Extended Reduced-Form Framework for Non-Life Insurance

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Abstract

In this paper we propose a general framework for modeling an insurance claims' information flow in continuous time, by generalizing the reduced-form framework for credit risk and life insurance. In particular, we assume a nontrivial dependence structure between the reference filtration and the insurance internal filtration. We apply these results for pricing non-life insurance liabilities in hybrid financial and insurance markets, while taking into account the role of inflation under the benchmark approach. This framework offers at the same time a general and flexible structure, and explicit and treatable pricing formula.

JEL Classification: C02, G10, G19

Key words: non-life insurance, reduced-form framework, marked point process, benchmark approach, filtration dependence.

1 Introduction

In this paper we propose a general framework for modeling an insurance claims' flow in continuous time, by extending the classic reduced-form setting for credit risk and life insurance. In particular, we consider a nontrivial dependence structure between the reference filtration $\mathbb{F}$ and insurance internal filtration $\mathbb{H}$. The global information flow available to the insurance company is represented by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. In this way, we obtain for the first time a framework in continuous time for non-life insurance, where filtration dependence is taken into account. In view of the development of insurance-linked derivatives, which offer the possibility of transferring insurance risks to the financial market, this bottom-up modeling approach

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can be used for pricing and hedging both life and non-life insurance liabilities in hybrid financial and insurance markets. As an application of the general framework structure, we derive treatable pricing formula for non-life insurance claims by taking into account the role of inflation under the benchmark approach.

Historically, the mathematical modeling of life and non-life insurance liabilities in continuous time is quite asymmetric and risk mitigation of non-life portfolios via asset allocation is scarcely practiced. While there are a lot of recent works concerning life insurance, see e.g [28], [13], [14], [2], [8], [9], [7], [10], non-life insurance is often studied in discrete time and/or state space, see e.g. [23], [24], [18]. We refer to e.g. [36] for a unified framework for life and non-life insurance in discrete time. Mathematical frameworks for non-life insurance in continuous time can be found in e.g. [27], [16], [3], [32], [31] and [35]. However, these settings do not consider a nontrivial dependence structure between reference filtration and insurance internal filtration. In particular, in e.g. [32] and [31], the insurance internal filtration is not distinguished from the reference filtration, and in e.g. [27], [16] and [3], reference and insurance internal filtrations are assumed to be independent. The importance of considering a nontrivial dependence structure between filtrations, which represent different information flows in a hybrid market, is discussed in [4] in view of the recent introduction of insurance-linked derivatives. Derivatives based on occurrence intensity index, such as mortality derivatives, weather derivatives etc., play an important role in mitigating risks of insurance companies in the case of life and non-catastrophe non-life business. This last one, which includes car insurance, theft insurance, home insurance, etc., as opposed to catastrophe non-life insurance[1], covers high-probability low-cost events, and is often neglected by the literature. This paper aims to fill this gap, as well as to provide analytical results which can be used for the non-life insurance reserving problem and the valuation of non-life non-catastrophe linked financial products, which are currently still not common but can be potentially attractive in the future.

Recent non-life insurance literature in continuous time, see e.g. [3], [32], [31] and [35], commonly assume the insurance internal information flow as given by the natural filtration of a marked point process, which describes the insurance claim movement. Pricing and hedging formulas are then obtained by using the compensator of this marked point process. However, as we discuss in Section 4, this approach can not be always followed in the case of multiple filtrations with nontrivial dependence. Indeed, with respect to a generic filtration, it is not always true that there exists a marked point process with a given compensator, and the compensator does not always determine uniquely the law of the process. To overcome these difficulties, we propose a new framewok, which uses a direct approach as in Section 5.1 and 9.1.2 of [11] and allows an explicit bottom-up construction to treat more general filtrations. We note that, when our general framework is reduced to the case of life insurance, the compensator approach and

[1]See e.g. [12] for the distinction between catastrophe and non-catastrophe insurance.

the direct modeling approach coincide, see the discussion in e.g. [11] for the classic reduced-form framework.

More precisely, in our new framework we consider an homogeneous insurance portfolio with n claims. We assume that the reference filtration $\mathbb{F}$ includes information related to financial market and environmental, social and economic indicators. Following the classic non-life insurance modeling approach as in e.g. [1] and [26], we assume the insurance internal filtration $\mathbb{H}$, which represents internal information of an insurance company given by the claim movements, to be generated by a family of marked point processes, describing sequences of reporting times and associated losses. As typically in the case of non-life insurance, accident times and their related damages are unknown until the moment of reporting. We are able to capture these features and at the same time to introduce a dependence structure between filtrations $\mathbb{F}$ and $\mathbb{H}$ by providing a nontrivial extension of the classical reduced-form framework. In particular, we model the accident times of the related insurance securities as $\mathbb{F}$-conditionally independent random variables with a common $\mathbb{F}$-adapted intensity process μ. Random delay between accident time and the first reporting is modelled in the first mark and subsequent development of the claim is modelled by a time shift of an independent marked point process with respect to the first reporting. This structure includes the life insurance case and allows to obtain analytical valuation formulas, which can be expressed in term of the accident intensity μ, the delay distribution and the updating distribution, as illustrated in the preliminary calculations in Section 3. We then apply these results for pricing insurance liabilities in a hybrid market under the benchmark approach. The hybrid nature of the combined market is given by the presence of derivatives related to the intensity process μ on the financial market and by the influence of inflation and benchmark portfolio in the valuation of insurance liabilities. Here we focus only on pricing non-life insurance claims and obtain analytical pricing formulas, which can also be useful for future design of insurance-linked derivatives, especially non-catastrophe non-life derivatives.

This paper is organised as follows. In Section 2 we construct a general framework for insurance claims' flow in continuous time under a nontrivial dependence between the reference and the insurance internal filtrations, applicable both to life and non-life insurance, and give a brief comparison with the existing insurance frameworks in the literature. In Section 3 we give some useful preliminary valuation results in this setting. In Section 4 we discuss the compensator approach. In Section 5 we describe the hybrid nature of the combined market and derive the real world pricing formula for non-life insurance reserving under the benchmark approach.

2 General framework

In this section we construct a general framework for modeling an insurance claims' flow. We consider a filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, P)$, where $\mathbb{G} := (\mathcal{G}_t)_{t\geqslant 0}$,

$\mathcal{G} = \mathcal{G}_\infty$, and $\mathcal{G}_0$ is trivial.

W assume that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} := (\mathcal{F}_t)_{t \geqslant 0}$ and $\mathbb{H} := (\mathcal{H}_t)_{t \geqslant 0}$ are filtrations representing respectively a reference information flow and the internal information flow only available to the insurance company. Hence $\mathbb{G}$ describes the global information flow available to the insurance company. The reference filtration $\mathbb{F}$ typically includes information related to the financial market, and to environmental, political and social indicators. While we do not specify the structure of the reference filtration $\mathbb{F}$, we assume that the insurance internal filtration $\mathbb{H}$ is generated by a family of marked point processes, representing the times and amounts of losses of the insurance portfolio, as in e.g. [1], [22], [29] and [30]. Filtrations $\mathbb{F}$ and $\mathbb{H}$ are not supposed be independent. Without loss of generality, we assume that all filtrations satisfy the conditions of completeness and right-continuity. If not otherwise specified, all relations in this paper hold in the P-a.s. sense. For a detailed background of marked point processes we refer to e.g. [25], [15] and [20]. In the following we use the classic terminology of non-life insurance, see e.g. [37] and [33], and specify the filtration $\mathbb{H}$ as follows.

We consider an insurance portfolio with n policies. For i-th policy with $i = 1, ..., n$, the insurance company is typically informed about the accident occurred at a random time τ_0^i only after a random delay θ^i, which can be very long especially in the case of non-life insurance. Once the accident is reported at time τ_1^i, where

$$\tau_1^i := \tau_0^i + \theta^i, \tag{2.1}$$

both the accident time τ_0^i, the reporting delay θ^i and the impact size of the accident, described by a nonnegative random variable X_1^i, become available information. In particular, we assume that for all $i = 1, ..., n$, $\tau_0^i > 0$ P-a.s.

Let $\mathbb{N}^+$ be the set of natural numbers without zero. We describe the i-th insurance policy movement by a marked point process $(\tau_j^i, \Theta_j^i)_{j \in \mathbb{N}^+}$ with 2-dimensional nonnegative marks. That is, the sequence $(\tau_j^i)_{j \in \mathbb{N}^+}$ is a point process, where

$$\tau_j^i : (\Omega, \mathcal{G}, P) \to (\overline{\mathbb{R}}_+, \mathcal{B}(\overline{\mathbb{R}}_+)), \qquad j \in \mathbb{N}^+,$$

and $(\Theta_j^i)_{j \in \mathbb{N}^+}$ is a sequence of 2-dimensional nonnegative random variables, with

$$\Theta_j^i : (\Omega, \mathcal{G}, P) \to (\mathbb{R}_+^2, \mathcal{B}(\mathbb{R}_+^2)), \qquad j \in \mathbb{N}^+.$$

For every $j \in \mathbb{N}_+$, the random time τ_j^i describes the reporting time of j-th event related to i-th policy. The mark components Θ_j^i describe the reporting delay and the impact size of the corresponding event, respectively, which are known only if the event is reported. More precisely, we set

$$\tau_1^i \quad \text{with mark} \quad \Theta_1^i = (\theta^i, X_1^i), \tag{2.2}$$

and

$$\tau_{j+1}^i = \tau_1^i + \tilde{\tau}_j^i \quad \text{with mark} \quad \Theta_{j+1}^i = (0, X_{j+1}^i) := (0, \tilde{X}_j^i), \tag{2.3}$$

for $j \geqslant 1$, where $(\tilde{\tau}_j^i, \tilde{X}_j^i)_{j\in\mathbb{N}_+}$ is an auxiliary marked point process, which describes updating and development after the first reporting at τ_1^i. Here we assume that only the first reporting delay is different from zero, since in this paper we focus on modeling the first accident times τ_0^i and their relation with the reference filtration. However our setting can be easily generalized by considering non zero random delays in (2.3). We set furthermore that the marked point process $(\tilde{\tau}_j^i, \tilde{X}_j^i)_{j\in\mathbb{N}_+}$ is simple, i.e.

$$\lim_{j\to\infty} \tilde{\tau}_j^i = \infty,$$

and $\tilde{\tau}_j^i < \tilde{\tau}_{j+1}^i$, if $\tilde{\tau}_j^i < \infty$, and satisfies the following integrability condition

$$E\left[\sum_{j=1}^{\infty} \mathbf{1}_{\{\tilde{\tau}_j^i \leqslant t\}} \tilde{X}_j^i\right] < \infty \quad \text{for all } t \geqslant 0, \tag{2.4}$$

for $i = 1, ..., n$. In particular, the random times $(\tau_j^i)_{j\in\mathbb{N}_+}$ are strictly ordered:

$$\begin{array}{l} \tau_1^1 < \tau_2^1 < \cdots < \tau_j^1 < \tau_{j+1}^1 < \cdots, \\ \tau_1^2 < \tau_2^2 < \cdots < \tau_j^2 < \tau_{j+1}^2 < \cdots, \\ \quad \vdots \\ \tau_1^n < \tau_2^n < \cdots < \tau_j^n < \tau_{j+1}^n < \cdots. \end{array} \tag{2.5}$$

Note that we may have $\infty = \tau_j^i = \tau_{j+1}^i = ...$, in such case infinite value stands for an event which never happens. For the sake of simplicity we assume also the following.

Assumption 2.1.

1. Homogeneous delay*: the random delays θ^i, $i = 1, ..., n$, have the same distribution.*
2. Homogeneous development*: the marked point processes $(\tilde{\tau}_j^i, \tilde{X}_j^i)_{j\in\mathbb{N}_+}$, $i = 1, ..., n$, have the same distribution.*
3. Independent first mark*: the first marks X_1^i, $i = 1, ..., n$, are mutually independent and independent from $\mathcal{F}_\infty \vee \sigma(\tau_0^1) \vee ... \vee \sigma(\tau_0^n)$.*
4. Independent delay*: the random delays θ^i, $i = 1, ..., n$, are mutually independent and independent from $\mathcal{F}_\infty \vee \sigma((\tau_0^1, X_1^1)) \vee ... \vee \sigma((\tau_0^n, X_1^n))$.*
5. Independent development*: the marked point processes $(\tilde{\tau}_j^i, \tilde{X}_j^i)_{j\in\mathbb{N}_+}$, $i = 1, .., n$ are mutually independent and independent from $\mathcal{F}_\infty \vee \sigma((\tau_1^1, \theta^1, X_1^1)) \vee ... \vee \sigma((\tau_1^n, \theta^n, X_1^n))$.*

We emphasize that the above assumptions are general enough. The homogeneity assumptions can be satisfied by subdividing opportunely the insurance portfolio. The independence assumptions reflect the fact that reporting delays θ^i, occurrences and size of the losses after the first reporting time, described by $(\tilde{\tau}^i_j, \tilde{X}^i_j)_{j\in\mathbb{N}_+}$, are typically idiosyncratic factors which are independent to each other and independent from the reference information. However, we introduce a dependence structure by modeling the occurrence intensities of the accidents, as we will present in (2.14) and (2.15). We assume furthermore that the distribution of delay variables θ^i, $i = 1, ..., n$ has the following structure.

Assumption 2.2. *The common cumulative distribution function G of θ^i, $i = 1, ..., n$, with*

$$G(x) := P(\theta^i \leqslant x), \qquad x \in \mathbb{R}, \tag{2.6}$$

satisfies

$$G(x) = \alpha_0 + \int_0^{x\vee 0} g(y)\mathrm{d}y, \qquad x \in \mathbb{R}, \tag{2.7}$$

where $\alpha_0 = P(\theta^i \leqslant 0) = P(\theta^i = 0)$[2], and g is a nonnegative Lebesgue-integrable function.

