Stochastic programs without duality gaps

Teemu Pennanen Ari-Pekka Perkkiö

October 22, 2011

Abstract

This paper studies dynamic stochastic optimization problems parameterized by a random variable. Such problems arise in many applications in operations research and mathematical finance. We give sufficient conditions for the existence of solutions and the absence of a duality gap. Our proof uses extended dynamic programming equations, whose validity is established under new relaxed conditions that generalize certain no-arbitrage conditions from mathematical finance.

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $(\mathcal{F}_t)_{t=0}^T$ (an increasing sequence of sub-sigma-algebras of $\mathcal{F}$) and consider the dynamic stochastic optimization problem

$$\text{minimize} \quad Ef(x(\omega), u(\omega), \omega) \quad \text{over } x \in \mathcal{N}, \tag{P}$$

where, for given integers n_t and m

$$\mathcal{N} = \{(x_t)_{t=0}^T \,|\, x_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^{n_t})\},$$

$u \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^m)$ and f is an extended real-valued convex *normal integrand* on $\mathbb{R}^n \times \mathbb{R}^m \times \Omega$, where $n = n_0 + \ldots + n_T$. Recall that $L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^{n_t})$ denotes the space of equivalence classes of $\mathcal{F}_t$-measurable $\mathbb{R}^{n_t}$-valued functions that coincide P-almost surely. That f is a normal integrand, means that the set-valued mapping $\omega \mapsto \{(x, u, \alpha) \,|\, f(x, u, \omega) \le \alpha\}$ is closed-valued and $\mathcal{F}$-measurable; see e.g. [23, Chapter 14]. This implies that f is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^m)$-measurable (see [23, Corollary 14.34]), so that $\omega \mapsto f(x(\omega), u(\omega), \omega)$ is $\mathcal{F}$-measurable for every $x \in \mathcal{N}$. Throughout this paper, the expectation is defined for any measurable function by setting it equal to $+\infty$ unless the positive part is integrable. We will also assume that $\mathcal{F}$ as well as $\mathcal{F}_t$ for $t = 0, \ldots, T$ are complete with respect to P.[1]

[1]This allows us to use certain results on conditional expectations of integrands which are not necessarily normal in the general case. This is based on [23, Corollary 14.34], which says that, when $\mathcal{F}$ is P-complete, then a function $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$ is a normal integrand if and only if it is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^m)$-measurable and $(x, u) \mapsto f(\omega, x, u)$ is lower semicontinuous for every ω.

The measurable function u may be interpreted as a parameter or a perturbation in a given stochastic optimization problem. It was shown in [17] that (P) covers many important problems in operations research and mathematical finance and how the conjugate duality framework of Rockafellar [21] allows for a unified treatment of many well-known duality frameworks. In that context, the lower-semicontinuity of the value function

$$\varphi(u) = \inf_{x \in \mathcal{N}} Ef(x(\omega), u(\omega), \omega)$$

over an appropriate space of measurable functions u is equivalent to the absence of a duality gap; see [17, Section 2] for a precise statement. In certain applications, most notably in mathematical finance, the objective in (P) lacks the inf-compactness properties required by the classical "direct method" of calculus of variations for establishing lower semicontinuity (and the existence of solutions). It was shown in [17, Section 5] how certain measure theoretic techniques from mathematical finance can be combined with classical techniques of convex analysis to obtain the lower semicontinuity of φ. It is essential for this that the strategies $x \in \mathcal{N}$ are allowed to be general measurable functions not restricted to be e.g. integrable. The lower semicontinuity result given in [17], however, applies to normal integrands f that take only the values 0 and $+\infty$. While that already covers some fundamental results in mathematical finance, as illustrated in [17, Section 6], it is far from satisfactory from the general point of view.

The main purpose of this paper is to establish the lower semicontinuity of φ for more general normal integrands. This will be done in Section 3. Our proof extends that of [17, Theorem 8], which employs a recursive argument reminiscent of dynamic programming. We clarify this connection in Section 2 by generalizing the dynamic programming equations proposed by Rockafellar and Wets [22] for stochastic convex optimization. The dynamic programming equations were substantially generalized already by Evstigneev [9] who removed many of the assumptions made in [22], including convexity. We will show that in the convex case, the inf-compactness assumption made in both [22] and [9] can be replaced by weaker "recession condition" which subsumes, in particular, various no-arbitrage conditions used in mathematical finance. An early application of recession analysis to utility maximization in financial markets can be found in Bertsekas [1]. Section 4 of this paper gives an application to an optimal consumption problem in illiquid markets.

2 Dynamic programming

The purpose of this section is to extend the *dynamic programming* recursion of [22, Section 3] which generalizes the classical Bellman equation for convex stochastic optimization. We will use the notion of a conditional expectation of a normal integrand much as in [9] where certain assumptions (convexity, nonanticipativity of the domain of f and the boundedness of the strategies) of [22] were relaxed. We show that, in the convex case, the inf-compactness assumption used in both [22] and [9] can be replaced by a milder condition on the directions of recession much like in the classical closedness results of finite-dimensional convex analysis; see [20, Section 8]. In certain financial applications, the new condition turns out to be equivalent to the classical no-arbitrage condition.

Let X be a nonnegative $\mathcal{F}$-measurable function and let $\mathcal{G} \subseteq \mathcal{F}$ be another sigma-algebra. Then, there is a $\mathcal{G}$-measurable nonnegative function $E^{\mathcal{G}}X$, unique up to sets of P-measure zero, such that

$$E[\chi_A X] = E[\chi_A (E^{\mathcal{G}} X)] \quad \forall A \in \mathcal{G}, \tag{1}$$

where χ_A denotes the characteristic function of A; see e.g. Shiryaev [24, II.7]. The function $E^{\mathcal{G}}X$ is called the $\mathcal{G}$-*conditional expectation* of X. For a general $\mathcal{F}$-measurable extended real-valued function X, we set

$$E^{\mathcal{G}}X := E^{\mathcal{G}}X_+ - E^{\mathcal{G}}X_-,$$

where again, the convention $\infty - \infty = \infty$ is used. It is easily checked that with the extended definition of the integral, (1) is then valid for any measurable function X. Our definition of conditional expectation extends [24, Definition II.7.1], which assumes that $\min\{E^{\mathcal{G}}X_+, E^{\mathcal{G}}X_-\} < \infty$ almost surely. Our choice of setting $\infty - \infty = \infty$ is not arbitrary but specifically directed towards minimization problems.

The $\mathcal{G}$-*conditional expectation* of a normal integrand h is a $\mathcal{G}$-measurable normal integrand $E^{\mathcal{G}}h$ such that

$$(E^{\mathcal{G}}h)(x(\omega), \omega) = E^{\mathcal{G}}[h(x(\cdot), \cdot)](\omega) \quad P\text{-a.s.}$$

for all $x \in L^0(\Omega, \mathcal{G}, P; \mathbb{R}^n)$. There are various conditions that guarantee the existence and uniqueness of a conditional expectation of a normal integrand; see e.g. Bismut [4], Dynkin and Evstigneev [8], Castaing and Valadier [5, Section VIII.9], Thibault [25], Truffert [26] or Choirat, Hess and Seri [6]. The following suffices for the purposes of this paper.

