PRICING OF COUNTERPARTY RISK AND FUNDING WITH CSA DISCOUNTING, PORTFOLIO EFFECTS AND INITIAL MARGIN.

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Abstract. In this paper we extend the existing literature on xVA along three directions. First, we enhance current BSDE-based xVA frameworks to include initial margin by following the approach of Crépey (2015a) and Crépey (2015b). Next, we solve the consistency problem that arises when the front-office desk of the bank uses trade-specific discount curves that differ from the discount curve adopted by the xVA desk. Finally, we address the existence of multiple aggregation levels for contingent claims in the portfolio between the bank and the counterparty, providing suitable extensions of our proposed single-claim xVA framework.

1. Introduction

As a consequence of the 2007-2009 financial crisis, academics and practitioners are revisiting the valuation of financial products in several aspects. In particular, the value of a product should account for the possibility of default of any agent involved in the transaction. Also the trading activity is funded by resorting on different sources of liquidity, which results in the interest rate multi-curve phenomenon, so that the existence of a unique source of funding growing at a risk-free interest rate no longer represents a realistic assumption. Financial regulations, such as Basel III/IV and Emir, are also driving the methodological development. Regulations on collateral imply an increasingly important role of central counterparties.

All these issues are represented at the level of valuation equations by introducing *value adjustments* (xVA), which are further terms to be added or subtracted to an idealized reference price, computed in the absence of the aforementioned frictions, in order to obtain the final value of the transaction.

The literature on counterparty credit risk and funding is large and we only attempt to provide insights on the main contributions. Possibly, the first contribution on the subject is Duffie and Huang (1996). Before the 2007-2009 financial crisis, we mention the works of Brigo and Masetti (2005) and Cherubini (2005), where the concept of *credit valuation adjustment* (CVA) is analyzed. The possibility of default of both counterparties involved in the transaction, represented by the introduction of the *debt valuation adjustment* (DVA), is investigated, among others, in Brigo et al. (2011) and Brigo et al. (2014).

Apart from the issue of default risk, another important source of concern for practitioners and academics is represented by funding costs. A parallel stream of literature emerged during and after the financial crisis, to generalize valuation equations to account for features such as the presence of collateralization agreements. In a Black-Scholes economy, Piterbarg (2010) provides valuation formulas in presence or absence of collateral agreements. Piterbarg (2012) generalizes the issue in a multi-currency economy, see also Fujii et al. (2010) and Fujii et al. (2011). The *funding valuation adjustment* (FVA)

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under several alternative assumptions on the Credit Support Annex (CSA) is derived in Pallavicini et al. (2011), while Brigo and Pallavicini (2014) also discusses the role of central counterparties in the context of funding costs. A general approach to funding issues in a semimartingale setting is provided by Bielecki and Rutkowski (2015).

Both funding and default risk need to be unified in a unique pricing framework. Contributions in this sense can be found in Brigo et al. (2018) by means of the so-called *discounting approach*. In a series of papers, Burgard and Kjaer generalize the classical Black-Scholes replication approach to include many effects, see Burgard and Kjaer (2011) and Burgard and Kjaer (2013). A more general BSDE approach is provided by Crépey (2015a), Crépey (2015b), Bichuch et al. (2018). The equivalence between the discounting and the BSDE-based replication approaches is demonstrated in Brigo et al. (2018).

The importance of the topic is reflected by the increasing number of monographs on the subject, see e.g. Brigo et al. (2013). An advanced BSDE-based treatment is provided by Crépey et al. (2014). A detailed analysis of how to construct large hybrid models for counterparty risk simulations are provided in Green (2015), Lichters et al. (2015) and Sokol (2014), while Gregory (2015) provides an accessible introduction to most aspects of the topic.

In this work, we propose an xVA framework using BSDEs techniques in a market described by diffusion. We revisit concepts such as the self-financing property, absence of arbitrage and replication of contingent claims in a market with frictions, due to the presence of counterparty risk and multiple funding curves. Our replication BSDE, introduced under a classical enlarged filtration, is specified up to a random time horizon given by the minimum between the default time of the counterparty, the default time of the bank, and the natural maturity of the contract. We discuss the well posedness of the BSDE by considering associated pre-default BSDEs under a reduced filtration along the lines of Crépey (2015a), Crépey (2015b), Bichuch et al. (2018), Bielecki and Rutkowski (2015) and Brigo et al. (2018).

Given the xVA framework for a single transaction, we then consider the consistency problem between xVA pricing equations and the CSA discounting rules. The latter originate from the quoting mechanism of market standard instruments. Such instruments are quoted under the assumption that they are perfectly collateralized transactions. Since a perfectly collateralized transaction is funded by the collateral provider, the discounting rate applied to evaluate market instruments is given by a collateral rate, which typically corresponds to an overnight interest rate. The presence of multiple assumptions on the collateral rate implies the co-existence of quotes with different discounting rates, which are in general at odds with the unique discounting rate dictated by the xVA pricing BSDE. We solve the consistency issue by relying on an invariance property of linear BSDEs.

Finally, we present incremental xVA charges for new potential trades under the proposed xVA framework: given the presence of portfolio effects in the computation of value adjustments, and given an existing portfolio of K trades, the xVA charge for a new potential $(K+1)$-th trade is computed as the difference of the xVA charges of the extended portfolio, consisting of $(K+1)$ trades, and the xVA charge of the base portfolio of K trades. Such an approach represents an effective way to describe the non-linearity effect existing in the financial industry framework.

Given our focus on discounting and aggregation levels, in this paper we do not discuss *capital valuation adjustment* (KVA). The issue is treated in recent papers such as Albanese and Crépey (2017), Albanese et al. (2016) and Albanese et al. (2017). This is beyond the scope of the present paper and leave it for future research.

The paper is organized as follows. In Section 2 we formalize in mathematical terms the main financial concepts related to the xVA framework. Section 3 describes the results related to the xVA evaluation when only one transaction is taken into account. In Section 4 we extend the xVA framework, as well as the related mathematical background, to the case in which the portfolio consists of multiple contracts. Section 5 provides an example illustrating most of the previously introduced concepts. In Appendix A we gather some results from the literature used to derive the main results.

2. The financial setting

We fix a time horizon $T < \infty$ for the trading activity. We consider two agents named the *bank* (B) and the *counterparty* (C). All processes are modeled over a probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$, where $\mathbb{G} = (\mathcal{G}_t)_{t \in [0,T]} \subseteq \mathcal{G}$ is a filtration satisfying the usual assumptions. Here $\mathcal{G}_0$ is assumed to be trivial. We denote by τ^B, resp. by τ^C, the *time of default* of the bank, resp. of the counterparty.

Remark 2.1. Unless otherwise stated, throughout the paper we assume the bank's perspective and refer to the bank as the *hedger.*

We assume that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is a reference filtration satisfying the usual hypotheses and $\mathbb{H} = \mathbb{H}^B \vee \mathbb{H}^C$, with $\mathbb{H}^j = \left(\mathcal{H}_t^j\right)_{t \in [0,T]}$ for $\mathcal{H}_t^j = \sigma\left(H_u |\, u \le t\right)$, and $H_t^j := 1_{\{\tau^j \le t\}}$, $j \in \{B, C\}$. We set

$$\tau = \tau^C \wedge \tau^B. \tag{2.1}$$

Remark 2.2. We use the following conventions: $x^+ := \max\{x, 0\}$, $x^- := \max\{-x, 0\}$ so that $x = x^+ - x^-$. Note that this is in contrast to the convention adopted e.g. in Burgard and Kjaer (2011, 2013).

In the present paper we will extensively make use of the so called *Immersion Hypothesis.*

Hypothesis 2.3. *Any local* $(\mathbb{F}, \mathbb{P})$*-martingale is a local* $(\mathbb{G}, \mathbb{P})$*-martingale.*

We introduce some useful spaces of processes.

Definition 2.4. *Let* $\mathbb{Q}$ *be a probability measure on* $(\Omega, \mathcal{G})$*. The subspace of all* $\mathbb{R}^d$*-valued,* $\mathbb{F}$*-adapted processes* X *such that*

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \|X_t\|^2\, dt\right] < \infty \tag{2.2}$$

is denoted by $\mathcal{H}^{2,d}(\mathbb{Q})$*. We set* $\mathcal{H}^2(\mathbb{Q}) := \mathcal{H}^{2,1}(\mathbb{Q})$*.*
The subspace of all $\mathbb{R}^d$*-valued, continuous* $\mathbb{F}$*-adapted processes* X *such that*

$$\mathbb{E}^{\mathbb{Q}}\left[\sup_{t \in [0,T]} \|X_t\|^2\right] < \infty \tag{2.3}$$

is denoted by $\mathcal{S}^{2,d}(\mathbb{Q})$*. We set* $\mathcal{S}^2(\mathbb{Q}) := \mathcal{S}^{2,1}(\mathbb{Q})$*.*

2.1. **Basic traded assets.**

2.1.1. *Risky assets.* For $d \ge 1$, we denote by S^i, $i = 1, \dots, d$ the *ex-dividend price* (i.e. the price) of risky securities with associated *cumulative dividend processes* D^i. All S^i are assumed to be càdlàg $\mathbb{F}$-semimartingales, while the cumulative dividend streams D^i are $\mathbb{F}$-adapted processes of finite variation with $D_0^i = 0$.

Let $W^{\mathbb{P}} = \left(W_t^{\mathbb{P}}\right)_{t \in [0,T]}$ be a d-dimensional $(\mathbb{F}, \mathbb{P})$-Brownian motion (hence a $(\mathbb{G}, \mathbb{P})$-Brownian motion, thanks to Hypothesis 2.3). We introduce the following coefficient functions:

$$\begin{aligned} \mu &: \left(\mathbb{R}_+ \times \mathbb{R}^d, \mathcal{B}\left(\mathbb{R}_+ \times \mathbb{R}^d\right)\right) \mapsto \left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right), \\ \sigma &: \left(\mathbb{R}_+ \times \mathbb{R}^d, \mathcal{B}\left(\mathbb{R}_+ \times \mathbb{R}^d\right)\right) \mapsto \left(\mathbb{R}^{d\times d}, \mathcal{B}\left(\mathbb{R}^{d\times d}\right)\right), \\ \kappa &: \left(\mathbb{R}_+ \times \mathbb{R}^d, \mathcal{B}\left(\mathbb{R}_+ \times \mathbb{R}^d\right)\right) \mapsto \left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right), \end{aligned} \tag{2.4}$$

which are assumed to satisfy standard conditions ensuring existence and uniqueness of strong solutions of SDEs driven by the Brownian motion $W^{\mathbb{P}}$. The matrix process σ is assumed to be invertible at every point in time. We assume that

$$\begin{cases} dS_t &= \mu(t, S_t)dt + \sigma(t, S_t)dW_t^{\mathbb{P}} \\ S_0 &= s_0 \in \mathbb{R} \end{cases} \tag{2.5}$$

on $[0, T]$. Note that we are not postulating that the processes S^i are positive. The dividend processes D^i are specified via

$$(D_t^1, \ldots, D_t^d)^\top = \int_0^t \kappa(u, S_u)du, \ t \in [0, T], \tag{2.6}$$

for κ given in (2.4) such that $\int_0^T |\kappa(u, S_u)|du < \infty$ $\mathbb{P}$-a.s.
Throughout the paper we assume that the market is complete for the sake of simplicity.

2.1.2. *Cash accounts.* We assume the existence of an indexed family of cash accounts $(B^x)_{x \in \mathcal{I}}$, where the stochastic process $r^x := (r_t^x)_{t \geq 0}$ is lower bounded, right-continuous and $\mathbb{F}$-adapted for all $x \in \mathcal{I}$. The set of indices $\mathcal{I}$ embodies the type of agreement the counterparties establish in order to mitigate the counterparty credit risk. We will specify the characteristics of the aforementioned indices later on. All cash accounts, with unitary value at time 0, are assumed to be strictly positive continuous processes of finite variation of the form

$$B_t^x := \exp\left\{\int_0^t r_s^x ds\right\}, \ t \in [0, T]. \tag{2.7}$$

In particular, $B^x := (B_t^x)_{t \in [0,T]}$ is also continuous and adapted for all $x \in \mathcal{I}$.

2.1.3. *Defaultable bonds.* Default times are assumed to be exponentially distributed random variables with time-dependent intensity

$$\Gamma_t^j = \int_0^t \lambda_s^j ds, \ j \in \{B, C\}, \ t \in [0, T],$$

where λ^j are non-negative measurable bounded deterministic functions such that

$$\int_0^T \lambda_s^j ds < \infty, \forall\, t \geq 0, \ j \in \{B, C\}.$$

We introduce two risky bonds with maturity $T^\star \leq T$ and rate of return $r^j + \lambda^j$, issued by the bank and the counterparty, with dynamics

$$dP_t^j = \left(r_t^j + \lambda_t^j\right) P_t^j dt - P_{t-}^j dH_t^j, \ P_0^j = e^{-\int_0^{T^\star \wedge \tau^j} (r_u^j + \lambda_u^j)du}, \ j \in \{B, C\}. \tag{2.8}$$

2.2. **Repo trading.** In line with the existing literature, we assume that the trading activity on the risky assets is *collateralized.* This means that borrowing and lending activities related to risky

securities are financed via security lending or *repo market.* We refer to Bichuch et al. (2018) for an illustration of cash-driven and security driven repo transactions. Since transactions on the repo market are collateralized by the risky assets, repo rates are lower than unsecured funding rates. As argued in Crépey (2015a), assuming that all assets are traded via repo markets is not restrictive.
We let $B^1, \dots, B^d$ be the cash accounts associated to the risky assets $S^1, \dots, S^d$.
In case that the transactions are fully collateralized, this translates in the following equality

$$\xi^i_t S^i_t + \psi^i_t B^i_t = 0,\ i = 1 \dots, d,\ t \in [0, T]. \tag{2.9}$$

Remark 2.5. It is worth noting that ξ^i_t, $i = 1, \dots, d$, may be either positive or negative. Here $\xi^i_t > 0$ means that we are in a long position, which has to be financed by collateralization. On the other hand, $\xi^i_t < 0$ implies that the i-th asset is shorted, so that the whole amount of collateral is deposited in the riskless asset.

Remark 2.6. Condition (2.9) plays an important role in precluding trivial arbitrage opportunities among different cash accounts. In fact, by assuming $\xi^i_t = 1$ for all $i = 1, \dots, d$ in (2.9), the *gain process* $G_t = (G_t)_{t \in [0,T]}$ has dynamics

$$dG_t = r^i_t \psi^i_t B^i_t dt + dS^i_t + dD^i_t = dS^i_t - r^i_t S^i_t dt + dD^i_t, \tag{2.10}$$

where D^i, $i = 1, \dots, d$, is defined in (2.6).

Analogously, positions in the bonds satisfy the following condition

$$\xi^j_t P^j_t + \psi^j_t B^j_t = 0,\ j \in \{B, C\},\ t \in [0, T]. \tag{2.11}$$

It is worth noting that in (2.9) and (2.11) ξ, ψ serve as portion of traded securities constituting the portfolio. The properties of such processes will be pointed out in Section 3.

2.3. **Unsecured funding account.** Within the bank, the trading desk borrows and lends money from/to the treasury desk. Borrowing and lending rates are allowed to differ, hence we denote by $r^{f,b}, r^{f,l}$ the rate at which the trading desk borrows from and lends to the treasury desk, respectively. Recalling the notation given in (2.7), we introduce the associated cash accounts $B^{f,b}, B^{f,l}$ and set

$$r^f_t := r^{f,b}_t 1_{\{\psi^f = \psi^{f,b} < 0\}} + r^{f,l}_t 1_{\{\psi^f = \psi^{f,l} > 0\}},\ B^f_t := \exp\left\{\int_0^t r^f_u du\right\}, \tag{2.12}$$

for $t \in [0, T]$. This means that if the position of the trading desk is negative, i.e. $\psi^f = \psi^{f,b} < 0$, the trading desk borrows from the treasury desk at the rate $r^{f,b}$. Conversely, if the position of the trading desk is positive, i.e. $\psi^f = \psi^{f,l} > 0$, the trading desk lends money to the treasury desk with remuneration $r^{f,l}$.

Remark 2.7. It is worth observing that simultaneously borrowing and lending from the treasury desk is precluded, so we set $\psi^{f,l}_t \psi^{f,b}_t = 0$ for all $t \in [0, T]$.

2.4. **Collateralization.** In the financial jargon, a *margin* represents an economic value, either in the form of cash or risky securities, exchanged between the counterparties of a financial transaction, in order to reduce their outstanding risk exposures. In line with the market practice, we distinguish between *initial margin* and *collateral* (or *variation margin*), that we present in what follows.

2.4.1. *Variation margin.* A collateral is posted between the bank and the counterparty to mitigate counterparty risk. The collateral process $C = (C_t)_{t \in [0,T]}$ is assumed to be $\mathbb{G}$-adapted. We follow the convention of Bichuch et al. (2018) and Crépey (2015a):

- If $C_t > 0$, we say that the bank is the *collateral provider.* It means that the counterparty measures a positive exposure towards the bank, so it is a potential lender to the bank, hence the bank provides/lends collateral to reduce its exposure.
- If $C_t < 0$, we say that the bank is the *collateral taker.* It means that the bank measures a positive exposure towards the counterparty, so it is a potential lender to the counterparty, hence the counterparty provides/lends collateral to reduce its exposure.

Let $\mathcal{V} = (\mathcal{V}_t)_{t\in[0,T]}$ be a generic $\mathbb{G}$-adapted process, representing either the value of the trade including counterparty risk and funding adjustments or the clean value process, as it will be clarified later on. We assume that $C_t := f(\mathcal{V}_t)$, $t \in [0,T]$, where $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function.

If there is a collateral agreement (or a multitude of agreements) between the bank and the counterparty, in evaluating portfolio dynamics we need to make a distinction between the value of the portfolio and the wealth of the bank, the two concepts being distinguished since the bank is not the legal owner of the collateral (prior to default).

In this paper collateral is always posted in the form of cash, in line with standard market practice. Moreover, we assume *rehypothecation,* meaning that the holder of collateral can use the cash to finance her trading activity. This is the opposite of *segregation,* where the received cash collateral must be kept in a separate account and can not be used to finance the purchase of assets.

In line with Section 2.3, we associate the following interest rates to the collateral account:

- $r^{c,l}$ with account $B^{c,l}$, representing the rate on the collateral amount *received* by the bank who posted collateral to the counterparty.
- $r^{c,b}$ with account $B^{c,b}$, representing the rate on the collateral amount *paid* by the bank who received collateral from the counterparty.

