r(s)\mathrm{d}s}E^{\mathbb{Q}^*}\left[\mathbb{I}_A|\mathcal{F}_t\right] - S_t E^{\mathbb{Q}^{*,X}}\left[\mathbb{I}_A|\mathcal{F}_t\right],
\end{aligned} \tag{10}$$

where by [16] we have

$$\frac{\mathrm{d}\mathbb{Q}^{*,X}}{\mathrm{d}\mathbb{Q}^*} = \frac{X_T}{X_0}.$$

b) It remains to compute the second term:

$$\begin{aligned}
E^{\mathbb{Q}^*}\left[1+(\delta(\omega)-1)H_T\big|\mathcal{G}_t\right] = \\
1+\underbrace{E^{\mathbb{Q}^*}\left[\delta(\omega)H_T\big|\mathcal{G}_t\right]}_{\mathbf{c)}} - E^{\mathbb{Q}^*}\left[H_T\big|\mathcal{G}_t\right].
\end{aligned} \tag{11}$$

Then, we have to examine the conditional expectation $E^{\mathbb{Q}^*}\left[H_T\big|\mathcal{G}_t\right]$. First we note that

$$E^{\mathbb{Q}^*}\left[H_T\big|\mathcal{G}_t\right] = E^{\mathbb{Q}^*}\left[H_T|\mathcal{H}_t\right].$$

Lemma 4.1. *The process M given by the formula*

$$M_t = \frac{1-H_t}{1-F_t}, \quad \forall t\in\mathbb{R}^+, \tag{12}$$

where F_t is defined in (1)*, follows a martingale with respect to the filtration $(\mathcal{H}_t)_{t\geq 0}$. Moreover, for any $t<s$, the following equality holds:*

$$E^{\mathbb{Q}^*}[1-H_s|\mathcal{H}_t] = (1-H_t)\frac{1-F_s}{1-F_t}. \tag{13}$$

Proof. We refer to Corollary 4.1.2 of [11]. □

Note that the cumulative distribution function of τ is the same both under $\mathbb{Q}^*$ and $\mathbb{Q}$ since $\mathbb{Q}^*(\tau\leq t) = \nu(\tau\leq t) = \mathbb{Q}(\tau\leq t)$. We apply (13) to get

$$\begin{aligned}
E^{\mathbb{Q}^*}[H_T|\mathcal{H}_t] &= 1-\underbrace{\left(\frac{1-H_t}{1-F_t}\right)}_{M_t}(1-F_T) \\
&= 1-(1-F_T)M_t.
\end{aligned} \tag{14}$$

To complete the computations, we evaluate the conditional expectation **c)**.

c) In view of the Corollary 4.1.3 and the Corollary 5.1.1 of [11], using (8) we have:

$$\begin{aligned}
E^{\mathbb{Q}^*}\left[\delta(\omega)H_T|\mathcal{G}_t\right] &= E^{\mathbb{Q}^*}\left[h(\tau\wedge T)H_T|\mathcal{G}_t\right] \\
&= h(\tau\wedge T)H_t + (1-H_t)e^{\int_0^t \lambda_u \mathrm{d}u}E^{\mathbb{Q}^*}\left[\mathbb{I}_{\{\tau>t\}}h(\tau\wedge T)H_T\right] \\
&= h(\tau\wedge T)H_t + (1-H_t)e^{\int_0^t \lambda_u \mathrm{d}u}E^{\mathbb{Q}^*}\left[\mathbb{I}_{\{t<\tau<T\}}h(\tau\wedge T)\right] \\
&= h(\tau\wedge T)H_t + (1-H_t)\int_t^T h(s)\lambda_s e^{-\int_t^T \lambda_u \mathrm{d}u}\mathrm{d}s.
\end{aligned}$$

Finally, gathering the results, we obtain the following Proposition.

Proposition 4.2. *In the market model outlined in Sections 2 and 3, we obtain that the discounted value at time t of the replicating portfolio according to the intensity-based approach is:*

$$\frac{S_t}{B_t} = E^{\mathbb{Q}^*}\left[\frac{X_1}{B_T}\mathbb{I}_{\{\tau>T\}} + \frac{X_2}{B_T}\mathbb{I}_{\{\tau\le T\}}\middle|\mathcal{G}_t\right]$$

$$= P_t\left[H_t h(\tau\wedge T) + (1-H_t)\left(\int_t^T h(s)\lambda_s e^{-\int_t^s \lambda_u \mathrm{d}u}\mathrm{d}s\right) + (1-F_T)M_t\right], \quad (15)$$

where P_t is the hedging portfolio value for a standard put option given in (10).