According to the above assumption, the delays may have a mixed distribution. In this way we cover both the case of life insurance with $\theta^i \equiv 0$, i.e. $g = 0$, and the case of non-life insurance with non-null delays.

For every $i = 1, ..., n$, we define the marked cumulative process N^i by

$$N^i(t, B)(\omega) := \sum_{j=1}^{\infty} \mathbf{1}_{\{\tau^i_j(\omega)\leqslant t\}} \mathbf{1}_{\{\Theta^i_j(\omega)\in B\}},$$

for every $t \geqslant 0$, $B \in \mathcal{B}(\mathbb{R}^2_+)$, $\omega \in \Omega$. The process $(\mathbf{N}^i_t)_{t\geqslant 0}$ defined by

$$\mathbf{N}^i_t := N^i(t, \mathbb{R}^2_+) = \sum_{j=1}^{\infty} \mathbf{1}_{\{\tau^i_j\leqslant t\}}, \qquad t \geqslant 0,$$

is called ground process associated to the marked point process. At any time $t \geqslant 0$, the random variable $\mathbf{N}^i_t$ counts the number of occurrence of τ^i_j up to time t. In the literature, the name marked point process refers sometimes to the process N^i. Indeed, there is a unique correspondence between the marked point process and its marked cumulative process. More precisely,

$$\{\tau^i_j \leqslant t\} = \{\mathbf{N}^i_t \geqslant j\}, \tag{2.8}$$

for all $t \geqslant 0$ and

$$\{\Theta^i_j \in B\} = \bigcup_{K'=1}^{\infty} \bigcap_{K=K'}^{\infty} \bigcup_{k=1}^{\infty} \{\mathbf{N}^i_{(k-1)/2^K} = n-1, N^i(k/2^K, B) - N^i((k-1)/2^K, B) = 1\}, \tag{2.9}$$

for all $B \in \mathcal{B}(\mathbb{R}^2_+)$. See equations (2.8), (2.9) of [20] and Lemma 2.2.2 of [25].

[2]The delays θ^i are assumed to be nonnegative.

We consider the filtrations $\mathbb{H}^{i,1} := (\mathcal{H}_t^{i,1})_{t\geqslant 0}$ with

$$\mathcal{H}_t^{i,1} := \sigma\left(\mathbf{1}_{\{\tau_1^i\leqslant s\}}\mathbf{1}_{\{(\theta^i,X_1^i)\in B\}}, 0\leqslant s\leqslant t,\ \text{for all } B\in\mathcal{B}(\mathbb{R}_+^2)\right),$$

for all $t\geqslant 0$, and $\mathbb{H}^{i,j} := (\mathcal{H}_t^{i,j})_{t\geqslant 0}$, $j>1$, with

$$\mathcal{H}_t^{i,j} := \sigma\left(\mathbf{1}_{\{\tau_j^i\leqslant s\}}\mathbf{1}_{\{X_j^i\in B\}}, 0\leqslant s\leqslant t,\ \text{for all } B\in\mathcal{B}(\mathbb{R}_+)\right),$$

for all $t\geqslant 0$. It holds clearly

$$\mathcal{H}_\infty^{i,j} = \sigma(\tau_j^i)\vee\sigma(X_j^i) \quad \text{for } j>1.$$

In particular, in view of (2.1) we have

$$\mathcal{H}_\infty^{i,1} = \sigma(\tau_1^i)\vee\sigma((\theta^i,X_1^i)) = \sigma(\tau_0^i)\vee\sigma((\theta^i,X_1^i)). \tag{2.10}$$

Let $\mathbb{H}^i := (\mathcal{H}_t^i)_{t\geqslant 0}$ be the natural filtration of the marked cumulative process N^i, that is for all $t\geqslant 0$,

$$\mathcal{H}_t^i = \sigma(N^i(s,B), 0\leqslant s\leqslant t,\ \text{for all } B\in\mathcal{B}(\mathbb{R}_+^2)).$$

The internal information flow of the insurance company is described by the filtration $\mathbb{H} := (\mathcal{H}_t)_{t\geqslant 0}$, where

$$\mathcal{H}_t := \mathcal{H}_t^1\vee\ldots\vee\mathcal{H}_t^n, \quad t\geqslant 0. \tag{2.11}$$

Similarly, for $i=1,..,n$, we call $\tilde{N}^i$ the corresponding marked cumulative processes associated to the marked point processes $(\tilde{\tau}_j^i,\tilde{X}_j^i)_{j\in\mathbb{N}_+}$ and $\tilde{\mathbb{H}}^i$ the corresponding filtration, respectively. Similarly, all other notations associated to these last processes will be denoted by the symbol "$\sim$".

Lemma 2.3. *For every $i=1,...,n$, we have $\mathbb{H}^i = \bigvee_{j\in\mathbb{N}_+}\mathbb{H}^{i,j}$.*

Proof. Clearly, we have

$$\mathcal{H}_t^i \subseteq \bigvee_{j\in\mathbb{N}_+}\mathcal{H}_t^{i,j}.$$

For the other inclusion, it is sufficient to show that for all $0\leqslant s\leqslant t$ and $B\in\mathcal{B}(\mathbb{R}_+^2)$,

$$\{\tau_j^i\leqslant s\}\cap\{\Theta_j^i\in B\}\in\mathcal{H}_t^i.$$

This follows directly from (2.8) and (2.9). $\square$

We now introduce the following notations, which is useful in the sequel. For $i=1,...,n$, $j\in\mathbb{N}_+$, we define

$$\mathbb{H}^{i,\leqslant j} := \bigvee_{k\leqslant j}\mathcal{H}^{i,k}, \qquad \mathbb{H}^{i,\geqslant j} := \bigvee_{k\geqslant j}\mathcal{H}^{i,k},$$

similarly for $\mathbb{H}^{i,>j}$ and $\mathbb{H}^{i,<j}$. In particular, in the case of $j=1$, we set $\mathcal{H}_t^{i,<1} := \{\emptyset,\Omega\}$ for every $t\geqslant 0$. The following corollary is a direct consequence of Lemma 2.3.

Corollary 2.4. *For every $i = 1, ..., n$, $j \in \mathbb{N}_+$, we have*

$$\mathbb{H}^i = \mathbb{H}^{i,\leqslant j} \vee \mathbb{H}^{i,>j} = \mathbb{H}^{i,<j} \vee \mathbb{H}^{i,\geqslant j}.$$

Similarly to the reduced form setting for credit risk and life insurance, we now model the accident times τ_0^i, $i = 1, ..., n$ and their relation with the reference filtration in the following way. As in Section 9.1.2 of [11], we assume that all random times $(\tau_j^i)_{j\in\mathbb{N}}$, $i = 1, ..., n$, are not $\mathbb{F}$-stopping times, and that accident times τ_0^i, $i = 1, ..., n$, are such that for $t \in [0, \infty]$ and $s \in [0, t] \cap [0, \infty)$,

$$P\left(\tau_0^i > s \middle| \mathcal{F}_t\right) = P\left(\tau_0^i > s \middle| \mathcal{F}_s\right), \tag{2.12}$$

and for $l, k = 1, .., n$ with $l \neq k$, τ_0^l and τ_0^k are $\mathbb{F}$-conditionally independent, i.e. if $t \in [0, \infty]$ and $r, s \in [0, t] \cap [0, \infty)$, we have

$$P\left(\tau_0^l > r, \tau_0^k > s \middle| \mathcal{F}_t\right) = P\left(\tau_0^l > r \middle| \mathcal{F}_t\right) P\left(\tau_0^k > s \middle| \mathcal{F}_t\right). \tag{2.13}$$

Remark 2.5. *If we define $\mathcal{H}_t^{i,0} := \sigma\left(\mathbf{1}_{\{\tau_0^i \leqslant s\}} : 0 \leqslant s \leqslant t\right)$, $i = 1, ..., n$, then condition (2.12) is equivalent to*

$$E[X|\mathcal{F}_t] = E[X|\mathcal{F}_s],$$

for each integrable $\mathcal{H}_s^{i,0}$-measurable random variable X. Condition (2.13) is equivalent to the $\mathcal{F}_t$-conditional independence between the σ-algebras $\mathcal{H}_t^{l,0}$ and $\mathcal{H}_t^{k,0}$.

Furthermore, if $F^i := (F_t^i)_{t\geqslant 0}$ is the $\mathbb{F}$-conditional cumulative function of τ_0^i,

$$F_t^i := P\left(\tau_0^i \leqslant t \middle| \mathcal{F}_t\right), \quad t \geqslant 0,$$

we assume that there exists a locally integrable and $\mathbb{F}$-progressively measurable process $\mu^i := (\mu_t^i)_{t\geqslant 0}$, such that

$$e^{-\int_0^t \mu_u^i \mathrm{d}u} = 1 - F_t^i \quad \text{for all } t \geqslant 0. \tag{2.14}$$

We define $\Gamma^i := (\Gamma_t^i)_{t\geqslant 0}$ as

$$\Gamma_t^i := \int_0^t \mu_u^i \mathrm{d}u, \quad t \geqslant 0. \tag{2.15}$$

The process μ^i is called *intensity process* of the random jump time τ_0^i and the process Γ^i is called *hazard process* of τ_0^i. An explicit construction in Example 9.1.5 of [11] shows that for a given family of locally integrable $\mathbb{F}$-progressively measurable process μ^i, $i = 1, ..., n$, it is always possible to construct random times τ_0^i, $i = 1, ..., n$, such that Γ^i is the hazard process of τ_0^i for every $i = 0, ..., n$, and all the assumptions above are satisfied. For the sake of simplicity, we assume that the insurance portfolio is homogeneous.

Assumption 2.6. *The accident times τ_0^i, $i = 1, ..., n$, have the same intensity process.*

Under the homogeneity condition, we denote the common $\mathbb{F}$-conditional cumulative function, hazard process and intensity process respectively by F, Γ and μ. The above assumption reflects the fact that, while the policy developments may not have direct link to the information flow $\mathbb{F}$, the accident occurrences τ_0^i, $i = 1, ..., n$, are influenced by some common systematic risk-factors, and the common conditional intensity μ is deducible from the reference information flow.

We now show how the general framework described above comprehends in a synthetic way both life and non-life insurance modeling, and compare our setting with the existing literature.

2.1 Life insurance

Life insurance policies typically do not have reporting delay and depend only on τ_0^i, $i = 1, ..., n$, which actually represent the decease times. This can be included in our framework by setting $\theta^i \equiv 0$, $\tau_j^i \equiv \infty$ for all $j > 1$ and $X_j^i \equiv 1$ for all $j \in \mathbb{N}_+$, and interpreting τ_0^i as the decease time of person i, where $i = 1, ..., n$. The filtration $\mathbb{G}$ is hence reduced to

$$\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee ... \vee \mathbb{H}^n,$$

where

$$\mathcal{H}_t^i = \sigma\left(\mathbf{1}_{\{\tau_0^i \leqslant s\}}, 0 \leqslant s \leqslant t\right), \qquad t \geqslant 0,\ i = 1, ..., n.$$

In particular, the $\mathbb{F}$-progressively measurable process μ is interpreted as mortality intensity in this context. The financial market is typically assumed to include mortality or longevity linked derivatives, such as longevity bond, which pays off the longevity index value $e^{-\int_0^T \mu_s \mathrm{d}s}$ at maturity T.

Life insurance within hybrid market under this setting has been intensively studied in the literature, see e.g. [2], [9], [7] and [10].

2.2 Non-life insurance

The framework in Section 2 in its full generality describes the case of non-life insurance. Indeed, non-life insurance policies typically have reporting delay, i.e. $\theta^i \neq 0$, which can also count to several years. For i-th policy, we interpret X_j^i as payment amount at the j-th random times τ_j^i; the exact accident time τ_0^i and first payment amount X_1^i is known only after reporting at time τ_1^i. Further developments may occur after the first reporting and before the settlement of claim. The total number of developments $(\tau_j^i)_{j \in \mathbb{N}_+}$ is unknown as well as the amount of corresponding payments $(X_j^i)_{j \in \mathbb{N}_+}$. The accident time τ_0^i admits an $\mathbb{F}$-progressively measurable intensity process μ related to the underlying risk. If liquidly traded derivatives related to the μ process are available on the financial market, they could be used for hedging systematic risks related to non-life portfolio.

The above described setting gives a nontrivial extention of the underlying frameworks in e.g. [16], [3], [32] and [31]. In e.g. [16] and [3], the reference filtration $\mathbb{F}$ is assumed to be independent from the insurance internal filtration $\mathbb{H}$ generated by the non-life portfolio movement. The interaction between the financial and the insurance markets is thus captured only by means of interest rate and/or inflation risk. On the contrary, in e.g. [32] and [31], it is assumed that $\mathbb{G} = \mathbb{H} = \mathbb{F}$. Financial products used for hedging purpose are in these cases liquidly traded catastrophe derivatives and/or reinsurance contracts, which share similar risk structure of the target non-life insurance portfolio. Considering a more general setting, where $\mathbb{F}$ and $\mathbb{H}$ are not necessarily independent or equal, is technically challenging, as we discuss in Section 4. However, the extended reduce-form framework proposed in this paper allows to consider a nontrivial dependence structure between filtrations $\mathbb{F}$ and $\mathbb{H}$ and still to derive analytical pricing formulas for non-life insurance liabilities. Furthermore, beside the financial instruments used in e.g. [16], [3], [32] and [31], it is possible to use intensity μ related derivatives as hedging instrument, see discussion in Section 5. This last type of derivatives is still not common but is potentially attractive for covering systematic risks arising from non-catastrophe non-life insurance.

