Optional projection in duality

Teemu Pennanen* Ari-Pekka Perkkiö†

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Abstract

This article characterizes topological duals of Fréchet spaces of stochastic processes. This is done by analyzing the optional projection on spaces of cadlag processes whose pathwise supremum norm belongs to a given Fréchet space of random variables. We unify many existing results in the duality theory of stochastic processes and give extensions to more general spaces of processes. In particular, we obtain an explicit characterization of dual of the Banach space of adapted cadlag processes of class (D). When specialized to spaces of continuous processes, we obtain a simple proof of a sharpened version of the main result of [3] on regular processes.

Keywords. stochastic process, topological dual, optional projection

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1 Introduction

Many important results in the theory of stochastic processes are based on characterizations of the topological dual of a space of stochastic processes. Examples include the Doob decomposition of a supermartingale, Bismut's characterization of regular processes and optimal stopping theorems; see e.g. [3, 4, 8] and the references therein. Moreover, duality theory and optimality conditions for general convex stochastic control problems are often derived in a functional analytic framework of paired spaces of stochastic processes; see e.g. [2]. To extend such a framework to singular stochastic control, one needs processes of bounded variation (BV) in separating duality with a space of cadlag processes; see e.g. [14].

This paper studies Fréchet spaces (in particular, Banach spaces) of stochastic processes whose dual can be identified with a space of optional Borel measures (and thus with BV-processes). When dealing with *raw* (not necessarily adapted) stochastic processes, the duality is fairly easy to establish. The dual of the

*Department of Mathematics, King's College London, Strand, London, WC2R 2LS, United Kingdom

†Mathematics Institute, Ludwig-Maximilian University of Munich, Theresienstr. 39, 80333 Munich, Germany.

space $\mathcal{Y}(D)$ of raw cadlag processes whose pathwise supremum norms belong to a Fréchet space $\mathcal{Y}$ turns out to be a space of pairs of random measures whose pathwise total variation belongs to the dual of $\mathcal{Y}$. When specialized to continuous processes, each dual element can be represented by a single random measure.

The case of adapted cadlag processes is more involved and requires additional techniques. This paper characterizes topological duals of spaces of adapted cadlag processes via functional analysis of the optional projection on the space $\mathcal{Y}(D)$ of raw stochastic processes. Our main results extend those of [8, Theorems VII.65], [1] and [14, Theorem 8], the first one of which characterizes the dual space of optional cadlag processes whose pathwise supremum norm is in L^p with $p > 1$. Theorem 3.1 of [1] extends this to Orlicz. Theorem 8 of [14] goes beyond the settings of [8] and [1] as it applies to "regular processes" whose pathwise supremum norm may fail to be integrable. Nevertheless, it turns out that the dual space can still be identified with a space of optional Borel measures.

The above results are derived by proving first that, as soon as the optional projection is continuous to a Fréchet space $\mathcal{D}$ of adapted cadlag processes, its surjectivity is equivalent the topological dual of $\mathcal{D}$ being identifiable with a space of optional random measures. The duality results of [8, Theorems VII.65] and [1] (and their extensions) are then obtained by establishing the surjectivity. On the other hand Theorem 8 of [14] is based on [3, Theorem 3] which states that regular processes (cadlag processes of class (D) whose predictable projections coincide with their left limits) are the optional projections of continuous processes whose pathwise supremum norm has finite expectation. Bismut's result follows easily from the above equivalence once the dual space has been identified. Besides being considerably simpler, our proof extends to more general spaces of processes. The proofs of our main results are based on the classical closed range and closed graph theorems which are valid in general Fréchet spaces; see e.g. [11].

The extensions are of interest e.g. in singular stochastic control where one optimizes over spaces of optional processes of bounded variation. Our results allow for formulations where the variation need not be essentially bounded. One can then develop dual problems and optimality conditions for convex singular stochastic control much like in [2] in the case of absolutely continuous trajectories. This will be developed in a separate article.

2 Fréchet lattices of random variables

Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\mathcal{P}$ be a countable collection of seminorms p on L^1 and define

$$\tilde{\mathcal{Y}} := \bigcap_{p \in \mathcal{P}} \operatorname{dom} p.$$

We endow $\tilde{\mathcal{Y}}$ with the locally convex topology generated by $\mathcal{P}$ and assume that each $p \in \mathcal{P}$ satisfies the following:

1. p is lower semicontinuous on L^1,
2. there exists a constant c such that $\frac{1}{c}\|\xi\|_{L^1} \leq p(\xi) \leq c\|\xi\|_{L^\infty}$ for all $\xi \in L^1$,
3. $p(\xi_1) \leq p(\xi_2)$ whenever $|\xi_1| \leq |\xi_2|$,
4. $p(\xi^\nu) \searrow 0$ whenever $(\xi^\nu) \subset L^\infty$ with $\xi^\nu \searrow 0$ almost surely.

We define $\mathcal{Y}$ as the closure of L^∞ in $\tilde{\mathcal{Y}}$. The above setting covers, in particular, L^p spaces with $p \in [1, \infty)$. Indeed, when $\mathcal{P} = \{\|\cdot\|_{L^p}\}$, we have $\mathcal{Y} = \tilde{\mathcal{Y}} = L^p$. More interesting examples will be given at the end of this section.

When restricted to L^∞, members of $\mathcal{P}$ are continuous with respect to the *Mackey topology* $\tau(L^\infty, L^1)$ that L^∞ has as the dual of L^1. Recall that $\tau(L^\infty, L^1)$ is the strongest locally convex topology on L^∞ under which every continuous linear functional l is expressible as $l(\eta) = E[\eta\xi]$ for some $\eta \in L^1$.

Lemma 1. *Under 3 and 4, p is $\tau(L^\infty, L^1)$-continuous on L^∞.*

Proof. By [6, Proposition 1.2], it suffices to show that $p(\xi^\nu) \to 0$ whenever (ξ^ν) is bounded and $\xi^\nu \to 0$ in probability. Since every subsequence has a further subsequence that converges to zero almost surely, we may assume that $\xi^\nu \to 0$ almost surely. By 3, we may assume that $\xi^\nu \geq 0$. Defining $\bar{\xi}^\nu := \sup_{\nu' \geq \nu} \xi^\nu$, we have $\bar{\xi}^\nu \searrow 0$ almost surely. By 3 and 4, $p(\xi^\nu) \to 0$. □

For each $p \in \mathcal{P}$, we define a seminorm p° on L^1 by

$$p^\circ(\eta) := \sup_{\xi \in L^\infty} \{E(\xi\eta) \,|\, p(\xi) \leq 1\}.$$

Theorem 2. *The space $\mathcal{Y}$ is Fréchet and its dual may be identified with the space*

$$\mathcal{U} := \bigcup_{p \in \mathcal{P}} \operatorname{dom} p^\circ$$

under the bilinear form

$$\langle \xi, \eta \rangle := E(\xi\eta).$$

For every $\xi \in L^1$ and $\eta \in L^1$,

$$E(\xi\eta) \leq p(\xi)p^\circ(\eta),$$

so p° is the polar of p on $\mathcal{U}$. Each p° satisfies 1,2, and 3.

Proof. Property 2 implies that the topology of $\tilde{\mathcal{Y}}$ is stronger than the L^1 topology. Thus, if (ξ^ν) is a Cauchy sequence in $\tilde{\mathcal{Y}}$, it is Cauchy also in L^1 so it L^1-converges to an $\xi \in L^1$. Being Cauchy in $\tilde{\mathcal{Y}}$ means that for every $\epsilon > 0$ and $p \in \mathcal{P}$, there is an N such that

$$p(\xi^\nu - \xi^\mu) \leq \epsilon \quad \forall \nu, \mu \geq N.$$

Property 1 then gives

$$p(\xi^\nu - \xi) \leq \epsilon \quad \forall \nu \geq N$$

so $\xi \in \tilde{\mathcal{Y}}$ and (ξ^ν) converges in $\tilde{\mathcal{Y}}$ to ξ. Thus $\tilde{\mathcal{Y}}$ is complete and since $\mathcal{Y}$ is a closed subspace of $\tilde{\mathcal{Y}}$, it is complete as well.

Given $\xi \in L^1$ and $\eta \in L^1$, define $\xi^\nu \in L^\infty$, $\nu = 1, 2, \dots$ as the pointwise projection of ξ to $[-\nu, \nu]$. By monotone convergence and property 3,

$$E(\xi\eta) \leq E(|\xi||\eta|) = \lim_{\nu\to\infty} E(|\xi^\nu||\eta|) \leq \lim_{\nu\to\infty} p(\xi^\nu)p^\circ(\eta) \leq p(\xi)p^\circ(\eta).$$

Thus, any $\eta \in \mathcal{U}$ defines a continuous linear functional on $\mathcal{Y}$. On the other hand, if l is a continuous linear functional on $\mathcal{Y}$, there is a $p \in \mathcal{P}$ and $q \in \mathbb{R}$ such that $l \leq qp$. By Lemma 1, there exists an $\eta \in L^1$ such that $l(\xi) = E(\xi\eta)$ for all $\xi \in L^\infty$. By the definition of p°, the inequality $l \leq qp$ implies $p^\circ(\eta) \leq q$, so $\eta \in \mathcal{U}$ and $\xi\eta \in L^1$.