Remark 4.3. *Since in our market there are non-defaultable primary assets, finding a self-financing portfolio that replicates our put option perfectly is not possible (see [9] for further details). Hence, we have restricted our attention to the pricing problem, according to [11].*

5 Local Risk-Minimization

In Section 4 we have computed in Proposition 4.2 the discounted portfolio value that replicates our defaultable option. The main idea of the intensity-based approach is to assume that the market is complete. However, due to the possibility of default, one cannot perfectly hedge a credit derivative in our model, since only non-defaultable assets are present in our market model. Then it is interesting to study defaultable markets by the means of hedging methods for incomplete markets, such as *local risk-minimization* and *mean-variance hedging.* We start with the local risk-minimization approach.

In this section we first provide a short review of the main results of the theory of local-risk minimization (see [15], [17], [25]) and then we see an application in the case of a default-put. The main feature of this approach is the fact that one has to work with strategies which are not self-financing.

Problem: in the financial market outlined in Section 2 and 3, we look for a *hedging strategy with minimal cost* which replicates the defaultable contingent claim H in (9).

We introduce the basic framework and some definitions. We recall that the assets prices dynamics are given by (4) and that

$$X_t := \frac{S_t}{B_t}$$

denotes the discounted risky asset price.

- We remark that in our model X belongs to the space $\mathcal{S}^2(\mathbb{Q})$ of semimartingales so that it can be decomposed as follows:

  $$X_t = X_0 + M_t^X + A_t^X, \qquad t \in [0,T],$$

  where M^X is a square-integrable local $\mathbb{Q}$-martingale null at 0 and A^X is a predictable process of finite variation null at 0. Moreover, in our case X_t is a continuous process.

- In our model we have that the so-called **Structure Condition (SC)** is satisfied, i.e. the *mean-variance tradeoff*

  $$\widehat{K}_t := \int_0^t \theta_s^2 \mathrm{d}s \tag{16}$$

  is almost surely finite, where θ_t is the market price of risk defined in (5), since X_t is continuous and $\mathcal{P}_e^2(X) \neq \emptyset$ by hypothesis (see [24]).
  In particular, from now on we assume that $\widehat{K}_t$ is uniformly bounded in t and ω, i.e. there exists k such that

  $$\widehat{K}_t(\omega) \leq k, \qquad \forall t \in [0,T], \text{ a.s.} \tag{17}$$

We want to find a hedging strategy φ with "minimal" cost C_t and value process

$$\bar{V}_t(\varphi) := \frac{V_t}{B_t}(\varphi) = V_0(\varphi) + \int_0^t \xi_s \mathrm{d}X_s + C_t(\varphi)$$

such that

$$\bar{V}_T(\varphi) = H \qquad \mathbb{Q}-\text{a.s.}$$

In which sense is the cost minimal?
We denote by Θ_s the space of $\mathcal{G}$-predictable processes ξ on $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$ such that

$$E^{\mathbb{Q}}\left[\int_0^T (\xi_s)^2 \mathrm{d}\langle M^X\rangle_s\right] + E^{\mathbb{Q}}\left[\left(\int_0^T |\xi_s| \mathrm{d}|A_s^X|\right)^2\right] < \infty. \tag{18}$$

Definition 5.1. *An* L^2-strategy *is a pair* $\varphi = (\xi, \eta)$ *such that*

1. *ξ is a $\mathcal{G}$-predictable process belonging to Θ_s.*
2. *η is a real-valued $\mathcal{G}$-adapted process such that $\bar{V}(\varphi) = \xi \cdot X + \eta$ is right-continuous and square-integrable.*

The cost process *is defined by:*

$$C_t = \bar{V}_t - \int_0^t \xi_s \mathrm{d}X_s, \ \ 0 \le t \le T. \tag{19}$$

Definition 5.2. *An L^2-strategy φ is called* mean-self-financing *if its cost process $C(\varphi)$ is a $\mathbb{Q}$-martingale.*

Following [25], we introduce an optimal replicating strategy:

Definition 5.3. *Let $H \in L^2(\mathcal{G}_T, \mathbb{Q})$. An L^2-strategy φ with $\bar{V}_T(\varphi) = H$ $\mathbb{Q}$-a.e. is* pseudo-locally risk minimizing (plrm) *for H if φ is mean-self-financing and the martingale $C(\varphi)$ is* strongly orthogonal *to M.*

For the reader's convenience we recall that two square-integrable martingales are said to be *strongly orthogonal* if their product is a (uniformly integrable) martingale.
In general how to characterize a pseudo-locally risk-minimizing strategy is shown in the next result due to Föllmer and Schweizer (see [15]):