We simply set $r^c = r^{c,l} = r^{c,b}$ in case there is no bid-offer spread in the collateral rate. Possible choices for the collateral rate are e.g. EONIA for EUR trades, Fed Fund for USD and SONIA for GBP trades. Such rates are overnight rates with a negligible embedded risk component. The choice of such approximately risk-free rates as collateral rates is motivated by market consensus. However, two counterparties might enter a collateral agreement that involves a remuneration of collateral at any other risky rate of their choice. Here we do not assume any requirements on collateral rates. This allows us to cover the quite common situation where the collateral rate agreed between the two counterparties in the CSA is defined by including a real valued spread over some market publicly observed rate, e.g. EONIA $-50\,bps$, where *bps* stands for *basis points.*

For the collateral account we have the following equations:

(i) if $C_t > 0$, then the bank has lent $\psi_t^c = \psi_t^{c,l} < 0$ units of the collateral account to the counterparty, i.e.,

$$\psi_t^{c,l} B_t^{c,l} = -C_t^+,\ t \in [0,T]; \tag{2.13}$$

(ii) if $C_t < 0$, then the bank has borrowed $\psi_t^c = \psi_t^{c,b} > 0$ units of the collateral account from the counterparty, i.e.,

$$\psi_t^{c,b} B_t^{c,b} = C_t^-,\ t \in [0,T]. \tag{2.14}$$

2.4.2. *Initial margin.* The collateralization represented by the variation margin is imperfect, due to the margin period of risk phenomenon: a defaulted counterparty stops posting collateral. However, bankruptcy procedure requires a certain time interval (typically 10 or 20 days) before the close-out payments are performed. This results in a period of time where the value of the transaction oscillates

in the absence of an adjustment of the collateral account, hence producing an exposure. This is one of the reasons for the introduction of initial margins, which constitutes a further form of collateral.
According to the EMIR regulation, starting from 2020, most agents participating in an OTC transaction will be forced to post initial margin, which constitutes an additional form of collateral. Initial margin, according to Garcia Trillos et al. (2016) is a misnomer, as an initial margin is not only initial, but it is periodically updated during the lifetime of the trade. It is *initial* in the sense that it is meant to provide a coverage from the initial point in time, where there is a default of the counterparty in a collateralized transaction.
It is important to stress that, differently from variation margin, an initial margin can not be rehypothecated, but it is instead segregated. From the point of view of the wealth dynamics, this means that initial margin received from the counterparty can not be used by the trading desk as a component of the value of the portfolio. However, the received initial margin represents a loan from the counterparty that must be remunerated, hence funding costs related to initial margin will appear in the self-financing condition, see Section 3.1 for further details.
We model initial margins with a $\mathbb{G}$-adapted process $I = (I_t)_{t\in[0,T]}$, and we denote by $B^{I,x}$, $x \in \{l, b\}$, the cash accounts associated to I. In particular, we write

$$I_t = I_t^{TC} - I_t^{FC},\ t \in [0, T],$$

where I^{TC} (TC = *to counterparty*) represents the initial margin posted by the bank to the counterparty and I^{FC} (FC = *from counterparty*) represents the initial margin posted by the counterparty to the bank. By using the same sign convention as for variation margin, we set

$$\psi_t^{I,l} B_t^{I,l} = -I_t^{TC},\ t \in [0, T], \tag{2.15}$$

or equivalently

$$-\psi_t^{I,l} dB_t^{I,l} = r_t^{I,l} I_t^{TC} dt\ , \tag{2.16}$$

and

$$\psi_t^{I,b} B_t^{I,b} = I_t^{FC},\ t \in [0, T], \tag{2.17}$$

i.e.,

$$\psi_t^{I,b} dB_t^{I,b} = r_t^{I,b} I_t^{FC} dt\ . \tag{2.18}$$

We highlight that, contrary to the case of variation margin and in line with market practice, I^{TC} and I^{FC} are simultaneously active and do not net each other.
More precisely, initial margins are computed via stochastic processes with values in the space of risk measures, such as *value at risk* or *expected shortfall,* as we will specify in (3.34). Expected shortfall is a popular choice to compute the initial margin for credit derivatives, since it is a coherent risk measure. Recently, the *International Swaps and Derivatives Association* (ISDA) has proposed a novel methodology, the so called *Standard Initial Margin Model* (SIMM), see ISDA (2018). SIMM provides some standardized formulae to evaluate initial margin on non-cleared derivatives, based on using portfolio sensitivities instead of historical simulations.
From a computational point of view, the presence of a risk measure inside portfolio dynamics results in an increased complexity both from a theoretical and computational point of view: since xVA equations are generally solved by means of Monte Carlo simulation, a brute force computation of future initial margin profiles requires nested historical simulation inside the risk neutral forward Monte Carlo simulation. There is a significant stream of research regarding efficient methodologies for the

estimation of future initial margin profiles, a popular technique being given by *adjoint algorithmic differentiation* (AAD), see e.g. Fries et al. (2018), Fries (2019b), Fries (2019a), Antonov et al. (2017), Henrard (2017), Capriotti (2011) and references therein.
Such issues are beyond the scope of the present study. For our purposes, we assume that the initial margin is a given real-valued process which is regular enough to guarantee existence and uniqueness of the BSDEs we are going to consider.
Another peculiar feature of initial margins is that, in case the counterparty is a clearing house, then the bank is always initial margin provider, i.e. $I^{FC} = 0 \ d\mathbb{P} \otimes dt$-a.s.

2.5. **Contingent claims.** We introduce the process $A = (A_t)_{t \in [0,T]}$ representing the payment stream of a financial contract. The process A is assumed to be an $\mathbb{F}$-adapted càdlàg process of finite variation, as in Crépey (2015b). We use the notation $\Delta A_t := A_t - A_{t-}$ for the jumps of A.
The following assumption will be useful later on.

Assumption 2.8. *Assume that $A \in \mathcal{S}^2(\mathbb{Q})$ and $A_T \in L^2(\mathcal{F}_T, \mathbb{Q})$.*

Remark 2.9. Differently from the standard literature, see e.g. Agarwal et al. (2018), we require stronger integrability conditions for the process A, because of the possible presence of jumps.

To include the more general case in which the presence of default events is assumed, we define the process $\bar{A} = \left(\bar{A}_t\right)_{t \in [0,T]}$ by setting

$$\bar{A}_t := 1_{\{t<\tau\}} A_t + 1_{\{t \geq \tau\}} A_{\tau-} \ , \tag{2.19}$$

where we recall that $\tau := \tau^C \wedge \tau^B$.

2.6. **The close-out condition.** In case of default, cashflows are exchanged between the surviving agent and the liquidators of the defaulted agent. Here we use the term *agent* as a placeholder for the bank or for the counterparty. Due to the exchange of cashflows at default time, agents need to perform a valuation of the position at a random time. The object of the analysis can be the value in the absence of counterparty risk (referred to in the literature as *risk-free close-out*) or the value of the trade including the price adjustments due to counterparty risk and funding (*risky close-out*), see e.g. Brigo and Morini (2018). A risky close-out condition guarantees that the surviving counterparty can ideally fully substitute the transaction with a new trade entered with another counterparty with the same credit quality. This comes at the price of a significant increase of the complexity of the valuation equations. Market practice and the existing literature mainly focus on the estimation of the risk-free close-out value.
We now provide the definition of *close-out condition,* in line with Bichuch et al. (2018) Section 3.4.

Definition 2.10. *Let $0 < R^j < 1, \ j \in \{B, C\}$, be the recovery rates of the bank and the counterparty, respectively. The close-out condition $\theta_\tau(\mathcal{V}, C, I)$, expressed from the bank's perspective, is defined by*

$$\begin{aligned}\theta_\tau(\mathcal{V}, C, I) &:= \mathcal{V}_\tau + \Delta A_\tau + 1_{\{\tau^C < \tau^B\}}(1 - R^C)\left(\mathcal{V}_\tau + \Delta A_\tau - C_{\tau-} + I^{FC}_{\tau-}\right)^- \\ &\quad - 1_{\{\tau^B < \tau^C\}}(1 - R^B)\left(\mathcal{V}_\tau + \Delta A_\tau - C_{\tau-} - I^{TC}_{\tau-}\right)^+ .\end{aligned} \tag{2.20}$$

The interpretation of $\theta_\tau(\mathcal{V}, C, I)$ is in line with Bichuch et al. (2018) Remark 3.3.

Remark 2.11. Eq. (2.20) already encodes two terms giving rise to the *credit valuation adjustment* (CVA) and *debt valuation adjustment* (DVA), that we will define in details in Section 3.

3. Single aggregation level xVA framework

3.1. **Trading strategies and the self-financing property.** In this section we proceed to adapt classical concepts relating to contingent claim valuation in the present multiple-curve and defaultable setting. We define the concept of self-financing trading strategies in this context. In the following section we then address the issue of viability of the unextended market model featuring only the basic traded assets, thus excluding trading on the contingent claim with dividend process $\bar{A}$.

Definition 3.1. *A dynamic portfolio, denoted by φ, is given by*

$$\varphi = \left(\xi^1, \dots, \xi^d, \xi^B, \xi^C, \psi^1, \dots, \psi^d, \psi^B, \psi^C, \psi^{f,b}, \psi^{f,l}, \psi^{c,b}, \psi^{c,l}, \psi^{I,b}, \psi^{I,l}\right),$$

where

(i) $\xi^1, \dots, \xi^d$ *are $\mathbb{G}$-predictable processes, denoting the number of shares of the risky primary assets $S^1, \dots, S^d$.*
(ii) ξ^B, ξ^C *are $\mathbb{G}$-predictable processes, denoting the number of shares of the risky bonds P^B and P^C.*
(iii) $\psi^1, \dots, \psi^d, \psi^B, \psi^C$ *are $\mathbb{G}$-adapted processes, denoting the number of shares of the repo accounts $B^1, \dots B^d, B^B, B^C$.*
(iv) $\psi^{f,b}$ *is a $\mathbb{G}$-adapted process, denoting the number of shares of the unsecured funding borrowing cash account $B^{f,b}$.*
(v) $\psi^{f,l}$ *is a $\mathbb{G}$-adapted process, denoting the number of shares of the unsecured funding lending cash account $B^{f,l}$.*
(vi) $\psi^{c,b}$ *is a $\mathbb{G}$-adapted process, denoting the number of shares of the collateral borrowing cash account $B^{c,b}$ for the received cash collateral.*
(vii) $\psi^{c,l}$ *is a $\mathbb{G}$-adapted process, denoting the number of shares of the collateral lending cash account $B^{c,l}$ for the posted cash collateral.*
(viii) $\psi^{I,b}$ *is a $\mathbb{G}$-adapted process, denoting the number of shares of the initial margin borrowing cash account $B^{I,b}$ for the initial margin received from the counterparty.*
(ix) $\psi^{I,l}$ *is a $\mathbb{G}$-adapted process, denoting the number of shares of the initial margin lending cash account $B^{I,l}$ for the initial margin posted to the counterparty.*

All processes introduced above are such that the stochastic integrals in the sequel are well defined.

Given a dynamic portfolio, we associate it to a financial contract, known in the literature as *Credit Support Annex (CSA),* see e.g. BCBS (2014).

Definition 3.2. *A CSA between the bank and the counterparty is represented through the pair (C, I), where C is the variation margin and I is the initial margin.*

Definition 3.3. *A* collateralized hedger's trading strategy *associated to the collateralized contract $\bar{A}$ and the CSA (C, I) is a quintuplet $\left(x, \varphi, \bar{A}, C, I\right)$, where $x \in \mathbb{R}$ is the initial endowment and φ is a dynamic portfolio.*

We can define the *wealth process* associated to a collateralized hedger's trading strategy $\left(x, \varphi, \bar{A}, C, I\right)$ as follows.

Definition 3.4. *The* wealth process *(or* value process*)* $V(\varphi) = (V_t(\varphi))_{t\in[0,T]}$ *associated to a collateralized hedger's trading strategy* $\left(x, \varphi, \bar{A}, C, I\right)$ *is given by*

$$
\begin{aligned}
V_t(\varphi) :=& \sum_{i=1}^{d} \left(\xi_t^i S_t^i + \psi_t^i B_t^i\right) + \sum_{j\in\{B,C\}} \left(\xi_t^j P_t^j + \psi_t^j B_t^j\right) + \psi_t^{f,b} B_t^{f,b} + \psi_t^{f,l} B_t^{f,l} \\
&- \left(\psi_t^{c,b} B_t^{c,b} + \psi_t^{c,l} B_t^{c,l} + \psi_t^{I,l} B_t^{I,l}\right) .
\end{aligned} \tag{3.1}
$$

Remark 3.5. The sign minus in (3.1) in front of the last term depends on our convention on the collateral.

Note that in (3.1) we are not including the cash account for the received initial margin. This is due to the fact that the received initial margin is posted in a segregated account and, hence, is not available as a funding asset to the trading desk. However, the received initial margin will generate funding costs that will appear in the self-financing condition we are going to introduce.

Definition 3.6. *Given the initial endowment* x*, a collateralized hedger's trading strategy* $\left(x, \varphi, \bar{A}, C, I\right)$ *associated to the collateralized contract* $\bar{A}$ *and the CSA* (C, I) *is said to be* self-financing *if, for any* $t \in [0, T]$*, the wealth process* $V_t(\varphi)$ *satisfies*

$$
\begin{aligned}
V_t(\varphi) =& x + \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i \left(\mu^i(u, S_u)du + \sum_{k=1}^{d} \sigma^{i,k}(u, S_u) dW_u^{k,\mathbb{P}} + \kappa^i(u, S_u)du\right) \\
&+ \sum_{i=1}^{d} \int_0^t \psi_u^i dB_u^i + \sum_{j\in\{B,C\}} \int_0^t \left(\xi_u^j dP_u^j + \psi_u^j dB_u^j\right) - \bar{A}_t \\
&+ \int_0^t \psi_u^{f,b} dB_u^{f,b} + \int_0^t \psi_u^{f,l} dB_u^{f,l} - \int_0^t \psi_u^{c,b} dB_u^{c,b} - \int_0^t \psi_u^{c,l} dB_u^{c,l} - \int_0^t \psi_u^{I,b} dB_u^{I,b} - \int_0^t \psi_u^{I,l} dB_u^{I,l}.
\end{aligned} \tag{3.2}
$$

The last two terms in (3.2) represent the cash for the received initial margin. In general, we assume zero initial endowment, $x = 0$, i.e., $V_t(\varphi) = V_t\left(0, \varphi, \bar{A}, C, I\right)$ for the sake of simplicity.

Definition 3.7. *A collateralized hedger's trading strategy is* admissible *if it is self-financing and the associated value process* $V(\varphi)$ *is bounded from below.*

Following Definition 5.1 in Bielecki and Rutkowski (2015) and the discussion thereafter, we can give the definition of *replicating strategy.*

Definition 3.8. *A self-financing collateralized hedger's trading strategy* $(0, \varphi, \bar{A}, C, I)$ *is said to* replicate *the collateralized contract* $\bar{A}$ *if* $V_{\hat{\tau}}(\varphi) = 0$*, where* $\hat{\tau} := \tau \wedge T$.

3.2. **Absence of arbitrage.** We provide the following definition of *arbitrage-free strategy.*

Definition 3.9. *Let the assumptions of Section 2 be in force. Then, the market is* arbitrage-free *if, for* $(0, \varphi, 0, 0, 0)$*, we have either*

$$\mathbb{P}\left[V_{\hat{\tau}}\left(0, \varphi, 0, 0, 0\right) = 0\right] = 1 \;\; \textit{or} \;\; \mathbb{P}\left[V_{\hat{\tau}}\left(0, \varphi, 0, 0, 0\right) < 0\right] > 0,$$

for some stopping time $\hat{\tau} > 0$*, where* $\hat{\tau}$ *is introduced in Def. 3.8.*

Remark 3.10. In Bielecki and Rutkowski (2015, Definition 3.3) the authors introduce the concept of a market which is said to be *arbitrage-free for the hedger* with respect to a class of contingent claims. Their definition is formulated in terms of a *netted wealth process*, which corresponds to a long-short strategy involving the claim $\bar{A}$, where the first position is hedged and the second is unhedged. On the other hand, in Bichuch et al. (2018) the question concerning absence of arbitrage is first answered in a setting where only the basic traded assets are considered. This is also referred to as *absence of arbitrage with respect to the null contract* in Bielecki et al. (2018). In our setting, the two approaches coincide.

We restate, in our notations, an analog of Assumption 4.2 from Bichuch et al. (2018).

Assumption 3.11. *We assume r^f_t bounded from below and $r^{f,l}_t \leq r^{f,b}_t$, $d\mathbb{P}\otimes dt$-a.s.*

Unlike Bichuch et al. (2018), we do not impose constraints between the unsecured funding rate and the returns of the risky bonds, since such securities are traded via repo markets. If the positions on the risky bonds were financed via unsecured funding, then we would need the same sort of restrictions between the rates, i.e., we would need to impose the assumption

$$r^{f,l}_t \leq (r^B_t + \lambda^B_t) \wedge (r^C_t + \lambda^C_t),\ t \in [0,T], \mathbb{P}-a.s.$$

Such assumptions would exclude the possibility for the trading desk to create trivial arbitrages between the unsecured funding accounts and the risky bonds.
To prove the absence of arbitrage for non-collateralized, non-defaultable contracts we define the *cumulative dividend price* process, see e.g. Bielecki and Rutkowski (2015).

Definition 3.12. *The cumulative dividend price associated to the i-th asset is given by*

$$S^{i,cld}_t := S^i_t + B^i_t \int_{(0,t]} \frac{dD^i_u}{B^i_u},\quad i = 1,\dots d\ ,\ t \in [0,T]. \tag{3.3}$$

Proposition 3.13. *Let Assumption 3.11 hold. Moreover, assume that $r^{f,l}_t \geq r^i_t$, $i = 1,\dots,d$, $r^{f,l}_t \geq r^j_t$, $j \in \{B, C\}$, $\mathbb{P}$-a.s., for all $t \in [0,T]$, and that there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that the discounted asset price processes*

$$\tilde{S}^{i,cld}_t := \frac{S^{i,cld}_t}{B^i_t},\ i = 1,\dots d, \qquad \tilde{P}^j_t := \frac{P^j_t}{B^j_t},\ j \in \{B, C\}, \tag{3.4}$$

are local martingales. Then, the market consisting of the basic traded assets $(0, \varphi, 0, 0, 0)$ is free of arbitrage opportunities.