For each p, p° satisfies 1 since it is defined as a supremum over lower semicontinuous functions on L^1. Property 2 follows from the fact that the polar operation is order-reversing and L^∞ is dense in L^1. Property 3 implies $p^\circ(\eta) = p^\circ(|\eta|)$, while for nonnegative η_1, η_2, 3 is clear. □

Remark 1. *When $\tilde{\mathcal{Y}} \neq \mathcal{Y}$, there exist continuous linear functionals l on $\tilde{\mathcal{Y}}$ that are not expressible as $l(\xi) = E[\xi\eta]$ with $\eta \in L^1$. Indeed, if $\tilde{\xi} \in \tilde{\mathcal{Y}} \setminus \mathcal{Y}$, the separation theorem gives the existence of a continuous linear functional l on $\tilde{\mathcal{Y}}$ such that*

$$\sup_{\xi\in\mathcal{Y}} l(\xi) < l(\tilde{\xi}).$$

Since $\mathcal{Y}$ is a linear space, this means that $l(\xi) = 0$ for all $\xi \in \mathcal{Y}$ and $l(\tilde{\xi}) > 0$. Since $L^\infty \subset \mathcal{Y}$, this cannot happen if $l(\xi) = E(\xi\eta)$ for some $\eta \in L^1$.

Remark 2. *For any $\xi \in \mathcal{Y}$, the pointwise projection ξ^ν of ξ to $[-\nu, \nu]$ converges to ξ in $\mathcal{Y}$ as $\nu \to \infty$. Indeed, given an $\epsilon > 0$, there exists $\bar{\xi} \in L^\infty$ such that $p(\xi - \bar{\xi}) < \epsilon$, so, by 3 and 4,*

$$p(\xi^\nu - \xi) \leq p(\mathbb{1}_{|\xi|\geq\nu}\xi) \leq p(\mathbb{1}_{|\xi|\geq\nu}(\xi - \bar{\xi})) + p(\mathbb{1}_{|\xi|\geq\nu}\bar{\xi}) \leq p(\xi - \bar{\xi}) + p(\mathbb{1}_{|\xi|\geq\nu}\bar{\xi}) < \epsilon$$

for ν large enough. In particular,

$$\mathcal{Y} = \{\xi \in \tilde{\mathcal{Y}} \mid \lim_{\nu\to\infty} p(\mathbb{1}_{|\xi|\geq\nu}\xi) \searrow 0\}.$$

As noted already, Theorem 2 covers L^p spaces with $p \in [1, \infty)$. Note that p° fails Property 4, for example, when p is the L^1-norm. The following example goes beyond Banach spaces.

Example 1. *The L^p-norms with $p = 1, 2, \dots$ satisfy properties 1-4, so the space $\mathcal{Y} := \cap_{p\geq1} L^p$ of measurable functions with finite moments is a Fréchet space and its dual may be identified with is $\mathcal{U} := \cup_{p\geq1} L^p$ under the bilinear form $\langle\xi, \eta\rangle = E(\xi\eta)$.*

Theorem 2 yields easy proofs of some fundamental facts about Orlicz spaces.

Example 2. *Let Φ be a nonzero nondecreasing finite convex function on $\mathbb{R}_+$ with $\Phi(0) = 0$ and let $\mathcal{P} = \{p\}$, where*

$$p(\xi) := \inf\{\beta > 0 \mid E\Phi(|\xi|/\beta) \leq 1\},$$

the Luxemburg norm. Then $\tilde{\mathcal{Y}}$ is the Orlicz space associated with Φ, p satisfies 1-4 and

$$\mathcal{Y} = \{\xi \in L^1 \mid E\Phi(|\xi|/\beta) < \infty \quad \forall \beta > 0\},$$

the associated Morse heart. The polar of p can be expressed as

$$p^\circ(\eta) = \sup_{\xi \in L^\infty}\{E(\eta\xi) \mid E\Phi(\xi) \leq 1\} = \inf_{\beta > 0}\{\beta E\Phi^*(\eta/\beta) + \beta\}$$

and, moreover,

$$\|\eta\|_{\Phi^*} \leq p^\circ(\eta) \leq 2\|\eta\|_{\Phi^*},$$

where $\|\cdot\|_{\Phi^*}$ is the Luxemburg norm associated with the conjugate of Φ. Thus, the dual of $\mathcal{Y}$ coincides with the Orlicz space

$$\mathcal{U} = \{\eta \in L^1 \mid \exists \beta > 0 : \; E\Phi^*(\eta/\beta) < \infty\}.$$

Proof. Let $\xi^\nu \to \xi$ in L^1 be such that $p(\xi^\nu) \leq \alpha$ or, in other words, $E\Phi(\xi^\nu/\alpha) \leq 1$. By Fatou, $E\Phi(\xi/\alpha) \leq 1$ so $p(\xi) \leq \alpha$. Thus, property 1 holds. As to the first inequality in 2, there exists $a, b \geq 0$ with $\Phi(\xi) \geq a\xi - b$. Thus, $E\Phi(|\xi|/\beta) \geq E[a|\xi|/\beta] - b$ for any $\xi \in L^1$ and $\beta > 0$, so

$$p(\xi) \geq \inf\{\beta > 0 \mid E[a|\xi|/\beta] - b \leq 1\} = \frac{a}{b+1}\|\xi\|_{L^1}.$$

The second inequality in 2 is clear when $\|\xi\|_{L^\infty} = \infty$. When $\|\xi\|_{L^\infty} < \infty$, it suffices to choose c such that $\Phi(1/c) \leq 1$ since then $\Phi(|\xi|/(c\|\xi\|_{L^\infty})) \leq 1$ almost surely. Property 3 is clear. If $\xi^\nu \searrow 0$ almost surely in the Morse heart, then

$$E\Phi(\xi^\nu/\beta) \searrow 0 \quad \forall \beta > 0,$$

by dominated convergence. This implies $p(\xi^\nu) \searrow 0$ so, in particular, 4 holds. This also shows that the Morse heart belongs to $\mathcal{Y}$. Indeed, for ξ in the heart and $\xi^\nu = \xi \mathbb{1}_{|\xi| \leq \nu} \in L^\infty$, we have $|\xi^\nu - \xi| \searrow 0$ almost surely. To prove that $\mathcal{Y}$ and the Morse heart coincide, it remains to show that the heart is closed in $\tilde{\mathcal{Y}}$. If (ξ^ν) is in the heart and converges to $\xi \in \tilde{\mathcal{Y}}$, we have for any $\beta > 0$,

$$E\Phi(\xi/(2\beta)) \leq \frac{1}{2}E\Phi(\xi^\nu/\beta) + \frac{1}{2}E\Phi((\xi - \xi^\nu)/\beta) \leq \frac{1}{2}\Phi(\xi^\nu/\beta) + \frac{1}{2}$$

for ν large enough, so $E\Phi(\xi/2\beta) < \infty$ and thus ξ belongs to the Morse heart.

Since the infimum in the definition of the Luxemburg norm is attained,

$$\begin{aligned} p^\circ(\eta) &= \sup_{\xi \in L^\infty}\{E(\xi\eta) \,|\, p(\xi) \leq 1\} \\ &= \sup_{\xi \in L^\infty}\{E(\eta\xi) \mid E\Phi(\xi) \leq 1\}. \end{aligned}$$

Lagrangian duality and the interchange rule for integral functionals (see [17, Theorem 14.60] give

$$p^\circ(\eta) = \inf_{\beta>0} \sup_{\xi\in L^\infty} \{E(\xi\eta) - \beta E\Phi(\xi) + \beta\}$$
$$= \inf_{\beta>0} \{\beta E\Phi^*(\eta/\beta) + \beta\}.$$

Clearly,

$$p^\circ(\eta) \le \inf_{\beta>0} \{\beta E\Phi^*(\eta/\beta) + \beta \mid E\Phi^*(\eta/\beta) \le 1\} \le 2 \inf_{\beta>0} \{\beta \mid E\Phi^*(\eta/\beta) \le 1\}.$$

On the other hand, the epigraph of p° is the convex cone generated by the epigraph of the function $g(\eta) = E\Phi^*(\eta)+1$, so it suffices to show that $g \ge \|\cdot\|_{\Phi^*}$. This is clear when $\|\eta\|_{\Phi^*} \le 1$, so assume $\|\eta\|_{\Phi^*} > 1$. Since $E\Phi^*(0) = 0$, convexity gives

$$E\Phi^*(\eta/\|\eta\|_{\Phi^*}) \le E\Phi^*(\eta)/\|\eta\|_{\Phi^*}.$$

By the definition of $\|\eta\|_{\Phi^*}$, the left side equals 1, so $\|\eta\|_{\Phi^*} \le E\Phi^*(\eta) \le g(\eta)$. □

The next example is concerned with a class of Banach spaces that arise naturally in the theory of *spectral risk measures*. Its proof is based on the results of [16].

Example 3. *Let $\sigma : [0,1] \to \mathbb{R}_+$ be unbounded and nondecreasing with $\int \sigma(u)du = 1$ and define*

$$f(\xi) := \sup_{\alpha\in(0,1)} \left\{ \frac{1}{1-\alpha} \int_\alpha^1 (q_\xi(u) - \sigma(u))du \right\},$$

where $u \to q_\xi(u)$ denotes the left-continuous quantile function of ξ. Let

$$p(\xi) := \inf\{\beta > 0 \mid f(|\xi|/\beta) \le 0\}.$$

and $\mathcal{P} = \{p\}$. If there exists $(0,1)$-uniformly distributed $\theta \in L^\infty$, then p satisfies 1-4. The polar of p can be expressed as $p^\circ(\eta) = \rho(|\eta|)$, where

$$\rho(\eta) := \int_0^1 \sigma(u)q_\eta(u)du,$$

Thus, the dual of $\mathcal{Y}$ coincides with the space

$$\mathcal{U} := \left\{\eta \in L^1 \;\middle|\; \rho(|\eta|) < \infty\right\}.$$

Functions ρ of the above form are known as spectral risk measures; see e.g. [9, 16]. Such functions satisfy properties 1-4. Moreover, they are comonotone additive *in the sense that $\rho(\eta^1 + \eta^2) = \rho(\eta^1) + \rho(\eta^2)$ whenever $\eta^1, \eta^2 \in \mathcal{U}$ are such that*

$$[\eta^1(\omega) - \eta^1(\omega')][\eta^2(\omega) - \eta^2(\omega')] \ge 0 \quad \forall\, \omega, \omega' \in \Omega.$$

Proof. By [16, Theorems 21 and 30], $\mathcal{U}$ is a Banach space under the norm $\|\eta\|_\sigma := \rho(|\eta|)$ and its dual can be identified with $(\tilde{\mathcal{Y}}, p)$. To prove property 1, let $\xi^\nu \to \xi$ in L^1 be such that $\limsup p(\xi^\nu) < \infty$. We may thus assume, by Banach-Alaoglu (and passing to a subsequence, if necessary), that (ξ^ν) converges in the weak topology $\sigma(\tilde{\mathcal{Y}}, \mathcal{U})$. Since $L^\infty \subseteq \mathcal{U}$, the sequence converges also in the weak topology of L^1 so the weak limit coincides with ξ. Property 1 thus follows from the $\sigma(\tilde{\mathcal{Y}}, \mathcal{U})$-lower semicontinuity of p. Equations (19)–(21) in [16] imply properties 2 and 3.