Proposition 5.4. *A contingent claim $H \in L^2(\mathbb{Q})$ admits a pseudo-locally risk-minimizing strategy φ if and only if H can be written as*

$$H = H_0 + \int_0^T \xi_s^H \mathrm{d}X_s + L_T^H \quad \mathbb{Q} - \text{a.s.} \tag{20}$$

with $H_0 \in \mathbb{R}$, $\xi^H \in \Theta_S$, $L^H \in \mathcal{M}_0^2(\mathbb{Q})$ strongly $\mathbb{Q}$-orthogonal to M^X. The plrm-strategy is given by

$$\xi_t = \xi_t^H, \quad 0 \le t \le T$$

with minimal cost

$$C_t(\varphi) = H_0 + L_t^H, \quad 0 \le t \le T.$$

If (20) holds, the optimal portfolio value is

$$V_t(\varphi) = C_t(\varphi) + \int_0^t \xi_s \mathrm{d}X_s = H_0 + \int_0^t \xi_s^H \mathrm{d}X_s + L_t^H,$$

and

$$\eta_t = \eta_t^H = V_t(\varphi) - \xi_t^H X_t.$$

Proof. For the proof, see [15]. □

Decomposition (20) is well known in literature as the *Föllmer-Schweizer decomposition* (in short FS decomposition). In the martingale case it coincides with the Galtchouk-Kunita-Watanabe decomposition. We see now how one can obtain the FS decomposition by choosing a convenient martingale measure for X following [15].

Definition 5.5 (The Minimal Martingale Measure). *A martingale measure $\widehat{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ with square-integrable density is called **minimal** if $\widehat{\mathbb{Q}} \equiv \mathbb{Q}$ on $\mathcal{G}_0$ and if* any square-integrable $\mathbb{Q}$-local martingale which is strongly orthogonal to M^X under $\mathbb{Q}$ remains a local martingale under $\widehat{\mathbb{Q}}$.

The minimal measure is the equivalent martingale measure that modifies the martingale structure as little as possible.

Theorem 5.6. *Suppose X is continuous and that it satisfies (SC). Suppose the strictly positive local $\mathbb{Q}$-martingale $\hat{Z}_t = \mathcal{E}(-\int \theta dW)_t$ is a square-integrable martingale and define the process $\widehat{V}^H$ as follows*

$$\widehat{V}_t^H := E^{\widehat{\mathbb{Q}}}[H|\mathcal{G}_t], \qquad 0 \le t \le T.$$

Let

$$\widehat{V}_T^H = E^{\widehat{\mathbb{Q}}}[H|\mathcal{G}_T] = \widehat{V}_0^H + \int_0^T \widehat{\xi}_s^H \mathrm{d}X_s + \widehat{L}_T^H \tag{21}$$

be the GKW decomposition of $\widehat{V}_t^H$ with respect to X under $\widehat{\mathbb{Q}}$. If either H admits a FS decomposition or $\hat{\xi}^H \in \Theta_s$ and $\hat{L}^H \in \mathcal{M}_0^2(\mathbb{Q})$, then (21) gives the FS decomposition of H and $\hat{\xi}^H$ gives a plrm strategy for H. A sufficient condition to guarantee that $\hat{Z} \in \mathcal{M}_0^2(\mathbb{Q})$ and the existence of a FS decomposition for H is that the mean-variance tradeoff process $\widehat{K}_t$ *is uniformly bounded.*

Proof. For the proof, see Theorem 3.5 of [25]. □

We apply these results to the case of defaultable claims.

5.1 Local-Risk-Minimization for defaultable claims

We focus on the particular case of a default put H defined in (9). For local risk minimization for a general defaultable claim, we refer to [1]. We wish to find a portfolio "with minimal cost" that perfectly replicates H according to the local risk-minimizing criterion.
We remark that we focus on the case of trading strategies *adapted to the full filtration* $\mathcal{G}_t$ (see [9]). For a further discussion on local risk-minimization with $\mathcal{F}_t$-adapted strategies, we refer to [2].

Lemma 5.7. *The minimal martingale measure for X_t with respect to $\mathcal{G}_t$ exists and coincides with $\mathbb{Q}^*$.*

Proof. Since W_t and $\hat{M}_t$ defined in (2) have the predictable representation property for the space of square-integrable local martingale on the product probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q}) = (\tilde{\Omega} \times \hat{\Omega}, \mathcal{F}^W \otimes \mathcal{H}, \mathcal{F}_t^W \otimes \mathcal{H}_t, \mathbb{P} \otimes \nu)$, the result follows by Definition 5.5. See also [3] and [20]. $\square$

Proposition 5.8. *Let $\hat{M}$ be the compensated process defined in (2) and X the discounted price process. The pair $(X, \hat{M})$ has the predictable representation property on $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q}^*)$, i.e. for every $H \in L^1(\Omega, \mathcal{G}_T, \mathbb{Q}^*)$, there exists a pair of $\mathcal{G}$-predictable processes $(\tilde{\Phi}, \tilde{\Psi})$ such that*

$$H = c + \int_0^T \tilde{\Phi}_s \mathrm{d}X_s + \int_0^T \tilde{\Psi}_s \mathrm{d}\hat{M}_s \tag{22}$$

and

$$\int_0^T \tilde{\Phi}_s^2 \mathrm{d}\langle X \rangle_s + \int_0^T \tilde{\Psi}_s^2 \mathrm{d}[\hat{M}]_s < \infty \quad a.s.$$