Lemma 1. *Let $\mathcal{G} \subseteq \mathcal{F}$ be a sigma-algebra and assume that h is an $\mathcal{F}$-normal integrand with an integrable lower bound i.e. an integrable function m such that $h(x, \omega) \geq m(\omega)$ for every x and ω. Then h has a well-defined conditional expectation $E^{\mathcal{G}}h$ which has the integrable lower bound $E^{G}m$.*

Proof. The integrable lower bound implies, for example, the quasi-integrability condition of Thibault [25] as well as the condition of Choirat, Hess and Seri [6], both of which give the existence and uniqueness of the conditional expectation. It follows from the monotonicity of the conditional expectation that if $f \geq m$ for an integrable function m, then $E^{\mathcal{G}}f \geq E^{\mathcal{G}}m$. □

We will study problem (P) for a fixed $u \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^m)$ so we will omit it from the notation and define

$$h(x, \omega) = f(x, u(\omega), \omega).$$

By [23, 14.45(c)], h is a normal integrand. The convexity of f implies that of h. We will use the notation $E_t = E^{\mathcal{F}_t}$ and $x^t = (x_0, \dots, x_t)$ and define extended real-valued functions $h_t, \tilde{h}_t : \mathbb{R}^{n_1+\cdots+n_t} \times \Omega \to \overline{\mathbb{R}}$ recursively for $t = T, \dots, 0$ by

$$\begin{aligned} \tilde{h}_T &= h, \\ h_t &= E_t \tilde{h}_t, \\ \tilde{h}_{t-1}(x^{t-1}, \omega) &= \inf_{x_t \in \mathbb{R}^{n_t}} h_t(x^{t-1}, x_t, \omega). \end{aligned} \tag{2}$$

This is essentially the dynamic programming recursion introduced in [22]. Our formulation with conditional expectations of normal integrands is closer to [9], where certain assumptions of [22] were relaxed. In the above formulation, one does not separate the decision variables x_t into “state” and “control” like in the classical dynamic programming models; see e.g. [2] and [3]. A formulation closer to the classical dynamic programming equations will be given in Corollary 4 below. A recent application of dynamic programming to mathematical finance can be found in Rásonyi and Stettner [19, Section 5].

In order to ensure that h_t and $\tilde{h}_t$ are well-defined it suffices to require that the function h has an integrable lower bound and that $h(\cdot, \omega)$ is inf-compact (i.e. $\{x \in \mathbb{R}^n \,|\, h(x, \omega) \le \alpha\}$ is compact for every $\alpha \in \mathbb{R}$) for every $\omega \in \Omega$; see [9, Theorem 5]. In the convex case, the compactness assumption can be replaced by a weaker condition stated in terms of the *recession function* of h. If $\operatorname{dom} h(\cdot, \omega)$ is nonempty, then the recession function has the expression

$$h^\infty(x, \omega) = \sup_{\lambda > 0} \frac{h(\lambda x + \bar{x}, \omega) - h(\bar{x}, \omega)}{\lambda},$$

which is independent of the choice of $\bar{x} \in \operatorname{dom} h(\cdot, \omega)$; see [20, Theorem 8.5] or [23, 3.21]. By [23, Exercise 14.54(a)], the function h^∞ is a convex normal integrand. If $h(\cdot, \omega)$ has an integrable lower bound, then $h^\infty(x, \omega) \ge 0$ for every $x \in \mathbb{R}^n$ as is easily seen by letting $\lambda \to \infty$.

Lemma 2. *Assume that h_t is a normal integrand and that the set-valued mapping*

$$N_t(\omega) = \{x_t \in \mathbb{R}^{n_t} \,|\, h_t^\infty(x^t, \omega) \le 0,\ x^{t-1} = 0\}$$

is linear-valued[2]*. Then $\tilde{h}_{t-1}$ is a normal integrand with*

$$\tilde{h}_{t-1}^\infty(x^{t-1}, \omega) = \inf_{x_t \in \mathbb{R}^{n_t}} h_t^\infty(x^{t-1}, x_t, \omega).$$

Moreover, given an $x \in \mathcal{N}$, there is an $\mathcal{F}_t$-measurable $\bar{x}_t$ such that $\bar{x}_t(\omega) \perp N_t(\omega)$ and

$$\tilde{h}_{t-1}(x^{t-1}(\omega), \omega) = h_t(x^{t-1}(\omega), \bar{x}_t(\omega), \omega).$$

Proof. By [20, Theorem 9.2], the linearity condition implies that the infimum in the definition of $\tilde{h}_{t-1}$ is attained and that $\tilde{h}_{t-1}(\cdot, \omega)$ is a lower semicontinuous convex function with

$$\tilde{h}_{t-1}^\infty(x^{t-1}, \omega) = \inf_{x_t \in \mathbb{R}^{n_t}} h_t^\infty(x^{t-1}, x_t, \omega).$$

[2] A set-valued mapping $\omega \mapsto S(\omega)$ is linear-valued if $S(\omega)$ is a linear space for every ω.

Indeed, we have $\tilde{h}_{t-1}(x^{t-1},\omega) = \inf\{h_t(x^t,\omega)\,|\,Ax^t = x^{t-1}\}$, where A is the linear mapping defined by $Ax^t = x^{t-1}$. The linearity condition can be written in terms of A as

$$h_t^\infty(x^t,\omega) \le 0,\ h_t^\infty(-x^t,\omega) > 0 \implies Ax^t \neq 0,$$

which is exactly the condition in [20, Theorem 9.2].

By [23, Proposition 14.47], the lower semicontinuity implies that $\tilde{h}_{t-1}$ is an $\mathcal{F}_t$-measurable convex normal integrand. By [23, Proposition 14.45(c)], the function $p(x,\omega) := h_t(x^{t-1}(\omega),x,\omega)$ is then also an $\mathcal{F}_t$-measurable normal integrand so, by [23, Theorem 14.37], there is an $\mathcal{F}_t$-measurable $\bar{x}_t$ that attains the minimum for every ω. By [20, Corollary 8.6.1], the value of $h_t(x^{t-1}(\omega),x,\omega)$ does not change if we replace $\bar{x}_t(\omega)$ by its projection to the orthogonal complement of $N_t(\omega)$. By [23, Exercise 14.17], such a projection preserves measurability. □

It is clear that if h_t has an integrable lower bound, then so will $\tilde{h}_{t-1}$. Applying Lemmas 1 and 2 recursively backwards for $t = T,\dots,0$, we then see that if h has an integrable lower bound, the functions $\tilde{h}_t$ and h_t are well-defined for every t provided that N_t is linear-valued at each step.

We now get the following refinement of the optimality conditions in [22, Theorem 1] and [9, Theorems 1 and 2] in the convex case.

Theorem 3. *Assume that h has an integrable lower bound and that N_t is linear-valued for $t = T,\dots,0$. The functions h_t are then well-defined normal integrands and we have for every $x \in \mathcal{N}$ that*

$$Eh_t(x_t(\omega),\omega) \ge \inf\,(\mathrm{P}) \quad t = 0,\dots,T. \tag{3}$$

Optimal solutions $x \in \mathcal{N}$ exist and they are characterized by the condition

$$x_t(\omega) \in \operatorname*{argmin}_{x_t} h_t(x^{t-1}(\omega),x_t,\omega) \quad P\text{-a.s.} \quad t = 0,\dots,T.$$

which is equivalent to having equalities in (3)*. Moreover, there is an optimal solution $x \in \mathcal{N}$ such that $x_t \perp N_t$ for every $t = 0,\dots,T$.*

Proof. As noted above, a recursive application of Lemmas 1 and 2 imply that the functions h_t and $\tilde{h}_t$ are well-defined normal integrands. Given an $x \in \mathcal{N}$, the law of iterated expectations (see e.g. Shiryaev [24, Section II.7]) gives

$$Eh_t(x^t(\omega),\omega) \ge E\tilde{h}_{t-1}(x^{t-1}(\omega),\omega) = Eh_{t-1}(x^{t-1}(\omega),\omega) \quad t = 1,\dots,T.$$

Thus,

$$Eh(x(\omega),\omega) = Eh_T(x^T(\omega),\omega) \ge Eh_0(x^0(\omega),\omega) \ge E \inf_{x_0\in\mathbb{R}^{n_0}} h_0(x_0,\omega),$$

where the inequalities hold as equalities if and only if

$$h_t(x^t(\omega),\omega) = \tilde{h}_{t-1}(x^{t-1}(\omega),\omega) \quad P\text{-a.s.} \quad t = 0,\dots,T.$$

The existence of such an $x \in \mathcal{N}$ with $x_t \perp N_t$ follows by applying Lemma 2 recursively for $t = 0,\dots,T$. □

When the normal integrand h has a separable structure, the dynamic programming equations (2) can be written in a more familiar form.