3 Valuation formulas

In this Section, we state several results under the above structure assumptions, by following Section 5.1 of [11] for the presentation. These preliminary calculations are fundamental for the pricing problem of non-life insurance claims in Section 5.1.

We start with extension of relation (2.12) and the $\mathbb{F}$-independence (2.13) of τ_0^i, $i = 1, ..., n$. We show that, if these relations hold for the filtrations $\mathbb{H}^{i,0}$, $i = 1, ..., n$, then they also hold for the filtrations $\mathbb{H}^i$, $i = 1, ..., n$.

Lemma 3.1. *For any $t \in [0, \infty]$ and $l, k = 1, ..., n$ with $l \neq k$, the σ-algebras $\mathcal{H}_t^l$ and $\mathcal{H}_t^k$ are $\mathcal{F}_t$-independent.*

Proof. In view of Lemma 2.3, it is sufficient to prove that $\mathcal{H}_t^{k,p}$ and $\mathcal{H}_t^{k,q}$ are $\mathcal{F}_t$-independent for all $p, q \in \mathbb{N}_+$. For notation simplicity, we only consider $p \neq 1$ and $q \neq 1$, since the other cases are similar. That is, we want to show

$$
\begin{aligned}
&E\left[\mathbf{1}_{\{\tau_p^l \leqslant s\}} \mathbf{1}_{\{X_p^l \in B^l\}} \mathbf{1}_{\{\tau_q^k \leqslant r\}} \mathbf{1}_{\{X_q^k \in B^k\}} \middle| \mathcal{F}_t\right] \\
&= E\left[\mathbf{1}_{\{\tau_p^l \leqslant s\}} \mathbf{1}_{\{X_p^l \in B^l\}} \middle| \mathcal{F}_t\right] E\left[\mathbf{1}_{\{\tau_q^k \leqslant r\}} \mathbf{1}_{\{X_q^k \in B^k\}} \middle| \mathcal{F}_t\right],
\end{aligned}
$$

where $s, r \in [0, t] \cap [0, \infty)$[3] and $B^l, B^k \in \mathcal{B}(\mathbb{R}_+)$. By (2.2) and (2.3), the above

[3] We note that t may assume the value ∞.

equality is equivalent to

$$
\begin{aligned}
&E\left[\mathbf{1}_{\{\tau_0^l+\theta^l+\tilde{\tau}_p^l\leqslant s\}}\mathbf{1}_{\{\tilde{X}_p^l\in B^l\}}\mathbf{1}_{\{\tau_0^k+\theta^k+\tilde{\tau}_q^k\leqslant r\}}\mathbf{1}_{\{\tilde{X}_p^k\in B^k\}}\middle|\,\mathcal{F}_t\right]\\
&=E\left[\mathbf{1}_{\{\tau_0^l+\theta^l+\tilde{\tau}_p^l\leqslant s\}}\mathbf{1}_{\{\tilde{X}_p^l\in B^l\}}\middle|\,\mathcal{F}_t\right]E\left[\mathbf{1}_{\{\tau_0^k+\theta^k+\tilde{\tau}_q^k\leqslant r\}}\mathbf{1}_{\{\tilde{X}_p^k\in B^k\}}\middle|\,\mathcal{F}_t\right].
\end{aligned}
$$

If we define the following deterministic functions

$$
f^l(x):=E\left[\mathbf{1}_{\{\theta^l+\tilde{\tau}_p^l\leqslant s-x\}}\mathbf{1}_{\{\tilde{X}_p^l\in B^l\}}\right],
$$

$$
f^k(x):=E\left[\mathbf{1}_{\{\theta^k+\tilde{\tau}_p^l\leqslant r-x\}}\mathbf{1}_{\{\tilde{X}_q^l\in B^k\}}\right],
$$

then

$$
\begin{aligned}
f^l(x)&=f^l(x)\mathbf{1}_{\{x\leqslant s\}},\\
f^k(x)&=f^l(x)\mathbf{1}_{\{x\leqslant r\}}.
\end{aligned}
$$

In particular, $f^l(\tau_0^l)$ and $f^l(\tau_0^k)$ are $\mathcal{H}_t^{l,0}$- and $\mathcal{H}_t^{k,0}$-measurable respectively. This together with Remark 2.5 and the independence conditions in Assumption 2.1 yields

$$
\begin{aligned}
&E\left[\mathbf{1}_{\{\tau_0^l+\theta^l+\tilde{\tau}_p^l\leqslant s\}}\mathbf{1}_{\{\tilde{X}_p^l\in B^l\}}\mathbf{1}_{\{\tau_0^k+\theta^k+\tilde{\tau}_q^k\leqslant r\}}\mathbf{1}_{\{\tilde{X}_p^k\in B^k\}}\middle|\,\mathcal{F}_t\right]\\
&=E\left[E\left[\mathbf{1}_{\{\tau_0^l+\theta^l+\tilde{\tau}_p^l\leqslant s\}}\mathbf{1}_{\{\tilde{X}_p^l\in B^l\}}\mathbf{1}_{\{\tau_0^k+\theta^k+\tilde{\tau}_q^k\leqslant r\}}\mathbf{1}_{\{\tilde{X}_p^k\in B^k\}}\middle|\,\mathcal{F}_t\vee\sigma(\tau_0^l)\vee\sigma(\tau_0^k)\right]\middle|\,\mathcal{F}_t\right]\\
&=E\left[E\left[\mathbf{1}_{\{x+\theta^l+\tilde{\tau}_p^l\leqslant s\}}\mathbf{1}_{\{\tilde{X}_p^l\in B^l\}}\mathbf{1}_{\{y+\theta^k+\tilde{\tau}_q^k\leqslant r\}}\mathbf{1}_{\{\tilde{X}_p^k\in B^k\}}\right]\Big|_{\substack{x=\tau_0^l\\ y=\tau_0^k}}\middle|\,\mathcal{F}_t\right]\\
&=E\left[E\left[\mathbf{1}_{\{x+\theta^l+\tilde{\tau}_p^l\leqslant s\}}\mathbf{1}_{\{\tilde{X}_p^l\in B^l\}}\right]\Big|_{x=\tau_0^l}E\left[\mathbf{1}_{\{y+\theta^k+\tilde{\tau}_q^k\leqslant r\}}\mathbf{1}_{\{\tilde{X}_p^k\in B^k\}}\right]\Big|_{y=\tau_0^k}\middle|\,\mathcal{F}_t\right]\\
&=E\left[f^l(\tau_0^l)f^l(\tau_0^k)\middle|\,\mathcal{F}_t\right]\\
&=E\left[f^l(\tau_0^l)\middle|\,\mathcal{F}_t\right]E\left[f^l(\tau_0^k)\middle|\,\mathcal{F}_t\right].
\end{aligned}
$$

The same calculation as above yields

$$
\begin{aligned}
&E\left[f^l(\tau_0^l)\middle|\,\mathcal{F}_t\right]E\left[f^l(\tau_0^k)\middle|\,\mathcal{F}_t\right]\\
&=E\left[\mathbf{1}_{\{\tau_0^l+\theta^l+\tilde{\tau}_p^l\leqslant s\}}\mathbf{1}_{\{\tilde{X}_p^l\in B_l\}}\middle|\,\mathcal{F}_t\right]E\left[\mathbf{1}_{\{\tau_0^k+\theta^k+\tilde{\tau}_q^k\leqslant r\}}\mathbf{1}_{\{\tilde{X}_p^k\in B_k\}}\middle|\,\mathcal{F}_t\right],
\end{aligned}
$$

which concludes the proof. □

Lemma 3.2. *For any $0\leqslant s\leqslant t\leqslant\infty$ and $i=1,...,n$, if X is $\mathcal{H}_s^i$-measurable, then*

$$
E\left[X|\,\mathcal{F}_t\right]=E\left[X|\,\mathcal{F}_s\right].
$$

Proof. The proof of the Lemma is similar to the one of Lemma 3.1. Indeed, it is sufficient to apply Remark 2.5. □

As a consequence of the above two lemmas, the $\mathbb{G}$-conditional expectation can be reduced to $\mathbb{F} \vee \mathbb{H}^i$-conditional expectation in most cases.

Corollary 3.3. *If $0 \leqslant t \leqslant T < \infty$, and Y is an integrable $(\mathcal{F}_T \vee \mathcal{H}_T^i)$-measurable random variable, then*

$$E\left[Y \mid \mathcal{G}_t\right] = E\left[Y \mid \mathcal{F}_t \vee \mathcal{H}_t^i\right].$$

Proof. It is enough to prove the statement for the indicator functions of the form $Y = \mathbf{1}_A \mathbf{1}_B$ where $A \in \mathcal{F}_T$ and $B \in \mathcal{H}_T^i$. We note that

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t^1 \vee \ldots \vee \mathcal{H}_t^n.$$

Let $C \in \mathcal{F}_t$, $D^j \in \mathcal{H}_t^j$, $j = 1, \ldots, n$. It is sufficient to show that

$$\int_{C \cap D^1 \cap \ldots \cap D^n} \mathbf{1}_A \mathbf{1}_B \mathrm{d}P = \int_{C \cap D^1 \cap \ldots \cap D^n} E\left[\mathbf{1}_A \mathbf{1}_B \mid \mathcal{F}_t \vee \mathcal{H}_t^i\right] \mathrm{d}P.$$

Clearly, it holds

$$\begin{aligned}\int_{C \cap D^1 \cap \ldots \cap D^n} \mathbf{1}_A \mathbf{1}_B \mathrm{d}P &= \int_{C \cap D^i \cap A \cap B} \prod_{\substack{j=1,\ldots,n \\ j \neq i}} \mathbf{1}_{D^j} \mathrm{d}P \\ &= \int_{C \cap D^i \cap A \cap B} E\left[\prod_{\substack{j=1,\ldots,n \\ j \neq i}} \mathbf{1}_{D^j} \,\middle|\, \mathcal{F}_T \vee \mathcal{H}_T^i\right] \mathrm{d}P.\end{aligned}$$

By Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}&E\left[\prod_{\substack{j=1,\ldots,n \\ j \neq i}} \mathbf{1}_{D^j} \,\middle|\, \mathcal{F}_T \vee \mathcal{H}_T^i\right] = E\left[\prod_{\substack{j=1,\ldots,n \\ j \neq i}} \mathbf{1}_{D^j} \,\middle|\, \mathcal{F}_T\right] \\ &= E\left[\prod_{\substack{j=1,\ldots,n \\ j \neq i}} \mathbf{1}_{D^j} \,\middle|\, \mathcal{F}_t\right] = E\left[\prod_{\substack{j=1,\ldots,n \\ j \neq i}} \mathbf{1}_{D^j} \,\middle|\, \mathcal{F}_t \vee \mathcal{H}_t^i\right].\end{aligned}$$

It follows,

$$
\begin{aligned}
\int_{C\cap D^1\cap\ldots\cap D^n} \mathbf{1}_A\mathbf{1}_B \mathrm{d}P &= \int_{C\cap D^i\cap A\cap B} E\left[\prod_{\substack{j=1,\ldots,n\\ j\neq i}} \mathbf{1}_{D^j}\middle|\, \mathcal{F}_t\vee\mathcal{H}^i_t\right]\mathrm{d}P\\
&= \int_{C\cap D^i} \mathbf{1}_A\mathbf{1}_B\, E\left[\prod_{\substack{j=1,\ldots,n\\ j\neq i}} \mathbf{1}_{D^j}\middle|\, \mathcal{F}_t\vee\mathcal{H}^i_t\right]\mathrm{d}P\\
&= \int_{C\cap D^i} E\left[\mathbf{1}_A\mathbf{1}_B\middle|\, \mathcal{F}_t\vee\mathcal{H}^i_t\right] E\left[\prod_{\substack{j=1,\ldots,n\\ j\neq i}} \mathbf{1}_{D^j}\middle|\, \mathcal{F}_t\vee\mathcal{H}^i_t\right]\mathrm{d}P\\
&= \int_{C\cap D^i} \prod_{\substack{j=1,\ldots,n\\ j\neq i}} \mathbf{1}_{D^j}\, E\left[\mathbf{1}_A\mathbf{1}_B\middle|\, \mathcal{F}_t\vee\mathcal{H}^i_t\right]\mathrm{d}P\\
&= \int_{C\cap D^1\cap\ldots\cap D^n} E\left[\mathbf{1}_A\mathbf{1}_B\middle|\, \mathcal{F}_t\vee\mathcal{H}^i_t\right]\mathrm{d}P.
\end{aligned}
$$

□

An other important corollary of Lemma 3.1 and Lemma 3.2 is the so called *H-hypothesis* between filtrations $\mathbb{F}$ and $\mathbb{G}$, i.e. the property that every $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale.