Proof. Since there exists a unique equivalent martingale measure $\mathbb{P}^*$ for the continuous asset process X_t on $(\tilde{\Omega}, \mathcal{F}, \mathcal{F}_t)$, then by Theorem 40 of Chapter IV of [21] we have that X_t has the predictable representation property for the local martingales on $(\tilde{\Omega}, \mathcal{F}, \mathcal{F}_t, \mathbb{P}^*)$.
By Proposition 4.1 of [4] the compensated default process $\hat{M}_t$ has the predictable representation property for the local martingales on $(\hat{\Omega}, \mathcal{H}, \mathcal{H}_t, \nu)$.
Hence, since X_t and $\hat{M}_t$ are strongly orthogonal, by Proposition A.2 of [3] and by using a limiting argument we obtain that $(X, \hat{M})$ has the predictable representation property on the product probability space

$$(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q}^*) = (\tilde{\Omega} \times \hat{\Omega}, \mathcal{F} \otimes \mathcal{H}, \mathcal{F}_t \otimes \mathcal{H}_t, \mathbb{P}^* \otimes \nu).$$

$\square$

We remark that the market is incomplete even if we trade with $\mathcal{G}_t$-adapted strategies since $\hat{M}$ does not represent the value of any tradable asset.
We can apply Proposition 5.8 to obtain the *plrm* strategy for $H \in L^2(\Omega, \mathcal{G}_T, \mathbb{Q})$.

Proposition 5.9. *Let $H \in L^2(\Omega, \mathcal{G}_T, \mathbb{Q})$ be the value of a defaultable claim. Then the plrm strategy for H exists and it is given by*

$$\Phi_t = \tilde{\Phi}_t, \quad C_t = c + \int_0^t \tilde{\Psi}_s \mathrm{d}\hat{M}_s,$$

where $\tilde{\Phi}_t, \tilde{\Psi}_t$ are the same as in Proposition 5.8.

Proof. Let $H \in L^2(\Omega, \mathcal{G}_T, \mathbb{Q})$. We note that since $\dfrac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} \in L^2(\mathbb{Q})$, then $L^2(\Omega, \mathcal{G}_T, \mathbb{Q}) \subset L^1(\Omega, \mathcal{G}_T, \widehat{\mathbb{Q}})$. Then $H \in L^1(\widehat{\mathbb{Q}})$ and we can apply Proposition 5.8 to obtain decomposition (22) for H given by

$$H = c + \int_0^T \tilde{\Phi}_s \mathrm{d}X_s + \int_0^T \tilde{\Psi}_s \mathrm{d}\hat{M}_s. \tag{23}$$

The martingale $\hat{M}$ is strongly orthogonal to the martingale part M^X of X, hence (23) gives the GKW decomposition of H under $\widehat{\mathbb{Q}}$. Since by hypothesis $\dfrac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = \dfrac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{Q}} \in L^2(\mathbb{Q})$ and X_t is continuous, then by Theorem 3.5 of [15] the associated density process

$$Z_t = E^{\mathbb{Q}}\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\middle|\mathcal{G}_t\right] = E^{\mathbb{Q}}\left[\frac{\mathrm{d}\widehat{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\middle|\mathcal{F}_t\right]$$

is a square-integrable martingale. Since hypothesis (17) is in force, we can apply Theorem 5.6 and conclude that (22) is the FS decomposition of H. $\square$

Remark 5.10. *It is possible to choose different hypotheses that guarantee that decomposition (22) gives the FS decomposition. Assumption (17) is the simplest condition that can be assumed. For a complete survey and a discussion of the others, we refer to [25].*

Under the equivalent martingale probability measure $\hat{\mathbb{Q}}$, the discounted price process $\hat{V}_t$ of the default put at time t, is given by:

$$\begin{aligned}
\hat{V}_t &= E^{\widehat{\mathbb{Q}}}[H|\mathcal{G}_t] \\
&= E^{\widehat{\mathbb{Q}}}\left[\frac{X_1}{B_T}\mathbb{I}_{\{\tau>T\}} + \frac{X_2}{B_T}\mathbb{I}_{\{\tau\le T\}}\middle|\mathcal{G}_t\right] \\
&= E^{\widehat{\mathbb{Q}}}\left[\frac{X_1}{B_T}\middle|\mathcal{G}_t\right]\cdot E^{\widehat{\mathbb{Q}}}\left[1+(\delta(\omega)-1)H_T\middle|\mathcal{G}_t\right] \\
&= \underbrace{E^{\widehat{\mathbb{Q}}}\left[\frac{(K-S_T)^+}{B_T}\middle|\mathcal{G}_t\right]}_{\textbf{a)}}\cdot\underbrace{E^{\widehat{\mathbb{Q}}}\left[(1+(\delta(\omega)-1)H_T)\middle|\mathcal{G}_t\right]}_{\textbf{b)}}. \qquad (24)
\end{aligned}$$