Proof. Since we are only trading in the basic risky assets, the position in the initial margin is zero hence, by (2.9) and (3.1), the value process is of the form

$$V_t(\varphi) = \psi^{f,b}_t B^{f,b}_t + \psi^{f,l}_t B^{f,l}_t,\ t \in [0,T]. \tag{3.5}$$

Recalling that simultaneous borrowing and lending at the same time is not allowed, we have by (3.5) that

$$\psi^{f,l} = (V_t(\varphi))^+ \left(B^{f,l}_t\right)^{-1},\ \psi^{f,b} = -\left(V_t(\varphi)\right)^- \left(B^{f,b}_t\right)^{-1},\ t \in [0,T].$$

Moreover, we can rewrite the funding term of the generic i-th risky assets as follows

$$\int_0^t \psi^i_u dB^i_u = -\int_0^t \frac{\xi^i_u S^i_u}{B^i_u} dB^i_u = -\int_0^t r^i_u \xi^i_u S^i_u du,\ t \in [0,T].$$

Upon substitution in the self-financing condition (3.2), we obtain

$$dV_t(\varphi) = \sum_{i=1}^d \xi^i_t \left(dS^i_t + dD^i_t - r^i_t S^i_t dt\right) + \sum_{j\in\{B,C\}} \xi^j_t \left(dP^j_t - r^j_t P^j_{t-} dt\right) - r^{f,b}_t \left(V_t(\varphi)\right)^- dt + r^{f,l}_t \left(V_t(\varphi)\right)^+ dt.$$

We now use the inequality $r^{f,l}_t \leq r^{f,b}_t$ from Assumption 3.11, hence

$$
\begin{aligned}
dV_t(\varphi) &= \sum_{i=1}^{d} \xi^i_t \left(dS^i_t + dD^i_t - r^i_t S^i_t dt \right) + \sum_{j\in\{B,C\}} \xi^j_t \left(dP^j_t - r^j_t P^j_{t-} dt \right) + r^{f,l}_t \left(V_t(\varphi)\right)^+ dt - r^{f,b}_t \left(V_t(\varphi)\right)^- dt \\
&\leq \sum_{i=1}^{d} \xi^i_t \left(dS^i_t + dD^i_t - r^i_t S^i_t dt \right) + \sum_{j\in\{B,C\}} \xi^j_t \left(dP^j_t - r^j_t P^j_{t-} dt \right) + r^{f,l}_t V_t(\varphi) dt.
\end{aligned}
$$

Introducing $\tilde{V}^l_t(\varphi) := \left(B^{f,l}_t\right)^{-1} V_t(\varphi)$, we have then the inequality

$$
d\tilde{V}^l_t(\varphi) \leq \sum_{i=1}^{d} \xi^i_t \left(B^{f,l}_t\right)^{-1} \left(dS^i_t + dD^i_t - r^i_t S^i_t dt \right) + \sum_{j\in\{B,C\}} \xi^j_t \left(B^{f,l}_t\right)^{-1} \left(dP^j_t - r^j_t P^j_{t-} dt \right)
$$

or equivalently,

$$
d\tilde{V}^l_t(\varphi) \leq \sum_{i=1}^{d} \xi^i_t \frac{B^i_t}{B^{f,l}_t} \frac{\left(dS^i_t + dD^i_t - r^i_t S^i_t dt \right)}{B^i_t} + \sum_{j\in\{B,C\}} \xi^j_t \frac{B^j_t}{B^{f,l}_t} \frac{\left(dP^j_t - r^j_t P^j_{t-} dt \right)}{B^j_t},
$$

and so, by (3.4), we arrive at the inequality

$$
d\tilde{V}^l_t(\varphi) \leq \sum_{i=1}^{d} \xi^i_t \frac{B^i_t}{B^{f,l}_t} d\tilde{S}^{i,cld}_t + \sum_{j\in\{B,C\}} \xi^j_t \frac{B^j_t}{B^{f,l}_t} d\tilde{P}^j_t. \tag{3.6}
$$

We observe that the right-hand side in (3.6) is a local martingale, which is bounded from below, by Definition 3.7 and Assumption 3.11 on r^f. This implies that the aforementioned right-hand side is a supermartingale. Absence of arbitrage follows along the usual lines. □

From now on, we assume the following.

Assumption 3.14. *There exists an equivalent martingale probability measure $\mathbb{Q} \sim \mathbb{P}$ under which the processes $\tilde{S}^{i,cld}_t$, $\tilde{P}^j_t$ in (3.4) are local martingales with dynamics*

$$
\begin{aligned}
d\tilde{S}^{i,cld}_t &= \frac{1}{B^i_t} \left(dS^i_t - r^i_t S^i_t dt + dD^i_t \right) = \sum_{k=1}^{d} \frac{\sigma^{i,k}(t,S_t)}{B^i_t} dW^{k,\mathbb{Q}}_t, \ i = 1, \ldots, d, \\
d\tilde{P}^j_t &= \frac{1}{B^j_t} \left(dP^j_t - r^j_t P^j_{t-} dt \right) = -\tilde{P}^j_{t-} dM^{j,\mathbb{Q}}_t, \ j \in \{B, C\}.
\end{aligned}
$$

More precisely, we assume the existence of an equivalent probability measure $\mathbb{Q} \sim \mathbb{P}$ with Radon-Nikodym density

$$
\left.\frac{\partial \mathbb{Q}}{\partial \mathbb{P}}\right|_{\mathcal{G}_t} = \mathcal{E}\left(\int_0^{\cdot} \beta_s dW^{\mathbb{P}}_s \right) \prod_{j\,\in\,\{B,C\}} \exp\left\{ \int_0^t \ln\left(1 + \frac{r^f_s - r^j_s}{\lambda^{j,\mathbb{P}}_s} \right) dH^j_s - \int_0^t (r^f_s - r^j_s) ds \right\}, \ t \in [0,T], \tag{3.7}
$$

where the process $\beta = (\beta_t)_{t\,\in\,[0,T]}$, with $\beta_t := (\sigma(t,S_t))^{-1} \left(\mu(t,S_t) - r_t\right)$, is such that the stochastic exponential $\mathcal{E}\left(\int_0^{\cdot} \beta_s dW^{\mathbb{P}}_s\right)$ is a martingale and the dynamics of defaultable bonds under $\mathbb{Q}$ are given by

$$
dP^{j,\mathbb{Q}}_t = r^j_t P^j_t dt - P^j_{t-} dM^{j,\mathbb{Q}}_t, \ j \in \{B, C\},
$$

where the process

$$M_t^{j,\mathbb{Q}} = M_t^{j,\mathbb{P}} + \int_0^t (1 - H_u^j)(\lambda_u^{j,\mathbb{P}} - \lambda_u^{j,\mathbb{Q}})du, \ \ \lambda_t^{j,\mathbb{Q}} = r_t^f - \lambda_u^{j,\mathbb{P}} - r_t^j, \ t \in [0,T], \tag{3.8}$$

is a $(\mathbb{G}, \mathbb{Q})$-martingale, for $j \in \{B, C\}$. Then $\mathbb{Q}$ is an ELMM for the discounted asset price process in (3.4).

3.3. **Contingent claim valuation.** In this section we consider the problem of pricing and hedging a financial contract with payment stream $\bar{A}$. To this purpose, we first write a BSDE for the candidate value process V as a consequence of our assumptions so far. After that, we proceed to address the issue of existence and uniqueness for the solutions of such BSDEs. Finally, we discuss if the process V, emerging as solution to such BSDEs, provides us with an arbitrage free price for the contingent claim with dividend process $\bar{A}$.

Under Assumption 3.14 the dynamics of a self-financing collateralized trading strategy $\left(x, \varphi, \bar{A}, C, I\right)$ is

$$\begin{aligned} dV_t(\varphi) &= \sum_{i=1}^d \xi_t^i B_t^i d\tilde{S}_t^{i,cld} + \sum_{j \in \{B,C\}} \xi_t^j B_t^j d\tilde{P}_t^j - d\bar{A}_t \\ &\quad + \psi_t^{f,l} dB_t^{f,l} + \psi_t^{f,b} dB_t^{f,b} - \psi_t^{c,l} dB_t^{c,l} - \psi_t^{c,b} dB_t^{c,b} - \psi_t^{I,l} dB_t^{I,l} - \psi_t^{I,b} dB_t^{I,b} \,. \end{aligned} \tag{3.9}$$

By using the repo constraints (2.9), (2.13), (2.14) and (2.16), the portfolio value satisfies

$$V_t(\varphi) = \psi_t^{f,l} B_t^{f,l} + \psi_t^{f,b} B_t^{f,b} + C_t + I_t^{TC}, \ t \in [0,T],$$

since I^{FC} is segregated. Thanks to Remark 2.7 we obtain the identities

$$\psi_t^{f,l} = \left(V_t(\varphi) - C_t - I_t^{TC}\right)^+ \left(B_t^{f,l}\right)^{-1}, \tag{3.10}$$

$$\psi_t^{f,b} = -\left(V_t(\varphi) - C_t - I_t^{TC}\right)^- \left(B_t^{f,b}\right)^{-1} \tag{3.11}$$

for $t \in [0,T]$. Observe that by (2.13) and (2.14)

$$-\psi_t^{c,l} dB_t^{c,l} = -\psi_t^{c,l} r_t^{c,l} B_t^{c,l} dt = +r_t^{c,l} C_t^+ dt, \tag{3.12}$$

$$-\psi_t^{c,b} dB_t^{c,b} = -\psi_t^{c,b} r_t^{c,b} B_t^{c,b} dt = -r_t^{c,b} C_t^- dt, \tag{3.13}$$

respectively. By (3.10), (3.11), (3.12), (3.13), (2.16) and (2.18), we can rewrite the wealth dynamics as follows

$$\begin{aligned} dV_t(\varphi) &= \sum_{i=1}^d \xi_t^i B_t^i d\tilde{S}_t^{i,cld} + \sum_{j \in \{B,C\}} \xi_t^j B_t^j d\tilde{P}_t^j - d\bar{A}_t \\ &\quad + \Big[r_t^{f,l} \left(V_t(\varphi) - C_t - I_t^{TC}\right)^+ - r_t^{f,b} \left(V_t(\varphi) - C_t - I_t^{TC}\right)^- \\ &\quad + r_t^{c,l} C_t^+ - r_t^{c,b} C_t^- + r_t^{I,l} I_t^{TC} - r_t^{I,b} I_t^{FC} \Big] dt. \end{aligned} \tag{3.14}$$

We now introduce for convenience an auxiliary artificial interest rate process $r = (r_t)_{t \in [0,T]}$, assumed to be right-continuous, bounded and $\mathbb{F}$-adapted. This rate is not necessarily linked to a traded asset, but it can be interpreted as an interest rate level, used to express all other rates as spreads over this artificial rate. When needed, we will explicitly state when the rate r becomes a market rate. Using the artificial rate r, we can conveniently rewrite the portfolio dynamics as follows

$$
\begin{aligned}
dV_t(\varphi) &= \sum_{i=1}^{d} \xi^i_t B^i_t d\tilde{S}^{i,cld}_t + \sum_{j\in\{B,C\}} \xi^j_t B^j_t d\tilde{P}^j_t - d\bar{A}_t \\
&\quad + \Big[(r^{f,l}_t - r_t) \left(V_t(\varphi) - C_t - I^{TC}_t\right)^+ - (r^{f,b}_t - r_t) \left(V_t(\varphi) - C_t - I^{TC}_t\right)^- \\
&\quad + (r^{c,l}_t - r_t) C^+_t - (r^{c,b}_t - r_t) C^-_t + (r^{I,l}_t - r_t) I^{TC}_t - r^{I,b}_t I^{FC}_t + r_t V_t(\varphi) \Big] dt \ ,
\end{aligned} \tag{3.15}
$$

where we added and subtracted the term $r_t V_t(\varphi) dt$.

Remark 3.15. The term $(r^{I,l}_t - r_t) I^{TC}_t$ measures a funding benefit from the posted initial margin over the reference rate level r. We would like to stress that, in general, spreads over r can be negative, representing that we may have funding costs, even when the bank is collateral provider. Such a situation is faced by banks, which clear swaps with the *London Clearing House* (LCH). If r is chosen to represent the EONIA overnight rate, then the rate applied by LCH is $r^{I,l} = r - 58bps$, where bps stands for basis points[1]. On top of such a negative benefit, the bank needs to take into account the cost of raising the amount I^{TC}, hence initial margin can generate funding costs in both directions, from the point of view of fund-raising and from the point of view of collateral remuneration, hence representing a significant source of costs for the bank.

We restate the portfolio dynamics in the form of a BSDE under the enlarged filtration $\mathbb{G}$. We set

$$
Z^k_t := \sum_{i=1}^{d} \xi^i_t \sigma^{i,k}(t, S_t), \tag{3.16a}
$$

$$
U^j_t := -\xi^j_t P^j_{t-}, \tag{3.16b}
$$

$$
\begin{aligned}
f(t, V, C, I) := &- \Big[(r^{f,l}_t - r_t) \left(V_t(\varphi) - C_t - I^{TC}_t\right)^+ - (r^{f,b}_t - r_t) \left(V_t(\varphi) - C_t - I^{TC}_t\right)^- \\
&+ (r^{c,l}_t - r_t) C^+_t - (r^{c,b}_t - r_t) C^-_t + (r^{I,l}_t - r_t) I^{TC}_t - r^{I,b}_t I^{FC}_t \Big] .
\end{aligned} \tag{3.16c}
$$

The full contract $\mathbb{G}$-BSDE for the portfolio's dynamics has then the form on $\{\tau > t\}$

$$
\begin{cases}
-dV_t(\varphi) = d\bar{A}_t + \left(f(t, V, C, I) - r_t V_t(\varphi)\right) dt - \sum_{k=1}^{d} Z^k_t dW^{k,\mathbb{Q}}_t - \sum_{j\in\{B,C\}} U^j_t dM^{j,\mathbb{Q}}_t \\
V_\tau(\varphi) = \theta_\tau(\mathcal{V}, C, I).
\end{cases} \tag{3.17}
$$

We prove in Theorem 3.30 that there exists a unique solution (V, Z, U) for the $\mathbb{G}$-BSDE (3.17), and the process V assumes the following form on $\{\tau > t\}$

$$
V_t(\varphi) = B^r_t \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,\tau\wedge T]} \frac{d\bar{A}_u}{B^r_u} + \int_t^{\tau\wedge T} \frac{f(u, V, C, I)}{B^r_u} du + 1_{\{\tau\le T\}} \frac{\theta_\tau(\mathcal{V}, C, I)}{B^r_\tau} \middle| \mathcal{G}_t \right], \tag{3.18}
$$

where $B^r_t := \exp\left(\int_0^t r_u du\right)$, $t \in [0, T]$.

Remark 3.16. Our BSDE formulation (3.17) is in line with Definition 1.2 in Crépey (2015b) with hedging error term identically zero, but with a specific choice of the driving martingales, given by the Brownian motions $\left(W^{1,\mathbb{Q}}, \dots, W^{d,\mathbb{Q}}\right)^\top$ and the compensated jump processes $M^{j,\mathbb{Q}}$, $j \in \{C, B\}$. As in Crépey (2015b), the full BSDE is defined up to a random time horizon. The formulation can be simplified by obtaining the equivalent $\mathbb{F}$-BSDE by means of the Hypothesis 2.3 between $\mathbb{F}$ and $\mathbb{G}$. The close-out condition is still expressed in terms of the general value process $\mathcal{V}$. We characterize the process $\mathcal{V}$ in Subsection 3.3.1.

[1]see https://www.lch.com/risk-collateral-management/ltd-collateral-management/ltd-fees-collateral

3.3.1. *Clean Value under* $\mathbb{F}$. A financial product can be traded between any two counterparties. Since every agent has a different credit quality and different funding costs, this means in general that a single product (e.g. a 10 year EUR swap) has as many potential values as the number of possible combinations of agents in the market. It would be highly impractical for a broker to publish all possible market quotes for all possible counterparties. In fact, when we look at market quotes, we typically see a single value (more precisely a bid and offer price). Such quotes are clean prices, i.e., they do not represent real market prices.
A *clean price* is an ideal value process that would be acceptable between two agents entering a perfectly collateralized transaction. Perfect collateralization however is not enough to produce a clean price: we also need to explicitly assume that the two agents entering the transaction are default-free. This is necessary because, even in the presence of a perfect ideal collateral agreement, counterparty risk is not perfectly annihilated: when a counterparty defaults, she stops posting collateral.
However, default is not automatically legally recognized: typically, bankruptcy procedures require some days (e.g. 10 or 20 days) before the close-out payments are exchanged. This creates a period of time where the counterparty is not officially defaulted but without any collateral adjustment. Such period of time is known as *margin period of risk.* During such interval of time the value of the claim deviates from the value of the collateral account thus creating a credit exposure.
Hence, to preclude margin period of risk and obtain the ideal clean price process, we need to consider a parallel fictious market, where there is perfect collateralization but no default risk.

Assumption 3.17 (Clean market)**.** *A clean market under* $\mathbb{F}$ *without bid-offer spreads is defined by*

(i) *no bid-offer spread in the funding accounts, i.e.,* $r_t^{f,l} = r_t^{f,b} = r^f$*;*
(ii) *no bid-offer spread in the collateral accounts, i.e.,* $r_t^{c,l} = r_t^{c,b} = r^c$*;*
(iii) *the collateral rate is equal to the fictious rate, i.e.,* $r^c = r$*;*
(iv) *there is no default, i.e.* $\hat{\tau} = T$*, and risky bonds are excluded from the market;*
(v) *there is no exchange of initial margin;*
(vi) *perfect collateralization, i.e.,* $\hat{V}_t \equiv C_t$*, for all* $t \in [0,T]$*, where we use* $\hat{V}$ *to denote the value process of a collateralized hedging strategy in the fictious market without default-risk.*

Note that *(vi)* in Assumption 3.17 implies that the portfolio weights in the cash accounts are of the form

$$\psi_t^c = -\frac{\hat{V}_t}{B_t^c}, \quad \psi_t^f \equiv 0, \text{ for all } t \in [0,T],$$

meaning that the position is totally funded by the collateralization scheme, and $\hat{V} = (\hat{V}_t)_{[0,T]}$ is an $\mathbb{F}$-adapted process.
The portfolio dynamics under $\mathbb{Q}$ resulting from (3.15) under Assumption 3.17 are given by

$$d\hat{V}_t(\varphi) = \sum_{k=1}^{d} \hat{Z}_t^k dW_t^{k,\mathbb{Q}} - dA_t + r_t \hat{V}_t(\varphi)dt, \text{ where } \hat{Z}_t^k := \sum_{i=1}^{d} \hat{\xi}_t^i \sigma^{i,k}(t, S_t). \tag{3.19}$$

Note the introduction in (3.19) of the $\mathbb{F}$-predictable processes $\hat{Z}^k$, $k = 1, \ldots, d$, that represent the hedging position only for the clean price process, opposed to the processes Z^k, $k = 1, \ldots, d$, from the full portfolio dynamics that represent hedging positions for the clean price and the value adjustments. Inserting the terminal condition $\hat{V}_T = 0$, we can rewrite (3.19) in the classical $\mathbb{F}$-BSDE form

$$\begin{cases} -d\hat{V}_t(\varphi) = dA_t - r_t\hat{V}_t(\varphi)dt - \sum_{k=1}^d \hat{Z}_t^k dW_t^{k,\mathbb{Q}} \\ \hat{V}_T(\varphi) = 0. \end{cases} \tag{3.20}$$

We now perform two different tasks. First, we show that, given the processes A and r, it is possible to find a family of control processes $\hat{Z}^k$, $k = 1, \dots, d$ and a process $\hat{V}$ satisfying the clean BSDE (3.20), i.e., we prove an existence and uniqueness result for the solution of (3.20). Then, we establish that the process $\hat{V}$ provides the arbitrage free clean price.