Let $\xi^\nu \searrow 0$ almost surely in L^∞. We have $q_{\xi^\nu}(u) \leq \|\xi^1\|_{L^\infty}$ and $q_{\xi^\nu}(u) \searrow 0$ for all $u < 1$. For any $\beta > 0$, there exists $\alpha' < 1$ such that

$$f(\xi^\nu/\beta) = \sup_{\alpha \in (0,\alpha']} \left\{ \frac{1}{1-\alpha} \int_\alpha^1 (q_{\xi^\nu/\beta}(u) - \sigma(u)) du \right\}.$$

As functions of α on $[0, \alpha']$, the supremands converge uniformly to $\frac{-1}{1-\alpha}\int_\alpha^1 \sigma(u)du$ and thus $f(\xi^\nu/\beta) \to f(0/\beta) = -1$. Since $\beta > 0$ was arbitrary, $p(\xi^\nu) \searrow 0$, so property 4 holds.

As noted in the beginning of the proof, p is the polar of $\eta \to \rho(|\eta|)$, so $p^\circ(\eta) = \rho(|\eta|)$ by the bipolar theorem. It is clear that ρ satisfies 4 while, by [9, Lemma 4.90], the quantile function is comonotone additive in η, so ρ inherits this property. □

Banach spaces associated with more general risk measures than the ones in Example 3 and [16] have been studied in [12].

3 Raw cadlag processes

The Banach space of cadlag functions on $[0, T]$ equipped with the supremum norm will be denoted by D. We allow $T = +\infty$ in which case $[0, T]$ is understood as the one point compactification of the positive reals. The spaces of Borel measures and purely discontinuous Borel measures on $[0, T]$ will be denoted by M and $\tilde{M}$, respectively. The dual of D can be identified with $M \times \tilde{M}$ through the bilinear form

$$\langle y, (u, \tilde{u}) \rangle := \int y du + \int y_- d\tilde{u}$$

and the dual norm is given by

$$\sup_{y \in D} \{ \int y du + \int y_- d\tilde{u} \,|\, \|y\| \leq 1 \} = \|u\| + \|\tilde{u}\|,$$

where $\|u\|$ denotes the total variation norm on M. This can be deduced from [15, Theorem 1] or seen as the deterministic special case of [8, Theorem VII.65] combined with [8, Remark VII.4(a)].

We assume that $\mathcal{Y}$ and $\mathcal{U}$ are as in Section 2 and define

$$\mathcal{Y}(D) := \{ y \in L^1(D) \mid \|y\| \in \mathcal{Y} \},$$

where $L^1(D)$ is the space of cadlag processes y with $E\|y\| < \infty$. Throughout, we identify processes that coincide almost surely everywhere on $[0,T]$. We equip $\mathcal{Y}(D)$ with the topology induced by the seminorms

$$y \mapsto p(\|y\|), \quad p \in \mathcal{P}.$$

Theorem 3. *The space $\mathcal{Y}(D)$ is Fréchet and its dual can be identified with*

$$\mathcal{U}(M \times \tilde{M}) := \{(u, \tilde{u}) \in L^1(M \times \tilde{M}) \mid \|u\| + \|\tilde{u}\| \in \mathcal{U}\}$$

through the bilinear form

$$\langle y, (u, \tilde{u}) \rangle := E\left[\int y du + \int y_- d\tilde{u}\right].$$

Moreover, $L^\infty(D)$ is dense in $\mathcal{Y}(D)$, for every $y \in L^1(D)$ and $(u, \tilde{u}) \in L^1(M \times \tilde{M})$,

$$E\left[\int y du + \int y_- d\tilde{u}\right] \le p(\|y\|) p^\circ(\|u\| + \|\tilde{u}\|) \tag{1}$$

and

$$p^\circ(\|u\| + \|\tilde{u}\|) = \sup_{y \in L^\infty(D)} \left\{ E\left[\int y du + \int y_- d\tilde{u}\right] \,\middle|\, p(\|y\|) \le 1 \right\}. \tag{2}$$

In particular, $(u, \tilde{u}) \mapsto p^\circ(\|u\| + \|\tilde{u}\|)$ is the polar of $y \mapsto p(\|y\|)$.

Proof. We start by showing that $\tilde{\mathcal{Y}}(D) := \{y \in L^1(D) \mid \|y\| \in \tilde{\mathcal{Y}}\}$ is complete under the topology induced by the seminorms $y \mapsto p(\|y\|)$. If (y^ν) is a Cauchy sequence in $\tilde{\mathcal{Y}}(D)$, it is, by Property 2, Cauchy also in $L^1(D)$ which is complete (see e.g. [8, Theorem VI.22]), so (y^ν) $L^1(D)$-converges to an $y \in L^1(D)$. Being Cauchy in $\tilde{\mathcal{Y}}(D)$ means that for every $\epsilon > 0$ and $p \in \mathcal{P}$, there is an N such that

$$p(\|y^\nu - y^\mu\|) \le \epsilon \quad \forall \nu, \mu \ge N.$$

By the triangle inequality and property 3 of p,

$$p(\|y^\nu - y\| - \|y - y^\mu\|) \le \epsilon \quad \forall \nu, \mu \ge N.$$

Letting $\mu \to \infty$ and using property 1 now gives

$$p(\|y^\nu - y\|) \le \epsilon \quad \forall \nu \ge N.$$

Since $p \in \mathcal{P}$ and $\epsilon > 0$ were arbitrary, we thus have $y \in \tilde{\mathcal{Y}}(D)$ and that (y^ν) converges in $\tilde{\mathcal{Y}}(D)$ to y. Thus $\tilde{\mathcal{Y}}(D)$ is complete. It is clear that $\mathcal{Y}(D)$ contains the closure of $L^\infty(D)$. On the other hand, given $y \in \mathcal{Y}(D)$, its pointwise projection y^ν to the interval $[-\nu, \nu]$ belongs to $L^\infty(D)$ and, by Remark 2, converges to y. Thus $\mathcal{Y}(D)$ is a closed subspace of a Fréchet space and thus, Fréchet as well.

We have

$$\left[\int y du + \int y_- d\tilde{u}\right] \le \|y\|(\|u\| + \|\tilde{u}\|)$$

almost surely, so (1) follows from Theorem 2. Every element of $\mathcal{U}(M \times \tilde{M})$ thus defines a continuous linear functional on $\mathcal{Y}(D)$. Conversely, a continuous linear functional J on $\mathcal{Y}(D)$ satisfies property (5.1) in [8, Section VII.5] so, as in the proof of [8, Theorem VII.65], there exists $(u, \tilde{u}) \in L^1(M \times \tilde{M})$ such that

$$J(y) = E\left[\int y du + \int y_- d\tilde{u}\right]$$

on $L^\infty(D)$. Given $\delta \in (0,1)$, a measurable selection argument gives the existence of a $y \in L^1(D)$ such that

$$\|y\| \leq 1 \quad \text{and} \quad \int y du + \int y_- d\tilde{u} \geq \delta(\|u\| + \|\tilde{u}\|)$$

almost surely[1]. Thus, for any $p \in \mathcal{P}$ and $\xi \in L^\infty_+$ such that $p(\xi) \leq 1$,

$$\begin{aligned} E[\xi(\|u\| + \|\tilde{u}\|)] &\leq E[\int (\xi y) du + \int (\xi y_-) d\tilde{u}]/\delta \\ &\leq \sup_{y \in L^\infty(D)} \left\{ E\left[\int y du + \int y_- d\tilde{u}\right] \,\middle|\, p(\|y\|) \leq 1 \right\} /\delta. \end{aligned}$$

The definition of p° now gives $p^\circ(\|u\| + \|\tilde{u}\|) \leq p^\circ_{\mathcal{Y}(D)}(J)/\delta$. Since $\delta \in (0,1)$ was arbitrary, we see that the left hand side of (2) is less than the right side. The reverse follows from (1). □