We need only to find the Föllmer-Schweizer decomposition of $\hat{V}_t$ as illustrated in (20).

a) By Section 5 of [5] and using the "*change of numéraire*" technique of [16], we have

$$\begin{aligned}
E^{\widehat{\mathbb{Q}}}\left[\frac{X_1}{B_T}\middle|\mathcal{G}_t\right] &= E^{\widehat{\mathbb{Q}}}\left[\frac{(K-S_T)^+}{B_T}\middle|\mathcal{G}_t\right] \\
&= E^{\widehat{\mathbb{Q}}}\left[\frac{(K-S_T)}{B_T}\mathbb{I}_{\underbrace{\{K\ge S_T\}}_{A}}\middle|\mathcal{G}_t\right] \\
&= KE^{\widehat{\mathbb{Q}}}\left[\frac{1}{B_T}\mathbb{I}_A\middle|\mathcal{G}_t\right] - E^{\widehat{\mathbb{Q}}}\left[\frac{S_T}{B_T}\mathbb{I}_A\middle|\mathcal{G}_t\right] \\
&= \frac{K}{B_T}E^{\widehat{\mathbb{Q}}}\left[\mathbb{I}_A|\mathcal{G}_t\right] - E^{\widehat{\mathbb{Q}}}[X_T\mathbb{I}_A|\mathcal{G}_t] \\
&= \frac{K}{B_T}E^{\widehat{\mathbb{Q}}}\left[\mathbb{I}_A|\mathcal{G}_t\right] - X_tE^{\widehat{\mathbb{Q}}^X}\left[\mathbb{I}_A|\mathcal{G}_t\right],
\end{aligned}$$

where

$$\frac{\mathrm{d}\widehat{\mathbb{Q}}^X}{\mathrm{d}\hat{\mathbb{Q}}} = \frac{X_T}{X_0}$$

is well-defined since $X_T \in L^2(\mathbb{Q})$ by hypothesis and hence $X_T \in L^1(\hat{\mathbb{Q}})$. In addition by (22) we obtain that $E^{\widehat{\mathbb{Q}}}\left[\frac{X_1}{B_T}\middle|\mathcal{G}_t\right]$ admits the decomposition

$$E^{\widehat{\mathbb{Q}}}\left[\frac{X_1}{B_T}\middle|\mathcal{G}_t\right] = c + \int_0^t \xi_s \mathrm{d}X_s. \qquad (25)$$

Since

$$E^{\widehat{\mathbb{Q}}^X}\left[\mathbb{I}_A|\mathcal{G}_t\right] = E^{\widehat{\mathbb{Q}}^X}\left[\mathbb{I}_A|\mathcal{F}_t\right]$$

because $\mathbb{I}_A$ is independent of τ, by [16] we have that

$$\xi_t = E^{\widehat{\mathbb{Q}}^X}\left[\mathbb{I}_A|\mathcal{F}_t\right]. \tag{26}$$

b) It remains to calculate the term $E^{\widehat{\mathbb{Q}}}\left[1 + (\delta(\omega) - 1)H_T\middle|\mathcal{G}_t\right]$. First we note that

$$\begin{aligned}
E^{\widehat{\mathbb{Q}}}\left[1 + (\delta(\omega) - 1)H_T\middle|\mathcal{G}_t\right] &= 1 + E^{\widehat{\mathbb{Q}}}[\delta(\omega)H_T|\mathcal{G}_t] - E^{\widehat{\mathbb{Q}}}[H_T|\mathcal{G}_t] \\
&= 1 + E^{\widehat{\mathbb{Q}}}[\delta(\omega)H_T|\mathcal{G}_t] - (1 - (1 - F_T)M_t) \\
&= E^{\widehat{\mathbb{Q}}}[\delta(\omega)H_T|\mathcal{G}_t] + (1 - F_T)M_t,
\end{aligned}$$

by (14). Since $\delta(\omega)H_T = f(\tau)$ for some integrable Borel function $f : \mathbb{R}^+ \to [0, 1]$, by Proposition 4.3.1 of [11], we have

$$E^{\widehat{\mathbb{Q}}}\left[1 + (\delta(\omega) - 1)H_T\middle|\mathcal{G}_t\right] = c_h + \int_0^t \hat{f}(s)\mathrm{d}\hat{M}_s + (1 - F_T)M_t,$$

where $c_h = E^{\widehat{\mathbb{Q}}}[f(\tau)]$ and the function $\hat{f} : \mathbb{R}^+ \to \mathbb{R}$ is given by the formula

$$\hat{f}(t) = f(t) - e^{\Gamma_t}E^{\widehat{\mathbb{Q}}}[\mathbb{I}_{\{\tau>t\}}f(\tau)]. \tag{27}$$

Note that

$$f(x) = h(x \wedge T)\mathbb{I}_{\{x<T\}},$$

where h is introduced in (8). We only need to find the relationship between M_t and $\hat{M}_t$.