Theorem 3.18. *Under Assumption 2.8 on A, there exists a unique solution $\left(\hat{V}, \hat{Z}\right) \in \mathcal{S}^2(\mathbb{Q}) \times \mathcal{H}^{2,d}(\mathbb{Q})$ to the clean BSDE* (3.20)*.*

Proof. We note that the clean BSDE (3.20) is similar to the linear BSDE studied e.g. in El Karoui et al. (1997), where the driver is the multidimensional Brownian motion $\left(W^{1,\mathbb{Q}}, \dots, W^{d,\mathbb{Q}}\right)^\top$.
We can apply Theorem A.7 by observing that $M = W^{\mathbb{Q}}$, $Q_t = t$, $U = A$, $\hat{V} = Y$ and $h(t, Y_t, Z_t) = -r_t\hat{V}_t$, which clearly fulfills the uniform Lipschitz condition. Also the condition $h(\cdot, 0, 0) \in \mathcal{S}^2(\mathbb{Q})$ is trivially satisfied. We also observe that $\mathbb{X} = S = diag(S^1, \dots, S^d)$, hence we have $m_t = \sigma(t, S_t)$, so that $\gamma_t = S_1^{-1}\sigma(t, S_t)$, for γ satisfying the ellipticity condition (A.2). According to Theorem A.7 we have $\hat{V} \in \mathcal{H}^2(\mathbb{Q})$ and $\hat{V} - A \in \mathcal{S}^2(\mathbb{Q})$. Now, Assumption 2.8 allows us to conclude that also $\hat{V} \in \mathcal{S}^2$. □

Next we show that the process $\hat{V}$ in Theorem 3.18 provides the arbitrage-free price for the contract with cashflow stream A.

Theorem 3.19. *Let $\mathbb{Q} \sim \mathbb{P}$ be an equivalent probability measure such that all processes $\tilde{S}^{i,cld}$, $i = 1, \dots, d$, are local $\mathbb{Q}$-martingales. Let $\left(\hat{V}, \hat{Z}\right)$ be the unique solution of* (3.20)*. Then, under Assumption 2.8 on A, we have*

$$\hat{V}_t(\varphi) := \mathbb{E}^{\mathbb{Q}}\left[B_t^r \int_{(t,T]} \frac{dA_u}{B_u^r} \middle| \mathcal{F}_t \right], \textit{ for all } t \in [0, T]. \tag{3.21}$$

Proof. Let $\left(\hat{V}, \hat{Z}\right)$ be the solution of (3.20). Then, Theorem 3.18 ensures that $\hat{Z} \in \mathcal{H}^{2,d}(\mathbb{Q})$, which implies that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{\left(\hat{Z}_u^k\right)^2}{\left(B_u^r\right)^2} du \right] < \infty, \text{ for all } k = 1, \dots, d, \tag{3.22}$$

since B^r is bounded. By Assumption 2.8, it follows that

$$B_t^r \int_{(t,T]} \frac{dA_u}{B_u^r} \in L^1(\mathbb{Q}), \tag{3.23}$$

for $t \in [0, T]$, because B^r is bounded.
Thus, by (3.22) and (3.23) the rescaled process $\hat{V}^r := \hat{V}\,(B^r)^{-1}$ satisfies

$$-\hat{V}_t^r(\varphi) = -\int_{(t,T]} \frac{dA_u}{B_u^r} + \sum_{k=1}^d \int_{(t,T]} \frac{\hat{Z}_u^k}{B_u^r} dW_u^{k,\mathbb{Q}}. \tag{3.24}$$

We conclude by taking the $\mathcal{F}_t$-conditional expectation on both sides of (3.24). □

Remark 3.20. Our concept of clean value is in line with the concept of *third-party valuation* of Bichuch et al. (2018). Here we introduce the concept of clean value by means of a replicating strategy in a fictious idealized market. Our constructive approach is in line with the market standard. Formula (3.21) encodes the idea of *CSA discounting.* Since the rate r is the remuneration of collateral in a stylized perfect collateral agreement, we do not need to postulate the existence of a risk-free rate. Bichuch et al. (2018) define the clean value by introducing an additional valuation measure different from $\mathbb{Q}$. Working with the pricing measure $\mathbb{Q}$ also avoids the issue of estimating parameters under different measures.

So far, our discussion of the clean market focused on a dividend process specified under the reference filtration $\mathbb{F}$. As stressed e.g. in Crépey (2015b), this assumption is too restrictive to e.g. cover credit derivatives or *wrong-way risk.* Though, our objective is to focus on multiple aggregation levels and different discounting regimes, hence we choose to avoid the technicalities that are involved in generalizations of the immersion hypothesis.

Lemma 3.21. *Let $\tilde{X}$ be an $\mathbb{F}$-adapted process. Under the hypothesis 2.3 between $\mathbb{F}$ and $\mathbb{G}$, we have $\Delta\tilde{X}_\tau = 0$-a.s.*

Proof. This follows by Lemma 2.2 in Crépey (2015b). □

The following assumption is crucial for next results.

Assumption 3.22. *We assume a risk-free close-out valuation under $\mathbb{F}$, namely we set $\mathcal{V}_t = \hat{V}_t(\varphi)$ in* (2.20).

3.3.2. *Full value $\mathbb{G}$-BSDE.*

Definition 3.23. *We define the following valuation adjustments:*

$$CVA_t := B_t^r \mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau \le T\}} 1_{\{\tau^C < \tau^B\}} (1 - R^C) \frac{1}{B_\tau^r} \left(\hat{V}_\tau(\varphi) - C_{\tau-} + I_{\tau-}^{FC} \right)^- \middle| \mathcal{G}_t \right],$$

$$DVA_t := B_t^r \mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau \le T\}} 1_{\{\tau^B < \tau^C\}} (1 - R^B) \frac{1}{B_\tau^r} \left(\hat{V}_\tau(\varphi) - C_{\tau-} - I_{\tau-}^{TC} \right)^+ \middle| \mathcal{G}_t \right],$$

$$FVA_t := B_t^r \mathbb{E}^{\mathbb{Q}}\left[\int_t^{\tau \wedge T} \frac{(r_u^{f,l} - r_u)\left(V_u(\varphi) - C_u - I_u^{TC}\right)^+ - (r_u^{f,b} - r_u)\left(V_u(\varphi) - C_u - I_u^{TC}\right)^-}{B_u^r} du \middle| \mathcal{G}_t \right],$$

$$ColVA_t := B_t^r \mathbb{E}^{\mathbb{Q}}\left[\int_t^{\tau \wedge T} \frac{(r_u^{c,l} - r_u)C_u^+ - (r_u^{c,b} - r_u)C_u^-}{B_u^r} du \middle| \mathcal{G}_t \right],$$

$$MVA_t := B_t^r \mathbb{E}^{\mathbb{Q}}\left[\int_t^{\tau \wedge T} \frac{(r_u^{I,l} - r_u)I_u^{TC} - r_u^{I,b} I_u^{FC}}{B_u^r} du \middle| \mathcal{G}_t \right].$$

On $\{\tau > t\}$, we define

$$XVA_t := -CVA_t + DVA_t + FVA_t + ColVA_t + MVA_t, \tag{3.25}$$

and set

$$XVA_\tau = -\theta_\tau + \hat{V}_\tau \quad \text{on } \{\tau \le t\},$$

where θ_τ is defined in (2.20).

Remark 3.24. Upon inspection of the FVA term in Definition 3.23, we observe that, in general, the xVA-BSDE has a recursive nature. The exposure is proportional to the full value of the transaction V and not only to the clean value $\hat{V}$. This implies a high complexity of the numerical scheme. Some practitioner's papers, such as Burgard and Kjaer (2013), avoid the recursivity issue by means of ad-hoc choices of the funding strategies, such as the funding strategy called *semi-replication with no shortfall* *on default.* However, the bank usually needs to fund the clean value and the value adjustments. Hence, this feature cannot be ignored in a comprehensive mathematical model.

Our recursive FVA representation in Definition 3.23 is in line with the one presented in Piterbarg (2010). To clarify the latter point, let us consider the following

Example 3.25. Set $I_t^{TC} = I_t^{FC} = 0$, $r^{f,b} = r^{f,l} = r^f$, $r^{c,b} = r^{c,l} = r^c$ and $\tau^C = \tau^B = \infty$. Then the driver of the full BSDE is given by

$$f(t,V,C,0) := -\left((r_t^f - r_t)\left(V_t(\varphi) - C_t\right) + (r_t^c - r_t)C_t\right),\ t \in [0,T]. \tag{3.26}$$

In this case, the integral representation (3.18) of V is of the form

$$V_t(\varphi) = B_t^r \mathbb{E}^{\mathbb{Q}}\left[\int_{(t,T]} \frac{dA_u}{B_u^r} + \int_t^T \frac{f(u,V,C,0)}{B_u^r} du \middle| \mathcal{F}_t\right],\ t \in [0,T]. \tag{3.27}$$

If we set $r_t = r_t^f$ $d\mathbb{P}\otimes dt$-a.s. then we obtain by (3.26) that

$$V_t(\varphi) = B_t^{r^f} \mathbb{E}^{\mathbb{Q}}\left[\int_{(t,T]} \frac{dA_u}{B_u^{r^f}} + \int_t^T (r_u^f - r_u^c)\frac{C_u}{B_u^{r^f}} du \middle| \mathcal{F}_t\right],\ t \in [0,T]. \tag{3.28}$$

This corresponds to equation (3) in Piterbarg (2010). If we set $r_t = r_t^c$ $d\mathbb{P}\otimes dt$-a.s. in (3.26), we obtain

$$V_t(\varphi) = B_t^{r^c} \mathbb{E}^{\mathbb{Q}}\left[\int_{(t,T]} \frac{dA_u}{B_u^{r^c}} - \int_t^T (r_u^f - r_u^c)\frac{(V_t(\varphi) - C_u)}{B_u^{r^c}} du \middle| \mathcal{F}_t\right],\ t \in [0,T], \tag{3.29}$$

which corresponds to equation (5) in Piterbarg (2010).

3.4. **Well Posedness of the pricing BSDE.** In this section we address the issue of existence and uniqueness for the solution of the $\mathbb{G}$-BSDE (3.17). We follow the approach of Crépey (2015b).
From Assumption 3.22 we have $\mathcal{V} = \hat{V}$. Since $\hat{V}$ is an $\mathbb{F}$-adapted process, we know from Lemma 3.21 that $\Delta\hat{V}_\tau = 0$. Following Crépey (2015b), an application of Theorem 67b in Dellacherie and Meyer (1982) implies that there exists an $\mathbb{F}$-predictable process with the same value as $\hat{V}$ in τ, hence $\hat{V}$ can be chosen to be $\mathbb{F}$-predictable. The same argument holds true for the collateral process C, which we assumed to be a Lipschitz function of the clean value, and for the initial margin I, be it posted or received. In summary, both exposures

$$\hat{V}_t - C_t + I_t^{FC},\ \hat{V}_t - C_t - I_t^{TC},$$

are assumed to be $\mathbb{F}$-predictable from now on. We set

$$\begin{aligned} \theta_t^C &:= (1 - R^C)\left(\hat{V}_t - C_t + I_t^{FC}\right)^-, \\ \theta_t^B &:= (1 - R^B)\left(\hat{V}_t - C_t - I_t^{TC}\right)^+, \end{aligned} \tag{3.30}$$

and rewrite the close-out condition as

$$XVA_\tau = 1_{\{\tau<T\}}\vartheta_\tau,$$

where

$$\vartheta_t := -1_{\{t\geq\tau^C\}}\theta_t^C + 1_{\{t\geq\tau^B\}}\theta_t^B,\ t \in [0,T]. \tag{3.31}$$

Definition 3.26. *We call* pre-default XVA-BSDE *the following $\mathbb{F}$-BSDE on $[0,T]$ with null terminal condition in T :*

$$\begin{cases} -d\overline{XVA}_t = \bar{f}(\hat{V}_t - \overline{XVA}_t)dt - \sum_{k=1}^d \bar{Z}_t^k dW_t^{k,\mathbb{Q}} \\ \overline{XVA}_T = 0, \end{cases} \tag{3.32}$$

where

$$\bar{f}(\hat{V}_t - \overline{XVA}_t) := -f(t, \hat{V} - \overline{XVA}, C, I) - (r_t + \lambda_t^{C,\mathbb{Q}} + \lambda_t^{B,\mathbb{Q}})\overline{XVA}_t - \lambda_t^{C,\mathbb{Q}}\theta_t^C + \lambda_t^{B,\mathbb{Q}}\theta_t^B, \tag{3.33}$$

for θ^B, θ^C defined as in (3.30) *and $\lambda^{B,\mathbb{Q}}$, $\lambda^{C,\mathbb{Q}}$ introduced in* (3.8)*.*

We now discuss existence and uniqueness for the solution of (3.32). First, we observe that the driver (3.33) also depends on the initial margin I. More precisely, the initial margin I is a function of the clean value $\hat{V}$, evaluated up to the contract's maturity, since it is used to measure the potential future exposure. In other words, we set

$$I_t := \rho_t(\hat{V}_{t:T})_{t \in [t,T]}, \tag{3.34}$$

where $\hat{V}_{t:T} := (\hat{V}_s)_{s \in [t,T]}$ and $\rho_t = \rho(\omega, t; x)$, $t \in [0,T]$ is a process with values in the space of risk measures. The presence of such a function implies that the BSDE (3.32) is an *anticipative* BSDE, see e.g. Peng and Yang (2009). We also assume the following

Assumption 3.27. *For any $X \in \mathcal{S}^2(\mathbb{Q})$, the process $(\rho_s(X_{t:T}))_{s \in [0,T]}$ is in $\mathcal{H}^2(\mathbb{Q})$. There exists a constant $C_{\bar{f}} > 0$ and a family of measures $(\nu_s)_{s \in [0,T]}$ on $\mathbb{R}$ such that $\nu_t([t;T]) = 1$, for every $t \in [0,T]$, and, for any y^1, $y^2 \in \mathcal{S}^2(\mathbb{Q})$, we have*

$$\left|\rho_t(y^1_{t:T}) - \rho_t(y^2_{t:T})\right| \leq C_\rho \mathbb{E}\left[\int_t^T |y_s^1 - y_s^2| \nu_t(ds) \middle| \mathcal{F}_t\right], \; dt \otimes d\mathbb{P} \text{ a.e.} \tag{3.35}$$

Moreover, there exists a constant $k > 0$ such that for every continuous path $x : [0,T] \to \mathbb{R}$, we have

$$\int_0^T \int_s^T |x_s| \nu_u(ds) du < k \sup_{t \in [0,T]} |x_t|.$$

We are able to prove the following result.

Proposition 3.28. *Under Assumptions 2.8 and 3.27, the $\mathbb{F}$-BSDE* (3.32) *is well posed and has a unique solution $(\overline{XVA}, \bar{Z}) \in \mathcal{S}^2(\mathbb{Q}) \times \mathcal{H}^{2,d}(\mathbb{Q})$.*

Proof. The proof is an application of Theorem A.10, that is, we have to verify that the function $\bar{f}$ in (3.33) and the risk measure ρ in (3.34) satisfy suitable Lipschitz properties, given by Assumption A.9 and Assumption A.8 in Section A, respectively.
More precisely, Assumption 3.27 guarantees that the initial margin I satisfies Assumption A.8 in Agarwal et al. (2018). Concerning Assumption (S), we observe that there are three terms appearing in (3.33). The first one is the full $\mathbb{G}$-BSDE driver f, given in (3.16c) and expressed in terms of the collateral C, which is a Lipschitz function of the clean value by definition, and the (posted/received) initial margin I, which is Lipschitz by (3.34) and (3.35). The second term depends on the short rate r and the jump intensities $\lambda^{B,\mathbb{Q}}$, $\lambda^{C,\mathbb{Q}}$, which are bounded by definition. The last term relies upon the close-out conditions θ^B, θ^C given in (3.30), which are Lipschitz functions, by following the same arguments as before. □

Now, given the uniqueness of the solution to (3.32) we proceed to construct the unique solution to (3.37) by means of the following result.

Proposition 3.29. *Let $\left(\overline{XVA}, \bar{Z}\right)$ be the unique solution of the pre-default XVA-BSDE (3.32). Define*

$$X_t := \overline{XVA}_t J_t + 1_{\{\tau \le t\}} \vartheta_\tau,\ \ t \in [0, \tau \wedge T], \tag{3.36}$$

where $J_t := 1_{\{t<\tau\}} = 1 - H_t$. Then, under Assumptions 2.8 and 3.27, the process $\left(X, \tilde{Z}, \tilde{U}\right)$ solves the $\mathbb{G}$-BSDE on $\{\tau > t\}$

$$\begin{cases} -dX_t &= -\left[f(t, \hat{V} - XVA, C, I) + r_t XVA_t\right] dt - \sum_{k=1}^d \tilde{Z}^k_t dW^{k,\mathbb{Q}}_t - \sum_{j \in \{B,C\}} \tilde{U}^j_t dM^{j,\mathbb{Q}}_t \\ X_\tau &= 1_{\{\tau \le T\}} \left(\hat{V}_\tau(\varphi) - \theta_\tau(\hat{V}, C, I)\right) \end{cases} \tag{3.37}$$

with respect to the filtration $\mathbb{G}$.
Moreover, the $(\mathbb{G}, \mathbb{Q})$-martingale components of the XVA-BSDE satisfy on $\{t < \tau\}$

$$\sum_{k=1}^d \int_0^t \tilde{Z}^k_u dW^{k,\mathbb{Q}}_u = \sum_{k=1}^d \int_0^t \bar{Z}^k_u dW^{k,\mathbb{Q}}_u, \tag{3.38}$$

$$\sum_{j \in \{B,C\}} \tilde{U}^j_t dM^{j,\mathbb{Q}}_t = -\left((\vartheta_t - \overline{XVA}_t) dJ_t + \lambda^{C,\mathbb{Q}}_t(-\theta^C_t - \overline{XVA}_t) dt + \lambda^{B,\mathbb{Q}}_t(\theta^B_t - \overline{XVA}_t) dt\right), \tag{3.39}$$

where $\tilde{Z} \in \mathcal{H}^{2,d}(\mathbb{Q})$ and $\tilde{U} \in \mathcal{H}^{2,2}(\mathbb{Q})$. In particular, $X_t = XVA_t$, $t \in [0, T]$, where XVA is introduced in Definition 3.23.