4 The projections

From now on, we fix a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. The set of stopping times will be denoted by $\mathcal{T}$. A measurable process y is said to be of *class* (D) if $\{y_\tau \mid \tau \in \mathcal{T}\}$ is uniformly integrable. Given such y, we will denote its *optional and predictable projections* by ${}^o y$ and ${}^p y$, respectively. That is, ${}^o y$ is the unique optional process satisfying

$$E[y_\tau \mathbb{1}_{\{\tau < \infty\}} \mid \mathcal{F}_\tau] = {}^o y_\tau \mathbb{1}_{\{\tau < \infty\}} \quad P\text{-a.s.}$$

for every $\tau \in \mathcal{T}$ while ${}^p y$ is the unique predictable process satisfying

$$E\left[y_\tau \mathbb{1}_{\{\tau < \infty\}} \mid \mathcal{F}_{\tau-}\right] = {}^p y_\tau \mathbb{1}_{\{\tau < \infty\}} \quad P\text{-a.s.}$$

for every predictable time τ. Here $\mathcal{F}_\tau := \sigma(A \mid A \cap \{\tau \leq t\} \in \mathcal{F}_t \ \forall\ t\}$ and $\mathcal{F}_{\tau-} := \mathcal{F}_0 \vee \sigma\{A \cap \{t < \tau\} \mid A \in \mathcal{F}_t,\, t \in \mathbb{R}_+\}$. Throughout the paper, we

[1]Indeed, $(D, \mathcal{B}(D)) = (S, \mathcal{B}(S))$, where S is the space of cadlag functions equipped with the Skorokhod topology. The set

$$G := \{(y,\omega) \in S \times \Omega \mid \|y\| \leq 1, \int y du(\omega) + \int y_- d\tilde{u}(\omega) \geq \delta(\|u(\omega)\| + \|\tilde{u}(\omega)\|\}$$

is $\mathcal{B}(S) \otimes \mathcal{F}$-measurable (see the proof of [13, Lemma 3]) and each ω-section of G is nonempty. Thus [5, Theorem III.18] gives the existence of a measurable selection.

identify processes that are equal almost surely everywhere, that is, $y^1 = y^2$ if, almost surely, $y^1_t = y^2_t$ for all t.

By [8, Remark VI.50.(f)], the optional projection of a cadlag process of class (D) is a cadlag process of class (D) while the predictable projection of a caglad process of class (D) is a caglad process of class (D).

Lemma 4. *For any cadlag process y of class (D), we have $({}^o y)_- = {}^p(y_-)$.*

Proof. Given bounded predictable time τ, it is enough to verify, by the predictable section theorem [10, Corollary 4.11], that $(({}^o y)_-)_\tau = {}^p(y_-)_\tau$. By [10, Thereom 4.16], there is a sequence (τ^ν) of stopping times with $\tau^\nu < \tau^{\nu+1}$ and $\tau^\nu \nearrow \tau$ almost surely. Let $A_\nu \in \mathcal{F}_{\tau^\nu}$, and define $\tau^\nu_j := \tau^{\nu+j} + \delta_{A_\nu}$. We have $A_\nu \in \mathcal{F}_{\tau^{\nu+j}}$ for each j [10, Theorem 3.4], so τ^ν_j are stopping times [10, Theorem 3.9]. Since y and ${}^o y$ are of class (D),

$$E[({}^o y)_{\tau-} \mathbb{1}_{A_\nu}] = \lim_j E[({}^o y)_{\tau^\nu_j} \mathbb{1}_{A_\nu}] = \lim_j E[y_{\tau^\nu_j} \mathbb{1}_{A_\nu}] = E[y_{\tau-} \mathbb{1}_{A_\nu}].$$

By [10, Theorem 3.6], $\mathcal{F}_{\tau-} = \bigvee_\nu \mathcal{F}_{\tau^\nu}$, which proves the claim, since $A \in \mathcal{F}_{\tau^\nu}$ was arbitrary. □

Given $(u, \tilde{u}) \in L^1(M \times \tilde{M})$, there exist $u^o \in L^1(M)$ and $\tilde{u}^p \in L^1(\tilde{M})$ such that for every bounded measurable process y,

$$E \int {}^o y du = E \int y du^o,$$
$$E \int {}^p y du = E \int y du^p.$$

The random measure u^o is called the *optional projection* of u while $\tilde{u}^p$ is called the *predictable projection* of $\tilde{u}$. One says that u is *optional* if $u = u^o$ and that $\tilde{u}$ is *predictable* if $\tilde{u} = \tilde{u}^p$.

From now on, $\mathcal{Y}$ and $\mathcal{U}$ are as in Section 2. The optional projection is a linear mapping from $\mathcal{Y}(D)$ to the space of adapted cadlag processes of class (D). We denote this linear mapping by π and its kernel by

$$\ker \pi := \{y \in \mathcal{Y}(D) \mid {}^o y = 0\}.$$

The space

$$\hat{\mathcal{M}} := \{(u, \tilde{u}) \in \mathcal{U}(M \times \tilde{M}) \mid u = u^o,\ \tilde{u} = \tilde{u}^p\}$$

will play a central role in the remainder of this paper. Indeed, we will find it as the topological dual of various spaces of adapted cadlag processes. The following characterizes it in terms of the pairing of $\mathcal{Y}(D)$ and $\mathcal{U}(M \times \tilde{M})$ obtained in Theorem 3 above.

Lemma 5. *The space $\hat{\mathcal{M}}$ is the orthogonal complement of $\ker \pi \cap L^\infty(D)$ and thus, weakly closed in $\mathcal{U}(M \times \tilde{M})$.*

Proof. We first show that $(u,\tilde u)\in\mathcal{U}(M\times\tilde M)$ belongs to $\hat{\mathcal{M}}$ if and only if

$$E\left[\int ydu+\int y_-d\tilde u\right]=E\left[\int {}^oydu+\int {}^p(y_-)d\tilde u\right]\quad\forall y\in L^\infty(D).\tag{3}$$

Sufficiency of the variational condition is clear. To prove the necessity, it suffices to note that the equation can be written as

$$\begin{aligned}0&=E\left[\int(y-{}^oy)du+\int(y_--{}^p(y_-))d\tilde u\right]\\&=E\left[\int yd(u-u^o)+\int y_-d(\tilde u-\tilde u^p)\right]\\&=E\left[\int yd(u-u^o-(\tilde u^p)^c)+\int y_-d(\tilde u-(\tilde u^p)^d)\right],\end{aligned}$$

where $(\tilde u^p)^c$ and $(\tilde u^p)^d$ denote the continuous and purely discontinuous parts of $\tilde u^p$, respectively. Since $L^\infty(D)$ is dense in $\mathcal{Y}(D)$, the variational condition implies, by Theorem 3, that $\tilde u-(\tilde u^p)^d=0$ and $u-u^o-(\tilde u^p)^c=0$. The first equation implies that $\tilde u$ is predictable and that $(\tilde u^p)^d=0$. The second equation then implies that u is optional.

By Lemma 4, the variational characterization (3) of $\hat{\mathcal{M}}$ means that $\hat{\mathcal{M}}$ is the orthogonal complement of $L:=\{{}^oy-y\mid y\in L^\infty(D)\}$, which equals $\ker\pi\cap L^\infty(D)$. ☐

The following is the first main result of this paper. It will be the basis, later on, of more concrete characterizations of dual spaces of adapted cadlag processes.

Theorem 6. *Assume that the optional projection is continuous from $\mathcal{Y}(D)$ to a Fréchet space $\mathcal{D}$ of adapted cadlag processes. The projection is a surjection if and only if the dual of $\mathcal{D}$ can be identified with $\hat{\mathcal{M}}$ under the bilinear form*

$$\langle y,(u,\tilde u)\rangle=E\left[\int ydu+\int y_-d\tilde u\right].$$

In this case, the adjoint of the projection is the embedding of $\hat{\mathcal{M}}$ to $\mathcal{U}(M\times\tilde M)$, $\hat{\mathcal{M}}=(\ker\pi)^\perp$ and the topology of $\mathcal{D}$ is generated by the seminorms

$$p_{\mathcal{D}}(y):=\inf_{z\in\mathcal{Y}(D)}\{p(\|z\|)\mid {}^oz=y\}\quad p\in\mathcal{P}$$

whose polars are given by

$$p^\circ_{\mathcal{D}}((u,\tilde u))=p^\circ(\|u\|+\|\tilde u\|).$$

Proof. Assume first that the dual of $\mathcal{D}$ is $\hat{\mathcal{M}}$. For any $(u,\tilde u)\in\hat{\mathcal{M}}$ and $y\in L^\infty(D)$, Lemma 4 gives

$$E\left[\int {}^oydu+\int({}^oy)_-d\tilde u\right]=E\left[\int {}^oydu+\int {}^p(y_-)d\tilde u\right]=E\left[\int ydu+\int y_-d\tilde u\right].$$

Since the optional projection is continuous and since $L^\infty(D)$ is dense in $\mathcal{Y}(D)$, this extends to all of $\mathcal{Y}(D)$ so the adjoint is indeed the embedding.

Since $(\operatorname{rge}\pi)^\perp = \ker\pi^* = \{0\}$, the bipolar theorem gives $\operatorname{cl}\operatorname{rge}\pi = \mathcal{D}$, so it suffices to show that $\operatorname{rge}\pi$ is closed. By the closed range theorem [11, Theorem 21.9], this is equivalent to $\operatorname{rge}\pi^*$ being closed in $\mathcal{U}(M\times\tilde{M})$. Since π^* is the embedding, its range is $\hat{\mathcal{M}}$ which is closed, by Lemma 5.

On the other hand, if π is a surjection, the closed graph theorem (see [11, Theorem 11.2]) implies that $\mathcal{D}$ is isomorphic to the quotient space $\mathcal{Y}(D)/\ker\pi$. The dual of $\mathcal{Y}(D)/\ker\pi$ can be identified with the orthogonal complement of $\ker\pi$ on the dual of $\mathcal{Y}(D)$ which, by Theorem 3, is $\mathcal{U}(M\times\tilde{M})$. By Lemma 5, the density of $L^\infty(D)$ in $\mathcal{Y}(D)$ and the continuity of π imply $\ker\pi^\perp = \hat{\mathcal{M}}$.