Corollary 4 (Bellman equations). *Assume that*

$$h(x,\omega)=\sum_{t=0}^{T}k_t(x_{t-1},x_t,\omega)$$

for some fixed initial state x_{-1} and $\mathcal{F}_t$-measurable normal integrands h_t with integrable lower bounds. Consider the functions $V_t:\mathbb{R}^{n_t}\times\Omega\to\overline{\mathbb{R}}$ given by

$$\begin{aligned}V_T(x_T,\omega)&=0,\\ \tilde V_{t-1}(x_{t-1},\omega)&=\inf_{x_t\in\mathbb{R}^{n_t}}\{k_t(x_{t-1},x_t,\omega)+V_t(x_t,\omega)\},\\ V_{t-1}&=E_{t-1}\tilde V_{t-1}\end{aligned}\tag{4}$$

and assume that the set-valued mappings

$$N_t(\omega)=\{x_t\in\mathbb{R}^{n_t}\mid k_t^\infty(0,x_t,\omega)+V_t^\infty(x_t,\omega)\le 0\}$$

are linear-valued for each $t=T,\dots,0$. The functions V_t are then well-defined normal integrands and we have for every $x\in\mathcal{N}$ that

$$E\left[\sum_{s=0}^{t}k_s(x_{s-1}(\omega),x_s(\omega),\omega)+V_t(x_t(\omega),\omega)\right]\ge\inf\text{(P)}\quad t=0,\dots,T.\tag{5}$$

Optimal solutions $x\in\mathcal{N}$ exist and they are characterized by the condition

$$x_t(\omega)\in\operatorname*{argmin}_{x_t\in\mathbb{R}^{n_t}}\{k_t(x_{t-1}(\omega),x_t,\omega)+V_t(x_t,\omega)\}\quad P\text{-a.s.}\quad t=0,\dots,T,$$

which is equivalent to having equalities in (5). Moreover, there is an optimal solution $x\in\mathcal{N}$ such that $x_t\perp N_t$ for every $t=0,\dots,T$.

Proof. By Theorem 3, it suffices to show that

$$h_t(x^t,\omega)=\sum_{s=0}^{t}k_s(x_{s-1},x_s,\omega)+V_t(x_t,\omega)\tag{6}$$

for every $t=0,\dots,T$. For $t=T$, (6) is obvious since $V_T=0$ by definition. Assuming that (6) holds for t, we get

$$\begin{aligned}\tilde h_{t-1}(x^{t-1},\omega)&=\inf_{x_t\in\mathbb{R}^{n_t}}h_t(x^{t-1},x_t,\omega)\\ &=\sum_{s=0}^{t-1}k_s(x_{s-1},x_s,\omega)+\inf_{x_t\in\mathbb{R}^{n_t}}\{k_t(x_{t-1},x_t,\omega)+V_t(x_t,\omega)\}\\ &=\sum_{s=0}^{t-1}k_s(x_{s-1},x_s,\omega)+\tilde V_{t-1}(x_{t-1},\omega)\end{aligned}$$

and then, since for $s = 0, \ldots, t-1$, k_s is $\mathcal{F}_{t-1}$-measurable,

$$h_{t-1}(x^{t-1}, \omega) = \sum_{s=0}^{t-1} k_s(x_{s-1}, x_s, \omega) + V_{t-1}(x_{t-1}, \omega),$$

where, by Lemma 1, V_t is a well-defined normal integrand when N_t is linear-valued. □

The rest of this section is devoted to the study of the linearity condition in Theorem 3. Recall that a *$\mathcal{G}$-measurable selector* of an $\mathbb{R}^n$-valued set-valued mapping C is a $\mathcal{G}$-measurable function x such that $x(\omega) \in C(\omega)$ almost surely.

Lemma 5. *Let $\mathcal{G} \subseteq \mathcal{F}$ be a sigma-algebra and assume that h is an $\mathcal{F}$-normal integrand with an integrable lower bound. If there is an $\bar{x} \in L^0(\Omega, \mathcal{G}, P; \mathbb{R}^n)$ such that $Eh(\bar{x}(\omega), \omega)$ is finite, then $(E^{\mathcal{G}}h)^\infty = E^{\mathcal{G}}h^\infty$ and the level sets*

$$\operatorname{lev}_0 h^\infty(\omega) = \{x \in \mathbb{R}^n \mid h^\infty(x, \omega) \le 0\},$$
$$\operatorname{lev}_0 (E^{\mathcal{G}}h)^\infty(\omega) = \{x \in \mathbb{R}^n \mid (E^{\mathcal{G}}h)^\infty(x, \omega) \le 0\}$$

have the same $\mathcal{G}$-measurable selectors.

Proof. By [23, Exercise 14.54], h^∞ is a well-defined $\mathcal{F}$-normal integrand. Moreover, the lower bound on h implies that h^∞ is nonnegative. By Lemma 1, $E^{\mathcal{G}}h$ and $E^{\mathcal{G}}h^\infty$ are thus well-defined. To show that the latter is the recession function of the former, let $x \in L^0(\Omega, \mathcal{G}, P; \mathbb{R}^n)$ and $A \in \mathcal{G}$. Convexity of h implies that the difference quotient

$$\frac{h(\bar{x}(\omega) + \lambda x(\omega), \omega) - h(\bar{x}(\omega), \omega)}{\lambda}$$

is increasing in λ for every ω; see e.g. [20, Theorem 23.1]. The lower bound on h and the integrability of $h(\bar{x}(\cdot), \cdot)$ thus imply that, for $\lambda \ge 1$, the quotients are minorized by a fixed integrable function. Monotone convergence theorem then gives for every $A \in \mathcal{G}$

$$\begin{aligned}
E[1_A h^\infty(x)] &= E[1_A \lim_{\lambda \nearrow \infty} (h(\bar{x} + \lambda x) - h(\bar{x}))/\lambda] \\
&= \lim_{\lambda \nearrow \infty} E[1_A (h(\bar{x} + \lambda x) - h(\bar{x}))/\lambda] \\
&= \lim_{\lambda \nearrow \infty} E[1_A ((E^{\mathcal{G}}h)(\bar{x} + \lambda x) - (E^{\mathcal{G}}h)(\bar{x}))/\lambda] \\
&= E[1_A \lim_{\lambda \nearrow \infty} ((E^{\mathcal{G}}h)(\bar{x} + \lambda x) - (E^{\mathcal{G}}h)(\bar{x}))/\lambda] \\
&= E[1_A (E^{\mathcal{G}}h)^\infty(x)],
\end{aligned}$$

which means that $(E^{\mathcal{G}}h)^\infty$ is the conditional expectation of h^∞.

To prove the last claim, let $x \in L^0(\Omega, \mathcal{G}, P; \mathbb{R}^n)$. By the first claim and the definition of a conditional integrand,

$$(E^{\mathcal{G}}h^\infty)(x(\cdot), \cdot) = E^{\mathcal{G}}h^\infty(x(\cdot), \cdot).$$

We have $h^\infty(x(\omega),\omega) \le 0$ almost surely if and only if $E^{\mathcal{G}} h^\infty(x(\cdot),\cdot) \le\ 0$ almost surely, since $h^\infty \ge 0$. □

The following result shows that the linearity condition of Theorem 3 can be stated in terms of the original normal integrand h directly. In the proof, we will denote the set of $\mathcal{G}$-measurable selectors of a set-valued mapping C by $L^0(\mathcal{G}; C)$. We will also use the fact that if C is closed-valued and $\mathcal{G}$-measurable, then it is almost surely linear-valued if and only if the set of its measurable selectors is a linear space. This follows easily by considering the Castaing representation of C; see e.g. [23, Theorem 14.5].

Lemma 6. *Assume that h has an integrable lower bound and that $Eh(\bar{x}(\omega),\omega) < \infty$ for some $\bar{x} \in \mathcal{N}$. Then h_t is well-defined and N_t is linear-valued for $t = T,\dots,0$ if and only if*

$$\mathcal{L} = \{x \in \mathcal{N} \,|\, h^\infty(x(\omega),\omega) \le 0 \ \text{a.s.}\}$$

is a linear space. If $x \in \mathcal{L}$ is such that $x^{t-1} = 0$ then $x_t \in N_t$ almost surely.