Proof. We start from (3.36) and apply the product rule, hence

$$\begin{aligned} dX_t &= d\left(\overline{XVA}_t J_t\right) + d\left(1_{\{\tau<T\}} \vartheta_\tau\right) \\ &= d\overline{XVA}_{t\wedge\tau} + \overline{XVA}_t dJ_t - \vartheta_t dJ_t. \end{aligned}$$

By (3.32) we obtain

$$\begin{aligned} dX_t = &\left[f(t, \hat{V} - \overline{XVA}, C, I) + (r_t + \lambda^{C,\mathbb{Q}}_t + \lambda^{B,\mathbb{Q}}_t)\overline{XVA}_t + \lambda^{C,\mathbb{Q}}_t \theta^C_t - \lambda^{B,\mathbb{Q}}_t \theta^B_t\right] dt \\ &+ \sum_{i=1}^d \bar{Z}^k_t 1_{\{t<\tau\}} dW^{k,\mathbb{Q}}_t - \left(\vartheta_t - \overline{XVA}_t\right) dJ_t \end{aligned}$$

We note that the process $\sum_{i=1}^d \int_0^\cdot \bar{Z}^k_u 1_{\{u<\tau\}} dW^{k,\mathbb{Q}}_u$ is a $(\mathbb{G}, \mathbb{Q})$-martingale, since $\bar{Z}$ is in $\mathcal{H}^{2,d}(\mathbb{Q})$ due to the immersion hypothesis. From Lemma 5.2.9 in Crépey et al. (2014) we deduce that the process, expressed in differential form

$$-\left((\vartheta_t - \overline{XVA}_t) dJ_t + \lambda^{C,\mathbb{Q}}_t(-\theta^C_t - \overline{XVA}_t) dt + \lambda^{B,\mathbb{Q}}_t(\theta^B_t - \overline{XVA}_t) dt\right) \tag{3.40}$$

is also a $(\mathbb{G}, \mathbb{Q})$-local martingale. Moreover, we observe that, since $\hat{V} \in \mathcal{S}^2(\mathbb{Q})$, also $C \in \mathcal{S}^2(\mathbb{Q})$, C being a Lipschitz function of $\hat{V}$. Additionally, the initial margin, be it posted or received, lies in $\mathcal{H}^2(\mathbb{Q})$ by assumption. Summing up, both θ^B and θ^C, and hence ϑ belong to the space $\mathcal{H}^2(\mathbb{Q})$. On the other hand, $\overline{XVA} \in \mathcal{S}^2(\mathbb{Q})$. Recalling that both $\lambda^{C,\mathbb{Q}}$ and $\lambda^{B,\mathbb{Q}}$ are bounded, it follows that the compensated jump term (3.40) is a square integrable martingale. Then, we have that (3.39) must hold for some $\tilde{U}^j$ and we conclude that the process XVA solves the XVA-BSDE (3.37) under the filtration $\mathbb{G}$. □

We can finally combine the solution of the BSDE (3.32) for the clean value with the result above to solve the $\mathbb{G}$-BSDE (3.17).

Theorem 3.30. *Let $V_t := \hat{V}_t - XVA_t$, $t \in [0, T]$, on $\{\tau > t\}$, where $\hat{V}$ and XVA are defined in (3.21) and (3.25), respectively. Then, under Assumptions 2.8 and 3.27, the triplet $(V, Z, U) \in$*

$\mathcal{S}^2(\mathbb{Q}) \times \mathcal{H}^{2,d}(\mathbb{Q}) \times \mathcal{H}^{2,2}(\mathbb{Q})$ *solves the* $\mathbb{G}$*-BSDE* (3.17)*, where* Z *and* U *are given by*

$$Z_t^k = \hat{Z}_t^k - \tilde{Z}_t^k,\ k = 1, \ldots, d, \tag{3.41}$$

$$U_t^j = -\tilde{U}_t^j,\ j \in \{B, C\}. \tag{3.42}$$

Moreover, the process V *satisfies* (3.18)*.*

Proof. We notice that the random variable $\int_{(t,T]} \frac{dA_u}{B_u^r}$ is $\mathcal{F}_\infty$-measurable, hence on $\{t < \tau\}$ we can write

$$\hat{V}_t(\varphi) = B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,T]} \frac{dA_u}{B_u^r} \middle| \mathcal{F}_t \right] = B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,T]} \frac{dA_u}{B_u^r} \middle| \mathcal{G}_t \right].$$

So we consider $\hat{V}$ under $\mathbb{G}$. We also observe that, on $\{t < \tau\}$, we have $\bar{A}_t = A_t$ and recall the price decomposition $V_t = \hat{V}_t - XVA_t$. Using (3.20) and (3.37), we write the dynamics of V on $\{t < \tau\}$

$$\begin{aligned} -dV_t = d\bar{A}_t &+ \left[f(t, \hat{V} - XVA, C, I)dt - r_t \left(\hat{V}_t - XVA_t \right) \right] dt \\ &- \sum_{k=1}^{d} \left(\hat{Z}_t^k - \tilde{Z}_t^k \right) dW_t^{k,\mathbb{Q}} - \sum_{j \in \{B,C\}} \left(-\tilde{U}_t^j \right) dM_t^{j,\mathbb{Q}} \end{aligned}$$

with terminal condition at τ

$$V_\tau = \hat{V}_\tau - XVA_\tau = \hat{V}_\tau - \left(\theta_\tau - \hat{V}_\tau \right) = \theta_\tau.$$

Since $Z = \hat{Z} - \tilde{Z} \in \mathcal{H}^{2,d}(\mathbb{Q})$ and $U = -\tilde{U} \in \mathcal{H}^{2,2}(\mathbb{Q})$ by Theorem 3.18 and Proposition 3.29, we obtain that (V, Z, U) solves the $\mathbb{G}$-BSDE (3.17) and satisfies the required integrability conditions.
Finally, we are now able to prove that (3.18) is equivalent to (3.17).
Here we assume to work only on $\{\tau > t\}$. Since $V_t = \hat{V} - XVA_t$ and thanks to Definition 3.23 we have

$$\begin{aligned} V_t(\varphi) = \hat{V}_t(\varphi) + B_t^r \mathbb{E}^{\mathbb{Q}} \Bigg[& 1_{\{\tau \leq < T\}} 1_{\{\tau^C < \tau^B\}} \frac{(1 - R^C) \left(\hat{V}_\tau(\varphi) - C_{\tau-} + I_{\tau-}^{FC} \right)^-}{B_\tau^r} \\ & - 1_{\{\tau \leq T\}} 1_{\{\tau^B < \tau^C\}} \frac{(1 - R^B) \left(\hat{V}_\tau(\varphi) - C_{\tau-} - I_{\tau-}^{TC} \right)^+}{B_\tau^r} \\ & - \int_t^{\tau \wedge T} \frac{(r_u^{f,l} - r_u) \left(V_u(\varphi) - C_u - I_u^{TC} \right)^+ - (r_u^{f,b} - r_u) \left(V_u(\varphi) - C_u - I_u^{TC} \right)^-}{B_u^r} du \\ & - \int_t^{\tau \wedge T} \frac{(r_u^{c,l} - r_u) C_u^+ - (r_u^{c,b} - r_u) C_u^-}{B_u^r} du - \int_t^{\tau \wedge T} \frac{(r_u^{I,l} - r_u) I_u^{TC} - r_u^{I,b} I_u^{FC}}{B_u^r} du \Big| \mathcal{G}_t \Bigg]. \end{aligned}$$

By (3.16c) we obtain

$$\begin{aligned} V_t(\varphi) = \hat{V}_t(\varphi) + B_t^r \mathbb{E}^{\mathbb{Q}} \Bigg[& 1_{\{\tau \leq T\}} 1_{\{\tau^C < \tau^B\}} \frac{(1 - R^C) \left(\hat{V}_\tau(\varphi) - C_{\tau-} + I_{\tau-}^{FC} \right)^-}{B_\tau^r} \\ & - 1_{\{\tau \leq T\}} 1_{\{\tau^B < \tau^C\}} \frac{(1 - R^B) \left(\hat{V}_\tau(\varphi) - C_{\tau-} - I_{\tau-}^{TC} \right)^+}{B_\tau^r} \\ & + \int_t^{\tau \wedge T} \frac{f(u, V, C, I)}{B_u^r} du \Big| \mathcal{G}_t \Bigg]. \end{aligned}$$

Assumption 3.22 and (2.19) ensure that

$$V_t(\varphi) = \hat{V}_t(\varphi) + B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_t^{\tau \wedge T} \frac{f(u, V, C, I)}{B_u^r} du + 1_{\{\tau \leq T\}} \frac{\theta_\tau(\hat{V}(\varphi), C, I) - \hat{V}_\tau(\varphi)}{B_\tau^r} \Big| \mathcal{G}_t \right].$$

Now, we apply (3.21), the tower property and Hypothesis 2.3, so that

$$\begin{aligned} V_t(\varphi) &= B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_t^{\tau \wedge T} \frac{f(u, V, C, I)}{B_u^r} du + 1_{\{\tau \leq T\}} \frac{\theta_\tau(\hat{V}(\varphi), C, I)}{B_\tau^r} \Big| \mathcal{G}_t \right] \\ &+ B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,T]} \frac{dA_u}{B_u^r} - 1_{\{\tau \leq T\}} \int_{(t,T]} \frac{dA_u}{B_\tau^r} \Big| \mathcal{G}_t \right]. \end{aligned}$$

Finally, again by (2.19), we have

$$V_t(\varphi) = B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,T]} \frac{d\bar{A}_u}{B_u^r} + \int_t^{\tau \wedge T} \frac{f(u, V, C, I)}{B_u^r} du + 1_{\{\tau \leq T\}} \frac{\theta_\tau(\hat{V}(\varphi), C, I)}{B_\tau^r} \Big| \mathcal{G}_t \right].$$

□

We now provide an explicit formula for the value adjustments under the filtration $\mathbb{F}$. This representation is particularly useful from a computational point of view: risk factors can be simulated under the smaller filtration $\mathbb{F}$ and the computation of value adjustment does not require the simulation of default times. It is an immediate consequence of Proposition 3.28.

Corollary 3.31. *Let $(\overline{XVA}, \bar{Z})$ be the unique solution to the pre-default XVA-BSDE under $\mathbb{F}$ (3.32). Define the process $\tilde{r} = (\tilde{r}_t)_{t \in [0,T]}$ by setting $\tilde{r} := r + \lambda^{C,\mathbb{Q}} + \lambda^{B,\mathbb{Q}}$. Under Assumptions 2.8 and 3.27 the stochastic process $\overline{XVA}$ admits the following representation.*

$$\overline{XVA}_t = -\overline{CVA}_t + \overline{DVA}_t + \overline{FVA}_t + \overline{ColVA}_t + \overline{MVA}_t, \tag{3.43}$$

where

$$\begin{aligned} \overline{CVA}_t &:= B_t^{\tilde{r}} \mathbb{E}^{\mathbb{Q}} \left[(1 - R^C) \int_t^T \frac{1}{B_u^{\tilde{r}}} \left(\hat{V}_u(\varphi) - C_u + I_u^{FC} \right)^- \lambda_u^{C,\mathbb{Q}} du \Big| \mathcal{F}_t \right], \\ \overline{DVA}_t &:= B_t^{\tilde{r}} \mathbb{E}^{\mathbb{Q}} \left[(1 - R^B) \int_t^T \frac{1}{B_u^{\tilde{r}}} \left(\hat{V}_u(\varphi) - C_u - I_u^{TC} \right)^+ \lambda_u^{B,\mathbb{Q}} du \Big| \mathcal{F}_t \right], \\ \overline{FVA}_t &:= B_t^{\tilde{r}} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T \frac{(r_u^{f,l} - r_u) \left(V_u(\varphi) - C_u - I_u^{TC} \right)^+ - (r_u^{f,b} - r_u) \left(V_u(\varphi) - C_u - I_u^{TC} \right)^-}{B_u^{\tilde{r}}} du \Big| \mathcal{F}_t \right], \\ \overline{ColVA}_t &:= B_t^{\tilde{r}} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T \frac{(r_u^{c,l} - r_u) C_u^+ - (r_u^{c,b} - r_u) C_u^-}{B_u^{\tilde{r}}} du \Big| \mathcal{F}_t \right], \\ \overline{MVA}_t &:= B_t^{\tilde{r}} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T \frac{(r_u^{I,l} - r_u) I_u^{TC} - r_u^{I,b} I_u^{FC}}{B_u^{\tilde{r}}} du \Big| \mathcal{F}_t \right]. \end{aligned}$$

To conclude the section we briefly mention the problem given by the possible overlap between FVA and DVA, due to poor accounting policies in the bank. Regarding this topic there has been an intense discussion in the literature, see e.g. Hull and White (2012), Andersen et al. (2019), Brigo et al. (2019) and references therein. We limit ourselves to mention that a sound treatment of the issue is provided by Brigo et al. (2019) and that their solution can be embedded in our framework at the cost of further notations.

4. Multiple aggregation level xVA framework

4.1. **Multiple discounting regimes.** In this section we analyze the market practice of *CSA discounting* in the context of our general $\mathbb{G}$-BSDE. CSA discounting means that a transaction is considered as a clean transaction, in line with our previous Assumption 3.17 in Section 3.

In Section 3.3.1 we assumed that the clean value refers to an idealized fully collateralized transaction where the collateral rate is simply r. The situation in practice is more complicated. The market practice adopted for the computation of clean prices involves a multitude of discount curves. Possible examples from the market practice are

- The (clean) value of a perfectly uncollateralized derivative might be discounted by a bank by means of a bank-specific funding curve with associated short rate r^f (this could correspond to the Libor rate for a bank belonging to the Libor panel), see e.g. Piterbarg (2010).
- The (clean) value of a derivative collateralized in a foreign currency is discounted on the market at a rate depending on cross currency bases, see the formulas and derivations in Table 1 in Moreni and Pallavicini (2017).

It is quite natural to ask why banks employ multiple discount regimes for clean values and, on top of that, xVA corrections. The main reason is purely pragmatic and non-mathematical: from the perspective of a trading desk it is convenient to treat multiple CSAs by means of different discount regimes, because this allows to deal with portfolio market risk via traditional trading-desk techniques, such as curve trades (i.e. e.g. buying/selling interest rate swaps on different buckets/maturities along the curve). Hedging the expectation of an integral such as the FVA term in practice is much more complicated. A possible approximate treatment involves discretizing the time integral and treating the resulting Riemann sum over time as a portfolio of claims. In view of the aforementioned difficulty, market operators prefer to obtain an additive price representation, where discount curves are used to reduce the magnitude of the (funding related) xVA terms, which are more difficult to hedge.

From now on, we shall assume that the bank has two internal desks, dubbed the *front-office desk* and the *xVA desk,* respectively. The front-office desk is responsible for the calculation of clean values and for the trading activity required to hedge market risk of the clean values. The xVA desk instead computes and hedges all the value adjustments and is forced, according to internal rules of the bank, to adopt for each transaction the clean value dictated by the front-office desk. The fact that the xVA desk is a *clean-value-taker* implies that care is needed when computing xVAs, in order to avoid double counting effects.

The xVA desk has to deal with two different clean values for the same transaction. On the one side, the clean value performs an arbitrage-free pricing under the $\mathbb{F}$-BSDE. On the other side, we have the clean value prescribed by the front office function, which constitutes the official clean value accepted within the bank. The xVA desk is then faced with the following challenge:

Problem 4.1 (xVA-CSA consistency problem)**.** *Produce a price decomposition of V in terms of clean value and xVA such that*

(i) the representation of V is coherent with the $\mathbb{G}$-BSDE (3.17)*, and*
(ii) the clean price in the representation corresponds to the one prescribed by the front-office function.

We assume a portfolio consisting of $K > 1$ claims, with dividend processes $A^m = (A^m_t)_{t\in[0,T]}$ and value processes $\hat{V}^m = \left(\hat{V}^m_t\right)_{t\in[0,T]}$, for $m = 1, \dots, K$, and provide a price representation in terms of multiple discounting rules. Based on Assumption 3.17, we treat each discounting rule as based on a different clean market: every (possibly) trade-specific clean valuation results from an underlying (possibly) trade-specific clean market. In line with Assumption 3.17, in every trade-specific clean market the collateralization scheme is perfect, but now the remuneration of collateral is performed at a different interest rate.

Assumption 4.2. *A clean market under $\mathbb{F}$ without bid-offer spreads with multiple CSAs is defined by*

(i) *No bid-offer spreads in the funding accounts, i.e., $r_t^{f,l} = r_t^{f,b} = r^f$.*
(ii) *No bid-offer spreads in the collateral accounts, i.e., $r_t^{c,l} = r_t^{c,b} = r^c$.*
(iii) *There is no default, i.e. $\hat{\tau} = T$, and risky bonds are excluded from the market.*
(iv) *There is no exchange of initial margin.*
(v) *$\hat{V}^m$ and A^m are $\mathbb{F}$-adapted processes.*
(vi) *Perfect collateralization, i.e., $\hat{V}_t^m = C_t^m$ $d\mathbb{P} \otimes dt$-a.s.*
(vii) *There exists a specific collateral rate $\hat{r}^m$ with cash account $B_t^{\hat{r}^m}$ for each claim A^m, $m = 1, \ldots, K$.*

Observe that *(vii)* in Assumption 4.2 ensures that the repo-like relation

$$\hat{V}_t^m + \psi_t^m B_t^{\hat{r}^m} = 0 \tag{4.1}$$

holds for each claim A^m, $m = 1, \ldots, K$. Note also that $\psi^f = 0$.

Remark 4.3. To provide a concrete example, Assumption 4.2 covers the situation where the trading desk of the bank enters into two perfectly collateralized transactions with two different counterparties, the first one being e.g. a clearing house such as LCH, the other one being another clearing house such as Eurex. Although the dividend process of the claim is the same for both transactions, the collateral remuneration provided by the trade with Eurex and the trade with LCH is different. The spread in the collateral remuneration between EUREX and LCH is called *Eurex-LCH basis,* see e.g. Mackenzie Smith (2017) for a more detailed discussion. This will result in the two clean values being computed by means of different discounting rates.

In summary, the market practice of discounting cashflows according to trade-specific collateral rates implies that, within the bank, a single transaction will be discounted at least according to two different regimes. Initially, the front-office determines the clean value by discounting cash flows through an ideal market collateral rate $\hat{r}^m$. Hence the front-office clean value $\hat{P}_t^m$, $m = 1, \ldots, K$, is obtained from the $\mathbb{F}$-BSDE

$$\begin{cases} -d\hat{P}_t^m = -\sum_{k=1}^d \hat{Z}_t^{k,m} dW_t^{k,\mathbb{Q}} + dA_t^m - \hat{r}^m \hat{P}_t^m dt, \\ \hat{P}_T^m = 0. \end{cases} \tag{4.2}$$

On the other side, the xVA desk first computes the clean value $\hat{V}_t^m$, $m = 1, \ldots, K$, as the solution to the $\mathbb{F}$-BSDE (3.20), i.e. by solving

$$\begin{cases} -d\hat{V}_t^m = -\sum_{k=1}^d \hat{Z}_t^{k,m} dW_t^{k,\mathbb{Q}} + dA_t^m - r_t \hat{V}_t^m dt, \\ \hat{V}_T^m = 0, \end{cases} \tag{4.3}$$

for each claim A^m. From a valuation perspective, if clean values represented the prices of real transactions, the presence of multiple discounting rules would immediately imply the presence of trivial arbitrage opportunities in the market. Only the endogenous price (4.3) is compatible with the arbitrage-free setting of Section 3. On the other hand, the xVA desk is forced to provide results in terms of the multiple discounting regimes imposed by the front-office. The two approaches can be combined in an arbitrage-free setting by means of the following invariance property of linear BSDEs.