The isomorphism of $\mathcal{D}$ with $\mathcal{Y}(D)/\ker\pi$ also implies that the topology of $\mathcal{D}$ is induced by the quotient space seminorms $p_{\mathcal{D}}$. Since the adjoint of the optional projection is the embedding of $\hat{\mathcal{M}}$ the polar of $p_{\mathcal{D}}$ can be expressed for every $(u,\tilde{u})\in\hat{\mathcal{M}}$ as

$$\begin{aligned}
p^\circ_{\mathcal{D}}((u,\tilde{u})) &= \sup_{y\in\mathcal{D}}\{\langle y,(u,\tilde{u})\rangle \mid \inf_{z\in\mathcal{Y}(D)}\{p(\|z\|)\mid {}^oz = y\}\le 1\}\\
&= \sup_{z\in\mathcal{Y}(D)}\{\langle {}^oz,(u,\tilde{u})\rangle \mid p(\|z\|)\le 1\}\\
&= \sup_{z\in\mathcal{Y}(D)}\{\langle z,(u,\tilde{u})\rangle \mid p(\|z\|)\le 1\}\\
&= p^\circ(\|u\|+\|\tilde{u}\|),
\end{aligned}$$

where the last equality follows from Theorem 3. □

We have $p_{\mathcal{D}}(y) = p(y_T)$ for any martingale y. Moreover, the set of martingales is a closed subspace of $\mathcal{D}$ and its dual can be identified with $\mathcal{U}$ via the bilinear form $\langle y,\eta\rangle = E[y_T\eta]$.

Note also that $y\in\mathcal{Y}(D)$ does not imply ${}^oy\in\mathcal{Y}(D)$, in general. In other words, the optional projection need not be a projection in the sense of functional analysis. Indeed, if y is a martingale, it is the optional projection of the constant process $\mathbb{1}y_T\in L^1(D)$ but it may happen that $\|y\|\notin L^1$. Similarly, $(u,\tilde{u})\in\mathcal{U}(M\times\tilde{M})$ does not imply $(u^o,\tilde{u}^p)\in\mathcal{U}(M\times\tilde{M})$, in general.

Example 4. *Let $y\in L^1(D)$ be nonnegative such that ${}^oy\notin L^1(D)$. Let τ be a random time such that $E\,{}^oy_\tau=\infty$ and define $u=\delta_\tau$. We have $u\in L^\infty(M)$, but*

$$E\int y\,du^o = E\int {}^oy\,du = E\,{}^oy_\tau = \infty,$$

so $u^o\notin L^\infty(M)$.

5 Optional projection under Doob property

This section studies the case when the optional projection is a continuous linear mapping of the space $\mathcal{Y}(D)$ to itself. Without loss of generality, we assume that the collection $\mathcal{P}$ of seminorms form a nondecreasing sequence. Continuity of the projection then means that for each $p \in \mathcal{P}$ there exists a $p' \in \mathcal{P}$ and a constant q such that

$$p(\|{}^o y\|) \le qp'(\|y\|)$$

for all $y \in \mathcal{Y}(D)$. It turns out that this is equivalent to the validity of the Doob inequality, which means that for each $p \in \mathcal{P}$ there exists a $p' \in \mathcal{P}$ and a constant q such that

$$p(\|m\|) \le qp'(m_T)$$

for every martingale m. This is known to hold e.g. when $\mathcal{Y}$ is an Orlicz space associated with a Young function whose conjugate satisfies the Δ_2-condition; see [8, Section VI.103] which generalizes the better known case of $\mathcal{Y} = L^p$ with $p > 1$. We will say that $\mathcal{Y}$ has the *Doob property* if the Doob inequality is valid.

In this section, we define $\mathcal{D}$ as the optional processes in $\mathcal{Y}(D)$. Since convergence in $\mathcal{Y}(D)$ implies convergence almost surely everywhere, $\mathcal{D}$ is a closed subspace of $\mathcal{Y}(D)$. The first statement of Theorem 3 thus gives the following.

Lemma 7. *The space $\mathcal{D}$ is Fréchet.*

When the optional projection π is continuous on $\mathcal{Y}(D)$, it has an *adjoint* π^* which is a continuous linear operator on the dual $\mathcal{U}(M \times \tilde{M})$ of $\mathcal{Y}(D)$ defined by

$$\langle \pi y, (u, \tilde{u})\rangle = \langle y, \pi^*(u, \tilde{u})\rangle \quad \forall y \in \mathcal{Y}(D),\ \forall (u, \tilde{u}) \in \mathcal{U}(M \times \tilde{M}).$$

Theorem 8. *The conditions*

(a) $\mathcal{Y}$ has the Doob property,

(b) the optional projection is continuous on $\mathcal{Y}(D)$,

are equivalent and imply that the adjoint of the optional projection is given by

$$\pi^*(u, \tilde{u}) = (u^o + (\tilde{u}^p)^c, (\tilde{u}^p)^d).$$

and that the dual of $\mathcal{D}$ can be identified with $\hat{\mathcal{M}}$ through the bilinear form

$$\langle y, (u, \tilde{u})\rangle = E\left[\int y du + \int y_- d\tilde{u}\right].$$

The topology of $\mathcal{D}$ is generated by the seminorms

$$p_{\mathcal{D}}(y) := \inf_{z \in \mathcal{Y}(D)} \{p(\|z\|) \mid {}^o z = y\} \quad p \in \mathcal{P}$$

whose polars are given by

$$p^{\circ}_{\mathcal{D}}((u, \tilde{u})) = p^{\circ}(\|u\| + \|\tilde{u}\|).$$

Proof. As already noted, (b) implies (a). To prove the converse, let $y \in \mathcal{Y}(D)$ and $m = {}^o(\mathbb{1}\|y\|)$. Since $|y| \le \mathbb{1}\|y\|$, we have $|{}^oy| \le {}^o|y| \le m$, so $\|{}^oy\| \le \|m\|$, while (a) gives

$$p(\|m\|) \le qp'(\|y\|).$$

The monotonicity of p now gives (b).

If $y \in L^\infty(D)$, Lemma 4 gives for all $(u, \tilde{u}) \in \mathcal{U}(M \times \tilde{M})$,

$$\begin{aligned} \langle {}^oy, (u, \tilde{u})\rangle &= E\left[\int {}^oy du + \int ({}^oy)_- d\tilde{u}\right] \\ &= E\left[\int {}^oy du + \int {}^p(y_-) d\tilde{u}\right] \\ &= E\left[\int y du^o + \int y_- d\tilde{u}^p\right] \\ &= E\left[\int y d(u^o + (\tilde{u}^p)^c) + \int y_- d(\tilde{u}^p)^d\right]. \end{aligned} \tag{4}$$

Let $p \in \mathcal{P}$ be such that $p^\circ(\|u\| + \|\tilde{u}\|) < \infty$. Under (b), (4) and Theorem 3 give

$$\begin{aligned} E\left[\int y d(u^o + (\tilde{u}^p)^c) + \int y_- d(\tilde{u}^p)^d\right] &\le p(\|{}^oy\|)p^o(\|u\| + \|\tilde{u}\|) \\ &\le qp'(\|y\|)p^o(\|u\| + \|\tilde{u}\|). \end{aligned}$$

Taking the supremum over $\{y \in L^\infty(D) \,|\, p'(\|y\|) \le 1\}$, gives, by Theorem 3,

$$(p')^\circ(\|u^o + (\tilde{u}^p)^c\| + \|(\tilde{u}^p)^d\|) \le qp^\circ(\|u\| + \|\tilde{u}\|).$$

Thus $\|u^o + (\tilde{u}^p)^c\| + \|(\tilde{u}^p)^d\| \in \mathcal{U}$, so $(u^o + (\tilde{u}^p)^c, \tilde{u}^p) \in \mathcal{U}(M \times \tilde{M})$. Thus the density of $L^\infty(D)$ in $\mathcal{Y}(D)$ implies that (4) extends to all of $\mathcal{Y}(D)$, so the adjoint is given by

$$\pi^*(u, \tilde{u}) = (u^o + (\tilde{u}^p)^c, (\tilde{u}^p)^d).$$

Clearly, π is a surjection to $\mathcal{D}$, so, by Theorem 6, the dual of $\mathcal{D}$ can be identified with $\hat{\mathcal{M}}$. □

The characterization of the dual of $\mathcal{D}$ in Theorem 6 generalizes [8, Theorem VII.65] and [1, Theorem 3.1] that dealt with L^p and Morse hearts of Orlicz spaces, respectively. Indeed, [8, pages 166–169] establish the Doob inequality when $\mathcal{Y}$ is the Orlicz space associated with a Young function whose conjugate has the Δ_2-property. In that case, we may apply Theorem 6 in the setting of Example 2. Example 1 provides a simple example beyond Banach spaces.

6 Optional projection under Choquet property

When $\mathcal{Y}$ fails to have the Doob property, it may happen that $y^o \notin \mathcal{Y}(D)$ for an $y \in \mathcal{Y}(D)$. Nevertheless, if

$$p(E_\tau \xi) \le p(\xi) \quad \forall\, \xi \in \mathcal{Y},\ \tau \in \mathcal{T} \tag{5}$$

for all $p \in \mathcal{P}$, then

$$\sup_{\tau \in \mathcal{T}} p({}^o y_\tau) \le p(\|y\|) \quad \forall p \in \mathcal{P} \tag{6}$$

for all $y \in \mathcal{Y}(D)$. Thus, the optional projection of a $y \in \mathcal{Y}(D)$ belongs to the space $\tilde{\mathcal{D}}$ of optional cadlag processes for which the seminorms

$$p_{\mathcal{T}}(y) := \sup_{\tau \in \mathcal{T}} p(y_\tau)$$

are finite for all $p \in \mathcal{P}$. We equip $\tilde{\mathcal{D}}$ with the topology induced by the seminorms

$$y \mapsto p_{\mathcal{T}}(y), \quad p \in \mathcal{P}.$$

The case $\mathcal{Y} = L^1$ was studied in [3] and [8, Section VI.1].

