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MINIMAL VARIANCE HEDGING FOR INSIDER TRADING

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In this paper, we first study the problem of minimal hedging for an insider trader in incomplete markets. We use the forward integral in order to model the insider portfolio and consider a general larger filtration. We characterize the optimal strategy in terms of a martingale condition. In the second part we focus on a problem of mean-variance hedging where the insider tries to minimize the variance of his wealth at time T given that this wealth has a fixed expected value A. We solve this problem for an initial enlargement of filtration by providing an explicit solution.

Keywords: Forward integral; Malliavin calculus; minimal variance; insider hedging; mean-variance insider hedging.

1. Introduction

The problem of modeling insider trading in finance has been addressed by several authors. By an *insider* we mean a person who has access to a filtration $\{\mathcal{H}_t\}_{0\leq t\leq T}$ which is strictly bigger than the filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ generated by the underlying assets. Therefore the question is how to interpret integrals of the form

$$\int_0^T \psi(t,\omega)dB(t), \tag{1.1}$$

where ψ is assumed to be adapted to $\mathcal{H}_t \supset \mathcal{F}_t$ and represents the self-financing portfolio of an insider.

A natural, and the most common, approach to this question is to assume that $\mathcal{H}_t$ is such that $B(t)$ is a semimartingale with respect to $\mathcal{H}_t$ $[1, 2, 7, 8, 12, 13, 16–19]$. In this case we can write

$$B(t) = \widehat{B}(t) + A(t), \quad 0 \le t \le T, \tag{1.2}$$

where $\widehat{B}(t)$ is a $\mathcal{F}_t$-Brownian motion and A_t is a continuous $\mathcal{F}_t$-adapted finite variation process. If $A(t)$ has the form

$$A(t) = \int_0^t \alpha(u)du, \tag{1.3}$$

then the process $\alpha(\cdot)$ is called the *information drift* [18]. If a relation of the form (1.2) holds, then it is natural to define

$$\int_0^T \psi(t,\omega)dB(t) = \int_0^T \psi(t,\omega)d\widehat{B}(t) + \int_0^T \psi(t,\omega)dA(t), \tag{1.4}$$

because both terms of the right-hand side are well-defined. In general, there are several difficulties with this approach. Partial answers to when decomposition (1.2) holds can be found in the contributions to the book of Jeulin and Yor [21]. See also [18].

In the first part of this paper we follow the approach of [6, 14, 26] and use the forward integral to model the insider portfolio. This allows us to consider a general larger filtration, i.e., we do not necessarily assume that the underlying $\mathcal{F}_t$-Brownian motion $B(t)$ is also a semimartingale with respect to the filtration $\mathcal{H}_t$. In the case when the enlargement of filtration holds, we remark that the use of forward integrals coincides with the enlargement of filtration approach.

In particular, here we assume that the market is influenced by an additional source of randomness. For example, one can suppose that there exist two assets but trading is allowed only on one of them. The resulting market is incomplete and here we select the minimal variance hedging approach as pricing and hedging criterion. The purpose of this paper is to study the minimal variance hedging problem from the point of view of an insider trader who has access to larger information and check if she can do any better than the honest trader. This has been recently studied also in [7] by using initial enlargement of filtration technique. His results concern only the case when $\mathcal{H}_t = F_t \vee \sigma(L)$, where L is a suitable random variable representing the additional information, while in this paper we consider a general larger filtration. For the honest trader case, we refer to [39] for a complete survey on mean-variance hedging and a complete bibliography.

In the first part of this paper, we characterize the optimal strategy and the approximation cost (when they exist) in terms of a martingale condition and, when further differentiability conditions hold, as an equation involving Malliavin derivatives.

In the second part of this paper, we focus on a problem of mean-variance hedging where the insider tries to minimize the variance of his wealth at time T given a fixed expected value of the wealth at this time. To the best of our knowledge this is the first time this problem is discussed in an insider context. Here we are able to provide an explicit solution of it in the case of an initial enlargement of filtration of the form $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(B(T_0))$.

2. Some Preliminaries

Throughout the first part of this paper, we will use the standard tools of Malliavin calculus. As a general reference, we refer to [28, 30].

Here we give the definition and some properties of the forward integral. For more information, see [3, 4, 24, 29, 36–38].

Let $B(t)$ be a standard Brownian motion on the stochastic base $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. In the sequel we denote respectively by $D_s\psi(t)$ the Malliavin derivative and by $\int_0^t \psi(s)\delta B(s)$ the Skorohod integral of a process $\psi(t)$, when they exist. For further details on Malliavin calculus, we refer to [28, 31].

Definition 2.1. Let $\psi(t,\omega)$ be a measurable process.
(i) The forward integral of ψ is defined by

$$\int_0^T \psi(t,\omega)d^-B(t) = \lim_{\epsilon\to 0}\int_0^T \psi(t,\omega)\frac{B(t+\epsilon)-B(t)}{\epsilon}dt, \tag{2.1}$$

if convergent in probability. If the limit exists in $L^2(P)$ we write $\psi \in Dom_2\delta^-$.
(ii) We say that ψ is forward integrable in the strong sense and write $\psi \in \mathbb{D}$ if ψ is càglàd, Skorohod integrable and $D_{t^+}\psi(t) = \lim_{s\to t^+} D_s\psi(t)$ exists in $L^2(dP\otimes dt)$. In particular,

$$E\left[\int_0^T |D_{t^+}\psi(t)|^2 dt\right] < \infty. \tag{2.2}$$

By Proposition 2.3 of [36] we get:

Lemma 2.1. *Let $\psi \in \mathbb{D}$. Then ψ is forward integrable and*

$$\int_0^T \psi(t,\omega)d^-B(t) = \int_0^T \psi(t,\omega)\delta B(t) + \int_0^T D_{t^+}\psi(t)dt. \tag{2.3}$$

In particular,

$$E\left[\int_0^T \psi(t,\omega)d^-B(t)\right] = E\left[\int_0^T D_{t^+}\psi(t)dt\right]. \tag{2.4}$$

Note that if ψ is càglàd and forward integrable, then by Eq. (2.2) of [6], (2.1) coincides with the limit in probability of the Riemann sums

$$\int_0^T \psi(t,\omega)d^-B(t) := \lim_{|\Delta t|\to 0}\sum_j \psi(t_j)\cdot\Delta B(t_j), \tag{2.5}$$

if convergent in probability for any partition $0 = t_0 < t_1 < \cdots < t_N = T$ of $[0,T]$, with $\Delta t_j = t_{j+1} - t_j$, $|\Delta t| = \sup_{j=0,\ldots,N-1}\Delta t_j$ and $\Delta B(t_j) = B(t_{j+1}) - B(t_j)$.

Remark 2.1. Consider the integrand $\psi(t,\omega) = I_{\{\tau_1 < t \le \tau_2\}}$, where τ_1, τ_2 are random variables with values in $[0,T]$. As an immediate consequence of (2.5) we obtain that

$$\int_0^T \psi(t,\omega) d^- B(t) = \lim_{\Delta t_j \to 0} \sum_j \psi(t_j) \cdot \Delta B(t_j) = \int_{\tau_1}^{\tau_2} dB(t) = B(\tau_2) - B(\tau_1). \tag{2.6}$$

In finance the process $\psi(t,\omega)$ is sometimes called the "buy-and-hold" portfolio. Note also that τ_1, τ_2 need not be $\mathcal{F}_t$-stopping times.

Moreover, we note that if ψ is a predictable process such that $E\big[\int_0^T \psi(t)^2 dt\big] < \infty$, then ψ is forward integrable, belongs to $Dom_2\delta^-$ and its forward integral coincides with the Itô integral $\int_0^T \psi(t) dB(t)$ (see Proposition 1.1 of [37] and Lemma 2.1).