Proof. Redefining $h(x,\omega) := h(x - \bar{x}(\omega),\omega)$, we may assume that $\bar{x} = 0$. Indeed, such a translation amounts to translating the functions $\tilde{h}_t$ and h_t accordingly and it does not affect the recession functions $\tilde{h}_t^\infty$ and h_t^∞. We proceed by induction on T. When $T = 0$, Lemma 5 gives

$$\mathcal{L} = \{x \in \mathcal{N} \,|\, h_T^\infty(x(\omega),\omega) \le 0 \ \text{a.s.}\} = L^0(\mathcal{F}_T; N_T).$$

Since N_T is $\mathcal{F}_T$-measurable, the linearity of $\mathcal{L}$ is equivalent to N_T being linear-valued. Let now T be arbitrary and assume that the claim holds for every $(T-1)$-period model.

If $\mathcal{L}$ is linear then $\mathcal{L}' = \{x \in \mathcal{N} \,|\, x_0 = 0,\ h^\infty(x(\omega),\omega) \le 0 \ \text{a.s.}\}$ is linear as well. Applying the induction hypothesis to the $(T-1)$-period model obtained by fixing $x_0 \equiv 0$, we get that N_t is linear for $t = T,\dots,1$. Applying Lemmas 1 and 2 backwards for $s = T,\dots,1$, we then see that h_0 is well defined. Lemmas 5 and 2 give

$$\begin{aligned}
L^0(\mathcal{F}_0; N_0) &= \{x_0 \in L^0(\mathcal{F}_0) \,|\, h_0^\infty(x_0(\omega),\omega) \le 0 \ \text{a.s.}\}\\
&= \{x_0 \in L^0(\mathcal{F}_0) \,|\, \tilde{h}_0^\infty(x_0(\omega),\omega) \le 0 \ \text{a.s.}\}\\
&= \{x_0 \in L^0(\mathcal{F}_0) \,|\, \inf_{x_1} h_1^\infty(x_0(\omega),x_1,\omega) \le 0 \ \text{a.s.}\}\\
&= \{x_0 \in L^0(\mathcal{F}_0) \,|\, \exists \tilde{x} \in \mathcal{N}:\ \tilde{x}_0 = x_0,\ h_1^\infty(\tilde{x}^1(\omega),\omega) \le 0 \ \text{a.s.}\},
\end{aligned}$$

where the last equality follows by applying the last part of Lemma 2 to the normal integrand h^∞. Repeating the argument for $t = 1,\dots,T$, we get

$$\begin{aligned}
L^0(\mathcal{F}_0; N_0) &= \{x_0 \in L^0(\mathcal{F}_0) \,|\, \exists \tilde{x} \in \mathcal{N} : \tilde{x}_0 = x_0,\ h_T^\infty(\tilde{x}(\omega),\omega) \le 0 \ \text{a.s.}\}\\
&= \{x_0 \in L^0(\mathcal{F}_0) \,|\, \exists \tilde{x} \in \mathcal{N} : \tilde{x}_0 = x_0,\ h^\infty(\tilde{x}(\omega),\omega) \le 0 \ \text{a.s.}\}\\
&= \{x_0 \in L^0(\mathcal{F}_0) \,|\, \exists \tilde{x} \in \mathcal{L} : \tilde{x}_0 = x_0\}. \qquad (7)
\end{aligned}$$

The linearity of $\mathcal{L}$ thus implies that of $L^0(\mathcal{F}_0; N_0)$ which is equivalent to N_0 being linear-valued.

Assume now that N_t is linear-valued for $t = T, \dots, 0$ and let $x \in \mathcal{L}$. Expression (7) for $L^0(\mathcal{F}_0; N_0)$ is again valid so, by linearity of N_0, there is an $\tilde{x} \in \mathcal{L}$ with $\tilde{x}_0 = -x_0$. Since h^∞ is sublinear, $\mathcal{L}$ is a cone, so that $x + \tilde{x} \in \mathcal{L}$. Since $x_0 + \tilde{x}_0 = 0$, we also have $x + \tilde{x} \in \mathcal{L}'$. Since, by the induction assumption, $\mathcal{L}'$ is linear and since $\mathcal{L}' \subseteq \mathcal{L}$, we get $-x - \tilde{x} \in \mathcal{L}$. Since $\mathcal{L}$ is a cone, we get $-x = \tilde{x} - x - \tilde{x} \in \mathcal{L}$. Thus, $\mathcal{L}$ is linear.

For $t = 0$, the last claim follows directly from expression (7). The general case follows by applying this to the $(T - t)$-period model obtained by fixing $x^{t-1} \equiv 0$. □

We now return to the parameterized problem (P). The following lemma shows that the linearity condition is independent of the choice of $u \in \operatorname{dom} \varphi$.

Lemma 7. *Consider problem* (P) *and let $u \in L^0(\Omega, \mathcal{F}, P; \mathbb{R}^m)$ be such that $h(\cdot, \omega) := f(\cdot, u(\omega), \omega)$ is proper. Then*

$$h^\infty(x, \omega) = f^\infty(x, 0, \omega),$$

where $f^\infty(\cdot, \cdot, \omega)$ is the recession function of $f(\cdot, \cdot, \omega)$. Thus, as soon as they are well defined, the recession functions $\tilde{h}_t^\infty$ and h_t^∞ and thus, the mappings N_t are independent of the choice of $u \in \operatorname{dom} \varphi$.

Proof. The expression for h^∞ comes directly from the definition of the recession function. If $u \in \operatorname{dom} \varphi$ there is an $x \in \mathcal{N}$ such that $h(x(\omega), \omega) = f(x(\omega), u(\omega), \omega) < \infty$ almost surely. Recursive application of Lemmas 2 and 5 then shows that h_t and $\tilde{h}_t$ can be expressed in terms of h^∞, which is independent of $u \in \operatorname{dom} \varphi$. □

When h is the indicator function of a convex set, the linearity condition in Lemma 6 becomes the linearity condition of [17, Theorem 8] which generalizes various no-arbitrage conditions that have been used in mathematical finance. The following example illustrates the situation in the classical perfectly liquid market model; see [17] for more general models.

Example 1 (Superhedging in liquid markets)**.** *Let $S = (S_t)_{t=0}^T$ be an $\mathbb{R}^d$-valued $(\mathcal{F}_t)_{t=0}^T$-adapted stochastic process, $n_t = d$, $m = 1$ and*

$$f(x, u, \omega) = \begin{cases} 0 & \text{if } \sum_{t=0}^{T-1} x_t \cdot \Delta S_t(\omega) \geq u, \\ +\infty & \text{otherwise.} \end{cases}$$

We get

$$\varphi(u) = \inf_{x \in \mathcal{N}} I_f(x, u) = \begin{cases} 0 & \text{if } u \in \mathcal{C}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{C} = \{u \in L^0 \mid \exists x \in \mathcal{N} : \sum_{t=0}^{T-1} x_t \cdot \Delta S_t \geq u\}$. In the classical perfectly liquid model of financial markets, where S gives the unit prices of the "risky *assets" and x_t is the portfolio held over $(t, t+1]$, the set $\mathcal{C}$ consists of the* contingent claims *that can be* superhedged *without a cost; see e.g. [7, Section 6.4]. Since f is a closed positively homogeneous function, we have $f^\infty = f$ and*

$$\{x \in \mathcal{N} \mid f^\infty(x(\omega), 0, \omega) \leq 0 \ a.s.\} = \{x \in \mathcal{N} | \sum_{t=0}^{T-1} x_t \cdot \Delta S_t \geq 0\}.$$

This set is linear, and thus, by Lemma 7, the function $h(x, \omega) = f(x, u(\omega), \omega)$ satisfies the linearity condition in Lemma 6, if and only if the price process S satisfies the no-arbitrage *condition.*

The following simple example goes beyond indicator functions and also of inf-compact integrands considered in [22, 9].