In this section, we define $\mathcal{D}$ as the closure in $\tilde{\mathcal{D}}$ of the space $\mathcal{D}^\infty$ of bounded optional cadlag processes.

Lemma 9. *The space $\mathcal{D}$ is Fréchet and its elements are of class (D).*

Proof. We start by showing that $\tilde{\mathcal{D}}$ is complete. If (y^ν) is a Cauchy sequence in $\tilde{\mathcal{D}}$, it is, by Property 2, Cauchy also in $\mathcal{D}^1$ of optional cadlag processes equipped with the norm $\sup_{\tau \in \mathcal{T}} E|y_\tau|$. By [8, Theorem VI.22]), $\mathcal{D}^1$ is complete, so (y^ν) $\mathcal{D}^1$-converges to an $y \in \mathcal{D}^1$. Being Cauchy in $\tilde{\mathcal{D}}$ means that for every $\epsilon > 0$ and $p \in \mathcal{P}$, there is an N such that

$$p_{\mathcal{T}}(y^\nu - y^\mu) \le \epsilon \quad \forall \nu, \mu \ge N.$$

By the triangle inequality and property 3 of p,

$$p_{\mathcal{T}}(|y^\nu - y| - |y - y^\mu|) \le \epsilon \quad \forall \nu, \mu \ge N.$$

Letting $\mu \to \infty$ and using property 1 (and the fact that pointwise supremum of lsc functions is lsc) now gives

$$p_{\mathcal{T}}(y^\nu - y) \le \epsilon \quad \forall \nu \ge N.$$

Since $p \in \mathcal{P}$ and $\epsilon > 0$ were arbitrary, we thus have $y \in \tilde{\mathcal{D}}$ and that (y^ν) converges in $\tilde{\mathcal{D}}$ to y. Thus $\tilde{\mathcal{D}}$ is complete. Since $\mathcal{D}$ is a closed subspace of a Fréchet space, it is Fréchet as well.

Given $y \in \mathcal{D}$ and $\epsilon > 0$, there exists $y^\epsilon \in \mathcal{D}^\infty$ such that $\sup_{\tau \in \mathcal{T}} E|y_\tau - y^\epsilon_\tau| < \epsilon/2$. By Chebyshev's inequality,

$$\begin{aligned}\sup_{\tau \in \mathcal{T}} E[|y_\tau| \mathbb{1}_{\{|y_\tau| \ge \nu\}}] &\le \sup_{\tau \in \mathcal{T}} E|y_\tau - y^\epsilon_\tau| + \sup_{\tau \in \mathcal{T}} E[|y^\epsilon_\tau| \mathbb{1}_{\{|y_\tau| \ge \nu\}}] \\ &\le \epsilon/2 + \|y^\epsilon\|_{L^\infty} \sup_{\tau \in \mathcal{T}} E|y_\tau|/\nu < \epsilon\end{aligned}$$

for ν large enough, which shows that y is of class (D). □

We say that $\mathcal{Y}$ has the *Choquet property* if for every positive $\eta \in \mathcal{U}$, the polar seminorms p° can be expressed as Choquet integrals, that is, if

$$p^\circ(\eta) = \int_0^\infty p^\circ(\mathbb{1}_{\{\eta \geq s\}})ds \quad \forall p \in \mathcal{P}.$$

This is clearly satisfied if $\mathcal{Y} = L^1$ or $\mathcal{Y} = L^\infty$ (although the latter fails property 4 in Section 2). More generally, we have the following extension of [18]; see also [9].

Lemma 10. *A real-valued function ρ on $\mathcal{U}$ with $\rho(1) = 1$ is a Choquet integral on $\mathcal{U}$ if and only if it is monotone, comonotone additive and*

$$\rho(\eta \wedge \nu) \nearrow \rho(\eta) \quad \forall\ \eta \in \mathcal{U}_+.$$

Proof. The necessity is proved as in the proof of [9, Theorem 4.88] (their argument does not require $\mathcal{U} = L^\infty$). As to sufficiency, [18, Theorem] says that, ρ is Choquet integral on L^∞. By monotone convergence,

$$\int_0^\infty \rho(\mathbb{1}_{\{\eta\wedge\nu \geq s\}})ds \nearrow \int_0^\infty \rho(\mathbb{1}_{\{\eta \geq s\}})ds$$

while $\rho(\eta \wedge \nu) \nearrow \rho(\eta)$, by assumption. □

Assumptions of Lemma 10 are satisfied, e.g., by the polar seminorms given in terms of spectral risk measures in Example 3 of Section 2.

Theorem 11. *Assume that $\mathcal{Y}$ has the Choquet property and that each $p \in \mathcal{P}$ satisfies* (5)*. Then the dual of $\mathcal{D}$ can be identified with $\hat{\mathcal{M}}$ under the bilinear form*

$$\langle y, (u, \tilde{u})\rangle := E\left[\int y du + \int y_- d\tilde{u}\right].$$

The optional projection is a continuous surjection of $\mathcal{Y}(D)$ to $\mathcal{D}$, its adjoint is the embedding of $\hat{\mathcal{M}}$ to $\mathcal{U}(M \times \tilde{M})$, and the topology of $\mathcal{D}$ is generated by the seminorms

$$p_{\mathcal{D}}(y) := \inf_{z\in\mathcal{Y}(D)} \{p(\|z\|) \mid {}^o z = y\} \quad p \in \mathcal{P}$$

whose polars are given by

$$p^\circ_{\mathcal{D}}((u, \tilde{u})) = p^\circ(\|u\| + \|\tilde{u}\|).$$

Proof. By (6), the optional projection is continuous from $\mathcal{Y}(D)$ to $\tilde{\mathcal{D}}$ with norm one. Since $L^\infty(D)$ is dense in $\mathcal{Y}(D)$, the continuity of the projection implies that its range is contained in $\mathcal{D}$. By Theorem 6, it suffices to show that $\hat{\mathcal{M}}$ is the dual of $\mathcal{D}$.

Let $y \in \mathcal{D}$ and $(u, \tilde{u}) \in \hat{\mathcal{M}}$ and denote the corresponding total variations processes by u^{TV} and $\tilde{u}^{TV}$. By [7, Theorem IV.50], $\tau_s = \inf\{t \,|\, u^{TV}_t \geq s\}$ is a stopping time, and, by [8, A on page xiii], $\tilde{\tau}_s = \inf\{t \,|\, \tilde{u}^{TV}_t \geq s\}$ is a predictable time. By [8, Theorem 55] and Fubini-Tonelli,

$$
\begin{aligned}
E[\int ydu + \int y_- d\tilde{u}] &\leq E\int (|y|du^{TV} + |y_-|d\tilde{u}^{TV}) \\
&= E\int_0^\infty (|y_{\tau_s}|\mathbb{1}_{\{\|u\|\geq s\}} + |y_{\tilde{\tau}_s-}|\mathbb{1}_{\{\|\tilde{u}\|\geq s\}})ds \\
&= \int_0^\infty (E\left[|y_{\tau_s}|\mathbb{1}_{\{\|u\|\geq s\}} + |y_{\tilde{\tau}_s-}|\mathbb{1}_{\{\|\tilde{u}\|\geq s\}})\right] ds \\
&\leq \int_0^\infty [p(y_{\tau_s})p^\circ(\mathbb{1}_{\{\|u\|\geq s\}}) + p(y_{\tilde{\tau}_s-})p^\circ(\mathbb{1}_{\{\|\tilde{u}\|\geq s\}})]ds.
\end{aligned}
$$

By [10, Thereom 4.16], there is a sequence (τ^ν) of stopping times with $\tau^\nu < \tau^{\nu+1}$ and $\tau^\nu \nearrow \tilde{\tau}_s$ almost surely. Since y is of class (D) and p is weakly lsc in L^1, we get $p(y_{\tilde{\tau}_s-}) \leq \liminf_\nu p(y_{\tau^\nu}) \leq \sup_{\tau\in\mathcal{T}} p(y_\tau)$. By Choquet property,

$$
\begin{aligned}
E[\int ydu + \int y_- d\tilde{u}] &\leq \sup_{\tau\in\mathcal{T}} p(y_\tau)\int_0^\infty [p^\circ(\mathbb{1}_{\{\|u\|\geq s\}}) + p^\circ(\mathbb{1}_{\{\|\tilde{u}\|\geq s\}})]ds \\
&= p_{\mathcal{T}}(y)[p^\circ(\|u\|) + p^\circ(\|\tilde{u}\|)] \\
&\leq 2p_{\mathcal{T}}(y)p^\circ(\|u\| + \|\tilde{u}\|).
\end{aligned}
$$

Thus $(u, \tilde{u})$ defines a continuous linear functional on $\mathcal{D}$.