The following Lemma illustrates why the forward integral appears naturally in insider modeling. Let $\mathcal{H}_t \supset \mathcal{F}_t$ and assume that $B(t)$ is a semimartingale with respect to $\mathcal{H}_t$, so that (1.2) holds, i.e.

$$B(t) = \widehat{B}(t) + A(t), \quad 0 \le t \le T, \tag{2.7}$$

where $\widehat{B}(t)$ is a $\mathcal{H}_t$-adapted Brownian motion, $A(t)$ is a $\mathcal{H}_t$-adapted finite variation continuous process. Using Riemann sum approximations we see that the following holds:

Lemma 2.2. *Suppose that* (2.7) *holds and that* $\psi(t,\omega)$ *is an* $\mathcal{H}_t$*-adapted càglàd process which is integrable with respect to* $\widehat{B}(t)$ *and* $A(t)$*. Then* $\psi(t,\omega)$ *is forward integrable and*

$$\int_0^T \psi(t) d^- B(t) = \int_0^T \psi(t) d\widehat{B}(t) + \int_0^T \psi(t) dA(t). \tag{2.8}$$

In view of Remark 2.1 and Lemma 2.2, following the approach initiated in [6], and subsequently applied in [9, 10, 14, 22, 23, 32, 33], we model the stochastic integral "$\int_0^T \psi(t,\omega) dB(t)$" of the insider portfolio as given by the forward integral $\int_0^T \psi(t,\omega) d^- B(t)$, when $\psi(t)$ is $\mathcal{H}_t$-adapted, without assuming that (1.2) holds.

3. Minimal Variance Hedging of an Insider

Let $B_1(t), B_2(t)$ be two independent standard Brownian motions on the filtered product probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{F}^1 \otimes \mathcal{F}^2, \mathcal{F}_t^1 \otimes \mathcal{F}_t^2, \mathbb{P}^1 \otimes \mathbb{P}^2)$. Here $\mathcal{F}_t = \mathcal{F}_t^1 \otimes \mathcal{F}_t^2$ is the natural filtration generated by B_1, B_2. In this framework we consider the following market:

1. A risk-free asset $S_0(t)$, which we assume constantly equal to 1;
2. A risky asset $S(t)$ with the following dynamics:

$$\begin{cases} dS(t) = S(t)[\mu(t)dt + \sigma(t) d^- B_1(t)] \\ S(0) = s_1. \end{cases} \tag{3.1}$$

As in the approach of [6], we suppose that there exist on Ω two other filtrations $\mathcal{G}_t, \mathcal{H}_t$ in addition to $\mathcal{F}_t$ such that

$$\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{H}_t \subset \mathcal{F}, \quad \forall t \in [0, T], \tag{3.2}$$

and that the coefficients $\mu(t), \sigma(t)$ are $\mathcal{G}_t$-adapted. Moreover, we assume that the coefficients satisfy the following conditions:

$$\sigma(t) \in \mathbb{D}, \tag{3.3}$$

$$E\left[\int_0^T \left\{\mu(t)^2(1 + S^2(t)) + \sigma(t)^2\right\} dt\right] < \infty, \tag{3.4}$$

$$\sigma(t) \neq 0 \quad \text{for a.a. } (t, \omega) \in [0, T] \times \Omega. \tag{3.5}$$

By [38], we get that condition (3.4) guarantees that the solution of Eq. (3.1) exists and is unique in the sense that two solutions coincide for a.a. t, ω. By the Itô formula for the forward integral (Theorem 2.2 of [37]), by Lemma 1.1 of [36] and Corollary 5.5 of [38], we obtain the following explicit form for $S(t)$:

$$S(t) = S(0) \exp\left(\int_0^t \sigma(s) d^- B_1(s) + \int_0^t \left\{\mu(s) - \frac{1}{2}\sigma^2(s)\right\} ds\right).$$

We interpret the corresponding anticipative integrals in the dynamics of S as forward integrals with respect to the Brownian motion B_1. Since the agent trades in the asset price $S(t)$, we need to introduce the following

Definition 3.1. Let $\phi(t, \omega)$ be a measurable process and let $S(t)$ be as in (3.1). Then we define the forward integral of ϕ with respect to S as follows

$$\int_0^T \phi(s) d^- S(s) := \int_0^T \phi(s)\mu(s)S(s) ds + \int_0^T \phi(s)S(s)\sigma(s) d^- B_1(s), \tag{3.6}$$

if the right-hand side exists. In this case, we say that $\phi(t)$ is *forward integrable* with respect to S.

By using this approach, we model a market influenced by large investors with access to insider information, i.e., with access to the information $\mathcal{G}_t$, or more generally, a market influence by other random events than those described by $\mathcal{F}_t$.

Here we also assume that the market is influenced by an additional source of randomness represented by the standard Brownian motion $B_2(t)$. This is an example of the so-called *"almost complete markets"*, where there exist two assets but trading is allowed only on one of them. In the classical case this has been studied by using the mean-variance hedging approach by [5, 25, 34]. The resulting market is incomplete and one must choose a suitable pricing and hedging criterion. The purpose of the first part of this paper is to study the minimal variance hedging problem from the point of view of an insider trader who has access to larger information and check if she can perform better than the honest trader. Now we introduce the set $\mathcal{A}$ of admissible trading strategies for the insider:

Definition 3.2. Let $\mathcal{A}$ be the space of all stochastic processes $\phi(t,\omega)$ such that

1. $\phi(t)$ is an $\mathcal{H}_t$-adapted process, forward integrable with respect to S such that $\phi(t)S(t)\sigma(t) \in Dom_2\delta^-$;
2. $E\big[\int_0^T |\mu(t)S(t)\phi(t)|dt\big] < \infty$, $E\big[\int_0^T (\sigma(t)S(t)\phi(t))^2 dt\big] < \infty$.

We call this space $\mathcal{A}$ the set of *admissible portfolios* for the insider.

Definition 3.3. A random variable $F \in L^2(\mathcal{F}_T, \mathbb{P})$ is called a *claim*.

In this framework we study the following problem.

Problem 3.1. Let $F \in L^2(\mathcal{F}_T, \mathbb{P})$ be a claim. Find $V_T^{\mathcal{H}} \in \mathbb{R}, x^* \in \mathbb{R}$ and $\phi^* \in \mathcal{A}$ such that

$$V_T^{\mathcal{H}} = \inf_{\phi\in\mathcal{A}, x\in\mathbb{R}} J(x,\phi) = J(x^*,\phi^*), \tag{3.7}$$

where

$$J(x,\phi) = E\left[\left(F - x - \int_0^T \phi(t)d^-S(t)\right)^2\right] \quad \text{for } \phi \in \mathcal{A}. \tag{3.8}$$

We call $V_T^{\mathcal{H}} < \infty$ the *value* of the minimal variance hedging problem, ϕ^* a *minimal variance hedging portfolio* and x^* an *approximation price* (if it exists), in the same notation as in [39]. Moreover, we call $x^* + \int_0^T \phi^*(t)d^-S(t)$ the *closest hedge*. The interpretation of this problem is the following: we assume that the insider can only trade in the asset with price $S(t)$. How close can she get (in terms of minimal variance) at time T to a given claim F if she has access to the insider information $\mathcal{H}_t \supset \mathcal{G}_t \supset \mathcal{F}_t$?

Example 3.1. Assume that the trader is honest, i.e. $\mathcal{H}_t = \mathcal{F}_t, \forall t \in [0,T]$, and that $\mu(t) = 0$. Let $F \in L^2(\mathcal{F}_T, \mathbb{P})$ be a claim. By the Itô representation theorem we know that $F \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ can be written as

$$F = E[F] + \int_0^T \beta_1(t,\omega)dB_1(t) + \int_0^T \beta_2(t,\omega)dB_2(t), \tag{3.9}$$

where $\beta_i, i = 1,2$ are $\mathcal{F}_t$-adapted processes such that

$$E\left[\int_0^T (\beta_1^2(t,\omega) + \beta_2^2(t,\omega))dt\right] < \infty. \tag{3.10}$$

This decomposition is unique. Then for any $x \in \mathbb{R}$ and $\mathcal{F}_t$-adapted square integrable process ϕ we have

$$\begin{aligned} & E\left[\left(F - x - \int_0^T \phi(t) d^- S(t)\right)^2\right] \\ & \quad = E\left[\left(E[F] - x - \int_0^T (\phi(t)S(t)\sigma(t) - \beta_1(t))dB_1(t) + \int_0^T \beta_2(t)dB_2(t)\right)^2\right] \\ & \quad = (E[F]-x)^2 + E\left[\int_0^T [(\phi(t)S(t)\sigma(t) - \beta_1(t))^2 + \beta_2(t)^2]dt\right]. \end{aligned}$$

Hence it is optimal to choose $x^* = E\big[F\big]$ and $\phi^*(t) = \frac{\beta_1(t)}{S(t)\sigma(t)}$, which corresponds to the minimal variance $E\big[\int_0^T \beta_2(t)^2 dt\big]$.