Example 2 (Variance optimal hedging)**.** *Let $S = (S_t)_{t=0}^T$ be an $\mathbb{R}^d$-valued $(\mathcal{F}_t)_{t=0}^T$-adapted stochastic process, $u \in L^0(\Omega, \mathcal{F}, P; \mathbb{R})$ and consider the problem of minimizing*

$$E(V_0 + \sum_{t=0}^{T-1} z_t \cdot \Delta S_{t+1} - u)^2$$

over $V_0 \in \mathbb{R}$ and $\mathcal{F}_t$-measurable $\mathbb{R}^d$-valued functions z_t. This corresponds to (P) *with $x_0 = (z_0, V_0)$, $x_t = z_t$ for $t = 1, \ldots, T$ and*

$$f(x, u, \omega) = (V_0 + \sum_{t=0}^{T-1} z_t \cdot \Delta S_{t+1}(\omega) - u)^2.$$

The above problem has been studied e.g. in Föllmer and Schied [10, Section 10.3], where V_0 is interpreted as an initial value of a self-financing trading strategy where z_t is the portfolio of risky assets held over period $[t, t+1]$. By [20, Theorem 9.4],

$$f^\infty(x, u, \omega) = \begin{cases} 0 & \text{if } V_0 + \sum_{t=0}^{T-1} z_t \cdot \Delta S_{t+1}(\omega) - u = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

By Lemma 7, the function $h(x, \omega) = f(x, u(\omega), \omega)$ then satisfies the linearity condition of Lemma 6, so the optimal solution is attained. This should be compared with the existence results in [10, Section 10.3], where it was assumed that $d = 1$.

3 Lower semicontinuity of the value function

We now return to the parameterized problem (P). Being the inf-projection of the convex integral functional

$$I_f(x, u) = Ef(x(\omega), u(\omega), \omega),$$

the value function

$$\varphi(u) = \inf_{x\in\mathcal{N}} Ef(x(\omega), u(\omega), \omega)$$

is convex on $L^0(\Omega, F, P; \mathbb{R}^m)$; see e.g. [21, Theorem 1]. Our aim is to give conditions under which φ is lower semicontinuous on certain locally convex topological vector subspaces of $L^0(\Omega, F, P; \mathbb{R}^m)$. The lower semicontinuity is equivalent to the absence of a duality gap in the duality framework of [17] (which is essentially an instance of the conjugate duality framework of Rockafellar [21]) which we now briefly recall (and slightly generalize).

Assume that $\mathcal{U}$ and $\mathcal{Y}$ are vector subspaces of $L^0(\Omega, F, P; \mathbb{R}^m)$ in *separating duality* under the bilinear form

$$\langle u, y\rangle = E[u(\omega)\cdot y(\omega)],$$

i.e. that $E[u(\omega)\cdot y(\omega)]$ is finite for every $u \in \mathcal{U}$ and $y \in \mathcal{Y}$ and that for every nonzero $u \in \mathcal{U}$ (resp. $y \in \mathcal{Y}$), there is at least one $y \in \mathcal{Y}$ (resp. $u \in \mathcal{U}$) such that $\langle u, y\rangle \neq 0$. The special case $\mathcal{U} = L^p$ and $\mathcal{Y} = L^q$ was studied in [17]. The weakest and the strongest locally convex topologies on $\mathcal{U}$ compatible with the pairing will be denoted by $\sigma(\mathcal{U}, \mathcal{Y})$ and $\tau(\mathcal{U}, \mathcal{Y})$, respectively. Since the value function φ is convex, we have by the classical separation argument, that φ is lower semicontinuous with respect to $\sigma(\mathcal{U}, \mathcal{Y})$ if it is merely lower semicontinuous with respect to $\tau(\mathcal{U}, \mathcal{Y})$. When $\mathcal{U} = L^p$ and $\mathcal{Y} = L^q$ for $p \in [1, \infty)$, $\tau(\mathcal{U}, \mathcal{Y})$ is simply the norm topology on $\mathcal{U}$ and $\sigma(\mathcal{U}, \mathcal{Y})$ the weak topology. A general treatment of topological spaces in separating duality can be found e.g. in Kelley and Namioka [14].

The *Lagrangian* associated with (P) is the extended real-valued function

$$L(x, y) = \inf_{u\in\mathcal{U}}\{I_f(x, u) - \langle u, y\rangle\}$$

on $\mathcal{N} \times \mathcal{Y}$. The Lagrangian is convex in x and concave in y. The *dual objective* is the extended real-valued function on $\mathcal{Y}$ defined by

$$g(y) = \inf_{x\in\mathcal{N}} L(x, y).$$

The basic duality result [21, Theorem 7] says, in particular, that $g = -\varphi^*$. When φ is lower semicontinuous and proper, the biconjugate theorem (see e.g. [21, Theorem 5]) then gives the dual representation

$$\varphi(u) = \sup\{\langle u, y\rangle + g(y)\}. \tag{8}$$

It was shown in [17] that this abstract result is behind many duality frameworks in stochastic optimization and mathematical finance.

It was assumed in [17] that $\mathcal{U} = L^p$ and $\mathcal{Y} = L^q$, but the main result [17, Theorem 3] remains valid as long as the space $\mathcal{U}$ is *decomposable* in the sense that

$$\chi_A u + \chi_{\Omega\setminus A} u' \in \mathcal{U}$$

whenever $A \in \mathcal{F}$, $u \in \mathcal{U}$ and $u' \in L^\infty(\Omega, \mathcal{F}, P; \mathbb{R}^m)$. Indeed, the decomposability property allows the use of the interchange rule for minimization and integration (see [23, Theorem 14.60]), which suffices for the proof of [17, Theorem 3]. We also note that decomposability of the spaces $\mathcal{U}$ and $\mathcal{Y}$ implies that the separation property holds automatically for the bilinear form defined above; see [27, Lemma 6]. Moreover, we have the following relations for relative topologies.

Lemma 8. *If $\mathcal{U}$ and $\mathcal{Y}$ are decomposable, then $L^\infty \subseteq \mathcal{U} \subseteq L^1$ and*

$$\sigma(L^1, L^\infty)|_{\mathcal{U}} \subseteq \sigma(\mathcal{U}, \mathcal{Y}), \quad \sigma(\mathcal{U}, \mathcal{Y})|_{L^\infty} \subseteq \sigma(L^\infty, L^1),$$
$$\tau(L^1, L^\infty)|_{\mathcal{U}} \subseteq \tau(\mathcal{U}, \mathcal{Y}), \quad \tau(\mathcal{U}, \mathcal{Y})|_{L^\infty} \subseteq \tau(L^\infty, L^1).$$

Proof. By [4, Lemme 1, p.??], $L^\infty \subseteq \mathcal{U} \subset L^1$ and $L^\infty \subseteq \mathcal{Y} \subseteq L^1$ which give the relations for the σ-topologies. Since, by symmetry, analogous relations are valid for the σ-topologies on $\mathcal{Y}$, we have that $\sigma(L^\infty, L^1)$-compact subsets of L^∞ are $\sigma(\mathcal{Y},\mathcal{U})$-compact. Since, by the Mackey-Arens theorem, $\tau(\mathcal{U}, \mathcal{Y})$ is generated by the support functions of $\sigma(\mathcal{Y},\mathcal{U})$-compact sets, we get $\tau(L^1, L^\infty)|_{\mathcal{U}} \subseteq \tau(\mathcal{U}, \mathcal{Y})$. The remaining inclusion is verified similarly. □

The traditional "direct method" for proving the lower semicontinuity would be to assume that the integral functional I_f is uniformly inf-compact in x with respect to an appropriate topology on $\mathcal{N}$. If the topology is strong enough to imply the almost sure convergence of a subsequence, the sequential lower semicontinuity can often be derived from Fatou's lemma. In certain applications, this purely topological argument fails because I_f lacks an appropriate inf-compactness property in x. In convex problems, the following measure theoretic result can sometimes be used as a substitute for compactness.

Lemma 9 (Komlós' theorem). *Let $(x^\nu)_{\nu=1}^\infty$ be a sequence in $L^0(\Omega, \mathcal{F}, P; \mathbb{R}^n)$ which is either*

1. *bounded in L^1,*
2. *almost surely bounded in the sense that*

$$\sup_\nu |x^\nu(\omega)| < \infty \quad P\text{-a.s.}$$

Then there is a sequence of convex combinations $\bar{x}^\nu \in \operatorname{co}\{x^\mu \,|\, \mu \geq \nu\}$ that converges almost surely to an $\mathbb{R}^n$-valued function.

Proof. See e.g. [7] or [12]. □

The following is our main result.