On the other hand, let J be a continuous linear functional on $\mathcal{D}$. The continuity implies that J is continuous on $\mathcal{D}^\infty \subseteq \mathcal{D}\cap\mathcal{Y}(D)$ also with respect to the relative topology of $\mathcal{Y}(D)$. By Hahn–Banach, J extends to a continuous linear functional on all of $\mathcal{Y}(D)$. Theorem 3 then gives the existence of a $(w, \tilde{w}) \in \mathcal{U}(M\times\tilde{M})$ such that

$$J(y) = E\left[\int ydw + \int y_- d\tilde{w}\right] \quad \forall y \in \mathcal{D}^\infty.$$

By the definitions of the projections,

$$J(y) = E\left[\int ydw^o + \int y_- d\tilde{w}^p\right] = E\left[\int ydu + \int y_- d\tilde{u}\right] \quad \forall y \in \mathcal{D}^\infty,$$

where $(u, \tilde{u}) := (w^o + (\tilde{w}^p)^c, (\tilde{w}^p)^d) \in L^1(M\times\tilde{M})$ with u optional and $\tilde{u}$ predictable. The continuity of J on $\mathcal{D}$ means that there is a $p \in \mathcal{P}$ such that

$$p^\circ_{\mathcal{T}}(J) := \sup_{y\in\mathcal{D}}\{J(y) \mid p_{\mathcal{T}}(y) \leq 1\} < \infty.$$

By (2), Lemma 4 and (6),

$$
\begin{aligned}
p^\circ(\|u\|+\|\tilde u\|) &= \sup_{y\in L^\infty(D)}\left\{E\left[\int y du + \int y_- d\tilde u\right] \,\middle|\, p(\|y\|)\le 1\right\}\\
&= \sup_{y\in L^\infty(D)}\left\{E\left[\int {}^o y du + \int {}^p(y_-) d\tilde u\right] \,\middle|\, p(\|y\|)\le 1\right\}\\
&\le \sup_{y\in L^\infty(D)}\left\{E\left[\int {}^o y du + \int ({}^o y)_- d\tilde u\right] \,\middle|\, p_{\mathcal T}({}^o y)\le 1\right\}\\
&= \sup_{y\in L^\infty(D)}\{J({}^o y)\mid p_{\mathcal T}({}^o y)\le 1\}\le p^\circ_{\mathcal T}(J),
\end{aligned}
$$

where the last equality holds since ${}^o y\in\mathcal D^\infty$ for all $y\in L^\infty(D)$. Thus, J is represented on $\mathcal D^\infty$ by a $(u,\tilde u)\in\hat{\mathcal M}$. By continuity, the representation is valid on all of $\mathcal D=\operatorname{cl}\mathcal D^\infty$. □

It is clear from the above proof that, instead of the Choquet property, it would suffice that $\int_0^\infty p^\circ(\mathbb 1_{\{\eta\ge s\}})ds$ is finite whenever $\eta\in\operatorname{dom}p^\circ$.

When $\mathcal Y=L^1$, Theorem 11 can be written as follows.

Corollary 12. *The space $\mathcal D^1$ of optional cadlag processes of class (D) equipped with the norm*

$$\|y\|_{\mathcal D^1}:=\sup_{\tau\in\mathcal T}E|y_\tau|$$

is Banach and its dual can be identified with $\hat{\mathcal M}^\infty$ through the bilinear form

$$\langle y,(u,\tilde u)\rangle = E\left[\int y du+\int y_- d\tilde u\right].$$

The optional projection is a continuous surjection of $L^1(D)$ to $\mathcal D^1$ and its adjoint is the embedding of $\hat{\mathcal M}^\infty$ to $L^\infty(M\times\tilde M)$. The topology of $\mathcal D^1$ is generated by the seminorm

$$p_{\mathcal D}(y):=\inf_{z\in L^1(D)}\{E\|z\|\mid {}^o z=y\}$$

whose polar is given by

$$p^\circ_{\mathcal D}((u,\tilde u))=\text{ess sup}(\|u\|+\|\tilde u\|).$$

Proof. Since $\mathcal Y=L^1$ has the Choquet property, it suffices, by Theorem 11, to check that $\mathcal D^1$ is the closure of $\mathcal D^\infty$ in $\tilde{\mathcal D}^1$. Let $y\in\mathcal D^1$ and define $y^\nu\in\mathcal D^\infty$ as the pointwise projection of y to the Euclidean unit ball of radius $\nu=1,2,\ldots$. By uniform integrability,

$$\sup_{\tau\in\mathcal T}E|y_\tau-y^\nu_\tau|\le\sup_{\tau\in\mathcal T}E[|y_\tau|\mathbb 1_{\{|y_\tau|\ge\nu\}}]\to 0,$$

so $y\in\operatorname{cl}\mathcal D^\infty$. □

Corollary 12 complements [8, Theorem 67] which characterizes the dual of the Banach space of cadlag processes whose pathwise sup-norm is integrable. The larger space $\mathcal{D}^1$ in Corollary 12 was studied in [8, Section VI.1]. The above characterization of its dual seems new. The surjectivity of the projection in Corollary 12 was stated in [3, Theorem 4] without a complete proof.

7 Regular processes

Following [3] we say that an adapted cadlag process y of class (D) is *regular* if

$$^p y = y_-.$$

According to [3, Theorem 3], regular processes are the optional projections of elements of $L^1(C)$. This section gives an easy derivation of Bismut's result while allowing for more general $\mathcal{Y}$ in place of L^1. We assume that $\mathcal{Y}$ is as in Section 2 and define

$$\mathcal{Y}(C) := \{y \in L^1(C) \mid \|y\| \in \mathcal{Y}\}.$$

The following specializes Theorem 3 to continuous processes.

Corollary 13. *The space $\mathcal{Y}(C)$ is Fréchet and its dual can be identified with*

$$\mathcal{U}(M) := \{u \in L^1(M) \mid \|u\| \in \mathcal{U}\}$$

through the bilinear form

$$\langle y, u\rangle := E\int y du.$$

For every $y \in L^1(C)$ and $u \in L^1(M)$,

$$E\int y du \le p(\|y\|)p^\circ(\|u\|)$$

and

$$p^\circ(\|u\|) = \sup_{y\in L^\infty(C)}\left\{E\int y du \,\middle|\, p(\|y\|) \le 1\right\}.$$

In particular, $u \mapsto p^\circ(\|u\|)$ is the polar of $y \mapsto p(\|y\|)$.

Proof. $\mathcal{Y}(C)$ is a closed subspace of $\mathcal{Y}(D)$ and thus Fréchet. The elements of $\mathcal{U}(M)$ define continuous linear functionals on $\mathcal{Y}(C)$. On the other hand, by Hahn-Banach, a continuous linear functional l on $\mathcal{Y}(C)$ extends to a continuous linear functional on $\mathcal{Y}(D)$, which, by Theorem 3, has the expression

$$l(y) = E[\int y du + \int y_- d\tilde{u}]$$

for some $(u, \tilde{u}) \in \mathcal{U}(M \times \tilde{M})$. On $\mathcal{Y}(C)$, this can be written as

$$l(y) = E\int y d(u + \tilde{u}),$$

where $u+\tilde{u}\in\mathcal{U}(M)$. The expression for the polar seminorm follows as in the proof of Theorem 3. $\square$

We will assume that one of the equivalent conditions in Theorem 6 is satisfied and denote

$$\mathcal{R} := \{y\in\mathcal{D} \mid {}^p y = y_-\}.$$

We endow $\mathcal{R}$ with the relative topology it has as a subspace of $\mathcal{D}$. Let

$$\mathcal{M} := \{u\in\mathcal{U}(M) \mid u \text{ optional}\}.$$

The following is proved like Lemma 5 except that instead of Theorem 3 one applies Corollary 13.

Lemma 14. *The space $\mathcal{M}$ is the orthogonal complement of $\ker\pi\cap L^\infty(C)$ and thus, weakly closed in $\mathcal{U}(M)$.*

Combining this with Theorem 6 and the Hahn–Banach theorem, gives the following.

Theorem 15. *Under the assumptions of Theorem 6, $\mathcal{R}$ is Fréchet and its dual can be identified with $\mathcal{M}$ under the bilinear form*

$$\langle y,u\rangle := E\int y du.$$

The optional projection is a continuous surjection of $\mathcal{Y}(C)$ to $\mathcal{R}$, its adjoint is the embedding of $\mathcal{M}$ to $\mathcal{U}(M)$ and $(\ker\pi)^\perp=\mathcal{M}$. Moreover, the topology of $\mathcal{R}$ is generated by the seminorms

$$p_{\mathcal{R}}(y) = \inf_{z\in\mathcal{Y}(C)}\{p(\|z\|) \mid {}^o z = y\}$$

the polars of which are given by

$$p^\circ_{\mathcal{R}}(u) = p^\circ(\|u\|).$$

Proof. We start by showing that $\mathcal{R}$ is the orthogonal complement (with respect to the pairing of $\mathcal{D}$ and $\hat{\mathcal{M}}$) of the linear space

$$\mathcal{L} = \{(u,\tilde{u})\in\hat{\mathcal{M}} \mid u+\tilde{u}=0\}.$$

If $y\in\mathcal{R}$ and $(u,\tilde{u})\in\hat{\mathcal{M}}$, we have

$$E\left[\int y du + \int y_- d\tilde{u}\right] = E\left[\int y du + \int {}^p y d\tilde{u}\right] = E\int y d(u+\tilde{u}),$$

so $\mathcal{R}\subseteq\mathcal{L}^\perp$. On the other hand, if $y\in\mathcal{D}\setminus\mathcal{R}$, there exists, by the predictable section theorem, a predictable time τ such that $E({}^p y_\tau - y_{\tau-})\neq 0$. Defining $u=-\tilde{u}=\delta_\tau$, we have $(u,\tilde{u})\in\mathcal{L}$ while $\langle y,(u,\tilde{u})\rangle = E({}^p y_\tau - y_{\tau-})$. Thus, $\mathcal{R}=\mathcal{L}^\perp$.

Being a closed subspace of a Fréchet space, $\mathcal{R}$ is Fréchet. Since $\mathcal{M}$ is isomorphic to a subspace of $\hat{\mathcal{M}}$, every $u \in \mathcal{M}$ defines a continuous linear functional on $\mathcal{R}$. On the other hand, by Hahn-Banach, a continuous linear functional on $\mathcal{R}$ extends to a continuous linear functional l on $\mathcal{D}$ which, by assumption, has the expression

$$l(y) = E\left[\int ydu + \int y_- d\tilde{u}\right]$$

for some $(u, \tilde{u}) \in \hat{\mathcal{M}}$. On $\mathcal{R}$, this can be expressed as

$$E\left[\int ydu + \int y_- d\tilde{u}\right] = E\left[\int ydu + \int {}^p y d\tilde{u}\right] = E\int yd(u + \tilde{u}),$$

so the dual of $\mathcal{R}$ can indeed be identified with $\mathcal{M}$.