Example 3.2. This example shows how an insider trader can actually obtain better results than the honest trader.

Assume that $\mu(t) = 0$ and $\sigma(t) = 1$ for every $t \in [0, T]$. We suppose that $\mathcal{H}_t = \mathcal{F}_T^1 \vee \mathcal{F}_T^2$. Then $B_1(t)$ is not a semimartingale with respect to $\mathcal{H}_t$. Consider $F = \int_0^T (S(s) - S(0))dB_2(s)$. Then F is replicable with $(x^*, \phi^*(t)) = (0, B_2(T) - B_2(t))$. To see this, it is sufficient to show that

$$\int_0^T (S(s) - S(0))dB_2(s) = \int_0^T (B_2(T) - B_2(s))S(s)d^- B_1(s) = \int_0^T \phi^*(s) d^- S(s).$$

The right-hand side is equal to

$$\begin{aligned} \int_0^T (B_2(T) - B_2(s))S(s)d^- B_1(s) &= B_2(T)\int_0^T S(s)dB_1(s) - \int_0^T B_2(s)S(s)dB_1(s) \\ &= B_2(T)(S(T) - S(0)) - \int_0^T B_2(s)S(s)dB_1(s). \end{aligned}$$

By the Itô formula, this is the same as the left-hand side.

This simple example shows the essential role of forward integrable when the enlargement of filtration does not hold. Moreover it shows a case when a solution for the minimal hedging problem exists and can be explicitly computed.

We now focus on the general insider case. Let $\mathbb{H}$ be the space generated by the integrals of the form $x + \int_0^T \phi(t)d^- S(t)$, $x \in \mathbb{R}, \phi \in \mathcal{A}$. We wish to use a classical Hilbert space argument, but unfortunately we don't know if $\mathbb{H}$ is closed in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Given $F \in L^2(\mathcal{F}_T, \mathbb{P})$, let $\hat{F}$ be the closest element to F in the closure $\bar{\mathbb{H}}$ in L^2 of $\mathbb{H}$. Then

$$E\big[(F - \hat{F})k\big] = 0 \quad \forall k \in \mathbb{H}$$

Hence, we see that Problem (3.1) is equivalent to finding $x^* \in \mathbb{R}$ and $\phi^* \in \mathcal{A}$ such that

$$E\left[\left(F - x^* - \int_0^T \phi^*(t)d^- S(t)\right)\left(y + \int_0^T \theta(t)d^- S(t)\right)\right] = 0, \tag{3.11}$$

for all $y \in \mathbb{R}$, $\theta = \theta(t,\omega) \in \mathcal{A}$. We split it into the following

$$\begin{aligned} &yE\left[F - x^* - \int_0^T \phi^*(t)d^- S(t)\right] \\ &\quad + E\left[\left(F - x^* - \int_0^T \phi^*(t)d^- S(t)\right)\left(\int_0^T \theta(t)d^- S(t)\right)\right] = 0. \end{aligned} \tag{3.12}$$

Since it holds for all y we get the two equations:

$$x^* = E\left[F - \int_0^T \phi^*(t)d^- S(t)\right], \tag{3.13}$$

(by using Eqs. (3.10) and (2.3)) and

$$E\left[\left(F - x^* - \int_0^T \phi^*(t)d^- S(t)\right)\left(\int_0^T \theta(t)d^- S(t)\right)\right] = 0, \tag{3.14}$$

for all $\theta = \theta(t,\omega) \in \mathcal{A}$. We now put

$$G = F - x^* - \int_0^T \phi^*(t)d^- S(t). \tag{3.15}$$

In particular, for $\theta(s,\omega) = \theta_0 \chi_{(t,t+h]}(s), h > 0$, where θ_0 is a bounded $\mathcal{H}_t$-measurable random variable, we obtain

$$E\left[G\int_0^T \theta(t)d^- S(t)\right] = E[G\theta_0(S(t+h) - S(t))] = 0 \quad \forall h > 0,\ \ \forall t \in [0,T]. \tag{3.16}$$

We can summarize this result in the following

Theorem 3.1. *Let $F \in L^2(\mathcal{F}_T, \mathbb{P})$ be a claim. Suppose that there exists an optimal solution $(x^*, \phi^*) \in \mathbb{R} \times \mathcal{A}$ for Problem* 3.1. *Define $G = F - x^* - \int_0^T \phi^*(t)d^- S(t)$. Then the process*

$$S(t)E\left[G|\mathcal{H}_t\right], \tag{3.17}$$

is an $\mathcal{H}_t$-martingale.

Conversely, if there exists $(\hat{x}, \hat{\phi}) \in \mathbb{R} \times \mathcal{A}$ such that (3.17) *holds, then $(\hat{x}, \hat{\phi})$ is optimal.*

Proof. We choose $\theta_0 = I_A$, $A \in \mathcal{H}_t$, in (3.16). Hence, by using the definition of conditional expectation and Eq. (3.16) we have

$$\begin{aligned}\int_A S_t E\left[G|\mathcal{H}_t\right] d\mathbb{P} &= \int_A G S_t d\mathbb{P} = \int_A G S_{t+h} d\mathbb{P} \\ &= \int_A S_{t+h} E[G|\mathcal{H}_{t+h}] d\mathbb{P} = \int_A E\left[S_{t+h} E[G|\mathcal{H}_{t+h}]|\mathcal{H}_t\right] d\mathbb{P}).\end{aligned}$$

Hence the process $S(t)E\left[G|\mathcal{H}_t\right]$ is an $\mathcal{H}_t$-martingale.

The second part of the Theorem follows by reversing the above argument. □

Corollary 3.1. *Let $F \in L^2(\mathcal{F}_T, \mathbb{P})$ be a claim. Suppose that there exists an optimal solution $(x^*, \phi^*) \in \mathbb{R} \times \mathcal{A}$ for Problem* 3.1. *If $E\left[G|\mathcal{H}_t\right] \neq 0$ for a.a. $(t, \omega) \in [0, T] \times \Omega$, then $S(t)$ is a semimartingale with respect to $\mathcal{H}_t$.*

Proof. This is an immediate consequence of the well-known fact that the product of two semimartingales is a semimartingale. □

Here there are some examples when the enlargement of filtration does not hold, i.e., when $S(t)$ is not a semimartingale with respect to $\mathcal{H}_t$. Assume $\mathcal{H}_t = \mathcal{F}_T$ for all $t \in [0, T]$. If S_t is an $\mathcal{H}_t$-semimartingale with decomposition $S_t = M_t + A_t$, where M_t is an $\mathcal{H}_t$-local martingale and A_t an $\mathcal{H}_t$-adapted finite variation process, then $M_t = E\left[M_T|\mathcal{H}_t\right] = E\big[M_T|\mathcal{F}_T\big] = M_T$, i.e., S_t coincide with a finite variation process. Since $S(t)$ is continuous and has positive quadratic variation in every open interval of $[0, T]$, we see that this is not possible. Hence in this case we cannot have any enlargement of filtration.

We can generalize now the same argument to the case when $\mathcal{H}_t = \mathcal{F}_{t+\delta(t)}$ where $\delta(t) > 0$ for all $t \in [0, T)$.

Corollary 3.2. *Let $F \in L^2(\mathcal{F}_T, \mathbb{P})$ be a claim and suppose that $\mathcal{H}_t = \mathcal{F}_{t+\delta(t)}$ where $\delta(t) > 0$ for all $t \in [0, T)$. If there exists an optimal solution $(x^*, \phi^*) \in \mathbb{R} \times \mathcal{A}$ for F, then F can be perfectly replicated by the insider trader.*

Proof. By Theorem 3.1 we know that for $h > 0$,

$$E\left[S(t+h)E[G|\mathcal{H}_{t+h}]|\mathcal{H}_t\right] = S(t)E\left[G|\mathcal{H}_t\right],$$

i.e.,

$$E\Big[S(t+h)E\big[G|\mathcal{F}_{t+h+\delta(t+h)}\big]\,|\mathcal{F}_{t+\delta(t)}\Big] = S(t)E\big[G|\mathcal{F}_{t+\delta(t)}\big].$$

If $0 \leq h \leq \delta(t)$ this gives

$$S(t+h)E\big[G|\mathcal{F}_{t+\delta(t)}\big] = S(t)E\big[G|\mathcal{F}_{t+\delta(t)}\big].$$

Since $S(t+h) - S(t) \neq 0$ for almost every $(t, \omega) \in (0, T) \times \Omega$, we get that

$$E\big[G|\mathcal{F}_{t+\delta(t)}\big] = 0 \quad \text{for a.e. } (t, \omega) \in (0, T) \times \Omega.$$

Letting $t \to T$ we get by the martingale convergence theorem that

$$G = E[G|\mathcal{F}_T] = 0,$$

and hence F is perfectly replicable. $\square$

If the optimal strategy for F does not exist, then we may still have replication in the limit.