Theorem 10. *Assume that there is a $y \in \mathcal{Y}$ and an $m \in L^1(\Omega, \mathcal{F}, P)$ such that for P-almost every ω,*

$$f(x, u, \omega) \geq u \cdot y(\omega) + m(\omega) \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m$$

and that $\{x \in \mathcal{N} \,|\, f^\infty(x(\omega), 0, \omega) \le 0 \text{ a.s.}\}$ *is a linear space. Then*

$$\varphi(u) = \inf_{x \in \mathcal{N}} I_f(x, u)$$

is lower semicontinuous on $\mathcal{U}$ *and the infimum is attained for every* $u \in \mathcal{U}$*.*

Proof. Let $h_u(x, \omega) = f(x, u(\omega), \omega)$. The lower bound on f implies that h_u has an integrable lower bound. As noted in Lemma 7, $h_u^\infty(x, \omega) = f^\infty(x, 0, \omega)$ for every u, so the linearity condition on f implies that h_u satisfies the linearity conditions in Lemma 6. By Theorem 3, the infimum in $\varphi(u) = \inf_{x \in \mathcal{N}} I_f(x, u)$ is thus attained for every $u \in \mathcal{U}$ by an $x \in \mathcal{N}$ with $x_t(\omega) \perp N_t(\omega)$ almost surely.

For lower semicontinuity, it suffices to show that φ is lower semicontinuous on the linear space $L^{1,y} = \{u \in L^1 \,|\, |E[u \cdot y]| < \infty\}$ with respect to the norm $||u||_{L^{1,y}} = E|u| + |E[u \cdot y]|$. Indeed, since $y \in \mathcal{Y}$, we have $\mathcal{U} \subseteq L^{1,y}$ and, by Lemma 8, the norm $||\cdot||_{L^{1,y}}$ is continuous on $\tau(\mathcal{U}, \mathcal{Y})$, which means that $\tau(\mathcal{U}, \mathcal{Y})$ is stronger than the norm topology restricted to $\mathcal{U}$. Since $L^{1,y}$ is a normed space, it suffices to prove sequential lower semicontinuity, which means that for any $\gamma \in \mathbb{R}$ and for any sequence $(u^\nu)_{\nu=1}^\infty$ such that

$$\varphi(u^\nu) \le \gamma$$

and $u^\nu \to u$ in $L^{1,y}$, we have $\varphi(u) \le \gamma$. We will prove this by establishing the existence of an $x \in \mathcal{N}$ such that $I_f(x, u) \le \gamma$.

As observed at the beginning of the proof, there is for every ν an $x^\nu \in \mathcal{N}$ such that $x_t^\nu \perp N_t$ and

$$I_f(x^\nu, u^\nu) \le \gamma.$$

Moreover, the mappings N_t are independent of u^ν; see Lemma 7. Since u^ν converges in $L^{1,y}$, the lower bound on f implies that the negative parts of the functions $\omega \mapsto f(x^\nu(\omega), u^\nu(\omega), \omega)$ are bounded in L^1. Since $I_f(x^\nu, u^\nu) \le \gamma$, the positive parts must be bounded as well. Thus, by Lemma 9, there is a sequence of convex combinations

$$\phi^\nu(\omega) := \sum_{\mu=\nu}^\infty \alpha^{\nu,\mu} f(x^\mu(\omega), u^\mu(\omega), \omega)$$

that converges almost surely to a real-valued measurable function. In particular, the function $\phi(\omega) := \sup_\nu \phi^\nu(\omega)$ is almost surely finite. Defining

$$(\bar{x}^\nu, \bar{u}^\nu) = \sum_{\mu=\nu}^\infty \alpha^{\nu,\mu}(x^\mu, u^\mu)$$

we have by convexity that

$$f(\bar{x}^\nu(\omega), \bar{u}^\nu(\omega), \omega) \le \phi^\nu(\omega) \le \phi(\omega) \quad P\text{-a.s.}$$

and $I_f(\bar{x}^\nu, \bar{u}^\nu) \le \gamma$. Moreover, we still have $\bar{x}_t^\nu \in N_t^\perp$ almost surely and $\bar{u}^\nu \to u$ in the $L^{1,y}$-norm.

Passing to a subsequence if necessary, we may assume that $\bar{u}^\nu \to u$ almost surely, so that the measurable function $\rho(\omega) := \sup_\nu |\bar{u}^\nu(\omega)|$ is almost surely finite. Each $(\bar{x}^\nu, \bar{u}^\nu)$ then belongs to the set

$$\mathcal{C} = \{(x,u) \in \mathcal{N} \times L^0 \,|\, (x,u) \in C \text{ a.s.}\},$$

where $C(\omega) = \{(x,u) \,|\, x_t \in N_t^\perp(\omega),\ u \in \rho(\omega)\mathbb{B},\ f(x,u,\omega) \le \phi(\omega)\}$. We will now apply [17, Theorem 6], which says that the sequence $(\bar{x}^\nu, \bar{u}^\nu)_{\nu=1}^\infty$ is almost surely bounded if

$$\{(x,u) \in \mathcal{N} \times L^0 \,|\, (x,u) \in C^\infty \text{ a.s.}\} = \{(0,0)\}. \tag{9}$$

By Corollary 8.3.3 and Theorem 8.7 of [20],

$$C^\infty(\omega) = \{(x,0) \,|\, x_t \in N_t^\perp(\omega),\ f^\infty(x,0,\omega) \le 0\}.$$

If $x \in \mathcal{N}$ is such that $f^\infty(x(\omega),0,\omega) \le 0$ then, by the last part of Lemma 6, we have $x_0 \in N_0$. The condition $x_0 \in N_0^\perp$ then implies that $x_0 = 0$. Repeating the argument for $t = 1,\dots,T$ gives (9) so $(\bar{x}^\nu, \bar{u}^\nu)_{\nu=1}^\infty$ is almost surely bounded.

By Lemma 9, there is a sequence $(\hat{x}^\nu, \hat{u}^\nu)_{\nu=1}^\infty$ of convex combinations of $(\bar{x}^\nu, \bar{u}^\nu)_{\nu=1}^\infty$ that converges almost surely to a point $(x, \hat{u})$, where necessarily $\hat{u} = u$ since $\bar{u}^\nu \to u$ almost surely. We still have $\hat{u}^\nu \to u$ in the $L^{1,y}$-norm and, by convexity, $I_f(\hat{x}^\nu, \hat{u}^\nu) \le \gamma$. By Fatou's lemma,

$$\begin{aligned} &E[f(x(\omega),u(\omega),\omega) - y(\omega)\cdot u(\omega) - m(\omega)] \\ &\qquad \le \liminf_{\nu\to\infty} E[f(\hat{x}^\nu(\omega),\hat{u}^\nu(\omega),\omega) - y(\omega)\cdot \hat{u}^\nu(\omega) - m(\omega)], \end{aligned}$$

where $E[y(\omega)\cdot\hat{u}^\nu(\omega)] \to E[y(\omega)\cdot u(\omega)]$, by the $L^{1,y}$-convergence, so that

$$I_f(x,u) \le \liminf_{\nu\to\infty} I_f(\hat{x}^\nu, \hat{u}^\nu) \le \gamma,$$

which completes the proof. □

4 An application to mathematical finance

We will illustrate Theorem 10 on the optimal consumption problem considered in [17, Section 5]. The problem is set in a generalization of the market model of Kabanov [11], where a finite number d of securities is traded over finite discrete time $t = 0,\dots,T$. At each time t and state $\omega \in \Omega$, the market is described by two closed convex sets, $C_t(\omega)$ and $D_t(\omega)$, both of which contain the origin. The set $C_t(\omega)$ consists of the portfolios that are freely available in the market and $D_t(\omega)$ consists of the portfolios that the investor is allowed to hold over the period $[t, t+1)$. For each t, the sets C_t and D_t are assumed to be $\mathcal{F}_t$-measurable.