Corollary 3.4. *The H-hypothesis holds between filtrations $\mathbb{F}$ and $\mathbb{G}$.*

Proof. By Lemma 6.1.1 of [11], H-hypothesis between two filtrations $\mathbb{F}\subseteq\mathbb{G}$ is equivalent to the property that for any $t\geqslant 0$ and any bounded, $\mathcal{G}_t$-measurable random variable η, it holds that

$$E\left[\eta|\,\mathcal{F}_\infty\right] = E\left[\eta|\,\mathcal{F}_t\right].$$

It is sufficient to check the above relation for indicator functions of the form $\mathbf{1}_A\mathbf{1}_{B^1}\ldots\mathbf{1}_{B^n}$, where $A\in\mathcal{F}_t$, $B^i\in\mathcal{H}^i_t$, $i=1,...,n$. By applying Lemma 3.1 and Lemma 3.2, we obtain

$$
\begin{aligned}
E\left[\mathbf{1}_A\mathbf{1}_{B^1}\ldots\mathbf{1}_{B^n}|\,\mathcal{F}_\infty\right] &= \mathbf{1}_A E\left[\mathbf{1}_{B^1}\ldots\mathbf{1}_{B^n}|\,\mathcal{F}_\infty\right]\\
&= \mathbf{1}_A\prod_{i=1}^n E\left[\mathbf{1}_{B^i}|\,\mathcal{F}_\infty\right]\\
&= \mathbf{1}_A\prod_{i=1}^n E\left[\mathbf{1}_{B^i}|\,\mathcal{F}_t\right]\\
&= \mathbf{1}_A E\left[\mathbf{1}_{B^1}\ldots\mathbf{1}_{B^n}|\,\mathcal{F}_t\right]\\
&= E\left[\mathbf{1}_A\mathbf{1}_{B^1}\ldots\mathbf{1}_{B^n}|\,\mathcal{F}_t\right].
\end{aligned}
$$

□

Now we would like to derive some more explicit representations. We note that for every integrable random variable Y, $t \geqslant 0$, $i = 1, ..., n$ and $j \in \mathbb{N}_+$, we have the decomposition

$$E\left[Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] + E\left[\mathbf{1}_{\{\tau_j^i \leqslant t\}} Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right]. \tag{3.1}$$

In the following we will evaluate separately the two components on the right-hand side of (3.1). The following lemma is important for deriving a representation of the first component.

Lemma 3.5. *For any $t \geqslant 0$, $i = 1, ..., n$ and $j \in \mathbb{N}_+$, we have*

$$\mathcal{H}_t^i \vee \mathcal{F}_t \subseteq \mathcal{G}_t^{i,j},$$

where

$$\mathcal{G}_t^{i,j} := \left\{A \in \mathcal{G} : \exists C \in \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t, A \cap \{\tau_j^i > t\} = C \cap \{\tau_j^i > t\}\right\}. \tag{3.2}$$

Proof. By Corollary 2.4, it holds that

$$\mathcal{H}_t^i = \mathcal{H}_t^{i,<j} \vee \mathcal{H}_t^{i,\geqslant j}.$$

Hence, it is sufficient to show that both $\mathcal{H}_t^{i,\geqslant j}$ and $\mathcal{H}_t^{i,<j} \vee \mathcal{F}_t$ belong to $\mathcal{G}_t^{i,j}$. In the first case, if $i > 1$ and $A = \{\tau_k^i \leqslant s\} \cap \{X_k^i \in B\}$ for some $k \geqslant j$, $0 \leqslant s \leqslant t$ and $B \in \mathcal{B}(\mathbb{R})$, we take $C = \emptyset$. Similarly for $i = 1$ and $A = \{\tau_k^1 \leqslant s\} \cap \{(\theta_k, X_k^1) \in B\}$ for $k \geqslant j$, $0 \leqslant s \leqslant t$ and $B \in \mathcal{B}(\mathbb{R}_+^2)$. In the second case, if $A \in \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t$ we take $C = A$. □

The following Proposition gives two representations of the first component on the right-hand side of (3.1). Representation (3.3) is analogue to Lemma 5.1.2. in [11], representation (3.4) is new and will be used for our further discussion.

Proposition 3.6. *For any $t \geqslant 0$, $i = 1, ..., n$, $j \in \mathbb{N}_+$ and any integrable $\mathcal{G}$-measurable random variable Y, we have*

$$E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = \mathbf{1}_{\{\tau_j^i > t\}} \frac{E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right]}{P\left(\tau_j^i > t \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right)} \tag{3.3}$$

$$= \mathbf{1}_{\{\tau_j^i > t\}} E\left[Y \middle| \mathcal{H}_t^{i,\leqslant j} \vee \mathcal{F}_t\right]. \tag{3.4}$$

Proof. Equality (3.3) is equivalent to

$$E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \, P\left(\tau^i > t \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right) \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = \mathbf{1}_{\{\tau_j^i > t\}} E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right].$$

We note that the right-hand side is $(\mathcal{H}_t^i \vee \mathcal{F}_t)$-measurable. Hence, it suffices to show that for any $A \in \mathcal{H}_t^i \vee \mathcal{F}_t$,

$$\int_A \mathbf{1}_{\{\tau_j^i > t\}} Y\, P\left(\tau_j^i > t \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right) \mathrm{d}P = \int_A \mathbf{1}_{\{\tau_j^i > t\}}\, E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \mathrm{d}P.$$

By Lemma 3.5, there is an event $C \in \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t$ such that

$$A \cap \{\tau_j^i > t\} = C \cap \{\tau_j^i > t\},$$

hence

$$\begin{aligned}
&\int_A \mathbf{1}_{\{\tau_j^i > t\}} Y\, P\left(\tau_j^i > t \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right) \mathrm{d}P \\
&= \int_{A\cap\{\tau_j^i > t\}} Y\, P\left(\tau_j^i > t \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right) \mathrm{d}P \\
&= \int_{C\cap\{\tau_j^i > t\}} Y\, P\left(\tau_j^i > t \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right) \mathrm{d}P \\
&= \int_C \mathbf{1}_{\{\tau_j^i > t\}} Y\, P\left(\tau_j^i > t \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right) \mathrm{d}P \\
&= \int_C E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] E\left[\mathbf{1}_{\{\tau_j^i > t\}} \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \mathrm{d}P \\
&= \int_C E\left[\mathbf{1}_{\{\tau_j^i > t\}}\, E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \mathrm{d}P \\
&= \int_C \mathbf{1}_{\{\tau_j^i > t\}}\, E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \mathrm{d}P \\
&= \int_{C\cap\{\tau_j^i > t\}} E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \mathrm{d}P \\
&= \int_{A\cap\{\tau_j^i > t\}} E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \mathrm{d}P \\
&= \int_A \mathbf{1}_{\{\tau_j^i > t\}}\, E\left[\mathbf{1}_{\{\tau_j^i > t\}} Y \middle| \mathcal{H}_t^{i,<j} \vee \mathcal{F}_t\right] \mathrm{d}P.
\end{aligned}$$

Equality (3.4) can be proved in the same way. We only need to observe that

$$\mathcal{G}_t^{i,j} \subseteq \left\{A \in \mathcal{G} : \exists C \in \mathcal{H}_t^{i,\leqslant j} \vee \mathcal{F}_t, A \cap \{\tau_j^i > t\} = C \cap \{\tau_j^i > t\}\right\}.$$

Hence, the σ-algebra $\mathcal{H}_t^{i,<j}$ in (3.3) can be replaced by $\mathcal{H}_t^{i,\leqslant j}$. This concludes the proof. □

Now we focus on the second component on the right-hand side of (3.1). The following lemma gives a slightly more general result.

Lemma 3.7. *For any $t \geqslant 0$, $i = 1, ..., n$, $j \in \mathbb{N}_+$, any σ-algebra $\mathcal{A} \subseteq \mathcal{G}$ and any integrable $\mathcal{G}$-measurable random variable Y, we have*

$$E\left[\mathbf{1}_{\{\tau^i_j \leqslant t\}} Y \middle| \mathcal{H}^{i,\leqslant j}_t \vee \mathcal{A}\right] = E\left[\mathbf{1}_{\{\tau^i_j \leqslant t\}} Y \middle| \mathcal{H}^{i,\leqslant j}_\infty \vee \mathcal{A}\right].$$

Proof. We note that the left-hand side is $(\mathcal{H}^{i,\leqslant j}_\infty \vee \mathcal{A})$-measurable. Since the marked point process $(\tau^i_j, \Theta^i_j)_{j\in\mathbb{N}^+}$ is simple, i.e. the strict monotonicy (2.5) holds, if $A \in \mathcal{H}^{i,\leqslant j}_\infty \vee \mathcal{A}$, then $A \cap \{\tau^i_j \leqslant t\} \in \mathcal{H}^{i,\leqslant j}_t \vee \mathcal{A}$, and

$$\begin{aligned}\int_A \mathbf{1}_{\{\tau^i_j \leqslant t\}} Y \mathrm{d}P = \int_{A\cap\{\tau^i_j\leqslant t\}} Y \mathrm{d}P &= \int_{A\cap\{\tau^i_j\leqslant t\}} E\left[Y \middle| \mathcal{H}^{i,\leqslant j}_t \vee \mathcal{A}\right] \mathrm{d}P \\ &= \int_A E\left[\mathbf{1}_{\{\tau^i_j \leqslant t\}} Y \middle| \mathcal{H}^{i,\leqslant j}_t \vee \mathcal{A}\right] \mathrm{d}P.\end{aligned}$$

This concludes the proof. □

Remark 3.8. *Since we have*

$$\mathcal{H}^{i,\leqslant j}_\infty = \sigma\left(\tau^i_h, h = 1, ..., j\right),$$

Lemma 3.7 shows that, if τ^i_j has occurred before time t, then partial information about τ^i_j up to t is equivalent to full information about all the random times τ^i_h, $h = 1, ..., j$. In particular, if Y is a function of $\tau^i_1, ..., \tau^i_j$, i.e. $Y = f(\tau^i_1, ..., \tau^i_j)$, then the conditional expectation is simply

$$E\left[\mathbf{1}_{\{\tau^i_j \leqslant t\}} Y \middle| \mathcal{H}^{i,\leqslant j}_t \vee \mathcal{A}\right] = \mathbf{1}_{\{\tau^i_j \leqslant t\}} Y.$$

We summarize the above results in the following representation theorem.

Theorem 3.9. *For any $t \geqslant 0$, $i = 1, ..., n$, $j \in \mathbb{N}_+$ and any integrable $\mathcal{G}$-measurable random variable Y, we have*

$$E\left[Y \middle| \mathcal{H}^i_t \vee \mathcal{F}_t\right] = \mathbf{1}_{\{\tau^i_j \leqslant t\}}\, E\left[Y \middle| \mathcal{H}^{i,\leqslant j}_\infty \vee \mathcal{H}^{i,>j}_t \vee \mathcal{F}_t\right] + \mathbf{1}_{\{\tau^i_j > t\}}\, E\left[Y \middle| \mathcal{H}^{i,\leqslant j}_t \vee \mathcal{F}_t\right].$$

If furthermore Y is $(\mathcal{H}^i_T \vee \mathcal{F}_T)$-measurable, then

$$E\left[Y \middle| \mathcal{G}_t\right] = \mathbf{1}_{\{\tau^i_j \leqslant t\}}\, E\left[Y \middle| \mathcal{H}^{i,\leqslant j}_\infty \vee \mathcal{H}^{i,>j}_t \vee \mathcal{F}_t\right] + \mathbf{1}_{\{\tau^i_j > t\}}\, E\left[Y \middle| \mathcal{H}^{i,\leqslant j}_t \vee \mathcal{F}_t\right].$$

Proof. Since

$$E\left[Y \middle| \mathcal{H}^i_t \vee \mathcal{F}_t\right] = E\left[\mathbf{1}_{\{\tau^i_j \leqslant t\}} Y \middle| \mathcal{H}^i_t \vee \mathcal{F}_t\right] + E\left[\mathbf{1}_{\{\tau^i_j > t\}} Y \middle| \mathcal{H}^i_t \vee \mathcal{F}_t\right],$$

the first part is a straightforward consequence of Proposition 3.6 and Lemma 3.7. For the second part, it suffices to apply Corollary 3.3. □

We now show some results which will play a key role for the reserve estimation problem in Section 5.1. Let $0 \leqslant t \leqslant T < \infty$ and $Z := (Z_t)_{t\in[0,T]}$ be a continuous, bounded and $\mathbb{F}$-adapted process. For $i = 1, ..., n$, we now consider

$$Y = \sum_{j=\mathbf{N}_t^i}^{\mathbf{N}_T^i} X_j^i Z_{\tau_j^i} = \sum_{j=1}^{\infty} \mathbf{1}_{\left\{t<\tau_j^i \leqslant T\right\}} X_j^i Z_{\tau_j^i}, \tag{3.5}$$

and compute

$$E\left[Y \middle| \mathcal{G}_t\right] = E\left[\sum_{j=\mathbf{N}_t^i}^{\mathbf{N}_T^i} X_j^i Z_{\tau_j^i} \middle| \mathcal{G}_t\right]. \tag{3.6}$$

In particular, similarly to before, we study separately the two components of the decomposition of (5.3) with respect to the first reporting time τ_1^i, i.e.

$$E\left[\sum_{j=\mathbf{N}_t^i}^{\mathbf{N}_T^i} X_j^i Z_{\tau_j^i} \middle| \mathcal{G}_t\right] = E\left[\mathbf{1}_{\left\{\tau_t^i>t\right\}} \sum_{j=\mathbf{N}_t^i}^{\mathbf{N}_T^i} X_j^i Z_{\tau_j^i} \middle| \mathcal{G}_t\right] + E\left[\mathbf{1}_{\left\{\tau_t^i \leqslant t\right\}} \sum_{j=\mathbf{N}_t^i}^{\mathbf{N}_T^i} X_j^i Z_{\tau_j^i} \middle| \mathcal{G}_t\right], \tag{3.7}$$

and derive more explicit formulas in terms of the intensity process μ, the distribution of delay θ^i, and the distribution of development N^i after the first reporting. We start with the $\mathbb{F}$-conditional expectation of τ_1^i.