4. A Further Characterization of the Optimal Strategy

In this section we show that if the optimal strategy satisfies some additional derivability conditions, then it can be obtained as the solution of an equation involving Malliavin derivatives. We denote by $\mathbb{D}^{1,2}$ the space of Malliavin differentiable random variables F such that $\|DF\|^2_{L^2([0,T]\times\Omega)} < \infty$.

Definition 4.1. (i) The set $\mathcal{A}_{\mathbb{D}}$ of *strongly admissible* portfolios is defined by

$$\mathcal{A}_{\mathbb{D}} = \mathcal{A} \cap \mathbb{D},$$

where $\mathbb{D}$ is as in Definition 2.1.

(ii) We say that a claim F is *smooth* if $F \in L^2(\mathcal{F}_T, \mathbb{P}) \cap \mathbb{D}^{1,2}$.

Suppose now that the optimal strategy $\phi \in \mathcal{A}_{\mathbb{D}}$ and that the claim F is smooth. If ϕ is optimal, then by Theorem 3.1 we obtain that for every bounded $\mathcal{H}_t$-measurable random variable θ we have

$$\begin{aligned} 0 &= E[G\theta(S(t+h) - S(t))] \\ &= E\left[G\theta\left(\int_t^{t+h} S(u)\mu(u)du + \int_t^{t+h} S(u)\sigma(u)d^- B(u)\right)\right]. \end{aligned} \tag{4.1}$$

By using the property of the forward integral, Eq. (4.1) can be rewritten as

$$0 = E\left[\theta\left(\int_t^{t+h} S(u)\mu(u)Gdu + \int_t^{t+h} S(u)\sigma(u)Gd^- B_1(u)\right)\right]. \tag{4.2}$$

If in addition $S(t)\sigma(t)G \in \mathbb{D}$, by Lemma 2.1 we have

$$\begin{aligned} \int_t^{t+h} S(u)\sigma(u)Gd^- B_1(u) = &\int_t^{t+h} S(u)\sigma(u)G\delta B_1(u) \\ &+ \int_t^{t+h} D_{1,u}(S(u)\sigma(u)G)du, \end{aligned} \tag{4.3}$$

where $D_{1,u}$ represents the Malliavin derivative with respect to B_1. Hence we can rewrite Eq. (4.2) as

$$0 = E\left[\theta \int_t^{t+h} (S(u)\mu(u)G + D_{1,u}(S(u)\sigma(u)G))\, du\right]. \tag{4.4}$$

Dividing by h and letting $h \to 0$, we have

$$0 = E[\theta\,(S(t)\mu(t)G + D_{1,t}(S(t)\sigma(t)G))]\,. \tag{4.5}$$

Since Eq. (4.5) holds for every bounded $\mathcal{H}_t$-measurable θ, we obtain the following Theorem.

Theorem 4.1. *Let F be a smooth claim. Suppose that there exists an optimal solution $(x^*, \phi^*) \in \mathbb{R} \times \mathcal{A}_{\mathbb{D}}$ for Problem* 3.1 *with $\sigma\phi^* S \in \mathbb{L}^{1,2}$. Then, with $G = F - x^* - \int_0^T \phi^*(t)d^- S(t)$, (x^*, ϕ^*) is the solution of*

$$0 = S(t)\mu(t)E\left[G|\mathcal{H}_t\right] + S(t)\sigma(t)E\left[D_{1,t}G|\mathcal{H}_t\right] + E\left[GD_{1,t}(S(t)\sigma(t))|\mathcal{H}_t\right]. \tag{4.6}$$

Conversely, if there exists $(\hat{x}, \hat{\phi}) \in \mathbb{R} \times \mathcal{A}$ such that (4.6) *holds, then $(\hat{x}, \hat{\phi})$ is optimal.*

We consider now the case when the enlargement of filtration holds and restate Theorem 4.1 in this special case. In addition, the following result provides a partial converse of Theorem 3.1. Suppose that $S(t)$ is an $\mathcal{H}_t$-semimartingale, i.e. that the enlargement of the filtration holds. This is equivalent to assuming that there exists an $\mathcal{H}_t$-Brownian motion $\hat{B}_1(t)$ and an $\mathcal{H}_t$-adapted integrable process $\alpha(t)$ such that

$$B_1(t) = \hat{B}_1(t) + \int_0^t \alpha(s)ds. \tag{4.7}$$

Fix $t \in [0, T)$ and choose an $\mathcal{H}_t$-adapted process $\theta(s) = \theta_0\chi_{[t,t+h)}(s) \in \mathcal{A}_{\mathbb{D}}$ where θ_0 is a $\mathcal{H}_t$-measurable random variable. From Theorem 3.1 we have that

$$\begin{aligned}
0 &= E[G(S(t+h) - S(t))\theta_0] \\
&= E\left[G\theta_0 \left(\int_t^{t+h} S(s)\mu(s)ds + \int_t^{t+h} S(s)\sigma(s)d^- B_1(s)\right)\right] \\
&= E\left[G\theta_0 \left(\int_t^{t+h} S(s)(\mu(s) + \sigma(s)\alpha(s))ds + \int_t^{t+h} S(s)\sigma(s)d\hat{B}_1(s)\right)\right] \\
&= E\left[\theta_0 \left(\int_t^{t+h} GS(s)(\mu(s) + \sigma(s)\alpha(s))ds + \int_t^{t+h} S(s)\sigma(s)\hat{D}_{1,s}Gds\right)\right],
\end{aligned}$$

where $\hat{D}_{1,t}$ is the Malliavin derivative with respect to $\hat{B}_1(t)$. Dividing by h and letting $h \to 0$ we obtain

$$E[\theta_0(GS(t)(\mu(t) + \sigma(t)\alpha(t)) + S(t)\sigma(t)\hat{D}_{1,t}G)] = 0. \tag{4.8}$$

Since this holds for any $\mathcal{H}_t$-measurable θ we conclude that

$$E[GS(t)(\mu(t) + \sigma(t)\alpha(t)) + S(t)\sigma(t)\hat{D}_{1,t}G|\mathcal{H}_t] = 0. \tag{4.9}$$

Hence we have proved:

Theorem 4.2. *Let F be a smooth claim. Suppose that there exists an optimal solution $(x^*, \phi^*) \in \mathbb{R} \times \mathcal{A}_{\mathbb{D}}$ for Problem* 3.1 *with $\sigma\phi^* S \in \mathbb{D}$. Then, with $G = F - x^* - \int_0^T \phi^*(t) d^- S(t)$, (x^*, ϕ^*) is the solution of*

$$(\mu(t) + \sigma(t)\alpha(t)) E\left[G|\mathcal{H}_t\right] + \sigma(t) E[\hat{D}_{1,t} G|\mathcal{H}_t] = 0, \quad 0 \le t \le T. \tag{4.10}$$

From now on we use Eq. (4.10) in order to study the optimal strategy ϕ^*. First note that $\hat{D}_{1,t} G = \hat{D}_{1,t} F - \hat{D}_{1,t}(\int_0^T \phi^*(s) dS(s))$ and

$$\begin{aligned}
&\hat{D}_{1,t}\left(\int_0^T \phi^*(s) dS(s)\right) \\
&\quad = \hat{D}_{1,t}\left(\int_0^T \phi^*(s) S(s)(\mu(s) + \sigma(s)\alpha(s)) ds + \int_0^T \phi^*(s) S(s)\sigma(s) d\hat{B}_1(s)\right) \\
&\quad = \int_t^T \hat{D}_{1,t}(S(s)(\mu(s) + \sigma(s)\alpha(s))\phi^*(s)) ds + \int_t^T \hat{D}_{1,t}(\phi^*(s) S(s)\sigma(s)) d\hat{B}_1(s)) \\
&\qquad + \phi^*(t) S(t)\sigma(t).
\end{aligned}$$