Lemma 5.11. *Let M and $\hat{M}$ be defined by (2) and (12) respectively. The following equality holds:*

$$\mathrm{d}M_t = -\frac{1}{1 - F_t}\mathrm{d}\hat{M}_t. \tag{28}$$

Proof. To obtain (28), it suffices to apply Itô's formula. For further details see Section 6.3 of [11]. □

Finally, gathering the results we obtain

$$\begin{aligned}
\hat{V}_t &= E^{\widehat{\mathbb{Q}}}[H|\mathcal{G}_t] \\
&= \Bigg(\underbrace{c+\int_0^t \xi_s \mathrm{d}X_s}_{\Phi_t}\Bigg)\cdot\left(E^{\widehat{\mathbb{Q}}}[f(\tau)]+\int_0^t \hat{f}(s)\mathrm{d}\hat{M}_s+(1-F_T)M_t\right) \\
&= \Phi_t \cdot \Bigg(\underbrace{E^{\widehat{\mathbb{Q}}}[f(\tau)]+\int_0^t\left(\hat{f}(s)-\frac{1-F_T}{1-F_s}\right)\mathrm{d}\hat{M}_s}_{\Psi_t}\Bigg).
\end{aligned}$$

Since

$$\mathrm{d}[\Phi,\Psi]_t = \xi_t\left(\hat{f}(t)-\frac{1-F_T}{1-F_t}\right)\mathrm{d}[X,\hat{M}]_t = 0,$$

applying Itô's formula we get

$$\begin{aligned}
\mathrm{d}\hat{V}_t &= \Phi_t \mathrm{d}\Psi_t + \Psi_{t-}\mathrm{d}\Phi_t + \mathrm{d}[\Phi,\Psi]_t \\
&= \left(c+\int_0^t \xi_s \mathrm{d}X_s\right)\left(\hat{f}(t)-\frac{1-F_T}{1-F_t}\right)\mathrm{d}\hat{M}_t + \left(E^{\widehat{\mathbb{Q}}}[f(\tau)]+\int_0^t\left(\hat{f}(s)+\right.\right. \\
&\quad \left.\left. -\frac{1-F_T}{1-F_s}\right)\mathrm{d}\hat{M}_s\right)\xi_t \mathrm{d}X_t.
\end{aligned} \tag{29}$$

Hence we can conclude that:

Proposition 5.12. *In the market model outlined in Sections 2 and 3, under hypothesis* (17) *the local risk-minimizing portfolio for H defined in* (9) *is given by*

$$\hat{V}_t = c_1 + \int_0^t \Phi_s^1 \mathrm{d}X_s + \hat{L}_t, \tag{30}$$

where the plrm strategy is

$$\Phi_t^1 = \left(E^{\widehat{\mathbb{Q}}}[f(\tau)]+\int_0^t\left(\hat{f}(s)-\frac{1-F_T}{1-F_s}\right)\mathrm{d}\hat{M}_s\right)\xi_t \tag{31}$$

and the minimal cost is

$$\hat{L}_t = \int_0^t\left(c+\int_0^s \xi_u \mathrm{d}X_u\right)\left(\hat{f}(s)-\frac{1-F_T}{1-F_s}\right)\mathrm{d}\hat{M}_s, \tag{32}$$

where ξ_t is given by (26), *$\hat{f}(s)$ by* (27) *and F_t by* (1).

Proof. Proposition 5.9 guarantees that (29) provides the FS decomposition for H, i.e. that Φ_t^1 and $\hat{L}_t$ satisfy the required integrability conditions. □

6 Mean-Variance Hedging

Finally we consider the mean-variance hedging approach. We refer to [25] for an exhaustive survey of relevant results. This method has been already applied to defaultable markets in [6], [7], [8] and [9]. Here we extend their results to the case of general coefficients in the dynamics of X_t and random recovery rate and compute explicitly the mean-variance strategy in the particular case of a put option.
Again we focus on the case of $\mathcal{G}$-adapted hedging strategy and denote by $L(X)$ the set of all $\mathcal{G}$-predictable X-integrable processes.