Lemma 3.10. *For any $i = 1, ..., n$ and $t \geqslant 0$, we have*

$$P\left(\tau_1^i > t \middle| \mathcal{F}_t\right) = e^{-\int_0^t \mu_u \mathrm{d}u} + \int_0^t \bar{G}(t-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u, \tag{3.8}$$

and

$$P\left(\tau_1^i \leqslant t \middle| \mathcal{F}_t\right) = \int_0^t G(t-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u, \tag{3.9}$$

where G is the cumulative distribution function of θ^i defined in (2.6) and

$$\bar{G}(x) := 1 - G(x) = P(\theta^i > x), \qquad x \in \mathbb{R}. \tag{3.10}$$

Proof. We prove only equality (3.8), since equality (3.9) directly follows. Note that by Assumption 2.1, θ^i is independent from $\mathcal{F}_t \vee \sigma(\tau_0^i)$. Furthermore, both θ^i and τ_0^i are P-a.s. nonnegative. Therefore, we have

$$\begin{aligned}
P\left(\tau_1^i > t \middle| \mathcal{F}_t\right) &= E\left[\mathbf{1}_{\left\{\tau_0^i+\theta^i>t\right\}} \middle| \mathcal{F}_t\right] \\
&= E\left[\mathbf{1}_{\left\{\tau_0^i>t\right\}} + \mathbf{1}_{\left\{\tau_0^i\leqslant t\right\}} \mathbf{1}_{\left\{\tau_0^i+\theta^i>t\right\}} \middle| \mathcal{F}_t\right] \\
&= e^{-\int_0^t \mu_u \mathrm{d}u} + E\left[E\left[\mathbf{1}_{\left\{\tau_0^i\leqslant t\right\}} \mathbf{1}_{\left\{\tau_0^i+\theta^i>t\right\}} \middle| \mathcal{F}_t \vee \sigma(\tau_0^i)\right] \middle| \mathcal{F}_t\right] \\
&= e^{-\int_0^t \mu_u \mathrm{d}u} + E\left[\mathbf{1}_{\left\{\tau_0^i\leqslant t\right\}} E\left[\mathbf{1}_{\{\theta^i>t-x\}}\right]\Big|_{x=\tau_0^i} \middle| \mathcal{F}_t\right] \\
&= e^{-\int_0^t \mu_u \mathrm{d}u} + E\left[\mathbf{1}_{\left\{\tau_0^i\leqslant t\right\}} \bar{G}(t-\tau_0^i) \middle| \mathcal{F}_t\right].
\end{aligned}$$

To conclude we only need to show

$$E\left[\mathbf{1}_{\{\tau_0^i\leqslant t\}}\bar{G}(t-\tau_0^i)\middle|\,\mathcal{F}_t\right]=\int_0^t\bar{G}(t-u)e^{-\int_0^u\mu_v\mathrm{d}v}\mu_u\mathrm{d}u. \tag{3.11}$$

This can be done in the same way as for Proposition 5.1.1 of [11], in view of relation (2.12) and the fact that G is continuous according to Assumption 2.2. □

Remark 3.11. *Note that (3.11) is the conditional probability that the accident has incurred, but not yet reported (IBNR events in the terminology used in the insurance sector).*

In expression (3.9) of Lemma 3.10, the parameter t appears also in the integrand. The following corollary improves relation (3.9) and shows that the process of conditional expectation $(P\left(\tau_1^i\leqslant t\middle|\,\mathcal{F}_t\right))_{t\geqslant 0}$ is absolutely continuous with respect to the Lebesgue measure.

Corollary 3.12. *For any $i=1,...,n$, we have*

$$P\left(\tau_1^i\leqslant t\middle|\,\mathcal{F}_t\right)=\int_0^t\left(\alpha_0e^{-\int_0^s\mu_v\mathrm{d}v}\mu_s+\int_0^s g(s-u)e^{-\int_0^u\mu_v\mathrm{d}v}\mu_u\mathrm{d}u\right)\mathrm{d}s, \tag{3.12}$$

where α_0 and g are defined in (2.7).

Proof. This follows immediately from Assumption 2.2, relation (3.9) and Leibniz integral rule. □

Lemma 3.13. *If the process*[4] *$Z:=(Z_u)_{u\in[t,T]}$ is left-continuous and bounded*[5] *and Z_t is $\mathcal{F}_T$-measurable for all $t\geqslant 0$, then we have*

$$E\left[\mathbf{1}_{\{t<\tau_1^i\leqslant T\}}Z_{\tau_1^i}\middle|\,\mathcal{F}_t\right]=E\left[\int_t^T Z_u\mathrm{d}P\left(\tau_1^i\leqslant u\middle|\,\mathcal{F}_u\right)\middle|\,\mathcal{F}_t\right],$$

for $i=1,...,n$ and $t\in[0,T]$.

Proof. The proof is similar to the one in Proposition 5.1.1 of [11]. We assume first that Z is stepwise constant, i.e. we assume without loss of generality that

$$Z_u=\sum_{j=0}^n Z_{t_j}\mathbf{1}_{\{t_j<u\leqslant t_{j+1}\}},$$

[4]We do not assume that Z is $\mathbb{F}$-adapted, see Remark 3.14.

[5]We emphasize that the boundedness condition can be generalized.

for $t < u \leqslant T$, where $t_0 = t < ... < t_{j+1} = T$, Z_{t_j} is $\mathcal{F}_T$-measurable for all $j = 0, ..., n$. Lemma 3.2 yields

$$
\begin{aligned}
&E\left[\mathbf{1}_{\{t<\tau_1^i\leqslant T\}} Z_{\tau_1^i}\middle|\,\mathcal{F}_t\right]\\
=&E\left[\sum_{j=0}^{n} Z_{t_j}\mathbf{1}_{\{t_j<\tau_1^i\leqslant t_{j+1}\}}\middle|\,\mathcal{F}_t\right]\\
=&E\left[\sum_{j=0}^{n} E\left[Z_{t_j}\mathbf{1}_{\{t_j<\tau_1^i\leqslant t_{j+1}\}}\middle|\,\mathcal{F}_T\right]\middle|\,\mathcal{F}_t\right]\\
=&E\left[\sum_{j=0}^{n} Z_{t_j}\left(E\left[\mathbf{1}_{\{\tau_1^i\leqslant t_{j+1}\}}\middle|\,\mathcal{F}_{t_{j+1}}\right]-E\left[\mathbf{1}_{\{\tau_1^i\leqslant t_j\}}\middle|\,\mathcal{F}_{t_j}\right]\right)\middle|\,\mathcal{F}_t\right] &(3.13)\\
=&E\left[\int_t^T Z_u \mathrm{d}P\left(\tau_1^i\leqslant u\middle|\,\mathcal{F}_u\right)\middle|\,\mathcal{F}_t\right]. &(3.14)
\end{aligned}
$$

In the general case, it is sufficient to find a stepwise constant approximation for Z. Since Z is bounded, we have the convergence of the Riemann sum under the sign of conditional expectation in (3.13) to the Lebesgue-Stieltjes integral in expression (3.14). Hence also the convergence of the conditional expectation follows. $\square$

Remark 3.14. *We stress that the above Lemma involves only Lebesgue-Stieltjes integral in (3.14), which coincides with Lebesque integral in view of Corollary 3.12. Hence, it is not necessary to assume that* Z *is* $\mathbb{F}$*-adapted.*

Now we are able to calculate the first component on the right-hand side of (3.7). We define

$$
\begin{aligned}
\tilde{m}(t) &:= E\left[\sum_{j=1}^{\tilde{\mathbf{N}}_t}\tilde{X}_j^i\right], &&\text{if } t\geqslant 0,\\
\tilde{m}(t) &:= 0, &&\text{if } t<0,
\end{aligned}
\tag{3.15}
$$

where $\tilde{\mathbf{N}}$ denotes the ground process of $(\tilde{\tau}_j^i, \tilde{X}_j^i)_{j\in\mathbb{N}_+}$, i.e.

$$
\tilde{\mathbf{N}}_t := \sum_{j=1}^{\infty}\mathbf{1}_{\{\tilde{\tau}_j^i\leqslant t\}}, \qquad t\geqslant 0. \tag{3.16}
$$

Note that $\tilde{m}$ does not depend on i because of Assumption 2.1 (2).

Proposition 3.15. *Let* $Z := (Z_t)_{t\in[0,T]}$ *be a continuous, bounded and* $\mathbb{F}$*-adapted*[6]

[6] Note that the result also holds under different integrability and measurability conditions.

process and Y *be as in (3.5), then for any* $t \in [0,T]$,

$$\begin{aligned}
&E\left[\mathbf{1}_{\{\tau_1^i>t\}}Y\middle|\mathcal{H}_t^i\vee\mathcal{F}_t\right]\\
&=\mathbf{1}_{\{\tau_1^i>t\}}\frac{E\left[\int_t^T\left(E[X_1^i]Z_u+\int_u^T Z_v\mathrm{d}\tilde{m}(v-u)\right)\mathrm{d}P\left(\tau_1^i\leqslant u\middle|\mathcal{F}_u\right)\middle|\mathcal{F}_t\right]}{P\left(\tau_1^i>t\middle|\mathcal{F}_t\right)}
\end{aligned}$$

where $\tilde{m}$ *is defined in (3.15).*

Proof. By applying (3.4) in Proposition 3.6 to Y defined in (3.5), we get

$$\begin{aligned}
&E\left[\mathbf{1}_{\{\tau_1^i>t\}}Y\middle|\mathcal{H}_t^i\vee\mathcal{F}_t\right]=\mathbf{1}_{\{\tau_1^i>t\}}E\left[Y\middle|\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right]\\
=&\mathbf{1}_{\{\tau_1^i>t\}}E\left[\sum_{j=1}^{\infty}\mathbf{1}_{\{\tau_j^i\leqslant T\}}X_j^iZ_{\tau_j^i}\middle|\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right]\\
=&\mathbf{1}_{\{\tau_1^i>t\}}E\left[\mathbf{1}_{\{\tau_1^i\leqslant T\}}X_1^iZ_{\tau_1^i}\middle|\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right]+\mathbf{1}_{\{\tau_1^i>t\}}E\left[\sum_{j=2}^{\infty}\mathbf{1}_{\{\tau_j^i\leqslant T\}}X_j^iZ_{\tau_j^i}\middle|\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right].
\end{aligned}\tag{3.17}$$

For the first component of (3.17), it is sufficient to use (3.3) in Proposition 3.6 and an argument similar to Proposition 5.1.1 of [11], taking into account the independence condition in Assumption 2.1 (3) and Lemma 3.2. We have hence

$$\begin{aligned}
&\mathbf{1}_{\{\tau_1^i>t\}}E\left[\mathbf{1}_{\{\tau_1^i\leqslant T\}}X_1^iZ_{\tau_1^i}\middle|\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right]\\
=&E\left[\mathbf{1}_{\{t<\tau_1^i\leqslant T\}}X_1^iZ_{\tau_1^i}\middle|\mathcal{H}_t^{i,1}\vee\mathcal{F}_t\right]\\
=&\mathbf{1}_{\{\tau_1^i>t\}}\frac{E\left[\mathbf{1}_{\{t<\tau_1^i\leqslant T\}}X_1^iZ_{\tau_1^i}\middle|\mathcal{F}_t\right]}{P\left(\tau_1^i>t\middle|\mathcal{F}_t\right)}\\
=&\mathbf{1}_{\{\tau_1^i>t\}}\frac{E\left[\int_t^T E[X_1^i]Z_u\mathrm{d}P\left(\tau_1^i\leqslant u\middle|\mathcal{F}_u\right)\middle|\mathcal{F}_t\right]}{P\left(\tau_1^i>t\middle|\mathcal{F}_t\right)}.
\end{aligned}$$

Now we focus on the second component of (3.17). We assume first that restricted on the interval $[t,T]$, Z is a bounded, stepwise, $\mathbb{F}$-predictable process, i.e.

$$Z_u=\sum_{i=0}^{n}Z_{t_i}\mathbf{1}_{\{t_i<u\leqslant t_{i+1}\}},\tag{3.18}$$

for $t<u\leqslant T$, where $t_0=t<...<t_{n+1}=T$ and Z_{t_i} is $\mathcal{F}_{t_i}$-measurable for all

$i = 0, ..., n$. In such case, we have

$$
\begin{aligned}
&\mathbf{1}_{\{\tau_1^i > t\}} E\left[\sum_{j=2}^{\infty} \mathbf{1}_{\{\tau_j^i \leqslant T\}} X_j^i Z_{\tau_j^i} \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] \\
=&\mathbf{1}_{\{\tau_1^i > t\}} E\left[\sum_{j=1}^{\infty} \mathbf{1}_{\{t < \tau_1^i + \tilde{\tau}_j^i \leqslant T\}} \tilde{X}_j^i Z_{\tau_j^i} \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] \\
=&\mathbf{1}_{\{\tau_1^i > t\}} E\left[\sum_{i=0}^{n} \sum_{j=1}^{\infty} \mathbf{1}_{\{t_i < \tau_1^i + \tilde{\tau}_j^i \leqslant t_{i+1}\}} \tilde{X}_j^i Z_{t_i} \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] \\
=&\mathbf{1}_{\{\tau_1^i > t\}} E\left[\sum_{i=0}^{n} Z_{t_i}\ E\left[\sum_{j=1}^{\infty} \mathbf{1}_{\{t_i < \tau_1^i + \tilde{\tau}_j^i \leqslant t_{i+1}\}} \tilde{X}_j^i \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_{t_i} \vee \sigma(\tau_1^i)\right] \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] \\
=&\mathbf{1}_{\{\tau_1^i > t\}} E\left[\sum_{i=0}^{n} Z_{t_i}\ E\left[\sum_{j=1}^{\infty} \mathbf{1}_{\{t_i < x + \tilde{\tau}_j^i \leqslant t_{i+1}\}} \tilde{X}_j^i \right]\Bigg|_{x=\tau_1^i} \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] \\
=&\mathbf{1}_{\{\tau_1^i > t\}} E\left[\sum_{i=0}^{n} Z_{t_i}\left(\tilde{m}(t_{i+1} - \tau_1^i) - \tilde{m}(t_i - \tau_1^i)\right) \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right], \qquad (3.19)
\end{aligned}
$$

where in the second last equality we use the independence between the marked point process $(\tilde{\tau}_j^i, \tilde{X}_j^i)_{j\in\mathbb{N}_+}$ and the σ-algebra $\mathcal{H}_\infty^{i,1} \vee \mathcal{F}_\infty$ in Assumption 2.1. This shows that for any bounded, stepwise, $\mathbb{F}$-predictable process Z, we have

$$
\mathbf{1}_{\{\tau_1^i > t\}} E\left[\sum_{j=2}^{\infty} \mathbf{1}_{\{\tau_j^i \leqslant T\}} X_j^i Z_{\tau_j^i} \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] = \mathbf{1}_{\{\tau_1^i > t\}} E\left[\int_t^T Z_u \mathrm{d}\tilde{m}(u - \tau_1^i) \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right].
$$