Hence

$$\begin{aligned}
&E\left[\hat{D}_{1,t}\left(\int_0^T \phi^*(s) dS(s)\right)\middle|\mathcal{H}_t\right] \\
&\quad = E\left[\int_t^T \hat{D}_{1,t}\left(\phi^*(s) S(s)(\mu(s) + \sigma(s)\alpha(s))\right) ds\middle|\mathcal{H}_t\right] + \phi^*(t) S(t)\sigma(t).
\end{aligned}$$

Corollary 4.1. *Let F be a smooth claim. Suppose that there exists an optimal solution $(x^*, \phi^*) \in \mathbb{R} \times \mathcal{A}_{\mathbb{D}}$ for Problem* 3.1. *Then*

$$\begin{aligned}
&(\mu(t) + \sigma(t)\alpha(t))\left(E\left[F|\mathcal{H}_t\right] - x^* - E\left[\int_0^T \phi^*(s) dS(s)\middle|\mathcal{H}_t\right]\right) \\
&\quad + \sigma(t)\left(E[\hat{D}_{1,t} F|\mathcal{H}_t] - E\left[\int_t^T \hat{D}_{1,t}\left(\phi^*(s) S(s)(\mu(s) + \sigma(s)\alpha(s))\right) ds\middle|\mathcal{H}_t\right]\right) \\
&\quad - \phi^*(t) S(t)\sigma^2(t) = 0.
\end{aligned} \tag{4.11}$$

Since the optimal price x^* (if it exists), it is given by Eq. (3.13), we have that Eq. (4.11) characterizes completely the optimal strategy $\phi^*(t)$ even if it cannot be easily solved. Indeed the problem of computing explicitly the mean-variance hedging strategy is already quite complicated also in the honest trader case. See for example [5, 25, 34, 39].

Corollary 4.2. *Suppose $(x^*, \phi^*) \in \mathbb{R} \times \mathcal{A}_{\mathbb{D}}$ is optimal and*

$$\mu(t) + \sigma(t)\alpha(t) = 0 \quad \textit{for a.a. } (t, \omega).$$

Then

$$\phi^*(t)S(t)\sigma(t) = E[\hat{D}_{1,t}F|\mathcal{H}_t].$$

This result is related to the Clark–Ocone Theorem, which states that any smooth claim F can be given by the representation

$$F = E[F] + \int_0^T E[\hat{D}_{1,t}F|\mathcal{H}_t]d\hat{B}_1(t) + \int_0^T E[\hat{D}_{2,t}F|\mathcal{H}_t]d\hat{B}_2(t).$$

If $\mu(t) + \sigma(t)\alpha(t) = 0$ then

$$dS(t) = \sigma(t)S(t)d\hat{B}_1(t).$$

Corollary 4.3. *If F is a smooth claim, then $(x^*, \phi^*) = (0, 0)$ is optimal if and only if F satisfies*

$$(\mu(s) + \sigma(s)\alpha(s))E\left[F|\mathcal{H}_t\right] + \sigma(t)E[\hat{D}_{1,t}F|\mathcal{H}_t] = 0.$$

Unfortunately even in the case when an optimal strategy exists, it is not in general unique as shown by the following example.

Example 4.1. Suppose that $\mu(t) = 0$ almost everywhere for every $t \in (0, T)$. We recall that $\mathcal{F}_t^1$ and $\mathcal{F}_t^2$ are the filtrations generated respectively by B_t^1 and B_t^2. Consider now the case when $\mathcal{H}_t \supset \mathcal{F}_T^1$ for every t. Then, the process $\psi(t) = \frac{B_1(T)-2B_1(t)}{\sigma(t)S(t)}$ belongs to the space $\mathcal{A}_{\mathbb{D}}$ of strongly admissible strategies and we have

$$\int_0^T \psi(t)d^- B_1(t) = \int_0^T (B_1(T) - 2B_1(t))d^- B_1(t) = T. \tag{4.12}$$

Hence, if (x^*, ϕ^*) is an optimal strategy for Problem 3.1, then $(x^* - T, \phi^* + \psi)$ is optimal too. We conclude that in general the optimal strategy may be not unique.

5. The Mean-Variance Portfolio Problem for an Insider

From now on let $\mathcal{A}$ be the set of all $\mathcal{H}_t$-adapted càglàd processes $\phi \in Dom_2\delta^-$. We now study in detail the following problem. Let $X^{(\phi)}(t)$ denote the wealth at time t of an insider using the portfolio $\phi(t) \in \mathcal{A}$. Consider the problem of finding a portfolio which minimizes the variance

$$\mathrm{Var}[X^{(\phi)}(T)] = E\big[(X^{(\phi)}(T) - E\big[X^{(\phi)}(T)\big])^2\big], \tag{5.1}$$

under the condition that

$$E\big[X^{(\phi)}(T)\big] = A, \tag{5.2}$$

where A is a given constant. An optimal portfolio for this problem is called an *"efficient strategy"* and the pair $(\mathrm{Var}[X^{\phi^*}(T)], A)$ an *"efficient point"*. The set of all efficient points is called the *"efficient frontier"*. See [15] for more information on this point. By the Lagrange multiplier method we are led to study the expression

$$E[(X^{(\phi)}(T)-A)^2-\lambda(X^{(\phi)}(T)-A)] = E\left[\left(X^{(\phi)}(T)-\left(\frac{\lambda}{2}+A\right)\right)^2\right]-\frac{\lambda^2}{4}, \tag{5.3}$$

for given $\lambda \in \mathbb{R}$. Thus the problem is equivalent to minimizing, for a given $a \in \mathbb{R}$,

$$J(\phi) = [(X^{(\phi)}(T)-a)^2], \tag{5.4}$$

over all admissible insider strategies $\phi \in \mathcal{A}$.

In the classical, honest trader, case this problem has been studied by many researchers. See for example [25, 27, 39].

To the best of our knowledge this is the first time this problem is discussed in an insider context. Here we address this problem using both the results of Sec. 4 and the properties of the initial enlargement of filtration. In [7] a feedback representation for the difference process between the honest trader and the insider optimal strategies is provided only in some particular stochastic volatility models, while in more general settings the author is able to compare only one of the terms in the mean-variance hedging backward equation characterizing the two optimal strategies.

For simplicity we assume that the market is simply given by

$$\begin{aligned}
&(\text{bond price}) \quad S_0(t) = 1, \qquad 0 \le t \le T,\\
&(\text{stock price}) \quad S_1(t) = B(t), \quad 0 \le t \le T,
\end{aligned}$$

where $B(t) = B_1(t)$ is 1-dimensional Brownian motion with filtration $\mathcal{F}_t$. We remark that all the following results hold with suitable modification and can be obtained with the same method also when $S_1(t) = \mathcal{E}\left(\int \sigma(s)dB(s)\right)_t = \exp\left(\int_0^t \sigma(s)dB(s) - \frac{1}{2}\int_0^t \sigma^2(s)ds\right)$, for deterministic $\sigma(t)$ such that the stochastic exponential $\mathcal{E}\left(\int \sigma(s)dB(s)\right)_t$ is well defined. The insider filtration is

$$\mathcal{H}_t = \mathcal{F}_t \vee \sigma(B(T_0)), \quad 0 \le t \le T, \tag{5.5}$$

where $T_0 \ge T$ is some given constant. Hence the problem is to find

$$J^*(T) := \inf_{\phi\in\mathcal{A}} J_T(\phi), \tag{5.6}$$

where

$$J_T(\phi) = E\left[\left(a - \int_0^T \phi(s)d^-B(s)\right)^2\right]. \tag{5.7}$$

If $T = T_0$, we see that by using the forward integral, as in Example 3.2, that problem (5.6) has an optimal solution given by

$$\hat{\phi}(t) := \frac{a}{T_0}(B(T_0) - 2B(t)), \quad 0 \le t \le T_0, \tag{5.8}$$

which corresponds to the variance $J^*(T_0) = J_{T_0}(\hat{\phi}) = Var(\int_0^{T_0} \hat{\phi}(u)d^- B(u)) = 0$, since

$$a = \int_0^{T_0} \hat{\phi}(s)d^- B(s). \tag{5.9}$$

For $T < T_0$, we see that the variance associated to $\hat{\phi}(t)$ is given by

$$\begin{aligned} J_T(\hat{\phi}) &= E\left[\left(a - \int_0^T \hat{\phi}(s)d^- B(s)\right)^2\right] \\ &= E\left[\left(\int_T^{T_0} \hat{\phi}(s)d^- B(s)\right)^2\right] = a^2\left(1 - \frac{T}{T_0}\right). \end{aligned} \tag{5.10}$$

Hence we see that the insider can always obtain a smaller variance than the honest trader. Unfortunately we cannot conclude that $\hat{\phi}(t)$ is also optimal for the case $T < T_0$ since by lengthy computations we can show that it does not satisfy Eq. (4.10). Hence we look for the optimal strategy of the insider by using the following method instead.