Definition 6.1. *An* admissible *hedging strategy is any pair $\varphi = (\theta, \eta)$, where θ is a $\mathcal{G}$-predictable process in $L(X)$ and η is a real-valued $\mathcal{G}$-adapted process such that the discounted value process $\bar{V}_t(\varphi) := \frac{V_t}{B_t}(\varphi) = \eta_t + \theta_t X_t,\ 0 \le t \le T$ is right-continuous.*

Note that if the discounted value process $\bar{V}(\varphi)$ is self-financing - that is $\bar{V}_t(\varphi) = V_0 + \int_0^t \theta_s \mathrm{d}X_s$, then η is completely determined by the pair (V_0, θ):

$$\eta_t = V_0 + \int_0^t \theta_s \mathrm{d}X_s - \theta_t X_t, \quad 0 \le t \le T.$$

Hence we can formulate the mean-variance problem as follows:
Problem: find an admissible strategy (V_0, θ) which solves the following minimization problem:

$$\min_{(V_0,\theta)} E\left[\left(H - V_0 - \int_0^T \theta_s \mathrm{d}X_s\right)^2\right],$$

where θ belongs to

$$\Theta = \left\{\theta \in L(X) : \int_0^t \theta_s \mathrm{d}X_s \in \mathcal{S}^2(\mathbb{Q})\right\}.$$

If such strategy exists, it is called *Mean-Variance Optimal Strategy* and denoted by $(\tilde{V}_0, \tilde{\theta})$.
Dual Problem: find an equivalent martingale measure $\tilde{\mathbb{Q}}$ such that its density is square-integrable and its norm:

$$\left\|\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\right\|^2 = E\left[\left(\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\right)^2\right]$$

is minimal over all the probability measures in $\mathcal{P}_e^2(X)$. By [13] this probability measure exists since X_t is continuous and $\mathcal{P}_e^2 \neq \emptyset$ and it is called *Variance-Optimal Measure* since:

$$\left\| \frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} \right\|^2 = 1 + Var\left[\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} \right].$$

The main result is given by the following Theorem:

Theorem 6.2. *Suppose Θ is closed and let X be a continuous process such that $\mathcal{P}_e^2(X) \neq \emptyset$. Let $H \in L^2(\mathbb{Q})$ be a contingent claim and write the Galtchouk-Kunita-Watanabe decomposition of H under $\tilde{\mathbb{Q}}$ with respect to X as*

$$H = E^{\tilde{\mathbb{Q}}}[H] + \int_0^T \tilde{\xi}_u^H \mathrm{d}X_u + \tilde{L}_T = \tilde{V}_T, \tag{33}$$

with

$$\tilde{V}_t := E^{\tilde{\mathbb{Q}}}[H|\mathcal{G}_t] = E^{\tilde{\mathbb{Q}}}[H] + \int_0^t \tilde{\xi}_u^H \mathrm{d}X_u + \tilde{L}_t, \quad 0 \leq t \leq T. \tag{34}$$

Then the mean-variance optimal Θ-strategy for H exists and it is given by

$$\tilde{V}_0 = E^{\tilde{\mathbb{Q}}}[H]$$

and

$$\begin{aligned} \tilde{\theta}_t &= \tilde{\xi}_t^H - \frac{\tilde{\zeta}_t}{\tilde{Z}_t} \left(\tilde{V}_{t-} - E^{\tilde{\mathbb{Q}}}[H] - \int_0^t \tilde{\theta}_u \mathrm{d}X_u \right) \\ &= \tilde{\xi}_t^H - \tilde{\zeta}_t \left(\frac{\tilde{V}_0 - E^{\tilde{\mathbb{Q}}}[H]}{\tilde{Z}_0} + \int_0^{t-} \frac{1}{\tilde{Z}_u} \mathrm{d}\tilde{L}_u \right), \qquad 0 \leq t \leq T, \end{aligned}$$

where

$$\tilde{Z}_t = E^{\tilde{\mathbb{Q}}}\left[\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} \middle| \mathcal{G}_t \right] = \tilde{Z}_0 + \int_0^t \tilde{\zeta}_u \mathrm{d}X_u, \qquad 0 \leq t \leq T \tag{35}$$

Proof. The proof can be found in [23]. □

Let us now turn to the case of the default put. We can interpret the presence on the market of a default possibility as a particular case of *"incomplete information"*. Hence the results of [3] and [4], where the variance-optimal measure is characterized as the solution of an equation between Doléans exponentials, can also be applied in this context to compute $\tilde{\mathbb{Q}}$. In particular by [3], Theorem 2.16 and Section 3 (α), it follows that the variance-optimal measure coincides with the minimal one. In this case

$$\tilde{\mathbb{Q}} = \widehat{\mathbb{Q}} = \mathbb{Q}^*. \tag{36}$$

First of all we check that the space Θ is closed.
By Proposition 4.2 of [3], we have that Θ is closed if and only if for every stopping time η, with $0 \leq \eta \leq T$, the following condition holds

$$E^{\bar{\mathbb{Q}}}\left[\exp\left(\int_\eta^T \theta_s^2 \mathrm{d}s\right)\bigg|\mathcal{G}_\eta\right] \leq M, \tag{37}$$