A continuous bounded process Z can be approximated by a sequence of bounded, stepwise and $\mathbb{F}$-predictable processes, i.e. there is a sequence Z^n of the form (3.18) such that

$$
Z^n \longrightarrow Z \quad \text{and} \quad |Z^n| \leqslant M,
$$

with $M > 0$. Since $\tilde{m}$ is right-continuous and monotone, the Lebesgue–Stieltjes integral

$$
\int_t^T Z_u \mathrm{d}\tilde{m}(u - \tau_1^i) \qquad (3.20)
$$

is well defined. It holds by Lebesgue Theorem

$$
\int_t^T Z_u^n \mathrm{d}\tilde{m}(u - \tau_1^i) \longrightarrow \int_t^T Z_u \mathrm{d}\tilde{m}(u - \tau_1^i).
$$

Furthermore,

$$
\left|\int_t^T Z_u^n \mathrm{d}\tilde{m}(u - \tau_1^i)\right| \leqslant M \left|\int_t^T \mathrm{d}\tilde{m}(u - \tau_1^i)\right| = M|\tilde{m}(T - \tau_1^i) - \tilde{m}(t - \tau_1^i)|. \quad (3.21)
$$

The right-hand side of (3.21) is uniformly bounded by (3.15) and (2.4). By applying again Lebesgue Theorem, we have also the convergence of the conditional expectations

$$\mathbf{1}_{\{\tau_1^i > t\}} E\left[\int_t^T Z_u^n \mathrm{d}\tilde{m}(u-\tau_1^i) \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] \longrightarrow E\left[\int_t^T Z_u \mathrm{d}\tilde{m}(u-\tau_1^i) \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right].$$

We note that $\tilde{m}(u) = 0$ for $u < 0$, hence,

$$\begin{aligned}
&\mathbf{1}_{\{\tau_1^i > t\}} E\left[\sum_{j=2}^{\infty} \mathbf{1}_{\{\tau_j^i \leqslant T\}} X_j^i Z_{\tau_j^i} \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] \\
&= \mathbf{1}_{\{\tau_1^i > t\}} E\left[\int_t^T Z_u \mathrm{d}\tilde{m}(u-\tau_1^i) \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] \\
&= E\left[\mathbf{1}_{\{t < \tau_1^i \leqslant T\}} \int_t^T Z_u \mathrm{d}\tilde{m}(u-\tau_1^i) \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right].
\end{aligned}$$

By applying again (3.3) in Proposition 3.6 to the above expression, we get

$$\begin{aligned}
&\mathbf{1}_{\{\tau_1^i > t\}} E\left[\sum_{j=2}^{\infty} \mathbf{1}_{\{\tau_j^i \leqslant T\}} X_j^i Z_{\tau_j^i} \middle| \mathcal{H}_t^{i,1} \vee \mathcal{F}_t\right] \\
&= \mathbf{1}_{\{\tau_1^i > t\}} \frac{E\left[\mathbf{1}_{\{t < \tau_1^i \leqslant T\}} \int_t^T Z_u \mathrm{d}\tilde{m}(u-\tau_1^i) \middle| \mathcal{F}_t\right]}{P\left(\tau_1^i > t \middle| \mathcal{F}_t\right)}.
\end{aligned}$$

Let $\tilde{Z}_s := \int_t^T Z_u \mathrm{d}\tilde{m}(u-s)$, $s \in [0,T]$. We note that $\tilde{m}$ is right-continuous and monotone. On one hand, for fixed $s \in [0,T]$, the function $d_s(u) := \tilde{m}(u-s)$, $u \in [0,T]$, is also right-continuous and monotone and defines the cumulative distribution function of a finite positive measure, in view of (2.4). One the other hand, for fixed $u \in [0,T]$, the function $\tilde{m}(u-s)$, $s \in [0,T]$, is left-continuous in s, i.e. for every series $s_n \nearrow s$, we have the pointwise convergence

$$\lim_{s_n \nearrow s} d_{s_n}(u) = d_s(u) \qquad \text{for all } u \in [0,T]$$

of the cumulative distribution functions, equivalent to the convergence in distribution or weak convergence in measure[7]. This yields the convergence

$$\tilde{Z}_{s_n} \longrightarrow \tilde{Z}_s, \qquad P-\text{a.s.},$$

[7]A series of positive finite measures $(\nu_n)_{n\in\mathbb{N}}$ converges weakly to a positive finite measure ν, if for all bounded continuous functions f, it holds

$$\int f \mathrm{d}\nu_n \longrightarrow \int f \mathrm{d}\nu$$

that is, $\tilde{Z}_s := \int_t^T Z_u \mathrm{d}\tilde{m}(u-s)$, $s \in [0,T]$, is left-continuous. Futhermore, it is also bounded. Now we apply Lemma 3.13 and obtain

$$
\begin{aligned}
&\mathbf{1}_{\{\tau_1^i > t\}} \frac{E\left[\mathbf{1}_{\{t<\tau_1^i \leqslant T\}} \int_t^T Z_u \mathrm{d}\tilde{m}(u-\tau_1^i) \middle| \mathcal{F}_t\right]}{P\left(\tau_1^i > t \middle| \mathcal{F}_t\right)} \\
&= \mathbf{1}_{\{\tau_1^i > t\}} \frac{E\left[\mathbf{1}_{\{t<\tau_1^i \leqslant T\}} \tilde{Z}_{\tau_1^i} \middle| \mathcal{F}_t\right]}{P\left(\tau_1^i > t \middle| \mathcal{F}_t\right)} \\
&= \mathbf{1}_{\{\tau_1^i > t\}} \frac{E\left[\int_t^T \tilde{Z}_u \mathrm{d}P\left(\tau_1^i \leqslant u \middle| \mathcal{F}_u\right) \middle| \mathcal{F}_t\right]}{P\left(\tau_1^i > t \middle| \mathcal{F}_t\right)} \\
&= \mathbf{1}_{\{\tau_1^i > t\}} \frac{E\left[\int_t^T \left(\int_t^T Z_v \mathrm{d}\tilde{m}(v-u)\right) \mathrm{d}P\left(\tau_1^i \leqslant u \middle| \mathcal{F}_u\right) \middle| \mathcal{F}_t\right]}{P\left(\tau_1^i > t \middle| \mathcal{F}_t\right)}.
\end{aligned}
$$

As the last step, we note that for $u < s$, $\int_t^T Z_u \mathrm{d}\tilde{m}(u-s) = \int_s^T Z_u \mathrm{d}\tilde{m}(u-s)$ since $\tilde{m}(u-s) = 0$. This concludes the proof. □

Remark 3.16. *The proof of Proposition 3.15 relies on assumption (3.15). Another sufficient condition would be the continuity of $\tilde{m}$, such as in the case of a compound Poisson process or a Cox process with continuous intensity process and integrable marks. Indeed, since $\tilde{m}(u) = 0$ for $u < 0$,*

$$
\begin{aligned}
\mathbf{1}_{\{\tau_1^i > t\}} \left| \int_t^T Z_u^n \mathrm{d}\tilde{m}(u-\tau_1^i) \right| &\leqslant \mathbf{1}_{\{\tau_1^i > t\}} M \left| \int_t^T \mathrm{d}\tilde{m}(u-\tau_1^i) \right| \\
&= \mathbf{1}_{\{t<\tau_1^i \leqslant T\}} M |\tilde{m}(T-\tau_1^i) - \tilde{m}(t-\tau_1^i)|,
\end{aligned}
$$

and the right-hand side is uniformly bounded if $\tilde{m}$ is continuous.

The following proposition gives a representation of the second component on the right-hand side of (3.7).

Proposition 3.17. *Under the same assumptions of Proposition 3.15, if for each $i = 1, ..., n$, the process $\left(\sum_{j=1}^{\tilde{\mathbf{N}}_t} \tilde{X}_j^i\right)_{t \in [0,T]}$, where $\tilde{\mathbf{N}}$ is defined in (3.16), is of independent increments with respect to its natural filtration $\tilde{\mathbb{H}}^i$, then for $t \in [0,T]$ and Y as in (3.5), it holds*

$$
E\left[\mathbf{1}_{\{\tau_1^i \leqslant t\}} Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = \mathbf{1}_{\{\tau_1^i \leqslant t\}}\ E\left[\int_t^T Z_u \mathrm{d}\tilde{m}(u-x) \middle| \mathcal{H}_\infty^{i,1} \vee \tilde{\mathcal{H}}_{t-x}^i \vee \mathcal{F}_t\right]\Bigg|_{x=\tau_1^i},
$$

for $i = 1, ..., n$.

Proof. It follows from Lemma 3.7 that

$$
E\left[\mathbf{1}_{\{\tau^i \leqslant t\}} Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = E\left[\mathbf{1}_{\{\tau_1^i \leqslant t\}} Y \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{F}_t\right].
$$

Similarly to the proof of Proposition 3.15, we assume first Z of the form (3.18). In such case, we have

$$
\begin{aligned}
& E\left[\mathbf{1}_{\{\tau_1^i \leqslant t\}} Y \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{H}_t^{i,>1} \vee \mathcal{F}_t\right] \\
= & \mathbf{1}_{\{\tau_1^i \leqslant t\}}\ E\left[\sum_{j=1}^{\infty} \mathbf{1}_{\{t<\tau_1^i+\tilde{\tau}_j^i \leqslant T\}} \tilde{X}_j^i Z_{\tau_j^i} \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{H}_t^{i,>1} \vee \mathcal{F}_t\right] \\
= & \mathbf{1}_{\{\tau_1^i \leqslant t\}}\ E\left[\sum_{i=0}^{n} \sum_{j=1}^{\infty} \mathbf{1}_{\{t_i<\tau_1^i+\tilde{\tau}_j^i \leqslant t_{i+1}\}} \tilde{X}_j^i Z_{t_i} \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{H}_t^{i,>1} \vee \mathcal{F}_t\right] \\
= & \mathbf{1}_{\{\tau_1^i \leqslant t\}}\ E\left[\sum_{i=0}^{n} \sum_{j=1}^{\infty} \mathbf{1}_{\{t_i<x+\tilde{\tau}_j^i \leqslant t_{i+1}\}} \tilde{X}_j^i Z_{t_i} \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{H}_t^{i,>1} \vee \mathcal{F}_t\right]\Bigg|_{x=\tau_1^i} \\
= & \mathbf{1}_{\{\tau_1^i \leqslant t\}}\ E\left[\sum_{i=0}^{n} \sum_{j=1}^{\infty} \mathbf{1}_{\{t_i<x+\tilde{\tau}_j^i \leqslant t_{i+1}\}} \tilde{X}_j^i Z_{t_i} \middle| \mathcal{H}_\infty^{i,1} \vee \tilde{\mathcal{H}}_{t-x}^{i} \vee \mathcal{F}_t\right]\Bigg|_{x=\tau_1^i},
\end{aligned}
$$

where the last step follows from the definitions of the filtrations. By using tower property, the independence between the marked point process $(\tilde{\tau}_j^i, \tilde{X}_j^i)_{j\in\mathbb{N}_+}$ and $\mathcal{F}_\infty \vee \mathcal{H}_\infty^{i,1}$ (see Assumption 2.1), and the independence of increments of the process the process $\left(\sum_{j=1}^{\tilde{\mathbf{N}}_t} \tilde{X}_j^i\right)_{t\in[0,T]}$, we get furthermore