By a result of Itô [20], $B(t)$ is a semimartingale with respect to $\mathcal{H}_t$. In fact, we can write

$$B(t) = \hat{B}(t) + \int_0^t \alpha(s)ds, \quad 0 \le t \le T, \tag{5.11}$$

where $\hat{B}(t)$ is an $\mathcal{H}_t$-Brownian motion and

$$\alpha(t) = \frac{B(T_0) - B(t)}{T_0 - t}. \tag{5.12}$$

By Lemma 2.2, the forward integral with respect to $B(t)$ coincides with

$$\int_0^t \phi(s)d^- B(s) = \int_0^t \phi(s)d\hat{B}(s) + \int_0^t \phi(s)\alpha(s)ds, \tag{5.13}$$

if $\phi \in \mathcal{A}$. If we rewrite Eq. (5.11) as

$$dB(t) = \frac{-B(t)}{T_0 - t}dt + \frac{B(T_0)}{T_0 - t}dt + d\hat{B}(t), \tag{5.14}$$

we see that dividing by $T_0 - t$ and integrating, we get, if $B(0) = 0$,

$$B(t) = \frac{t}{T_0}B(T_0) + (T_0 - t)\int_0^t \frac{1}{T_0 - s}d\hat{B}(s), \quad 0 \le t \le T. \tag{5.15}$$

Since for the insider the value $y = B(T_0)$ is a "known" quantity, by using (5.13) and (5.15) we reduce problem (5.6) to a classical stochastic control problem for each value of the parameter $y = B(T_0) \in \mathbb{R}$ as follows. Note that

$$
\begin{aligned}
&E\left[\left(a - \int_0^T \phi(t)d^- B(t)\right)^2\right] \\
&\quad = E\left[E\left[\left(a - \int_0^T \phi(t)d^- B(t)\right)^2 \middle| B(T_0)\right]\right] \\
&\quad = E\left[E\left[\left(a - \int_0^T \phi(t)d^- B(t)\right)^2 \middle| B(T_0) = y\right]_{y=B(T_0)}\right] \\
&\quad = E\left[E\left[\left(a - \int_0^T \phi(t)dB^{(y)}(t)\right)^2\right]_{y=B(T_0)}\right],
\end{aligned} \tag{5.16}
$$

where

$$
B^{(y)}(t) = B^{(y)}(0) + \frac{ty}{T_0} + (T_0 - t)\int_0^t \frac{1}{T_0 - s} d\hat{B}(s), \quad 0 \le t \le T. \tag{5.17}
$$

We can solve this by minimizing the inner part of this variance point-wise in y for each $y \in \mathbb{R}$ and then take the expectation when evaluated at $y = B(T_0)$.

More precisely, for each y we let $\mathcal{A}(y)$ be the set of $\mathcal{F}_t$-adapted portfolios $\phi \in \mathcal{A}$ (which are allowed to depend on y) and we study the problem

Problem 5.1. (Mean-variance insider portfolio problem) For each $y \in \mathbb{R}$ find $\phi^* = \phi_y^* \in \mathcal{A}(y)$ and $J_y^*(T)$ such that

$$
J_y^*(T) = \inf_{\phi \in \mathcal{A}(y)} E\left[\left(a - \int_0^T \phi(t)dB^{(y)}(t)\right)^2\right] = E\left[\left(a - \int_0^T \phi^*(t)dB^{(y)}(t)\right)^2\right], \tag{5.18}
$$

where

$$
dB^{(y)}(t) = \left(\frac{y}{T_0} - \int_0^t \frac{1}{T_0 - s} d\hat{B}(s)\right)dt + d\hat{B}(t), \tag{5.19}
$$

where a is a known constant.

Remark 5.1. By the above we conclude that the corresponding solution of the problem (5.6) is

$$
J^*(T) = \inf_{\phi \in \mathcal{A}} J(\phi) = J(\phi^*_{B(T_0)}), \tag{5.20}
$$

where $\phi^*_{B(T_0)} = (\phi_y^*)_{y=B(T_0)}$.

We can make Problem 5.1 Markovian by introducing the system

$$dZ(t) = \begin{pmatrix} dZ_1(t) \\ dZ_2(t) \\ dZ_3(t) \\ dZ_4(t) \end{pmatrix} = \begin{pmatrix} dt \\ dB^{(y)}(t) \\ dM(t) \\ dX(t) \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ \frac{y}{T_0} - M(t) \\ 0 \\ \phi(t)(\frac{y}{T_0} - M(t)) \end{pmatrix} dt + \begin{pmatrix} 0 \\ 1 \\ \frac{1}{T_0 - t} \\ \phi(t) \end{pmatrix} d\hat{B}(t), \tag{5.21}$$

with

$$Z(0) = \begin{pmatrix} s \\ b \\ m \\ x \end{pmatrix} = z, \tag{5.22}$$

where $\phi(t)$ is $\mathcal{F}_t$-adapted (but may depend on y). This transforms Eq. (5.18) into $J_y^*(T) = F_y(0, 0, 0, 0)$, where

$$F_y(z) = F_y(s, b, m, x) = \inf_{\phi \in \mathcal{A}(y)} E^{s,b,m,x} \left[\left(a - \int_0^{T-s} \phi(u) dB^{(y)}(u) \right)^2 \right]. \tag{5.23}$$

Thus we have transformed the original insider mean-variance portfolio problem into a classical stochastic control problem which can be approached by dynamic programming, as we now describe. In the following we will write F in the place of F_y for the sake of simplicity.

The Hamilton–Jacobi–Bellman (HJB) equation for the problems (5.18)–(5.23) is, with $z = (s, b, m, x)$,

$$\begin{aligned} \inf_{\psi \in \mathbb{R}} \Bigg\{ & \frac{\partial F}{\partial t} + \left(\frac{y}{T_0} - m \right) \frac{\partial F}{\partial b} + \psi \left(\frac{y}{T_0} - m \right) \frac{\partial F}{\partial x} + \frac{1}{2} \frac{\partial^2 F}{\partial b^2} + \frac{1}{2} \frac{1}{(T_0 - t)^2} \frac{\partial^2 F}{\partial m^2} \\ & + \frac{1}{2} \psi^2 \frac{\partial^2 F}{\partial x^2} + \frac{1}{T_0 - t} \frac{\partial^2 F}{\partial b \partial m} + \psi \frac{\partial^2 F}{\partial b \partial x} + \frac{\psi}{T_0 - t} \frac{\partial^2 F}{\partial m \partial x} \Bigg\} = 0, \quad t < T, \end{aligned} \tag{5.24}$$

$$F(T, b, m, x) = (a - x)^2. \tag{5.25}$$

It is easy to see that the infimum in (5.24) is attained when

$$\psi = \hat{\psi} = -\left(\frac{\partial^2 F}{\partial x^2} \right)^{-1} \left[\left(\frac{y}{T_0} - m \right) \frac{\partial F}{\partial x} + \frac{\partial^2 F}{\partial b \partial x} + \frac{1}{T_0 - t} \frac{\partial^2 F}{\partial m \partial x} \right], \tag{5.26}$$

if we assume that $\frac{\partial^2 F}{\partial x^2} \neq 0$. Substituted in (5.24) this gives

$$\frac{\partial F}{\partial t} + \left(\frac{y}{T_0} - m\right)\frac{\partial F}{\partial b} + \frac{1}{2}\frac{\partial^2 F}{\partial b^2} + \frac{1}{2}\frac{1}{(T_0-t)^2}\frac{\partial^2 F}{\partial m^2} + \frac{1}{2}\psi^2\frac{\partial^2 F}{\partial x^2} + \frac{1}{T_0-t}\frac{\partial^2 F}{\partial b\partial m}$$
$$-\frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}\right)^{-1}\left[\left(\frac{y}{T_0} - m\right)\frac{\partial F}{\partial x} + \frac{\partial^2 F}{\partial b\partial x} + \frac{1}{T_0-t}\frac{\partial^2 F}{\partial m\partial x}\right]^2 = 0, \quad t < T. \tag{5.27}$$