Consider the problem

$$\begin{aligned} &\underset{(z,c)\in\mathcal{N}}{\text{maximize}} \quad E\sum_{t=0}^{T} U_t(c_t) \\ &\text{subject to} \quad z_t - z_{t-1} + c_t \in C_t,\ z_t \in D_t \quad P\text{-a.s. } t = 0,\dots,T, \end{aligned} \tag{10}$$

where $z_{-1} := 0$, $D_T(\omega) := \{0\}$ and $-U_t$ is a convex $\mathcal{F}_t$-measurable normal integrand on $\mathbb{R}^d \times \Omega$. This models an optimal consumption problem where at each time t and stage ω we can consume some of the assets and update the existing portfolio z_{t-1}. The combined process (z, c) is required to be *self-financing* in the sense that the sum of the portfolio update $\Delta z_t := z_t - z_{t-1}$ and the consumption vector c_t has to be freely available in the market, i.e. it belongs to $C_t(\omega)$. In addition, the portfolio constraint $z_t(\omega) \in D_t(\omega)$ is required to hold almost surely at each time. Problem (10) generalizes the classical optimal consumption problem where the numeraire asset is consumed in a perfectly liquid market model (see Examples 1 and 2). A general treatment of the continuous-time model can be found in Karatzas and Žitković [13].

Defining

$$\mathcal{C} = \{c \in \mathcal{A} \,|\, \Delta z_t + c_t \in C_t,\ z_t \in D_t \quad P\text{-a.s. } t = 0, \dots, T\},$$

where $\mathcal{A}$ denotes the set of $\mathbb{R}^d$-valued adapted processes (so that $\mathcal{N} = \mathcal{A} \times \mathcal{A}$), we can write problem (10) compactly as

$$\underset{c \in \mathcal{A}}{\text{maximize}} \quad E \sum_{t=0}^{T} U_t(c_t) \quad \text{over } c \in \mathcal{C}. \tag{11}$$

The set $\mathcal{C}$ can be interpreted as the set of consumption processes that can be super-replicated without a cost in the market given by the pair (C, D); compare with the definition of the set $\mathcal{C}$ in Example 1.

In order to dualize the problem, we embed it in the general duality framework with, $x_t = (z_t, c_t)$, $u = (u_t)_{t=0}^T$ and

$$f(x, u, \omega) = \begin{cases} -\sum_{t=0}^T U_t(c_t, \omega) & \text{if } \Delta z_t + c_t + u_t \in C_t(\omega),\ z_t \in D_t(\omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Here $u_t \in \mathbb{R}^d$ so that the dimension of u equals $m = (T + 1)d$. The value function $\varphi(u)$ thus gives the optimal value in the perturbed problem

$$\text{maximize} \quad E \sum_{t=0}^{T} U_t(c_t) \quad \text{over} \quad c \in \mathcal{C} - u. \tag{12}$$

The Lagrangian integrand becomes

$$
\begin{aligned}
l(x,y,\omega) &= \inf_{u\in\mathbb{R}^m}\{f(x,u,\omega)-u\cdot y\}\\
&= \inf_{u\in\mathbb{R}^m}\{-\sum_{t=0}^T[U_t(c_t,\omega)+u_t\cdot y_t]\,|\,\Delta z_t+c_t+u_t\in C_t(\omega),\ z_t\in D_t(\omega)\}\\
&= \inf_{\tilde u\in\mathbb{R}^m}\{-\sum_{t=0}^T[U_t(c_t,\omega)+(\tilde u_t-\Delta z_t-c_t)\cdot y_t]\,|\,\tilde u_t\in C_t(\omega),\ z_t\in D_t(\omega)\}\\
&= \begin{cases}-\sum_{t=0}^T[U_t(c_t,\omega)+\sigma_{C_t(\omega)}(y_t)-(\Delta z_t+c_t)\cdot y_t] & \text{if } z_t\in D_t(\omega)\\ +\infty & \text{otherwise}\end{cases}\\
&= \begin{cases}-\sum_{t=0}^T[U_t(c_t,\omega)+\sigma_{C_t(\omega)}(y_t)+z_t\cdot\Delta y_{t+1}-c_t\cdot y_t] & \text{if } z_t\in D_t(\omega)\\ +\infty & \text{otherwise,}\end{cases}
\end{aligned}
$$

where $\sigma_{C_t(\omega)}$ denotes the support function of $C_t(\omega)$. In the last equality we have used the "integration by parts" formula

$$\sum_{t=0}^T \Delta z_t\cdot y_t = -\sum_{t=0}^T z_t\cdot\Delta y_{t+1}$$

where $y_{T+1}:=0$.

Recall that, by Lemma 8, $\mathcal{Y}\subset L^1$. On the other hand, by [17, Theorem 3][3],

$$g(y)=\inf_{x\in\mathcal{N}^\infty} El(x(\omega),y(\omega),\omega),$$

where $\mathcal{N}^\infty=\mathcal{N}\cap L^\infty$. We can then use the law of iterated expectations (see e.g. [24, Section II.7]) and the interchange rule for integration and minimization (see e.g. [23, Theorem 14.60]) to write the dual objective as

$$
\begin{aligned}
g(y) = \inf_{(c,z)\in\mathcal{N}^\infty}\Bigg\{E\sum_{t=0}^T[&-U_t(c_t)-\sigma_{C_t}(y_t)\\
&-z_t\cdot E_t\Delta y_{t+1}+c_t\cdot E_t y_t]\ |\ z_t\in D_t \text{ a.s.}\Bigg\}\\
= E\sum_{t=0}^T\inf_{c_t,z_t\in\mathbb{R}^d}\{&-U_t(c_t,\omega)-\sigma_{C_t(\omega)}(y_t(\omega))\\
&-z_t\cdot(E_t\Delta y_{t+1})(\omega)+c_t\cdot(E_t y_t)(\omega)|\,z_t\in D_t(\omega)\}\\
= E\sum_{t=0}^T[&U_t^*(E_t y_t)-\sigma_{C_t}(y_t)-\sigma_{D_t}(E_t\Delta y_{t+1})].
\end{aligned}
$$

[3] Theorem 3 of [17] is stated for the case $\mathcal{U}=L^p$ and $\mathcal{Y}=L^q$, but its proof goes through in the general case without a change.

where

$$U_t^*(y,\omega) = \inf_{c\in\mathbb{R}^d}\{c\cdot y - U_t(c,\omega)\}$$

is the conjugate of U_t in the concave sense.

When $C_t(\omega)$ and $D_t(\omega)$ are convex cones, we have

$$g(y) = \begin{cases} E\sum_{t=0}^T U_t^*(E_t y_t) & \text{if } y\in\mathcal{D},\\ -\infty & \text{otherwise,}\end{cases}$$

where

$$\mathcal{D} = \{y\in\mathcal{Y}\,|\,y_t\in C_t^*,\ E_t\Delta y_t\in D_t^*\},$$

where $C_t^*(\omega)$ and $D_t^*(\omega)$ are the polar cones of $C_t(\omega)$ and $D_t(\omega)$, respectively. The dual problem can then be written as

$$\text{maximize}\quad E\sum_{t=0}^T U_t^*(y_t)\quad \text{over } y\in\mathcal{D} \tag{13}$$

in symmetry with the primal problem (11). The $(\mathcal{F}_t)_{t=0}^T$-adapted elements of $\mathcal{D}$ are called *consistent price systems* for the market model (C,D); see [17, Example 4.2]. It follows from Jensen's inequality that the value of the dual objective does not decrease when replacing a general $y\in\mathcal{Y}$ by its $(\mathcal{F}_t)_{t=0}^T$-adapted projection; see [17, Example 3.6].

In [17], the lower semicontinuity of the value function associated with the optimal consumption problem was left open so it could not be claimed that the optimal values of (11) and (13) are equal. With the help of Theorem 10, we can now derive simple sufficient conditions. We will assume that the utility function U_t satisfies the growth condition

$$U_t^\infty(c,\omega) = \begin{cases} 0 & \text{if } c\in\mathbb{R}^d_+,\\ -\infty & \text{otherwise,}\end{cases} \tag{14}$$

for every ω. When $d=1$ and $U_t(\cdot,\omega)$ is smooth, this is equivalent to the conditions $\lim_{c\to\infty}U_t'(c,\omega)=0$ and $\lim_{c\to-\infty}U_t'(c,\omega)=+\infty$ which generalize the Inada conditions; see e.g. [15]. We will say that the market model (C,D) satisfies the condition of *no-scalable arbitrage* if

$$\{c\in\mathcal{A}_+\,|\,\exists z\in\mathcal{A}:\ \Delta z_t + c_t\in C_t^\infty,\ z_t\in D_t^\infty\} = \{0\}, \tag{15}$$

where $\mathcal{A}_+$ denotes the set of componentwise nonnegative claim processes. This condition is related to arbitrage opportunities that can be scaled up by arbitrarily large positive numbers; see [16, 18].