$$
\begin{aligned}
& E\left[\mathbf{1}_{\{\tau_1^i \leqslant t\}} Y \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{H}_t^{i,>1} \vee \mathcal{F}_t\right] \\
= & \mathbf{1}_{\{\tau_1^i \leqslant t\}}\ E\Bigg[\sum_{i=0}^{n} Z_{t_i} \Bigg(E\left[\sum_{j=1}^{\infty} \mathbf{1}_{\{\tilde{\tau}_j^i \leqslant t_{i+1}-x\}} \tilde{X}_j^i \middle| \mathcal{H}_\infty^{i,1} \vee \tilde{\mathcal{H}}_{t-x}^{i} \vee \mathcal{F}_{t_i}\right] \\
& - \ E\left[\sum_{j=1}^{\infty} \mathbf{1}_{\{\tilde{\tau}_j^i \leqslant t_i-x\}} \tilde{X}_j^i \middle| \mathcal{H}_\infty^{i,1} \vee \tilde{\mathcal{H}}_{t-x}^{i} \vee \mathcal{F}_{t_i}\right]\Bigg) \Bigg| \mathcal{H}_\infty^{i,1} \vee \tilde{\mathcal{H}}_{t-x}^{i} \vee \mathcal{F}_t\Bigg]\Bigg|_{x=\tau_1^i} \\
= & \mathbf{1}_{\{\tau_1^i \leqslant t\}}\ E\Bigg[\sum_{i=0}^{n} Z_{t_i} \Bigg(E\left[\sum_{j=1}^{\infty} \mathbf{1}_{\{\tilde{\tau}_j^i \leqslant t_{i+1}-x\}} \tilde{X}_j^i \middle| \tilde{\mathcal{H}}_{t-x}^{i}\right] \\
& - \ E\left[\sum_{j=1}^{\infty} \mathbf{1}_{\{\tilde{\tau}_j^i \leqslant t_i-x\}} \tilde{X}_j^i \middle| \tilde{\mathcal{H}}_{t-x}^{i}\right]\Bigg) \Bigg| \mathcal{H}_\infty^{i,1} \vee \tilde{\mathcal{H}}_{t-x}^{i} \vee \mathcal{F}_t\Bigg]\Bigg|_{x=\tau_1^i} \\
= & \mathbf{1}_{\{\tau_1^i \leqslant t\}}\ E\left[\sum_{i=0}^{n} Z_{t_i} \left(\tilde{m}(t_{i+1}-x) - \sum_{j=1}^{\tilde{\mathbf{N}}_{t-x}} \tilde{X}_j^i - \tilde{m}(t_i-x) + \sum_{j=1}^{\tilde{\mathbf{N}}_{t-x}} \tilde{X}_j^i\right) \middle| \mathcal{H}_\infty^{i,1} \vee \tilde{\mathcal{H}}_{t-x}^{i} \vee \mathcal{F}_t\right]\Bigg|_{x=\tau_1^i} \\
= & \mathbf{1}_{\{\tau_1^i \leqslant t\}}\ E\left[\sum_{i=0}^{n} Z_{t_i} \left(\tilde{m}(t_{i+1}-x) - \tilde{m}(t_i-x)\right) \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{F}_t\right]\Bigg|_{x=\tau_1^i} \qquad (3.22)
\end{aligned}
$$

This yields that for any bounded, stepwise, $\mathbb{F}$-predictable process Z, we have

$$E\left[\mathbf{1}_{\{\tau_1^i \leqslant t\}} Y \middle| \mathcal{H}_t^i \vee \mathcal{F}_t\right] = \mathbf{1}_{\{\tau_1^i \leqslant t\}} \; E\left[\int_t^T Z_u \mathrm{d}\tilde{m}(u-x) \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{F}_t\right]\Big|_{x=\tau_1^i}.$$

If Z is continuous, bounded and $\mathbb{F}$-adapted, then Z can be approximated by a sequence of bounded, stepwise and $\mathbb{F}$-predictable processes. This together with the fact that $\tilde{m}$ is right-continuous and monotone guarantees that the Riemann sum in (3.22) under the sign of conditional expectation converges to Lebesgue-Stieltjes integral, by using the same arguments of Proposition 3.15. □

We summarize the results in the following theorem, which gives an explicit representation of $\mathbb{G}$-conditional expectation with respect to the first reporting time τ_1^i.

Theorem 3.18. *Let $Z := (Z_t)_{t\in[0,T]}$ be a continuous, bounded and $\mathbb{F}$-adapted process*[8]*, Y be of the form (3.5). If the process $\left(\sum_{j=1}^{\tilde{\mathbf{N}}_t} \tilde{X}_j^i\right)_{t\in[0,T]}$, has independent increments and $\tilde{m}$ is defined in (3.15), then*

$$\begin{aligned} E\left[Y \middle| \mathcal{G}_t\right] =& \mathbf{1}_{\{\tau_1^i \leqslant t\}} \; E\left[\int_t^T Z_u \mathrm{d}\tilde{m}(u-x) \middle| \mathcal{H}_\infty^{i,1} \vee \tilde{\mathcal{H}}_{t-x}^i \vee \mathcal{F}_t\right]\Big|_{x=\tau_1^i} \\ &+ \mathbf{1}_{\{\tau_1^i > t\}} \frac{E\left[\int_t^T \left(E[X_1^i] Z_u + \int_u^T Z_v \mathrm{d}\tilde{m}(v-u)\right) \mathrm{d}P\left(\tau_1^i \leqslant u \middle| \mathcal{F}_u\right) \middle| \mathcal{F}_t\right]}{P\left(\tau_1^i > t \middle| \mathcal{F}_t\right)}, \end{aligned}$$

for $i = 1, ..., n$, where

$$P\left(\tau_1^i \leqslant t \middle| \mathcal{F}_t\right) = \int_0^t \left(\alpha_0 e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u + \int_0^u g(u-v) e^{-\int_0^v \mu_s \mathrm{d}s} \mu_v \mathrm{d}v\right) \mathrm{d}u,$$

and

$$P\left(\tau_1^i > t \middle| \mathcal{F}_t\right) = e^{-\int_0^t \mu_u \mathrm{d}u} + \int_0^t \bar{G}(t-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u,$$

with α_0 and g defined in (2.7) and $\bar{G}$ defined in (3.10).

Proof. It is enough to combine Corollary 3.3, Lemma 3.10, Corollary 3.12, Proposition 3.15 and Proposition 3.17. □

Compared to Theorem 3.9, Theorem 3.18 is more explicit and has the advantage that the representation is expressed as function of μ, the distribution of θ^i and the distribution of $(\tilde{\tau}_j^i, \tilde{X}_j^i)_{j\in\mathbb{N}_+}$. This result will be useful for the concrete reserving problem in hybrid market in Section 5.

[8] Note that the result of Theorem 3.18 also holds under different integrability and measurability conditions.

4 Comparison with the compensator approach

In this section, we compare our framework with the compensator approach for non-life insurance in the existing literature. Within this section, the filtration $\mathbb{H}$ denotes the natural filtration of a marked point process $(\tau_n, X_n)_{n\in\mathbb{N}^+}$, with marked cumulative process N, and $\mathbb{G}$ is a generic enlargement of $\mathbb{H}$. We set $\mathcal{H} := \mathcal{H}_\infty$ and $\mathcal{G} := \mathcal{G}_\infty$.

In most of the current literature, e.g. [3], [32], [31] and [35], the study of non-life insurance contracts is based on modeling the $\mathbb{G}$-compensator of N, since the $\mathbb{G}$-compensator is involved in the pricing formula and in the calculation of the hedging strategy. In the reduced-form framework for life insurance, the direct modeling approach and the compensator approach coincide, see e.g. [11]. However, the compensator approach presents several difficulties in a non-life insurance setting with nontrivial filtrations' dependence.

Definition 4.1. *The* $\mathbb{G}$-mark-predictable σ-algebra *on the product space* $\mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+) \times \Omega$ *is the* σ*-algebra generated by sets of the form* $(s,t] \times B \times A$ *where* $0 < s < t$*,* $B \in \mathcal{B}(\mathbb{R}_+)$ *and* $A \in \mathcal{G}_s$*.*

Definition 4.2. *The* $\mathbb{G}$*-compensator of a marked point process* $(\tau_n, X_n)_{n\in\mathbb{N}^+}$ *is any* $\mathbb{G}$*-mark-predictable, cumulative process* $\Lambda(t, B, \omega)$ *such that,* $(\Lambda(t,B))_{t\geqslant 0}$ *with* $\Lambda(t,B)(\cdot) := \Lambda(t,B,\cdot)$ *is the* $\mathbb{G}$*-compensator of the point process* $(N(t,B))_{t\geqslant 0}$*. We use the notation* $(\Lambda_t)_{t\geqslant 0}$*,* $\Lambda_t := \Lambda(t, \mathbb{R}_+)$*, to denote the* $\mathbb{G}$*-compensator of the ground process* $(\mathbf{N}_t)_{t\geqslant 0}$*.*

Theorem 14.2.IV(a) of [15] shows that given a marked point process $(\tau_n, X_n)_{n\in\mathbb{N}^+}$ with finite first moment measure, its $\mathbb{G}$-compensator Λ always exists and is $(l \otimes P)$-a.e. unique, where l denotes the Lebesgue measure on $\mathbb{R}_+$. In particular, for all $(t, B, \omega) \in \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+) \times \Omega$, the following relation holds

$$\Lambda(t, B, \omega) = \int_0^t \kappa(B|s,\omega)\Lambda(\mathrm{d}s, \omega), \tag{4.1}$$

where $\kappa(B|s,\omega)$, $B \in \mathcal{B}(\mathbb{R}_+)$, $s \geqslant 0$, $\omega \in \Omega$, is the unique predictable kernel such that for all $A \in \mathcal{G}_s, 0 < s < t, B \in \mathcal{B}(\mathbb{R}_+)$,

$$\int_A \int_s^t N(u,B)(\omega)\mathrm{d}uP(\mathrm{d}\omega) = \int_A \int_s^t \kappa(B|u,\omega)\mathbf{N}_u(\omega)\mathrm{d}uP(\mathrm{d}\omega).$$

However, under general conditons it is not always true that given a $\mathbb{G}$-mark-predictable and cumulative process Λ, there exists a marked point process $(\tau_n, X_n)_{n\in\mathbb{N}^+}$ with $\mathbb{G}$-compensator Λ. The problem is first mentioned in [21], where the case with $\mathbb{G} = \mathbb{H}$ is solved. An extention of the existence theorem to the case of $\mathbb{G} = \mathbb{F} \otimes \mathbb{H}$, i.e. when the filtrations $\mathbb{F}$ and $\mathbb{H}$ are independent, is provided in [17]. Furthermore while the law of N is uniquely determined by the $\mathbb{H}$-compensator, this is not true for the $\mathbb{G}$-compensator. See discussion in [21] and Section 4.8 of [20].

Consequently, the literature with the compensator approach is mostly limited to the cases of $\mathbb{G} \equiv \mathbb{H}$, see e.g. [32], [31], or $\mathbb{G} = \mathbb{F} \otimes \mathbb{H}$, see e.g. [3].

In the following we provide a sufficient condition in the general case of $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, such that the law of N is uniquely determined by Λ. Similarly to e.g. [32] and [31], we assume that the $\mathbb{G}$-compensator of $(\tau_n, X_n)_{n\in\mathbb{N}^+}$ has the following form

$$\Lambda(t, B) = \int_0^t \int_B \lambda_s \eta_s(\mathrm{d}x)\mathrm{d}s \quad \text{for all } t \geqslant 0,\ B \in \mathcal{B}(\mathbb{R}_+), \tag{4.2}$$

where $\lambda := (\lambda_t)_{t\geqslant 0}$ is a $\mathbb{G}$-progressively measurable process and the mapping η

$$\begin{aligned} \eta : \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+) \times \Omega &\longrightarrow (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \\ (t, B, \omega) &\mapsto \eta_t(B)(\omega), \end{aligned}$$

is such that for every $t \geqslant 0$, $\omega \in \Omega$, $\eta(t, \cdot, \omega)$ is a probability measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, and for every $B \in \mathcal{B}(\mathbb{R}_+)$, $(\eta_t(B))_{t\geqslant 0}$ is a $\mathbb{G}$-progressively measurable process. Clearly, we have

$$\Lambda_t = \int_0^t \lambda_s \mathrm{d}s \quad \text{for all } t \geqslant 0.$$

In particular, we can choose a predictable version of both λ and η, see Section 14.3 of [15] for details. The processes λ and η can be interpreted respectively as jump intensity and jump size intensity. We recall that a marked point process $(\tau_n, X_n)_{n\in\mathbb{N}^+}$ has *independent marks* if the marks $(X_n)_{n\in\mathbb{N}}$ are mutually independent given $\mathbf{N}$.

Proposition 4.3. *The law of a simple marked point process $(\tau_n, X_n)_{n\in\mathbb{N}^+}$ on $(\Omega, \mathcal{H})$ with finite first moment measure, independent marks and of the form (4.2) is uniquely determined by λ and η. If furthermore λ is $\mathbb{H}$-measurable, then also the law of N on $(\Omega, \mathcal{G})$ is uniquely defined.*

Proof. By Proposition 6.4.IV(a) of [15], the law of marked point process with independent marks is uniquely determined by the kernel κ and the distribution of N. According to relations (4.1) and (4.2), the kernel κ is given by

$$\kappa(B|t, \omega) = \eta_t(B)(\omega), \qquad (t, B, \omega) \in \mathbb{R}_+ \times \mathcal{B}(\mathbb{R}_+) \times \Omega.$$

Corollary 4.8.5 of [20] and Theorem 14.2.IV(c) of [15] show that, if N is simple and of the form (4.2), the process $(E[\lambda_t|\mathcal{H}_t])_{t\geqslant 0}$ determines uniquely the distribution of N on $(\Omega, \mathcal{H})$. If in addition λ is $\mathbb{H}$-adapted, then by Theorem 4.8.1 of [20], also the distribution of N on $(\Omega, \mathcal{G})$ is uniquely determined. $\square$

Nevertheless, Proposition 4.3 requires the jump intensity process λ to be $\mathbb{H}$-adapted in order to have N uniquely defined in law, which is an unnatural condition in our context.

On the contrary, the approach proposed in Section 2 allows to take into account a dependence structure between the filtrations $\mathbb{H}$ and $\mathbb{G}$ by directly modeling the $\mathbb{F}$-adapted intensity process μ. Furthermore, this allows to obtain analytical results for valuation formulas as shown in Section 3.