We try a solution of the form

$$F(t,b,m,x) = (a-x)^2 \exp[\theta(t,b,m)], \tag{5.28}$$

for some function $\theta : \mathbb{R}^3 \to \mathbb{R}$ to be determined. This transforms (5.26) into

$$\psi = \hat{\phi} = (a-x)\left[\frac{y}{T_0} - m + \frac{\partial\theta}{\partial b} + \frac{1}{T_0-t}\frac{\partial\theta}{\partial m}\right], \tag{5.29}$$

and (5.27) becomes

$$\frac{\partial\theta}{\partial t} + \left(\frac{y}{T_0} - m\right)\frac{\partial\theta}{\partial b} + \frac{1}{2}\left[\frac{\partial^2\theta}{\partial b^2} + \left(\frac{\partial\theta}{\partial b}\right)^2\right] + \frac{1}{2}\frac{1}{(T_0-t)^2}\left[\frac{\partial^2\theta}{\partial m^2} + \left(\frac{\partial\theta}{\partial m}\right)^2\right]$$
$$+\frac{1}{T_0-t}\left[\frac{\partial^2\theta}{\partial b\partial m} + \frac{\partial\theta}{\partial b}\frac{\partial\theta}{\partial m}\right] - \left[\frac{y}{T_0} - m + \frac{\partial\theta}{\partial b} + \frac{1}{T_0-t}\frac{\partial\theta}{\partial m}\right]^2 = 0, \quad t < T, \tag{5.30}$$

with the boundary condition [from (5.25)]

$$\theta(T,b,m) = 0. \tag{5.31}$$

Equation (5.30) may be written as

$$\frac{\partial\theta}{\partial t} - \frac{1}{T_0-t}\left(\frac{y}{T_0} - m\right)\frac{\partial\theta}{\partial m} + \frac{1}{2}\frac{\partial^2\theta}{\partial b^2} + \frac{1}{2}\frac{1}{(T_0-t)^2}\frac{\partial^2\theta}{\partial m^2} + \frac{1}{T_0-t}\frac{\partial^2\theta}{\partial b\partial m}$$
$$-\frac{1}{2}\left(\frac{y}{T_0} - m\right)^2 - \frac{1}{2}\left[\frac{y}{T_0} - m + \frac{\partial\theta}{\partial b} + \frac{1}{T_0-t}\frac{\partial\theta}{\partial m}\right]^2 = 0, \quad t < T. \tag{5.32}$$

By using the approach of [35], we note that

$$\min_{v\in\mathbb{R}}\left\{v\left(\frac{y}{T_0} - m + \frac{\partial\theta}{\partial b} + \frac{1}{T_0-t}\frac{\partial\theta}{\partial m}\right) + \frac{1}{2}v^2\right\}$$
$$= -\frac{1}{2}\left[\frac{y}{T_0} - m + \frac{\partial\theta}{\partial b} + \frac{1}{T_0-t}\frac{\partial\theta}{\partial m}\right]^2. \tag{5.33}$$

Therefore equation (5.32) can be written as

$$\min_{v\in\mathbb{R}}\left\{\frac{\partial\theta}{\partial t}-\frac{1}{T_0-t}\left(\frac{y}{T_0}-m\right)\frac{\partial\theta}{\partial m}+\frac{1}{2}\frac{\partial^2\theta}{\partial b^2}+\frac{1}{2}\frac{1}{(T_0-t)^2}\frac{\partial^2\theta}{\partial m^2}+\frac{1}{T_0-t}\frac{\partial^2\theta}{\partial b\partial m}\right.$$
$$\left.-\frac{1}{2}\left(\frac{y}{T_0}-m\right)^2+v\left(\frac{y}{T_0}-m+\frac{\partial\theta}{\partial b}+\frac{1}{T_0-t}\frac{\partial\theta}{\partial m}\right)+\frac{1}{2}v^2\right\}=0,\quad t<T. \tag{5.34}$$

We note that

$$-\frac{1}{2}\left(\frac{y}{T_0}-m\right)^2+v\left(\frac{y}{T_0}-m\right)+\frac{1}{2}v^2=-\left(\frac{y}{T_0}-m\right)^2+\frac{1}{2}\left(\frac{y}{T_0}-m+v\right)^2 \tag{5.35}$$

and rewrite equation (5.34) as

$$\min_{v\in\mathbb{R}}\left\{\frac{\partial\theta}{\partial t}+\frac{1}{T_0-t}\left(m-\frac{y}{T_0}+v\right)\frac{\partial\theta}{\partial m}+v\frac{\partial\theta}{\partial b}+\frac{1}{2}\frac{\partial^2\theta}{\partial b^2}+\frac{1}{2}\frac{1}{(T_0-t)^2}\frac{\partial^2\theta}{\partial m^2}\right.$$
$$\left.+\frac{1}{T_0-t}\frac{\partial^2\theta}{\partial b\partial m}-\left(\frac{y}{T_0}-m\right)^2+\frac{1}{2}\left(\frac{y}{T_0}-m+v\right)^2\right\}=0,\quad t<T. \tag{5.36}$$

We recognize (5.36) as the HJB equation of the following stochastic control problem:

$$\theta(t,b,m)=\lambda(t,m), \tag{5.37}$$

$$\lambda(s,m)=\inf_{u\in\hat{\mathcal{A}}}E^{s,m}\left[\int_0^{T-s}\left\{\frac{1}{2}\left(u(t)+\frac{y}{T_0}-\tilde{m}(t)\right)^2-\left(\frac{y}{T_0}-\tilde{m}(t)\right)^2\right\}dt\right], \tag{5.38}$$

where the state process $\tilde{m}(t)$ is given by

$$d\tilde{m}(t)=\frac{1}{T_0-t}\left(\tilde{m}(t)-\frac{y}{T_0}+u(t)\right)dt+\frac{1}{T_0-t}d\tilde{W}(t),\quad \tilde{m}(0)=m, \tag{5.39}$$

where $\tilde{W}(t)$ is an auxiliary Brownian with filtration $\tilde{\mathcal{F}}_t$ and $\tilde{\mathcal{A}}$ is the set of all $\tilde{\mathcal{F}}_t$-adapted processes $u(t)$ such that (5.39) has a unique solution with $E\big[\int_0^T\tilde{m}(t)^2dt\big]<\infty$. If we consider $v(t)=u(t)-\tilde{m}(t)+\frac{y}{T_0}$ and $\zeta(t)=\tilde{m}(t)-\frac{y}{T_0}$, we simplify problem (5.38) in the following way:

$$\lambda(s,\zeta)=\inf_{v\in\tilde{\mathcal{A}}}E^{s,\zeta}\left[\int_0^{T-s}\left\{\frac{1}{2}v(t)^2-\zeta(t)^2\,dt\right\}\right]. \tag{5.40}$$

This is a linear quadratic control (LQC) problem, hence we can solve it for example by using the approach described in Example 11.2.4 of [31]. The solution of (5.40) is given by

$$\lambda(t,\zeta)=\psi(t)\zeta^2+a(t), \tag{5.41}$$

where $\psi(t)$ is the solution of the following Riccati equation:

$$\psi'(t)=-\frac{4}{T_0-t}\psi(t)+\frac{2}{(T_0-t)^2}\psi(t)^2+1, \tag{5.42}$$

and $a(t)$ is given by:

$$a(t) = \int_t^T \sigma(s)^2 \psi(s) ds = \int_t^T \frac{\psi(s)}{(T_0 - s)^2}\, ds. \tag{5.43}$$