If $y \in \mathcal{Y}(C)$, we have ${}^o y \in \mathcal{D}$, by assumption, and then, by Lemma 4,

$$({}^o y)_- = {}^p(y_-) = {}^p y = {}^p({}^o y),$$

so ${}^o y \in \mathcal{R}$. By Lemma 14, the density of $L^\infty(C)$ in $\mathcal{Y}(C)$ and the continuity of π imply $(\ker \pi)^\perp = \mathcal{M}$.

The claims about the surjectivity of π, its adjoint and the seminorms are established like in the proof of Theorem 6. □

When $\mathcal{Y} = L^1$, Corollary 12 implies that the assumptions of Theorem 6 hold, so Theorem 15 gives the following refinement of Corollary 12, first derived in [14] using the main result of [3].

Corollary 16. *The space $\mathcal{R}^1$ of regular processes equipped with the norm*

$$\|y\|_{\mathcal{D}^1} := \sup_{\tau \in \mathcal{T}} E|y_\tau|$$

is Banach and its dual can be identified with $\mathcal{M}^\infty$ through the bilinear form

$$\langle y, u\rangle = E\int ydu.$$

The optional projection is a continuous surjection of $L^1(C)$ to $\mathcal{R}^1$ and its adjoint is the embedding of $\mathcal{M}^\infty$ to $L^\infty(M)$. The topology of $\mathcal{R}^1$ is generated by the seminorm

$$p_{\mathcal{D}}(y) := \inf_{z \in L^1(C)} \{E\|z\| \mid {}^o z = y\}$$

whose polar is given by

$$p^\circ_{\mathcal{D}}(u) = \text{ess sup}(\|u\|).$$

Corollary 16 applies Theorem 15 to $\mathcal{Y}$ with the Choquet property. Likewise, Theorem 15 could be applied to cases when $\mathcal{Y}$ has the Doob property. This would cover appropriate Orlicz spaces and the Fréchet space of random variables with finite moments.

8 Doob decomposition

We will say that an optional cadlag process Z is a $\mathcal{U}$*-quasimartingale* if for some $p \in \mathcal{P}$,

$$\mathrm{Var}_p(Z) := \sup_{(\tau_i)_{i=0}^n \subset \mathcal{T}} p^\circ \left(\sum_{i=0}^{n-1} |E_{\tau_i}[Z_{\tau_i} - Z_{\tau_{i+1}}]| + |Z_{\tau_n}| \right) < \infty.$$

When $\mathcal{U} = L^1$ and $p^\circ(\xi) = \|\xi\|_{L^1}$, this reduces to the usual definition of a quasimartingale; see e.g. [8, Definition VI.38]. The space of quasimartingales contains e.g. supermartingales and their differences. The classical Doob-decomposition expresses a quasimartingale of class (D) as a sum of a martingale and a predictable process of integrable variation; see e.g. [8, Appendix II.4]. Choosing $n = 0$, we see that a $\mathcal{U}$-quasimartingale is of class (D) as soon as the level sets of some p° are uniformly integrable.

We will denote by $\tilde{\mathcal{N}}_0^{\mathcal{U}}$ the linear space of predictable cadlag processes that start at 0 and whose pathwise variation is in $\mathcal{U}$. The theorem below gives a refined Doob decomposition for $\mathcal{U}$-quasimartingales. It assumes that the seminorms satisfy the "Jensen inequality" (5).

Lemma 17. *If p satisfies* (5)*, then p° satisfies it as well.*

Proof. By properties of the conditional expectation,

$$\begin{aligned} p^\circ(E_\tau \eta) &= \sup_{\xi \in L^\infty} \{E[\xi E_\tau \eta] \mid p(\xi) \le 1\} = \sup_{\xi \in L^\infty} \{E[\eta E_\tau \xi] \mid p(\xi) \le 1\} \\ &\le \sup_{\xi \in L^\infty} \{E[\eta E_\tau \xi] \mid p(E_\tau \xi) \le 1\} \le \sup_{\xi \in L^\infty} \{E[\eta \xi] \mid p(\xi) \le 1\} = p^\circ(\eta), \end{aligned}$$

for any $\tau \in \mathcal{T}$. □

Theorem 18. *Assume that the conditions of Theorem 6 are satisfied, that the seminorms $p \in \mathcal{P}$ satisfy* (5) *and that the level sets of some p° are uniformly integrable. Then a process Z is a $\mathcal{U}$-quasimartingale if and only if there exists a $\mathcal{U}$-martingale M and an $A \in \tilde{\mathcal{N}}_0^{\mathcal{U}}$ such that*

$$Z_t = M_t - A_t.$$

Proof. If $Z = M - A$ for a $\mathcal{U}$-martingale M and $A \in \tilde{\mathcal{N}}_0^{\mathcal{U}}$, then the monotonicity of p°, the Jensen's inequality with $|\cdot|$ and Lemma 17 give

$$\begin{aligned} \mathrm{Var}_p(Z) &= \sup_{(\tau_i)_{i=0}^n \subset \mathcal{T}} p^\circ \left(\sum_{i=0}^{n-1} |E_{\tau_i}[A_{\tau_i} - A_{\tau_{i+1}}]| + |M_{\tau_n} - A_{\tau_n}| \right) \\ &\le \sup_{(\tau_i)_{i=0}^n \subset \mathcal{T}} p^\circ \left(\sum_{i=0}^{n-1} E_{\tau_i}|A_{\tau_i} - A_{\tau_{i+1}}| + |M_{\tau_n}| + |A_{\tau_n}| \right) \\ &\le p^\circ(2\|A\|_{TV} + |M_{\tau_n}|) \\ &\le p^\circ(2\|A\|_{TV}) + p^\circ(M_\infty) < \infty, \end{aligned}$$

where $\|A\|_{TV}$ denotes the total variation of A. On the other hand, let $\mathcal{D}_s \subset \mathcal{D}$ be the space of *simple processes* of the form

$$y = \sum_{i=0}^{n} \mathbb{1}_{[\tau_i, \tau_{i+1})} \eta^i,$$

where $(\tau_i)_{i=0}^n$ is an increasing sequence of stopping times with $\tau_0 = 0$ and $\tau_{n+1} = \infty$ and $\eta^i \in L^\infty(\mathcal{F}_{\tau_i})$. Define a linear functional l on $\mathcal{D}_s$ by

$$l(y) = E\left[\sum_{i=0}^{n-1} y_{\tau_i} E_{\tau_i}[Z_{\tau_i} - Z_{\tau_{i+1}}] + y_{\tau_n} E_{\tau_n} Z_{\tau_n}\right].$$

Given $\bar{y} \in \mathcal{D}_s$,

$$l(\bar{y}) = E \int \bar{y} d\bar{u}^{(\tau_i)},$$

where the measure $\bar{u}^{(\tau_i)}$ is given by

$$\bar{u}^{(\tau_i)} := \sum_{i=0}^{n} E_{\tau_i}[Z_{\tau_i} - Z_{\tau_{i+1}}]\delta_{\tau_i} + E_{\tau_n} Z_{\tau_n} \delta_{\tau_n}.$$

Thus, by Theorem 6,

$$l(\bar{y}) \leq p_{\mathcal{D}}(\bar{y}) p^\circ(\|\bar{u}^{(\tau_i)}\|) \leq p_{\mathcal{D}}(\bar{y}) \operatorname{Var}_p(Z)$$

so, l is continuous in the relative topology of $\mathcal{D}_s$. By Hahn–Banach, l extends to all of $\mathcal{D}$ so by, Theorem 6, there exists $(u, \tilde{u}) \in \hat{\mathcal{M}}$ such that

$$l(y) = E\left[\int y du + \int y_- d\tilde{u}\right].$$

Given $\tau \in \mathcal{T}$, let $y^\nu = \mathbb{1}_{[\tau, \tau+1/\nu)} \in \mathcal{D}_s$. Since Z is cadlag and of class (D), we have $l(y^\nu) = E(Z_\tau - Z_{\tau+1/\nu}) \to 0$. On the other hand,

$$l(y^\nu) = E\left[\int y^\nu du + \int y^\nu_- d\tilde{u}\right] = E[u([\tau, \tau+1/\nu)) + \tilde{u}((\tau, \tau+1/\nu])] \to Eu(\{\tau\})$$

so $Eu(\{\tau\}) = 0$ for every $\tau \in \mathcal{T}$. Thus the purely discontinuous part of u is zero and, in particular, u is predictable. We can thus express l in terms of the predictable measure $\bar{u} := u + \tilde{u}$ as

$$l(y) = \int y_- d\bar{u}.$$

It now suffices to take $A_t = u((0, t])$ and $M_t = E[A_\infty \,|\, \mathcal{F}_t]$. Indeed, taking $y = \mathbb{1}_{[\tau, \infty)} \in \mathcal{D}_s$, gives

$$E(Z_\tau \mathbb{1}_{\{\tau < \infty\}}) = E(A_\infty - A_\tau).$$

Taking $\tau = \tau_B$ for $B \in \mathcal{F}_\tau$ gives $Z_\tau = E[A_\infty - A_\tau \,|\, \mathcal{F}_\tau]$. Here $\tau_B := \tau$ on B and $+\infty$ otherwise. Finally, Lemma 17 gives $p^\circ(M_t) = p^\circ(E_t A_\infty) \leq p^\circ(A_\infty)$, so M is a $\mathcal{U}$-martingale. □

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