Theorem 11. *Assume that the optimal value of* (10) *or, equivalently,* (11) *is less than* ∞*, that* U_t *satisfy the growth condition* (14) *and that there is an integrable function* m *such that* $U_t(c,\omega)\le m(\omega)$ *almost surely for every* $c\in\mathbb{R}^d$

and $t = 0, \dots, T$. *If the market model* (C, D) *satisfies the no-scalable arbitrage condition* (15) *and if the set*

$$\{z \in \mathcal{A} \,|\, \Delta z_t \in C_t^\infty,\ z_t \in D_t^\infty\},$$

is linear, then the optimal value of (12) *is lower semicontinuous as a function of* $u \in \mathcal{U}$ *and the infimum is always attained. In particular, if* C *and* D *are conical, then the optimal value of the primal problem* (11) *is the negative of the optimal value of the dual problem* (13)*.*

Proof. By [20, Theorems 9.3 and 9.5],

$$f^\infty(x, u, \omega) = \begin{cases} -\sum_{t=0}^T U_t^\infty(c_t, \omega) & \text{if } \Delta z_t + c_t \in C_t^\infty(\omega),\ z_t \in D_t^\infty(\omega) \\ +\infty & \text{otherwise,} \end{cases}$$

so, by Lemma 6, the linearity condition of Theorem 10 means that the set

$$\mathcal{L} = \{(z, c) \in \mathcal{N} \,|\, \sum_{t=0}^T U_t^\infty(c_t) \geq 0,\ \Delta z_t + c_t \in C_t^\infty,\ z_t \in D_t^\infty\}$$

is linear. Under the growth condition (14),

$$\mathcal{L} = \{(z, c) \in \mathcal{N} \,|\, c_t \geq 0,\ \Delta z_t + c_t \in C_t^\infty,\ z_t \in D_t^\infty\}$$

If $(z, c) \in \mathcal{L}$, condition (15) implies that $c = 0$, and then, by linearity of the set $\{z \in \mathcal{A} \,|\, \Delta z_t \in C_t^\infty,\ z_t \in D_t^\infty\}$, we have $(-z, -c) \in \mathcal{L}$. Since $\mathcal{L}$ is also a cone, it has to be a linear space. □

Remark 1. *The conclusions of Theorem 11 remain valid if, instead of the growth condition* (14) *and the no-scalable arbitrage condition* (15)*, we assume that the set*

$$\{c \in \mathcal{A} \,|\, \exists z \in \mathcal{A} :\ \sum_{t=0}^T U_t^\infty(c_t) \geq 0,\ \Delta z_t + c_t \in C_t^\infty,\ z_t \in D_t^\infty\}, \tag{16}$$

is linear. Indeed, if $\mathcal{L}$ *is as in the above proof and* $(z, c) \in \mathcal{L}$*, condition* (16) *gives the existence of a* $z^- \in \mathcal{A}$ *such that* $(z^-, -c) \in \mathcal{L}$*. Since* $\mathcal{L}$ *is a cone, we get* $(z + z^-, 0) \in \mathcal{L}$ *and then, the linearity condition of Theorem 11 gives* $(-(z + z^-), 0) \in \mathcal{L}$*. Since* $-(z, c) = (-(z + z^-), 0) + (z^-, -c)$*, we get that* $-(z, c) \in \mathcal{L}$*, i.e.,* $\mathcal{L}$ *is linear.*

Above, the lower semicontinuity of the value function φ of (12) was used to prove the equality of the optimal values of the primal and dual problems (11) and (13). The lower semicontinuity of φ is essential also when studying pricing and hedging of contingent claims. Such issues will be studied in a separate article.

References

[1] D. P. Bertsekas. Necessary and sufficient conditions for existence of an optimal portfolio. *Journal of Economic Theory*, 8(2):235–247, 1974.

[2] D. P. Bertsekas. *Dynamic programming and stochastic control*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976. Mathematics in Science and Engineering, 125.

[3] D. P. Bertsekas and S. E. Shreve. *Stochastic optimal control*, volume 139 of *Mathematics in Science and Engineering*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978. The discrete time case.

[4] J.-M. Bismut. Intégrales convexes et probabilités. *J. Math. Anal. Appl.*, 42:639–673, 1973.

[5] C. Castaing and M. Valadier. *Convex analysis and measurable multifunctions*. Springer-Verlag, Berlin, 1977. Lecture Notes in Mathematics, Vol. 580.

[6] C. Choirat, C. Hess, and R. Seri. A functional version of the birkhoff ergodic theorem for a normal integrand: A variational approach. *Annals of probability*, 31(1):63–92, 2003.

[7] F. Delbaen and W. Schachermayer. *The Mathematics of Arbitrage*. Springer Finance. Springer-Verlag, Berlin Heidelberg, 2006.

[8] E. B. Dynkin and I. V. Evstigneev. Regular conditional expectation of correspondences. *Teor. Verojatnost. i Primenen.*, 21(2):334–347, 1976.

[9] I. V. Evstigneev. Measurable selection and dynamic programming. *Math. Oper. Res.*, 1(3):267–272, 1976.

[10] H. Föllmer and A. Schied. *Stochastic finance*, volume 27 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, extended edition, 2004. An introduction in discrete time.

[11] Y. M. Kabanov. Hedging and liquidation under transaction costs in currency markets. *Finance and Stochastics*, 3(2):237–248, 1999.

[12] Y. M. Kabanov and M. Safarian. *Markets with transaction costs*. Springer Finance. Springer-Verlag, Berlin, 2009. Mathematical theory.

[13] I. Karatzas and Žitković G. Optimal consumption from investment and random endowment in incomplete semimartingale markets. *The Annals of Probability*, 31(4):18211858, 2003.

[14] J. L. Kelley and I. Namioka. *Linear topological spaces*. Springer-Verlag, New York, 1976. With the collaboration of W. F. Donoghue, Jr., Kenneth R. Lucas, B. J. Pettis, Ebbe Thue Poulsen, G. Baley Price, Wendy Robertson, W. R. Scott, and Kennan T. Smith, Second corrected printing, Graduate Texts in Mathematics, No. 36.

[15] D. Kramkov and W. Schachermayer. The condition on the asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, 9(3):904–950, 1999.

[16] T. Pennanen. Arbitrage and deflators in illiquid markets. *Finance and Stochastics*, 15(1):57–83, 2011.

[17] T. Pennanen. Convex duality in stochastic optimization and mathematical finance. *Mathematics of Operations Research*, 36(2):340–362, 2011.

[18] T. Pennanen and I. Penner. Hedging of claims with physical delivery under convex transaction costs. *SIAM Journal on Financial Mathematics*, 1:158–178, 2010.

[19] M. Rásonyi and L. Stettner. On utility maximization in discrete-time financial market models. *Ann. Appl. Probab.*, 15(2):1367–1395, 2005.

[20] R. T. Rockafellar. *Convex analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[21] R. T. Rockafellar. *Conjugate duality and optimization.* Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1974.

[22] R. T. Rockafellar and R. J.-B. Wets. Nonanticipativity and L^1-martingales in stochastic optimization problems. *Math. Programming Stud.*, (6):170–187, 1976. Stochastic systems: modeling, identification and optimization, II (Proc. Sympos., Univ Kentucky, Lexington, Ky., 1975).

[23] R. T. Rockafellar and R. J.-B. Wets. *Variational analysis*, volume 317 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1998.

[24] A. N. Shiryaev. *Probability*, volume 95 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1996. Translated from the first (1980) Russian edition by R. P. Boas.

[25] L. Thibault. Espérances conditionnelles d'intégrandes semi-continus. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 17(4):337–350, 1981.

[26] A. Truffert. Conditional expectation of integrands and random sets. *Ann. Oper. Res.*, 30(1-4):117–156, 1991. Stochastic programming, Part I (Ann Arbor, MI, 1989).

[27] M. Valadier. Convex integrands on Souslin locally convex spaces. *Pacific J. Math.*, 59(1):267–276, 1975.