Lemma 4.4. *Let $(\hat{V}^m, \hat{Z}^{1,m}, \ldots, \hat{Z}^{d,m})$ be the unique solution of the $\mathbb{F}$-BSDE* (4.3)*. Under Assumption 4.2 for A^m, $m = 1, \ldots, K$, the value process $\hat{V}^m$ admits the two equivalent representations*

i) *xVA-discounting representation*

$$\hat{V}_t^m = B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,T]} \frac{dA_t^m}{B_u^r} \Big| \mathcal{F}_t \right], \tag{4.4}$$

ii) *CSA-discounting representation*

$$\hat{V}_t^m = \hat{P}_t^m - DiscVA_t^m, \tag{4.5}$$

where $DiscVA_t^m$ represents the discounting valuation adjustment, *defined as*

$$DiscVA_t^m := B_t^{\hat{r}^m} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T (r_u - \hat{r}^m) \frac{\hat{V}_u^m}{B_u^{\hat{r}^m}} du \Big| \mathcal{F}_t \right], \tag{4.6}$$

and $\hat{P}^m$ is the value process in the solution $(\hat{V}^m, \hat{Z}^{1,m}, \ldots, \hat{Z}^{d,m})$ of the $\mathbb{F}$-BSDE (4.2)

$$\hat{P}_t^m = B_t^{\hat{r}^m} \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,T]} \frac{dA_t^m}{B_u^{\hat{r}^m}} \Big| \mathcal{F}_t \right]. \tag{4.7}$$

Proof. The integral representation (4.4) is immediate. To obtain (4.5) we rewrite the $\mathbb{F}$-BSDE (4.3) adding and subtracting the term $\hat{r}^m \hat{V}_t^m$, i.e.,

$$\begin{cases} -d\hat{V}_t^m = -\sum_{k=1}^d \hat{Z}_t^{k,m} dW_t^{k,\mathbb{Q}} + dA_t^m - (r_t - \hat{r}_t^m)\hat{V}_t^m dt - \hat{r}_t^m \hat{V}_t^m dt \\ \hat{V}_T^m = 0, \ m = 1, \ldots, K. \end{cases}$$

The value process of the solution is given by

$$\hat{V}_t^m = B_t^{\hat{r}^m} \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,T]} \frac{dA_t^m}{B_u^{\hat{r}^m}} \Big| \mathcal{F}_t \right] - B_t^{\hat{r}^m} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T (r_u - \hat{r}_u^m) \frac{\hat{V}_u^m}{B_u^{\hat{r}^m}} du \Big| \mathcal{F}_t \right],$$

where we recognize the first expectation as $\hat{P}^m$, whereas the second one provides $DiscVA^m$. □

This lemma gives a price decomposition which is compatible with the presence of multiple discounting rules for different claims. The full contract $\mathbb{G}$-BSDE for the portfolio of claims $(A^m)_{m \in \{1,\ldots,K\}}$ can be written as

$$\begin{cases} -dV_t(\varphi) & = \sum_{m=1}^K d\bar{A}_t^m + (f(t, V, C, I) - r_t V_t(\varphi))\, dt \\ & \quad - \sum_{k=1}^d \mathcal{Z}_t^k dW_t^{k,\mathbb{Q}} - \sum_{j \in \{B,C\}} \mathcal{U}_t^j dM_t^{j,\mathbb{Q}}, \\ V_\tau(\varphi) & = \theta_\tau \left(\sum_{m=1}^N \hat{V}^m, C, I \right), \end{cases} \tag{4.8}$$

where $\bar{A}^m_t$ is defined in (2.19), $\mathcal{Z}_t = \left(Z^1_t, \dots, Z^d_t\right)$, and $\mathcal{U}_t = \left(U^B_t, U^C_t\right)$, represent the control processes given by $\mathbb{G}$-predictable processes, and $f(t, V, C, I)$ is the $\mathbb{G}$-BSDE driver given by (3.16c). The close-out condition is

$$
\begin{aligned}
V_\tau(\varphi) = \sum_{m=1}^K \hat{P}^m_\tau + 1_{\{\tau^C<\tau^B\}}(1-R^C)\left(\sum_{m=1}^K \hat{P}^m_\tau - C_{\tau-} + I^{FC}_{\tau-} - \sum_{m=1}^K DiscVA^m_\tau\right)^- \\
- 1_{\{\tau^B<\tau^C\}}(1-R^B)\left(\sum_{m=1}^K \hat{P}^m_\tau - C_{\tau-} - I^{TC}_{\tau-} - \sum_{m=1}^K DiscVA^m_\tau\right)^+ .
\end{aligned}
\tag{4.9}
$$

By using the same arguments given for Theorem 3.30 and taking into account Definition 3.23, we obtain the following result, with the help of Lemma 4.4.

Proposition 4.5. *Under Assumption 4.2 and 3.27, the* $\mathbb{G}$*-BSDE* (4.8) *admits the following integral representation*

$$
V_t(\varphi) = \sum_{m=1}^K B^{\hat{r}^m}_t \mathbb{E}^{\mathbb{Q}}\left[\int_{(t,T]} \frac{dA^m_u}{B^{\hat{r}^m}_u}\Big|\mathcal{F}_t\right] - XVA_t = \sum_{m=1}^K \hat{P}^m_t - \widehat{XVA}_t, \tag{4.10}
$$

on the event $\{\tau > t\}$, $t \in [0, T]$, *where*

$$
\widehat{XVA} := XVA + DiscVA, \tag{4.11}
$$

with

$$
\begin{aligned}
XVA_t &:= FVA_t + ColVA_t + MVA_t - CVA_t + DVA_t \\
&= FVA_t + ColVA_t + MVA_t \\
&- B^r_t \mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau<T\}}1_{\{\tau^C<\tau^B\}}(1-R^C)\frac{1}{B^r_\tau}\left(\sum_{m=1}^K \hat{P}^m_\tau - C_{\tau-} + I^{FC}_{\tau-} - \sum_{m=1}^K DiscVA^m_\tau\right)^- \Bigg| \mathcal{G}_t\right] \\
&+ B^r_t \mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau<T\}}1_{\{\tau^B<\tau^C\}}(1-R^B)\frac{1}{B^r_\tau}\left(\sum_{m=1}^K \hat{P}^m_\tau - C_{\tau-} + I^{TC}_{\tau-} - \sum_{m=1}^K DiscVA^m_\tau\right)^+ \Bigg| \mathcal{G}_t\right]
\end{aligned}
$$

and

$$
DiscVA_t := \sum_{m=1}^K B^{\hat{r}^m}_t \mathbb{E}^{\mathbb{Q}}\left[\int_t^T (r_u - \hat{r}^m_u)\frac{\hat{V}^m_u}{B^{\hat{r}^m}_u} du \Big| \mathcal{F}_t\right],
$$

where FVA, ColVA, MVA are defined in line with Definition 3.23.

Remark 4.6. Looking at the price representation in Proposition 4.5 we note the following.

- Existence and uniqueness for the solution of (4.8) follow along the lines of Section 3.4.
- The presence of $\hat{P}^m$, $m = 1, \dots, K$, in the CVA and DVA terms explicitly represents the impact on derivative exposures of CSA discounting. It is possible to explicitly observe the exposure profile of every claim in the portfolio either under r discounting or $\hat{r}^m$ discounting.
- Note that $\frac{V(\varphi)}{B^r}$ is a $\mathbb{Q}$-martingale, while both $\frac{B^{\hat{r}^m}_t}{B^r_t}\mathbb{E}^{\mathbb{Q}}\left[\int_{(t,T]}(B^{\hat{r}^m}_u)^{-1}dA^m_u \Big| \mathcal{F}_t\right]$, $t \in (0, T]$, and $\left(\frac{\widehat{XVA}}{B^r}\right)$ fail to be martingales separately.
- From a computational point of view, we observe that the price representation in term of CSA specific discount factors can be obtained at a reasonable computational cost: the xVA desk computes the price in terms of r discounting, i.e. $\hat{V}^m$, $m = 1, \dots K$. Such value then enters the computation of the $DiscVA$ terms.

4.2. **Multiple aggregation levels.** We can use our setting to analyze multiple aggregation levels. We start from the example illustrated in Figure 1. The set of trades between the bank and the counterparty can be typically split into several subsets reflecting multiple aggregation levels.

One can distinguish between funding/margin sets and netting sets. Funding/margin sets are traded between the bank and the counterparty that share the same funding policy. This corresponds to different CSAs: for example, one CSA (Margin Set 2) could group all trades for which collateral is exchanged in USD (e.g. for foreign exchange derivatives), whereas another CSA (Margin Set 3) could be relevant for all instruments collateralized in EUR. Finally, trades that are not collateralized, but whose exposures are netted among each other, can be also grouped in a separate margin/funding set, corresponding to Margin Set 1 in Figure 1.

The protection provided by collateralization agreements might however be imperfect, hence a legal agreement between the bank and the counterparty might allow for the netting of residual post collateral exposures arising from different margin sets. This corresponds to Netting Set 1 in Figure 1.

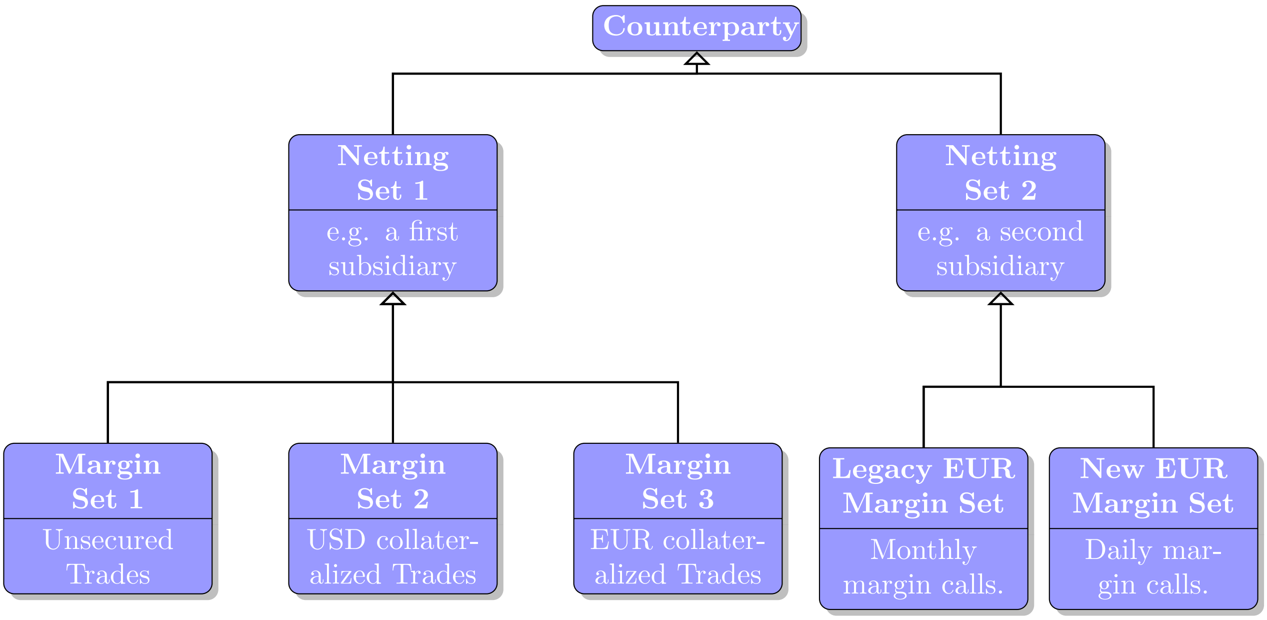

FIGURE 1. A possible hierarchical structure of aggregation levels.

Another typical source of multiple aggregation levels is the historical stratification of legal agreements: in Figure 1 we have a second netting set, corresponding to a second subsidiary of the parent counterparty, where legacy trades are covered by an old CSA agreement involving monthly margin calls, whereas all trades entered after a certain date are covered by a newer CSA agreement involving daily margin calls.

A further level of complexity could arise when the parent and the subsidiaries have different default times: this introduces further complications when modeling the close-out condition because one might have e.g. a situation where the default of a subsidiary is covered by the parent. Such issues are left for future research. From a practical point of view it is also difficult to find calibration instruments for default probabilities, since subsidiaries typically do not enjoy a liquid CDS market.

The example we discussed highlights the fact that the portfolio-wide $\mathbb{G}$-BSDE depends on the structure of all legal agreements between the bank and the counterparty.

In line with the previous sections, we assume that the portfolio $\mathcal{P}$ of trades between the bank and the counterparty consists of K trades, that we identify by means of the respective payment processes, i.e., $\mathcal{P} = \left\{A^1, \ldots A^K\right\}$. We use again $\hat{V}^m$ to denote the clean reference value of the claims A^m, $m =$

$1, \dots K$, representing also their credit exposure before collateral is applied. We construct a bottom-up aggregation hierarchy of claims by means of the following definitions.

Definition 4.7. *A margin (or funding) set $\mathcal{M}$ is a set of claims whose aggregated clean values (exposures) are fully or partially covered by a CSA (collateral agreement). We let $N_{\mathcal{M}}$ denote the number of margin sets in the portfolio $\mathcal{P}$.*

Assumption 4.8. *For every claim $A^m \in \mathcal{P}$, $m = 1, \dots K$, we assume that the margin set for initial and variation margin coincide.*

Remark 4.9. It is worth noting that uncollateralized trades that can be netted among each other can be treated as margin sets with zero initial and variation margin. Moreover, trades that can not be aggregated with other trades can be treated as separate margin sets consisting of the single trade themselves. Finally, we observe that all trades within a margin set share the same funding source.

Definition 4.10. *A netting set $\mathcal{N}$ is a set of margin sets whose post-margin exposures can be aggregated. We let $N_{\mathcal{N}}$ denote the number of netting sets in the portfolio $\mathcal{P}$.*

The structure of the portfolio is illustrated in Figure 1, where the first row illustrates the composition of all margin sets as groups of claims and the second line describes the netting sets as groups of margin sets,

$$
\begin{aligned}
\mathcal{P} &= \underbrace{\{A^1, \dots, A^{N_1}\}}_{\mathcal{M}_1} \cup \underbrace{\{A^{N_1+1}, \dots, A^{N_2}\}}_{\mathcal{M}_2} \cup \dots \cup \underbrace{\{A^{N_{N_{\mathcal{M}}-1}+1}, \dots, A^{N_{N_{\mathcal{M}}}}\}}_{\mathcal{M}_{N_{\mathcal{M}}}} \\
&= \underbrace{\{\mathcal{M}_1, \dots, \mathcal{M}_{M_1}\}}_{\mathcal{N}_1} \cup \underbrace{\{\mathcal{M}_{M_1+1}, \dots, \mathcal{M}_{M_2}\}}_{\mathcal{N}_2} \cup \dots \cup \underbrace{\{\mathcal{M}_{M_{N_{\mathcal{M}}-1}+1}, \dots, \mathcal{M}_{M_{N_{\mathcal{M}}}}\}}_{\mathcal{N}_{N_{\mathcal{N}}}}
\end{aligned}
\tag{4.12}
$$

where we have $N_{N_{\mathcal{M}}} = K$.

Given the structure we introduced, we can generalize the CVA and DVA formulas at the portfolio-wide level as follows. The portfolio exposure within a margin set is given by

$$\sum_{m=1}^{|\mathcal{M}_{m_1}|} \left(\hat{P}^m_\tau - DiscVA^m_\tau\right) - C_{\tau-} - I^{TC}_{\tau-},\ m_1 = 1, \dots, N_{\mathcal{M}}.$$

We aggregate the margin-set-level exposure at the netting set level to obtain the netting-set-level positive exposure

$$\left[\sum_{m_1=1}^{|\mathcal{N}_{m_2}|} \sum_{m=1}^{|\mathcal{M}_{m_1}|} \left(\hat{P}^{m,m_1}_\tau - DiscVA^{m,m_1}_\tau\right) - C^{\mathcal{M}_{m_1}}_{\tau-} - I^{TC,\mathcal{M}_{m_1}}_{\tau-}\right]^-,\ m_2 = 1, \dots, N_{\mathcal{N}},$$

and similarly for the netting-set-level negative exposure.