where $\dfrac{\mathrm{d}\bar{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} := \mathcal{E}\left(-\int 2\theta \mathrm{d}W\right)_T$. Note that since we are assuming that $\widehat{\mathbb{Q}}$ exists and it is square-integrable, then $\bar{\mathbb{Q}}$ also exists and $\exp\left(\int_0^T \theta_t^2 \mathrm{d}t\right)$ is $\bar{\mathbb{Q}}$-integrable ([3], Section 3(α)). Here we obtain that condition (37) is a verified for every $\mathcal{G}$-stopping time η such that $0 \leq \eta \leq T$ as a consequence of our assumption (17). Then we can use Theorem 6.2 to obtain the mean-variance optimal Θ-strategy for H.
The process $\tilde{V}_t$ at time t, is given by:

$$\begin{aligned}
\tilde{V}_t &= E^{\tilde{\mathbb{Q}}}[H|\mathcal{G}_t] \\
&= E^{\tilde{\mathbb{Q}}}\left[\frac{X_1}{B_T}\mathbb{I}_{\{\tau>T\}} + \frac{X_2}{B_T}\mathbb{I}_{\{\tau\leq T\}}\bigg|\mathcal{G}_t\right] \\
&= E^{\tilde{\mathbb{Q}}}\left[\frac{(K-S_T)^+}{B_T}\left(1+(\delta(\omega)-1)\mathbb{I}_{\{\tau\leq T\}}\right)\bigg|\mathcal{G}_t\right].
\end{aligned}$$

By Section 3 (α) in [3], we also obtain that

$$\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = \mathcal{E}\left(-\int \beta \mathrm{d}X\right)_T \frac{1}{E\left[(-\beta \mathrm{d}X)\right]},$$

where $\beta_t = \dfrac{\theta_t - h_t}{\sigma_t X_t}$ and h_t solves the equation

$$\mathcal{E}\left(\int h \mathrm{d}\bar{W}\right)_T = \frac{\exp(\int_0^T \theta_t^2 \mathrm{d}t)}{\bar{E}\left[\exp\left(\int_0^T \theta_t^2 \mathrm{d}t\right)\right]}$$

with $\bar{W}_t := W_t + 2\int_0^t \theta_s \mathrm{d}s$ and $\frac{\mathrm{d}\bar{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}} = \mathcal{E}\left(-\int 2\theta \mathrm{d}W\right)_T$. Hence we have that

$$\tilde{Z}_t = E^{\tilde{\mathbb{Q}}}\left[\frac{\mathrm{d}\tilde{\mathbb{Q}}}{\mathrm{d}\mathbb{Q}}\bigg|\mathcal{F}_t\right] = \frac{\mathcal{E}\left(-\int \beta \mathrm{d}X\right)_t}{E\left[\mathcal{E}\left(-\int \beta \mathrm{d}X\right)_T\right]} \tag{38}$$

and $\mathrm{d}\tilde{Z}_t = \beta_t \tilde{Z}_t \mathrm{d}X_t$. Consequently we can compute decomposition (35) and obtain

$$\tilde{\zeta}_t = \tilde{Z}_t \beta_t. \tag{39}$$

Since $\tilde{\mathbb{Q}} = \widehat{\mathbb{Q}} = \mathbb{Q}^*$, we can use (30), (31) and (32), to obtain the mean-variance optimal Θ-strategy $(\tilde{V}_0, \tilde{\theta})$ for H.

Proposition 6.3. *In the market model outlined in Sections 2 and 3, under hypothesis* (17) *the mean-variance hedging strategy for H defined in* (9) *is given by:*

- ***Optimal Price***

$$\tilde{V}_0 = E^{\tilde{\mathbb{Q}}}[H] = E^{\tilde{\mathbb{Q}}}\left[\frac{X_1}{B_T}\mathbb{I}_{\{\tau > T\}} + \frac{X_2}{B_T}\mathbb{I}_{\{\tau \leq T\}}\right].$$

  We note that the optimal price for the mean-variance hedging criterion coincides with the optimal price for the locally risk-minimizing criterion.

- ***Mean-Variance Optimal Strategy***

$$\tilde{\theta}_t = \Phi^1_t - \tilde{\zeta}_t \int_0^{t-} \frac{1}{\tilde{Z}_u} \mathrm{d}\tilde{L}_u, \tag{40}$$

  where Φ^1, $\tilde{Z}$ and $\tilde{\zeta}$ are given by (31), (38) and (39) respectively and

$$\mathrm{d}\tilde{L}_t = \mathrm{d}\hat{L}_t = \left(c + \int_0^t \xi_s \mathrm{d}X_s\right)\left(\hat{f}(t) - \frac{1 - F_T}{1 - F_t}\right)\mathrm{d}\hat{M}_t. \tag{41}$$

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