5 Pricing in hybrid market

In this section we consider a general structure for a hybrid insurance and financial market and address the issue of pricing non-life insurance liabilities. We fix a time horizon T with $0 < T < \infty$, and denote the inflation index process by $I := (I_t)_{t\in[0,T]}$, which represents the percentage increments of the Consumer Price Index (CPI) and follows a nonnegative $(P, \mathbb{F})$-semimartingale. We distinguish real price value, i.e. inflation adjusted, from nominal price value, which can be converted in real value at any time $t \in [0, T]$, if divided by the inflation index I_t. If not otherwise specified, all price values are expressed in nominal value.

We consider d liquidly traded primary assets on the financial market described by price process vector $S := (S_t^1, ..., S_t^d)_{t\in[0,T]}$, which follows a real-valued $(P, \mathbb{F})$-semimartingale. We assume that there is a publicly accessible index, based on the intensity process μ and modelled by the process $L := (L_t)_{t\in[0,T]}$ with

$$L_t := e^{-\Gamma_t}, \qquad t \in [0, T],$$

see e.g. [13]. This index reflects the underlying systematic risk-factor related to the insurance portfolio, such as mortality risk, weather risk, car accident risk, etc. We distinguish three kinds of primary assets as elements of the vector S:

1. traditional financial assets, such as the zero-coupon bond, call and put options, futures etc.;
2. inflation linked derivatives, such as inflation linked zero-coupon bond (called also zero-coupon Treasury Inflation Protected Security, TIPS), which pays off I_T (equivalent to 1 real unit) at time T, inflation linked call and put options, etc.;
3. macro risk-factor linked derivatives based on the index L, such as longevity bond which pays off L_T at time T, weather index-based derivatives, etc.

We denote by $L(S, P, \mathbb{G})$ the space of $\mathbb{R}^d$-valued $\mathbb{G}$-predictable S-integrable processes. We call portfolio or value process $S^\delta := (S_t^\delta)_{t\in[0,T]}$ associated to a trading strategy $\delta := (\delta_t)_{t\in[0,T]}$ in $L(S, P, \mathbb{G})$ the following càdlàg optional process[9]

$$S_{t-}^\delta = \delta_t^\top S_t = \sum_{i=1}^{d} \delta_t^i S_t^i, \qquad t \in [0, T].$$

It is called self-financing if

$$S_t^\delta = S_0^\delta + \int_0^t \delta_{u-}^\top \mathrm{d}S_u = S_0^\delta + \sum_{i=1}^{l} \int_0^t \delta_{u-}^i \mathrm{d}S_u^i, \qquad t \in [0.T].$$

We introduce the following set

$$\mathcal{V}_x^+ = \{S^\delta \text{ self-financing } : \ \delta \in L(S, \mathcal{P}, \mathbb{G}),\ S_0^\delta = x > 0,\ S^\delta > 0\}.$$

[9] We follow the definitions in [6].

Definition 5.1. *A* benchmark *or* numéraire portfolio $S^* := (S_t^*)_{t\in[0,T]}$ *is an element of* $\mathcal{V}_1^+$*, such that*

$$\frac{S_s^\delta}{S_s^*} \geqslant E\left[\left. \frac{S_t^\delta}{S_t^*} \right| \mathcal{G}_s \right], \qquad s,t \in [0,T], \quad t \geqslant s.$$

We follow the approach of [34] and work under the following assumption.

Assumption 5.2. *There exists a benchmark portfolio* S^**.*

In [19], it is shown that Assumption 5.2 is weaker than assuming the existence of an equivalent martingale measure. As discussed in [4], this weak no-arbitrage assumption is more suitable for modeling a hybrid market as in our case. Given a generic random variable or process X, we denote by $\hat{X} := X/S^*$ the *benchmarked* value of X. The following lemma is proved in [5].

Lemma 5.3. *If the vector process of primary assets* S *is continuous, then the benchmarked vector process* $\hat{S} := S/S^*$ *is a* $(P, \mathbb{G})$*-local martingale.*

For the sake of simplicity, we assume the following conditions similar to the ones in [10].

Assumption 5.4. *The inflation index process* $I = (I_t)_{t\in[0,T]}$ *and the vector process of primary assets* S *are continuous. The benchmark portfolio* $S^* = (S_t^*)_{t\in[0,T]}$ *is continuous,* $\mathbb{F}$*-adapted, and the benchmarked value process* $\hat{S} := S/S^*$ *is an* $(\mathbb{F}, P)$*-local martingale. Inflation linked zero-coupon bond (or TIPS) is a primary asset, i.e. an element of the vector* S*.*

The payment stream in real unit of the insurance company towards policyholders is modelled by a nonnegative $(P, \mathbb{G})$-semimartingale $D := (D_t)_{t\in[0,T]}$. We denote by $A := (A_t)_{t\in[0,T]}$ the nominal benchmarked cumulative payment, namely

$$A_t := \int_0^t \frac{I_u}{S_u^*} \, \mathrm{d}D_u, \qquad t \in [0,T]. \tag{5.1}$$

Definition 5.5. *We call* real world pricing formula *associated to* A *the following formula*

$$V_t := \frac{S_t^*}{I_t} E\left[A_T - A_t \right| \mathcal{G}_t] = \frac{S_t^*}{I_t} E\left[\left. \int_{]t,T]} \frac{I_u}{S_u^*} \, \mathrm{d}D_u \right| \mathcal{G}_t \right], \tag{5.2}$$

for $t \in [0,T]$*.*

The value of V_t in (5.2) is expressed in real value, i.e. inflation adjusted value. In particular, we note that it corresponds to the *benchmarked risk-minimizing price* for the payment process A at time t, if A is square integrable, i.e.

$$\sup_{t\in[0,T]} \mathbb{E}\left[A_t^2\right] < \infty.$$

This can be shown in the same way as in Appendix A of [10].

5.1 Pricing non-life insurance claims

In the setting outlined above, we now apply the results of Section 3 to compute the real-world pricing formula for non-life insurance claims, under the interpretation of Section 2.2. The cumulative payment at time t related to i-th policy expressed in real value is given by

$$\sum_{j=1}^{\infty} \mathbf{1}_{\{\tau_j^i \leqslant t\}} X_j^i = \sum_{j=1}^{\mathbf{N}_t^i} X_j^i.$$

The nominal benchmarked cumulative payment process $A := (A_t)_{t\in[0,T]}$ is hence

$$A_t := \int_0^t \frac{I_s}{S_s^*} \mathrm{d}D_s = \sum_{i=1}^{n} \sum_{j=1}^{\mathbf{N}_t^i} \frac{I_{\tau_j^i}}{S_{\tau_j^i}^*} X_j^i, \qquad t \in [0,T]. \tag{5.3}$$

The estimation of A is called *reserving problem* in the contest of non-life insurance, see [1]. Unlike the life insurance case, the risk related to non-life insurance policies is hence not only related to the accident itself, but also to the first reporting delay (this is the case of incurred but not reported claims, called IBNR claims), to the time and the size of developments after the first reporting. We now focus on pricing and hedging the nominal remaining payment $A_T - A_t$, for $t \in [0,T]$. We assume that the process I/S^* is $\mathbb{F}$-conditionally independent from τ_1^i, for all $i = 1, ..., n$, and that the cumulative payments related to marked point processes $(\tilde{\tau}^i, \tilde{X}_j^i)_{j\in\mathbb{N}_+}$, $i = 1, ..., n$,

$$\sum_{j=1}^{\tilde{\mathbf{N}}_t^i} \tilde{X}_j^i, \qquad t \in [0,T], \qquad i = 1, ..., n,$$

are i.i.d. compound Poisson processes, i.e. $\tilde{\mathbf{N}}^i$ are mutually independent Poisson processes with parameter λ, and $\tilde{X}_j^i$ are i.i.d. integrable nonnegative random variables independent from $\tilde{\mathbf{N}}^i$ with expectation $E[\tilde{X}_j^i] = m$. In this case, we have

$$\tilde{m}(t) = \lambda m t, \qquad t \in [0,T],$$

where $\tilde{m}$ is defined in (3.15).

In view of the above assumptions, all conditions in Theorem 3.18 are satisfied in the case of $Y = A_T - A_t$, for $t \in [0,T]$. Let R_t be the number of reported claims at time t, i.e.

$$R_t := \sum_{1}^{n} \mathbf{1}_{\{\tau_1^i \leqslant t\}}, \qquad t \in [0,T].$$

The real world pricing formula (5.2) together with Corollary 3.3, Theorem 3.18

and Assumption 5.4 yields

$$
\begin{aligned}
V_t \frac{I_t}{S_t^*} =& E\left[A_T - A_t \middle| \mathcal{G}_t\right] = E\left[\sum_{i=1}^{n} \sum_{j=\mathbf{N}_t^i}^{\mathbf{N}_T^i} \frac{I_{\tau_j^i}}{S_{\tau_j^i}^*} X_j^i \middle| \mathcal{G}_t\right] \\
=& \sum_{i=1}^{n} E\left[\sum_{j=\mathbf{N}_t^i}^{\mathbf{N}_T^i} \frac{I_{\tau_j^i}}{S_{\tau_j^i}^*} X_j^i \middle| \mathcal{F}_t \vee \mathcal{H}_t^i\right] \\
=& \lambda m R_t\, E\left[\int_t^T \frac{I_u}{S_u^*} \mathrm{d}u \middle| \mathcal{H}_\infty^{i,1} \vee \mathcal{F}_t\right] \\
&+ (n - R_t) \frac{E\left[\int_t^T \left(E[X_1^i] \frac{I_u}{S_u^*} + \lambda m \int_u^T \frac{I_v}{S_v^*} \mathrm{d}v\right) \mathrm{d}P\left(\tau_1^i \leqslant u \middle| \mathcal{F}_u\right) \middle| \mathcal{F}_t\right]}{e^{-\int_0^t \mu_u \mathrm{d}u} + \int_0^t \bar{G}(t-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u} \\
=& \lambda m R_t \int_t^T E\left[\frac{I_u}{S_u^*} \middle| \mathcal{F}_t\right] \mathrm{d}u \\
&+ (n - R_t) \frac{E\left[\int_t^T \left(E[X_1^i] \frac{I_u}{S_u^*} + \lambda m \int_u^T \frac{I_v}{S_v^*} \mathrm{d}v\right) \mathrm{d}P\left(\tau_1^i \leqslant u \middle| \mathcal{F}_u\right) \middle| \mathcal{F}_t\right]}{e^{-\int_0^t \mu_u \mathrm{d}u} + \int_0^t \bar{G}(t-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u} \\
=& \lambda m R_t (T - t) \frac{I_t}{S_t^*} \\
&+ (n - R_t) \frac{E\left[\int_t^T \left(E[X_1^i] \frac{I_u}{S_u^*} + \lambda m \int_u^T \frac{I_v}{S_v^*} \mathrm{d}v\right) \mathrm{d}P\left(\tau_1^i \leqslant u \middle| \mathcal{F}_u\right) \middle| \mathcal{F}_t\right]}{e^{-\int_0^t \mu_u \mathrm{d}u} + \int_0^t \bar{G}(t-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u}, \quad (5.4)
\end{aligned}
$$

where the conditional probability function $P\left(\tau_1^i \leqslant t \middle| \mathcal{F}_t\right)$ is given in (3.12), i.e.

$$
P\left(\tau_1^i \leqslant t \middle| \mathcal{F}_t\right) = \int_0^t \left(\alpha_0 e^{-\int_0^s \mu_v \mathrm{d}v} \mu_s + \int_0^s g(s-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u\right) \mathrm{d}s.
$$

The first component on the left-hand side of (5.4)

$$
\lambda m R_t (T - t) \frac{I_t}{S_t^*}
$$

corresponds to already reported claims. We observe that the valuation of this part does not involve any more the updating information after the first reporting. The second component on the right-hand side of (5.4)

$$
(n - R_t) \frac{E\left[\int_t^T \left(E[X_1^i] \frac{I_u}{S_u^*} + \lambda m \int_u^T \frac{I_v}{S_v^*} \mathrm{d}v\right) \mathrm{d}P\left(\tau_1^i \leqslant u \middle| \mathcal{F}_u\right) \middle| \mathcal{F}_t\right]}{e^{-\int_0^t \mu_u \mathrm{d}u} + \int_0^t \bar{G}(t-u) e^{-\int_0^u \mu_v \mathrm{d}v} \mu_u \mathrm{d}u}, \quad (5.5)
$$

which can be further explicitly computed, corresponds to not reported claims and includes both cases of incurred but not reported (IBNR) claims as well as not yet incurred claims. The standard literature of non-life insurance is mainly focused on IBNR claims. However, for the pricing problem it is more appropriate to consider the entire expression (5.4). As already mentioned in at the beginning of this section, this price equals the benchmarked risk-minimizing price, if we assume square integrability of the claim. In particular, using the same arguments of Proposition 4.11 in [2] and Section 4.1 of [10], we can calculate the associated benchmarked risk-minimizing strategy. The form of V suggests how to design derivatives which can be used to hedge risks in this market model. In particular, since V is expressed in terms of the intensity process μ, the distribution of θ^i and the distribution of $(\tilde{\tau}_j^i, \tilde{X}_j^i)_{j \in \mathbb{N}_+}$, the benchmarked risk-minimizing strategy can be explicitly calculated. For further details on the benchmarked risk-minimization method for non-life insurance liabilities, we refer to [38]. One method to derive the distribution of μ can be found in [10].

6 Conclusion

In this paper, we introduce a general framework for modeling an insurance claims' flow in continuous time by extending the reduced-form setting. This framework allows to consider a nontrivial dependence between the reference information flow and the internal insurance information flow. In this setting, we compute explicit valuation formulas, which can be used for pricing non-life insurance products under the benchmark approach.

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