Finally, we sum exposures over netting sets to obtain the portfolio-wide CVA over all K claims as

$$
\begin{aligned}
CVA^K_t &:= \sum_{m_2=1}^{N_{\mathcal{N}}} B^r_t \mathbb{E}^{\mathbb{Q}} \Bigg[1_{\{\tau<T\}} 1_{\{\tau^C<\tau^B\}} \frac{(1-R^C)}{B^r_\tau} \\
&\times \left[\sum_{m_1=1}^{|\mathcal{N}_{m_2}|} \sum_{m=1}^{|\mathcal{M}_{m_1}|} \left(\hat{P}^{m,m_1,m_2}_\tau - DiscVA^{m,m_1,m_2}_\tau\right) - C^{\mathcal{M}_{m_1},\mathcal{N}_{m_2}}_{\tau-} - I^{TC,\mathcal{M}_{m_1},\mathcal{N}_{m_2}}_{\tau-}\right]^- \Bigg| \mathcal{G}_t \Bigg],
\end{aligned}
\tag{4.13}
$$

and similarly for the DVA

$$
\begin{aligned}
DVA_t^K := \sum_{m_2=1}^{N_{\mathcal{N}}} B_t^r \mathbb{E}^{\mathbb{Q}} \Bigg[& 1_{\{\tau<T\}} 1_{\{\tau^B<\tau^C\}} \frac{(1-R^B)}{B_\tau^r} \\
& \times \left[\sum_{m_1=1}^{|\mathcal{N}_{m_2}|} \sum_{m=1}^{|\mathcal{M}_{m_1}|} \left(\hat{P}_\tau^{m,m_1,m_2} - DiscVA_\tau^{m,m_1,m_2} \right) - C_{\tau-}^{\mathcal{M}_{m_1},\mathcal{N}_{m_2}} + I_{\tau-}^{FC,\mathcal{M}_{m_1},\mathcal{N}_{m_2}} \right]^+ \Bigg| \mathcal{G}_t \Bigg].
\end{aligned}
\tag{4.14}
$$

We partition the portfolio between the bank and the counterparty over netting sets by writing $V_t(\varphi) = \sum_{m_2=1}^{N_{\mathcal{M}}} V_t^{m_2}(\varphi)$. The presence of multiple margin sets within a single netting set is reflected by the appearence of multiple variation margin and initial margin accounts in the following portfolio-wide expression for FVA:

$$
\begin{aligned}
FVA_t^K := & \sum_{m_2=1}^{N_{\mathcal{N}}} B_t^r \\
& \times \mathbb{E}^{\mathbb{Q}} \Bigg[\int_t^{\tau\wedge T} \frac{1}{B_u^r} \Bigg[(r_u^{f,l,\mathcal{N}_{m_2}} - r_u) \Bigg(V_u^{m_2}(\varphi) - \sum_{m_1=1}^{|\mathcal{N}_{m_2}|} \left(C_u^{\mathcal{M}_{m_1},\mathcal{N}_{m_2}} + I_u^{TC,\mathcal{M}_{m_1},\mathcal{N}_{m_2}} \right) \Bigg)^+ \\
& - (r_u^{f,b,\mathcal{N}_{m_2}} - r_u) \Bigg(V_u^{m_2}(\varphi) - \sum_{m_1=1}^{|\mathcal{N}_{m_2}|} \left(C_u^{\mathcal{M}_{m_1},\mathcal{N}_{m_2}} + I_u^{TC,\mathcal{M}_{m_1},\mathcal{N}_{m_2}} \right) \Bigg)^- \Bigg] du \Bigg| \mathcal{G}_t \Bigg].
\end{aligned}
\tag{4.15}
$$

Similar expressions are obtained for ColVA and MVA, again over all K claims in the portfolio:

$$
\begin{aligned}
ColVA_t^K := \sum_{m_2=1}^{N_{\mathcal{N}}} \sum_{m_1=1}^{|\mathcal{N}_{m_2}|} B_t^r \mathbb{E}^{\mathbb{Q}} \Bigg[\int_t^{\tau\wedge T} \frac{1}{B_u^r} \Bigg[& (r_u^{c,l,\mathcal{M}_{m_1},\mathcal{N}_{m_2}} - r_u) \left(C_u^{\mathcal{M}_{m_1},\mathcal{N}_{m_2}} \right)^+ \\
& - (r_u^{c,b,\mathcal{M}_{m_1},\mathcal{N}_{m_2}} - r_u) \left(C_u^{\mathcal{M}_{m_1},\mathcal{N}_{m_2}} \right)^- \Bigg] du \Bigg| \mathcal{G}_t \Bigg],
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
MVA_t^K := \sum_{m_2=1}^{N_{\mathcal{N}}} \sum_{m_1=1}^{|\mathcal{N}_{m_2}|} B_t^r \mathbb{E}^{\mathbb{Q}} \Bigg[\int_t^{\tau\wedge T} \frac{1}{B_u^r} \Bigg[& (r_u^{I,l,\mathcal{M}_{m_1},\mathcal{N}_{m_2}} - r_u) I_u^{TC,\mathcal{M}_{m_1},\mathcal{N}_{m_2}} \\
& - r_u^{I,b,\mathcal{M}_{m_1},\mathcal{N}_{m_2}} I_u^{FC,\mathcal{M}_{m_1},\mathcal{N}_{m_2}} \Bigg] du \Bigg| \mathcal{G}_t \Bigg].
\end{aligned}
\tag{4.17}
$$

By regrouping all portfolio-wide value adjustments (4.13), (4.14), (4.15), (4.16) and (4.17) we obtain an xVA correction for the entire portfolio that accounts for multiple discounting regimes and multiple aggregation levels. We set

$$
XVA_t^K := FVA_t^K + ColVA_t^K + MVA_t^K - CVA_t^K + DVA_t^K, \tag{4.18}
$$

$$
\widehat{XVA}_t^K := XVA_t^K + \sum_{m_2=1}^{N_{\mathcal{N}}} \sum_{m_1=1}^{|\mathcal{N}_{m_2}|} \sum_{m=1}^{|\mathcal{M}_{m_1}|} DiscVA_t^{m,m_1,m_2}, \tag{4.19}
$$

and finally write the whole portfolio value as

$$
\mathscr{V}_t^K(\varphi) := \sum_{m=1}^{K} \hat{P}_t^m - \widehat{XVA}_t^K. \tag{4.20}
$$

To avoid further heavy notations, we omit the statement of such BSDE.

A natural question involves the well-posedness of the portfolio-wide valuation BSDE for $\mathscr{V}$. Upon direct inspection of (4.13)- (4.17) we observe the following:

(i) The presence of multiple margin sets is represented by the introduction of multiple collateral accounts. If we assume that each margin account (be it of variation margin or initial margin type) satisfies the same assumptions from Section 2.4.1 and Section 2.4.2, then existence and uniqueness for a portfolio wide-BSDE in the context of one netting set and multiple margin sets immediately follow from our discussion so far as an application of our arguments from Section 3.4.

(ii) Multiple netting sets are simply accounted for by summing value adjustments over all netting sets, each netting set possibly featuring multiple margin sets. Being a sum of well-posed netting set specific BSDEs, the well posedness of the full portfolio BSDE is then again an immediate consequence of our arguments from Section 3.4.

In other words, netting sets correspond to different BSDEs, whereas margin sets appear as additional terms in the driver.

In summary, our assumptions underlying the single-claim xVA framework from Section 3 are sufficient to guarantee the well posedness of the BSDE for $\mathscr{V}$ also in the presence of multiple aggregation levels.

4.3. **Incremental xVA charge.** Consider the situation where the portfolio of contingent claims between the bank and the counterparty consists of K claims, so that the full portfolio value is given by (4.20). From the discussion so far it is evident that the portfolio-wide value adjustment $\widehat{XVA}^K$ does not coincide with the sum of K distinct xVA processes for the K distinct claims. This is due both to the presence of different aggregation levels (margin sets and netting sets) and the non-linear effects induced by different rates for borrowing and lending.

Let us assume now that the counterparty wishes to enter into a further $(K+1)$-th trade with the bank. If entered, the newly introduced $(K+1)$-th claim would contribute to the global riskiness of the portfolio between the bank and the counterparty. It is natural to ask then what is the price the bank should charge to the newly introduced $(K+1)$-th claim given the presence of the already existing K claims. One could consider two different approaches.

(i) **Stand-alone scenario:** the $(K+1)$-th contingent claim and the corresponding xVA are evaluated in isolation. This corresponds to computing the integral representation with discounting adjustment (4.10) from Proposition 4.5 for the case of the single $(K+1)$-th contingent claim, i.e. only for $m = K+1$. This scenario underestimates diversification benefits, due to existing deposited margins and netting agreements.

(ii) **Incremental xVA charge:** to account for portfolio effects involving margin and netting sets, two different scenarios are compared.

   (a) *Base scenario:* The value of the portfolio is given by $\mathscr{V}_t^K(\varphi)$ as in formula (4.20). This corresponds to the value of the portfolio before the inclusion of the candidate new trade.

   (b) *Full scenario:* The value of the portfolio is given by $\mathscr{V}_t^{K+1}(\varphi)$, computed in line with formula (4.20). This corresponds to the value of the portfolio after the inclusion of the candidate $(K+1)$-th contingent claim.

   The bank determines the price to be charged to the counterparty as the difference between the value of the portfolio under the full and the base scenario, i.e. the bank charges the *incremental value* ΔV_t^{K+1}, defined as

$$\Delta V_t^{K+1}(\varphi) := \mathscr{V}_t^{K+1}(\varphi) - \mathscr{V}_t^K(\varphi). \tag{4.21}$$

The incremental value (4.21) represents the prevailing market practice. From the perspective of the counterparty it has the interesting implication that the counterparty, who wishes to invest in the $(K+1)$-th claim, when setting up an auction on the $(K+1)$-th claim, will be offered different pricing proposals by the different banks participating in the auction, due to the different structures of the existing portfolios.
By analyzing (4.21) we can isolate the impact of the $(K+1)$-th trade as follows.

$$
\begin{aligned}
\Delta V_t^{K+1}(\varphi) &:= \mathscr{V}_t^{K+1}(\varphi) - \mathscr{V}_t^{K}(\varphi) \\
&= \sum_{m=1}^{K+1} \hat{P}_t^m - \widehat{XVA}_t^{K+1} - \sum_{m=1}^{K} \hat{P}_t^m + \widehat{XVA}_t^{K} \\
&= \hat{P}_t^{K+1} - \left(XVA_t^{K+1} - XVA_t^{K} \right) - DiscVA_t^{K+1} \\
&= \hat{P}_t^{K+1} - \Delta XVA_t - DiscVA_t^{K+1},
\end{aligned}
\tag{4.22}
$$

where, in the last step, we implicitly defined the *incremental xVA charge*

$$
\Delta XVA_t := XVA_t^{K+1} - XVA_t^{K} \tag{4.23}
$$

as the adjustment to be charged on the $(K+1)$-th claim, given the presence of the already existing K claims in the portfolio.
Our discussion motivates the introduction of the concept of *non-linearity effect.*

Definition 4.11. *The* non-linearity effect *on the $(K+1)$-th contingent claim is defined as*

$$
NL_t\left(V^{K+1}\right) := V_t^{K+1}(\varphi) - \Delta V_t^{K+1}(\varphi), \tag{4.24}
$$

where $V_t^{K+1}(\varphi)$ is determined by solving the stand-alone $\mathbb{G}$-BSDE and $\Delta V_t^{K+1}(\varphi)$ is the incremental charge as defined in (4.22)*.*

The non-linearity effect coincides with the difference of the incremental xVA charge and the stand-alone xVA, in fact:

$$
\begin{aligned}
NL_t\left(V^{K+1}\right) &:= V_t^{K+1}(\varphi) - \Delta V_t^{K+1}(\varphi) \\
&= \left(\hat{P}_t^{K+1} - XVA_t - DiscVA_t^{K+1} \right) - \left(\hat{P}_t^{K+1} - \Delta XVA_t - DiscVA_t^{K+1} \right) \\
&= \Delta XVA_t - XVA_t.
\end{aligned}
\tag{4.25}
$$

Remark 4.12. Let us observe the following.

- In the present setting the clean valuation of the contingent claim is still linear, hence the clean value of the portfolio still corresponds to the sum of the clean values of the single claims.
- We typically have $\Delta XVA_t - XVA_t \neq 0$. The stand-alone xVA of the $(K+1)$-th claim is higher than ΔXVA.
- $NL_t\left(V^{K+1}\right) = 0$ only when there are no portfolio/netting effects.

5. Example

We conclude the paper by presenting a simple example using a lognormal model for a single risky asset. Under the setting and assumptions of the previous sections we consider a single risky asset $S = (S_t)_{t\in[0,T]}$ that pays dividends at a rate $\kappa = (\kappa_t)_{t\in[0,T]}$, so that the dividend process of the asset is $D_t = \int_0^t \kappa_s S_s ds$.

The asset price is assumed to evolve according to the $\mathbb{P}$-dynamics

$$dS_t = S_t\left(\mu_t dt + \sigma_t dW_t^{\mathbb{P}}\right), \tag{5.1}$$

where μ_t, σ_t are deterministic functions of time such that the SDE (5.1) has a unique strong solution. Under the martingale measure $\mathbb{Q}$ defined by (3.7) the risky asset evolves according to

$$dS_t = S_t\left((r_t^r - \kappa_t)dt + \sigma_t dW_t^{\mathbb{P}}\right), \tag{5.2}$$

where $r^r = (r_t^r)_{t\in[0,T]}$ is the repo rate associated to the asset S. We now consider a simple contingent claim, namely a forward written on the asset S. The dividend process of the claim $A^1 = (A_t^1)_{t\in[0,T]}$, is given by

$$A_t^1 = 1_{\{t=T\}}(S_T - K_1), \tag{5.3}$$

for K_1 a positive constant. We recall that $\hat{V}$, the clean value satisfying (3.20), represents a fictious value process for the claim under the assumption of a perfect collateralization scheme that annihilates counterparty risk, see Assumption 3.17.
According to Theorem 3.19 the arbitrage free price of the forward is

$$\hat{V}_t^1(\varphi) = \mathbb{E}^{\mathbb{Q}}\left[B_t^r\int_{(t,T]}\frac{dA_u^1}{B_u^r}\middle|\mathcal{F}_t\right] = B_t^r\mathbb{E}^{\mathbb{Q}}\left[\frac{S_T-K_1}{B_T^r}\middle|\mathcal{F}_t\right] \tag{5.4}$$

Assume now that the bank enters a forward with a counterparty without any collateral agreement and without any previous existing trade: there is no exchange of variation or initial margin, meaning that $C = I^{TC} = I^{FC} = 0$, $d\mathbb{Q}\otimes dt$-a.s. Exposures on such a transactions are to be funded by the internal treasury desk of the bank, hence, due to internal rules of the bank, the front office desk decides to discount cashflows via a synthetic unsecured discount curve with associated short rate process $r^f = (r_t^f)_{t\in[0,T]}$ defined via $r^f = \frac{r^{f,c}+r^{f,b}}{2}$.
Such choice implies that the official clean price from the bank perspective is

$$\hat{P}_t^1 = \mathbb{E}^{\mathbb{Q}}\left[B_t^f\int_{(t,T]}\frac{dA_u^1}{B_u^f}\middle|\mathcal{F}_t\right] = B_t^f\mathbb{E}^{\mathbb{Q}}\left[\frac{S_T-K_1}{B_T^f}\middle|\mathcal{F}_t\right]. \tag{5.5}$$

The xVA desk is forced by the internal policy of the bank to employ (5.5) as the official clean price for the transaction. However, using Proposition 4.5 it is possible to compute a consistent price which is then given by

$$\mathscr{V}_t^1(\varphi) = V_t^1(\varphi) = \hat{P}_t^1 - XVA_t^1 - DiscVA_t^1,$$

where

$$\begin{aligned}XVA_t^1 &= -CVA_t^1 + DVA_t^1 + FVA_t^1\\ &= -B_t^r\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau<T\}}1_{\{\tau^C<\tau^B\}}(1-R^C)\frac{1}{B_\tau^r}\left(\hat{P}_\tau^1 - DiscVA_\tau^1\right)^-\middle|\mathcal{G}_t\right],\\ &+ B_t^r\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau<T\}}1_{\{\tau^B<\tau^C\}}(1-R^B)\frac{1}{B_\tau^r}\left(\hat{P}_\tau^1 - DiscVA_\tau^1\right)^+\middle|\mathcal{G}_t\right],\\ &+ B_t^r\mathbb{E}^{\mathbb{Q}}\left[\int_t^{\tau\wedge T}\frac{(r_u^{f,l}-r_u)\left(\mathscr{V}_u^1(\varphi)\right)^+ - (r_u^{f,b}-r_u)\left(\mathscr{V}_u^1(\varphi)\right)^-}{B_u^r}du\middle|\mathcal{G}_t\right],\end{aligned} \tag{5.6}$$

while the discounting adjustment is

$$DiscVA_t^1 := B_t^f\mathbb{E}^{\mathbb{Q}}\left[\int_t^T (r_u - r_u^f)\frac{\hat{V}_u^1}{B_u^f}du\middle|\mathcal{F}_t\right]. \tag{5.7}$$

The $\mathbb{G}$-BSDE solved by (5.6) is given by

$$\begin{cases} -dXVA^1_t &= -\left[f(t,\hat{V}^1 - XVA^1,0,0) + r_t XVA^1_t\right]dt \\ &\quad - \sum_{k=1}^d \tilde{Z}^k_t dW^{k,\mathbb{Q}}_t - \sum_{j\in\{B,C\}} \tilde{U}^j_t dM^{j,\mathbb{Q}}_t \\ XVA^1_\tau &= 1_{\{\tau\le T_1\}}\left(\hat{V}^1_\tau(\varphi) - \theta_\tau(\hat{V}^1,0,0)\right) \end{cases} \tag{5.8}$$

and we observe that the non-linearity effect $NL_t(V^1) = 0$ is of course zero, since the portfolio between the bank and the counterparty consists of a single contingent claim.
Assume now that the counterparty is interested in a second product, e.g. a second forward contract on the risky asset S with maturity T and opposite direction, so that

$$A^2_t = 1_{\{t=T\}}(K_2 - S_T). \tag{5.9}$$

In line with the previous reasoning, the clean values from the perspective of the xVA desk and the front-office desk are respectively

$$\hat{V}^2_t(\varphi) = B^r_t \mathbb{E}^{\mathbb{Q}}\left[\frac{K_2 - S_T}{B^r_T}\middle|\mathcal{F}_t\right], \quad \hat{P}^2_t = B^r_t \mathbb{E}^{\mathbb{Q}}\left[\frac{K_2 - S_T}{B^f_T}\middle|\mathcal{F}_t\right]. \tag{5.10}$$

Given the presence of the first forward contract in the portfolio, the full value of the portfolio, now including the second claim, is

$$\mathscr{V}^2_t(\varphi) = \hat{P}^1_t + \hat{P}^2_t - XVA^2_t - DiscVA^1_t - DiscVA^2_t,$$

where

$$\begin{aligned} XVA^2_t &= -CVA^2_t + DVA^2_t + FVA^2_t \\ &= -B^r_t\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau<T\}}1_{\{\tau^C<\tau^B\}}(1-R^C)\frac{1}{B^r_\tau}\left(\hat{P}^1_\tau + \hat{P}^2_\tau - DiscVA^1_\tau - DiscVA^2_\tau\right)^-\middle|\mathcal{G}_t\right], \\ &+ B^r_t\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau<T\}}1_{\{\tau^B<\tau^C\}}(1-R^B)\frac{1}{B^r_\tau}\left(\hat{P}^1_\tau + \hat{P}^2_\tau - DiscVA^1_\tau - DiscVA^2_\tau\right)^+\middle|\mathcal{G}_t\right], \\ &+ B^r_t\mathbb{E}^{\mathbb{Q}}\left[\int_t^{\tau\wedge T}\frac{(r^{f,l}_u - r_u)\left(\mathscr{V}^2_u(\varphi)\right)^+ - (r^{f,b}_u - r_u)\left(\mathscr{V}^2_u(\varphi)\right)^-}{B^r_u}du\middle|\mathcal{G}_t\right], \end{aligned} \tag{5.11}$$

and $DiscVA^2$ is of the same form as (5.7).
The solution to the $\mathbb{G}$-BSDE

$$\begin{cases} -dXVA^2_t &= -\left[f(t,\hat{V}^2 - XVA^2,0,0) + r_t XVA^2_t\right]dt \\ &\quad - \sum_{k=1}^d \tilde{Z}^k_t dW^{k,\mathbb{Q}}_t - \sum_{j\in\{B,C\}} \tilde{U}^j_t dM^{j,\mathbb{Q}}_t \\ XVA^2_\tau &= 1_{\{\tau\le T_1\}}\left(\hat{V}^1_\tau(\varphi) + \hat{V}^2_\tau(\varphi) - \theta_\tau(\hat{V}^1 + \hat{V}^2,0,0)\right) \end{cases} \tag{5.12}$$

is given by (5.6).
Given the presence of the first claim in the portfolio, the XVA charge on the second claim is $\Delta XVA = XVA^2 - XVA^1$, whereas the non-linearity is

$$\begin{aligned} NL_t(V^2) = XVA^2_t &+ B^r_t\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau<T\}}1_{\{\tau^C<\tau^B\}}(1-R^C)\frac{1}{B^r_\tau}\left(\hat{P}^2_\tau - DiscVA^2_\tau\right)^-\middle|\mathcal{G}_t\right], \\ &- B^r_t\mathbb{E}^{\mathbb{Q}}\left[1_{\{\tau<T\}}1_{\{\tau^B<\tau^C\}}(1-R^B)\frac{1}{B^r_\tau}\left(\hat{P}^2_\tau - DiscVA^2_\tau\right)^+\middle|\mathcal{G}_t\right], \\ &- B^r_t\mathbb{E}^{\mathbb{Q}}\left[\int_t^{\tau\wedge T}\frac{(r^{f,l}_u - r_u)\left(V^2_u(\varphi)\right)^+ - (r^{f,b}_u - r_u)\left(V^2_u(\varphi)\right)^-}{B^r_u}du\middle|\mathcal{G}_t\right]. \end{aligned} \tag{5.13}$$

Observe that in the last FVA term appearing in (5.13) we have the presence of V^2, i.e. a portfolio consisting only of the second claim: all expectations in (5.13) represent the stand-alone xVA correction for the second contingent claim. In (5.11) we observe instead the presence of $\mathscr{V}^2$, i.e. a portfolio consisting of the first and the second claim. The role of netting effects in reducing the overall impact of value adjustments can be seen by observing that

$$\begin{aligned}&\hat{P}_t^1 + \hat{P}_t^2 - (DiscVA_t^1 + DiscVA_t^2)\\ &= \left(B_t^f \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_T^f} \middle| \mathcal{F}_t \right] - B_t^f \mathbb{E}^{\mathbb{Q}}\left[\int_t^T (r_u - r_u^f) \frac{B_u^r}{B_u^f} \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{B_T^r} \middle| \mathcal{F}_u \right] du \middle| \mathcal{F}_t \right] \right) (K^2 - K^1).\end{aligned} \tag{5.14}$$

This shows that the combined exposure of the two forward contracts is obviously independent of the volatility of the asset S.

We finally stress that, given a numerical scheme that allows to estimate the evolution of the conditional expectation $\hat{V}$, e.g. a regression estimator in the context of a Monte Carlo simulation, the xVA desk can immediately estimate the DiscVA, hence only a simulation in terms of r discounting is required for the implementation.

References

Agarwal, A., De Marco, S., Gobet, E., López-Salas, J., Noubiagain, F., and Zhou, A. (2018). Backward stochastic differential equations arising in initial margin requirements. Preprint hal-01686952v2.

Albanese, C., Caenazzo, S., and Crépey, S. (2016). Capital Valuation Adjustment and Funding Valuation Adjustment. *ArXiv e-prints*, page arXiv:1603.03012.

Albanese, C., Caenazzo, S., and Crépey, S. (2017). Credit, funding, margin, and capital valuation adjustments for bilateral portfolios. *Probability, Uncertainty and Quantitative Risk*, 2(1):7.

Albanese, C. and Crépey, S. (2017). XVA Analysis From the Balance Sheet. *Preprint*.

Andersen, L., Duffie, D., and Song, Y. (2019). Funding value adjustments. *The Journal of Finance*, 74(1):145–192.

Antonov, A., Issakov, S., Konikov, M., McClelland, A., and Mechkov, S. (2017). PV and XVA greeks for callable exotics by algorithmic differentiation. *SSRN Electronic Journal*.

BCBS (2014). The standardized approach for measuring counterparty credit risk exposures. Technical Report 279. Basel Committee Paper.

Bichuch, M., Capponi, A., and Sturm, S. (2018). Arbitrage-free xva. *Mathematical Finance*, 28(2):582–620.

Bielecki, T. and Rutkowski, M. (2015). Valuation and hedging of contracts with funding costs and collateralization. *SIAM J. Finan. Math.*, 6(1):594–655.

Bielecki, T. R., Cialenco, I., and Rutkowski, M. (2018). Arbitrage-free pricing of derivatives in nonlinear market models. *Probability, Uncertainty and Quantitative Risk*, 3(1):2.

Brigo, D., Buescu, C., Francischello, M., Pallavicini, A., and Rutkowski, M. (2018). Risk-neutral valuation under differential funding costs, defaults and collateralization. *ArXiv e-prints*.

Brigo, D., Capponi, A., and Pallavicini, A. (2014). Arbitrage-free bilateral counterparty risk valuation under collateralization and application to credit default swaps. *Mathematical Finance*, 24(1):125–146.

Brigo, D., Francischello, M., and Pallavicini, A. (2019). Nonlinear valuation under credit, funding and margins: existence uniqueness, invariance and disentanglement. *European Journal of Operational Research*, 274(2):788–805.

Brigo, D. and Masetti, M. (2005). Risk neutral pricing of counterparty risk. In Pykhtin, M., editor, *Counterparty Credit Risk Modeling: Risk Management, Pricing and Regulation*. Risk Books, London.

Brigo, D. and Morini, M. (2018). Dangers of bilateral counterparty risk: the fundamental impact of closeout conventions. http://arxiv.org/abs/1011.3355v1.

Brigo, D., Morini, M., and Pallavicini, A. (2013). *Counterparty Credit Risk, Collateral and Funding.* Wiley Finance. Wiley, first edition.

Brigo, D. and Pallavicini, A. (2014). Nonlinear consistent valuation of ccp cleared or csa bilateral trades with initial margins under credit, funding and wrong-way risks. *Journal of Financial Engineering*, 01(01):1450001.

Brigo, D., Pallavicini, A., and Papatheodorou, V. (2011). Arbitrage-free valuation of bilateral counterparty risk for interest-rate products: Impact of volatilities and correlations. *International Journal of Theoretical and Applied Finance*, 14(06):773–802.

Burgard, C. and Kjaer, M. (2011). Partial differential equation representations of derivatives with bilateral counterparty risk and funding costs. *The Journal of Credit Risk*, 7(3):1–19.

Burgard, C. and Kjaer, M. (2013). Funding costs, funding strategies. *Risk*, Dec 2013:82–87.

Capriotti, L. (2011). Fast greeks by algorithmic differentiation. *The Journal of Computational Finance*, 14(3):3–35.

Carbone, R., Ferrario, B., and Santacroce, M. (2008). Backward stochastic differential equations driven by càdlàg martingales. *Theory Probab. Appl.*, 52(2):304–314.

Cherubini, U. (2005). Counterparty risk in derivatives and collateral policies: The replicating portfolio approach. In Tilman, L., editor, *ALM of Financial Institutions*. Institutional Investor Books.

Crépey, S. (2015a). Bilateral counterparty risk under funding constraints—part i: Pricing. *Mathematical Finance*, 25(1):1–22.

Crépey, S. (2015b). Bilateral counterparty risk under funding constraints—part ii: Cva. *Mathematical Finance*, 25(1):23–50.

Crépey, S., Bielecki, T. S., and Brigo, D. (2014). *Counterparty risk and funding: a tale of two puzzles*, volume 31 of *Chapman and Hall/CRC Press Series in Financial Mathematics*. Chapman and Hall/CRC, first edition.

Dellacherie, C. and Meyer, P. A. (1982). *Probabilities and Potential B*, volume 72 of *North-Holland Mathematics Studies*. North-Holland.

Duffie, D. and Huang, M. (1996). Swap rates and credit quality. *The Journal of Finance*, 51(3):921–949.

El Karoui, N., Peng, S., and Quenez, M. C. (1997). Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71.

Fries, C. (2019a). Fast stochastic forward sensitivities in monte carlo simulations using stochastic automatic differentiation (with applications to initial margin valuation adjustments). *Journal of Computational Finance*, 22(4):103–125.

Fries, C., Khol-Landgraf, P., and Viehmann, M. (2018). Melting sensitivities - exact and approximate margin valuation adjustments. Preprint, https://ssrn.com/abstract=3095619.

Fries, C. P. (2019b). Stochastic automatic differentiation: automatic differentiation for monte-carlo simulations. *Quantitative Finance*, 0(0):1–17.

Fujii, M., Shimada, A., and Takahashi, A. (2010). Note on construction of multiple swap curves with and without collateral. Preprint (available at `http://ssrn.com/abstract=1440633`).

Fujii, M., Shimada, A., and Takahashi, A. (2011). A market model of interest rates with dynamic basis spreads in the presence of collateral and multiple currencies. *Wilmott*, 54:61–73.

Garcia Trillos, C. A., Henrard, M. P. A., and Macrina, A. (2016). Estimation of future initial margins in a multi-curve interest rate framework. *SSRN eLibrary*.

Green, A. (2015). *XVA: Credit, Funding and Capital Valuation Adjustments*. Wiley Finance. John Wiley and Sons Inc, Chichester.

Gregory, J. (2015). *The xVA challenge*. Wiley Finance. Wiley, third edition.

Henrard, M. (2017). *Algorithmic Differentiation in Finance Explained*. Financial Engineering Explained. Palgrave Macmillan.

Hull, J. and White, A. (2012). The fva debate. *Risk*, 25(July):83–85.

ISDA (2018). Isda simm methodology, version 2.1. Technical report.

Lichters, R., Stamm, R., and Gallagher, D. (2015). *Modern Derivatives Pricing and Credit Exposure Analysis: Theory and Practice of CSA and XVA Pricing, Exposure Simulation and Backtesting*. Applied Quantitative Finance. Palgrave Macmillan, first edition.

Mackenzie Smith, R. (2017). Banks calm on eurex-lch basis volatility.

Moreni, N. and Pallavicini, A. (2017). Fx modelling in collateralized markets: foreign measures, basis curves, and pricing formulae. Preprint (available at https://arxiv.org/abs/1508.04321).

Nie, T. and Rutkowski, M. (2016). Bsdes driven by multidimensional martingales and their applications to markets with funding costs. *Theory Probab. Appl.*, 60(4):604–630.

Pallavicini, A., Perini, D., and Brigo, D. (2011). Funding Valuation Adjustment: a consistent framework including CVA, DVA, collateral,netting rules and re-hypothecation. *ArXiv e-prints*, page arXiv:1112.1521.

Peng, S. and Yang, Z. (2009). Anticipated backward stochastic differential equations. *The Annals of Probability*, 37(3):877–902.

Piterbarg, V. (2010). Funding beyond discounting: collateral agreements and derivatives pricing. *Risk Magazine*, 2:97–102.

Piterbarg, V. (2012). Cooking with collateral. *Risk Magazine*, 2:58–63.

Sokol, A. (2014). *Long-Term Portfolio Simulation: For XVA, Limits, Liquidity and Regulatory Capital*. Risk Books, London.

Appendix A. Existence and Uniqueness of BSDEs

In this section we review some results on existence and uniqueness for some BSDEs. Our main references are Nie and Rutkowski (2016), which in turn extends results from Carbone et al. (2008), and Agarwal et al. (2018).

Let $M = \left(M^1, \dots M^d\right)^\top$ be a d-dimensional, real-valued, continuous and square integrable martingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, where the filtration is assumed to satisfy the usual hypotheses and we assume that the predictable representation property holds with respect to M for $(\mathbb{F}, \mathbb{Q})$-martingales. We use $\langle M \rangle$ to denote the quadratic variation of M.

Assumption A.1 (Nie and Rutkowski (2016) Assumption 3.1)**.** *There exists an* $\mathbb{R}^{d\times d}$*-valued process* m *and an* $\mathbb{F}$*-adapted, continuous, bounded, increasing process* Q *with* $Q_0 = 0$ *such that, for all* $t \in [0, T]$,

$$\langle M \rangle_t = \int_0^t m_u m_u^\top dQ_u. \tag{A.1}$$

If $M = W$ is a one-dimensional standard Brownian motion, then $Q_t = t$, whereas m corresponds to the identity matrix. Next we introduce the driver of the BSDE via the following

Assumption A.2 (Nie and Rutkowski (2016) Assumption 3.2)**.** *Let* $h : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ *be an* $\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d)$*-measurable function such that* $h(\cdot, \cdot, y, z)$ *is an* $\mathbb{F}$*-adapted process for any fixed* $(y, z) \in \mathbb{R} \times \mathbb{R}^d$.

The BSDEs of interest in view of financial applications are forward-backward SDEs (FBSDEs). Following Nie and Rutkowski (2016), we introduce a generic (forward) factor matrix-valued process given by

$$\mathbb{X}_t := \begin{pmatrix} X_t^1 & 0 & \dots & 0 \\ 0 & X_t^2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & X_t^d \end{pmatrix}, \ t \in [0, T],$$

where the auxiliary processes X^i, $i = 1, \dots, d$, are assumed to be $\mathbb{F}$-adapted. The processes X^i represent market risk factors or traded assets. We assume that the function h of Assumption A.2 can be written as $h(\omega, t, y, z) = g(\omega, t, y, \mathbb{X}_t z)$, for g satisfying Assumption A.2.

Definition A.3 (Nie and Rutkowski (2016) Definition 4.1)**.** *We say that an* $\mathbb{R}^{d\times d}$*-valued process* γ *satisfies the ellipticity condition if there exists a constant* $\Lambda > 0$ *such that*

$$\sum_{i=1}^d \left(\gamma_t \gamma_t^\top\right)_{ij} a_i a_j \geq \Lambda \|a\| \tag{A.2}$$

for all $a \in \mathbb{R}^d$ *and* $t \in [0, T]$.

Assumption A.4 (Nie and Rutkowski (2016) Assumption 4.2)**.** *The* $\mathbb{R}^{d\times d}$*-valued* $\mathbb{F}$*-adapted process* m *in* (A.1) *is given by*

$$m_t m_t^\top = \mathbb{X}_t \gamma_t \gamma_t^\top \mathbb{X}_t^\top,$$

where $\gamma = [\gamma]_{ij}$ *is a* d*-dimensional square matrix of* $\mathbb{F}$*-adapted processes satisfying the ellipticity condition* (A.2).

In the following we recall some definitions from Nie and Rutkowski (2016).

Definition A.5. *We say that the function* $h : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ *satisfies*

- *the* uniform Lipschitz condition *if there exists a constant L such that for any $t \in [0,T]$ and all $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$*
$$|h(t,y_1,z_1) - h(t,y_2,z_2)| \leq L\left(|y_1 - y_2| + \|z_1 - z_2\|\right);$$
- *the* uniform m-Lipschitz condition *if there exists a constant $\hat{L}$ such that for any $t \in [0,T]$ and all $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$*
$$|h(t,y_1,z_1) - h(t,y_2,z_2)| \leq \hat{L}\left(|y_1 - y_2| + \left\|m_t^\top (z_1 - z_2)\right\|\right);$$
- *the* uniform $\mathbb{X}$-Lipschitz condition *if there exists a constant $\tilde{L}$ such that for any $t \in [0,T]$ and all $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$*
$$|h(t,y_1,z_1) - h(t,y_2,z_2)| \leq \tilde{L}\left(|y_1 - y_2| + \|\mathbb{X}_t(z_1 - z_2)\|\right).$$

Lemma A.6 (Nie and Rutkowski (2016) Lemma 4.2)**.** *If Assumption A.4 holds and the generator h is uniform $\mathbb{X}$-Lipschitz, then h is uniform m-Lipschitz with $\hat{L} = \tilde{L}\max\left\{1, \Lambda^{-\frac{1}{2}}\right\}$, where Λ is the constant defined in* (A.2)*.*

Theorem A.7 provides the existence and uniqueness result, which is relevant for our purposes.

Theorem A.7 (Nie and Rutkowski (2016) Theorem 4.1)**.** *Assume that the function h can be represented as $h(t,y,z) = g(t,y,\mathbb{X}_t z)$, where the function $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ satisfies the uniform Lipschitz condition. Let the process $h(\cdot,0,0)$ belong to the space $\mathcal{H}^2(\mathbb{Q})$, the random variable η belong to $L^2(\mathcal{F}_T,\mathbb{Q})$ and U be a real-valued $\mathbb{F}$-adapted process such that $U \in \mathcal{H}^2(\mathbb{Q})$ and $U_T \in L^2(\mathcal{F}_T,\mathbb{Q})$. Assume that the process m satisfies Assumption A.4 for some constant $\Lambda > 0$. Then the BSDE*

$$\begin{cases} dY_t &= Z_t^\top dM_t - h(t,Y_t,Z_t)dQ_t + dU_t, \\ Y_T &= \eta, \end{cases} \tag{A.3}$$

has a unique solution (Y,Z) such that $(Y, m^\top Z) \in \mathcal{H}^2(\mathbb{Q}) \times \mathcal{H}^{2,d}(\mathbb{Q})$. Moreover the processes Y and U satisfy

$$\mathbb{E}^{\mathbb{Q}}\left[\sup_{t\in[0,T]} |Y_t - U_t|^2\right] < \infty.$$

We now recall the results of Agarwal et al. (2018).

Assumption A.8 (Agarwal et al. (2018) Assumption (A))**.** *For any $X \in \mathcal{S}^2(\mathbb{Q})$, $(\Lambda_t(X_{t:T}))_{t\in[0,T]}$ defines a stochastic process that belongs to $\mathcal{H}^2(\mathbb{Q})$. There exists a constant $C_\Lambda > 0$ and a family of measures $(\nu_t)_{t\in[0,T]}$ on $\mathbb{R}$ such that for every $t \in [0,T]$ ν_t has support included in $[t,T]$, $\nu([t,T]) = 1$, and for any y^1, $y^2 \in \mathcal{S}^2(\mathbb{Q})$, we have*

$$|\Lambda_t(y^1_{t:T}) - \Lambda_t(y^2_{t:T})| \leq C_\Lambda \mathbb{E}\left[\int_t^T |y^1_s - y^2_s|\nu_t(ds)\Big|\mathcal{F}_t\right],\ dt \otimes d\mathbb{P}\ a.e.$$

Moreover, there exists a constant $k > 0$ such that for every $\beta \geq 0$ and every continuous path $x : [0,T] \to \mathbb{R}$,

$$\int_0^T e^{\beta s}\int_s^T |x_u|\nu_s(du)ds \leq k \sup_{t\in[0,T]} e^{\beta s}|x_t|.$$

Assumption A.9 (Agarwal et al. (2018) Assumption (S))**.** *For any y, z, $\lambda \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}$, $f(\cdot,y,z,\lambda)$ is an $\mathbb{F}$-adapted stochastic process with values in $\mathbb{R}$ and there exists a constant $C_f > 0$ such that $\mathbb{P}$-a.s.,* *for all* (s, y_1, z_1, λ_1), $(s, y_2, z_2, \lambda_2) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$,

$$|f(s, y_1, z_1, \lambda_1) - f(s, y_2, z_2, \lambda_2)| \leq C_f \left(|y_1 - y_2| + |z_1 - z_2| + |\lambda_1 - \lambda_2|\right).$$

Moreover, $\mathbb{E}\left[\int_0^T |f(s, 0, 0, 0)|^2\right] < \infty$.

Theorem A.10 (Agarwal et al. (2018) Theorem 2.1)**.** *Under Assumptions A.8 and A.9, for any terminal condition* $\xi \in L_T^2(\mathcal{F}_T, \mathbb{Q})$, *the BSDE*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \Lambda(Y_{s:T}))ds - \int_t^T Z_s dW_s,\ t \in [0, T]$$

has a unique solution $(Y, Z) \in \mathcal{S}^2(\mathbb{Q}) \times \mathcal{H}^{2,d}(\mathbb{Q})$.

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