By following the approach of [11], Sec. 3.2, we consider the change of variable

$$\psi(t) = \frac{(T_0 - t)^2}{2} \frac{d}{dt}(\log|\eta(t)|), \tag{5.44}$$

to obtain the following linear second order differential equation (since $t \le T < T_0$)

$$(T_0 - t)^2 \eta''(t) + 2(T_0 - t)\eta'(t) + 2\eta(t) = 0. \tag{5.45}$$

This is an Euler equation, with solution

$$\eta(t) = A_1 (T_0 - t)^{\gamma_1} + A_2 (T_0 - t)^{\gamma_2}, \tag{5.46}$$

where A_1, A_2 are arbitrary constants and $\gamma = \gamma_i$, $i = 1, 2$ solves the equation

$$\gamma(\gamma - 1) - 2\gamma - 2 = 0, \tag{5.47}$$

i.e., $\gamma_1 = 1$ and $\gamma_2 = 2$. Since $\eta(t)$ is only determined up to a multiplicative constant, we may choose $A_2 = \frac{1}{2}$ and put

$$\eta(t) = A_1(T_0 - t) + \frac{1}{2}(T_0 - t)^2. \tag{5.48}$$

Since we have required that $\psi(T) = 0$, by (5.44) we get

$$A_1 = -(T_0 - T), \tag{5.49}$$

and consequently

$$\psi(t) = -\frac{1}{2}(T_0 - t)^2 \frac{-A_1 - (T_0 - t)}{A_1(T_0 - t) + \frac{1}{2}(T_0 - t)^2} = \frac{(T_0 - t)(T - t)}{2T - T_0 - t}. \tag{5.50}$$

The solution $\psi(t)$ presents a singularity if $T > \frac{T_0}{2}$. Hence, we restrict ourselves to the case when $T < \frac{T_0}{2}$. If $T < \frac{T_0}{2}$, we can gather all our results together and get

$$\lambda(t, \zeta) = \psi(t)\zeta^2 + a(t), \tag{5.51}$$

i.e.,

$$\theta(t, b, m) = \psi(t)\left(m - \frac{y}{T_0}\right)^2 + a(t), \tag{5.52}$$

where

$$a(t) = \int_t^T \frac{\psi(s)}{(T_0 - s)^2}\, ds = -\frac{1}{2} \log\left[\frac{(T_0 - T)^2}{(T_0 - t)(T_0 + t - 2T)}\right]. \tag{5.53}$$

By (5.28) the optimal value function for our mean-variance hedging problem is

$$\begin{aligned} F = F_y(t,b,m,x) &= (x-a)^2 \exp[\theta(t,b,m)] \\ &= (x-a)^2 \exp\left[\psi(t)\left(m - \frac{y}{T_0}\right)^2 + a(t)\right]. \end{aligned} \tag{5.54}$$

In feedback form, by (5.29) we see that the optimal strategy for the insider is given by

$$\begin{aligned} \phi_y^* &= (a-x)\left[\frac{y}{T_0} - m + \frac{\partial\theta}{\partial b} + \frac{1}{T_0-t}\frac{\partial\theta}{\partial m}\right] \\ &= (a-x)\left[\frac{y}{T_0} - m + \frac{\psi(t)}{T_0-t}\right]. \end{aligned} \tag{5.55}$$

By Remark 5.1 we obtain that the optimal value function for (5.6) is given by

$$\begin{aligned} J^*(T) = \inf_{y\in\mathcal{A}} J(\phi) &= E\big[F_y(0,0,0,0)_{y=B(T_0)}\big] \\ &= a^2 E\left[\exp\left[-\psi(0)\frac{B^2(T_0)}{T_0^2} + a(0)\right]\right] \\ &= a^2\left(1 - \frac{T}{T_0-T}\right), \end{aligned} \tag{5.56}$$

since $B(T_0)$ is a Gaussian random variable $N(0,T_0)$.

Analogously, if we put $y = B(T_0)$ in (5.55), the optimal strategy for the insider trader is given by (in a feedback form):

$$\begin{aligned} \phi^*(t) &= (a - X^*(t))\left[\frac{B(T_0)}{T_0} - \int_0^t \frac{d\hat{B}(s)}{T_0-s} + \frac{\psi(t)}{T_0-t}\right] \\ &= \frac{a-X^*(t)}{T_0-t}\left[B(T_0) - B(t) + \psi(t)\right], \end{aligned} \tag{5.57}$$

with

$$\begin{aligned} X^*(t) &= \int_0^t \phi^*(u)\left(\frac{B(T_0)}{T_0} - M(u)\right)du + \int_0^t \phi^*(u)d\hat{B}(u) \\ &= \int_0^t \phi^*(u)d^-B(u), \end{aligned} \tag{5.58}$$

where the last equality follows by (5.14). One can check that a process $\phi^*(t)$ that solves (5.57) is admissible. We summarize our results in the following

Theorem 5.1. *Consider the initial enlargement of filtration $\mathcal{H}_t = \mathcal{F}_t \vee \sigma(B(T_0))$. Then for problem* (5.6) *the following holds*:

1. *For $T = T_0$, there exists an optimal portfolio $\phi^*(t)$ given by*

$$\phi^*(t) = \hat{\phi}(t) = \frac{a}{T_0}(B(T_0) - 2B(t)), \quad 0 \le t \le T_0, \tag{5.59}$$

which corresponds to the minimal variance $J^*(T_0) = J_{T_0}(\phi^*) = 0$.

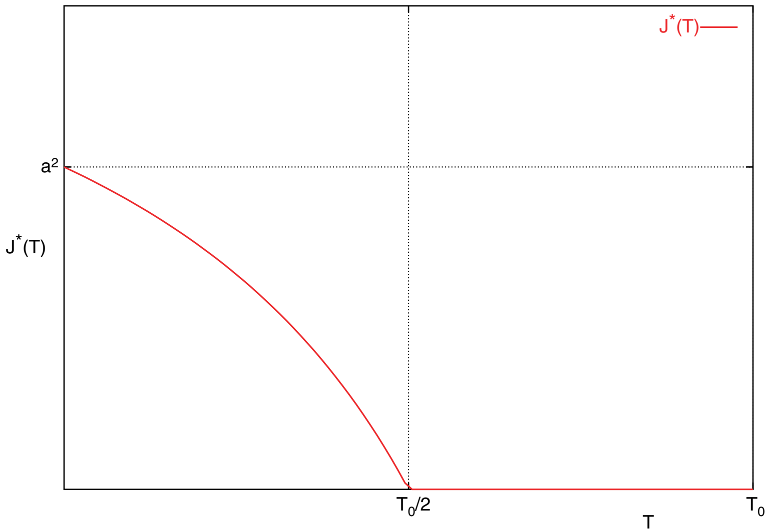

Fig. 1. The value function $J^*(T)$.

2. *For* $0 < T < \frac{T_0}{2}$, *there exists an optimal portfolio* $\phi^*(t)$ *given in feedback form by*

$$\phi^*(t) = \frac{a - X^*(t)}{T_0 - t}[B(T_0) - B(t) + \psi(t)], \tag{5.60}$$

where $X^*(t) = \int_0^t \phi^*(u)d^- B(u)$ *and* $\psi(t) = \frac{(T_0-t)(T-t)}{2T-T_0-t}$. *This corresponds to the minimal variance*

$$\begin{aligned} J^*(T) = J_T(\phi^*) &= \inf_{y\in\mathcal{A}} J(\phi) = E\big[F_y(0,0,0,0)_{y=B(T_0)}\big] \\ &= a^2\left(1 - \frac{T}{T_0 - T}\right). \end{aligned} \tag{5.61}$$

3. *For* $\frac{T_0}{2} \le T < T_0$, *we have*

$$J^*(T) = J_T(\phi^*) = \inf_{y\in\mathcal{A}} J(\phi) = 0, \tag{5.62}$$

but an optimal portfolio ϕ^* *does not exist.*

For an illustration of these cases, see Fig. 1.

Proof. We have already proved statements 1 and 2. It remains to prove 3. First note that by (5.61) we have

$$\lim_{S\to\frac{T_0}{2}^-} J^*(S) = 0. \tag{5.63}$$

Next, note that with T_0 fixed the function $T \to J^*(T)$ is decreasing, since the space of admissible strategies for the interval $[0,T]$ increases with T. Hence we have

$$J^*(T) \le J^*(S), \quad \text{for all } S < \frac{T_0}{2} \le T \le T_0, \tag{5.64}$$

and therefore

$$J^*(T) \leq \lim_{S \to \frac{T_0}{2}^-} J^*(S) = 0 \quad \text{for } \frac{T_0}{2} \leq T \leq T_0, \tag{5.65}$$

as claimed.

Finally, we claim that if $T < T_0$ then there does not exist $\phi_y^* \in \mathcal{A}(y)$ such that

$$\int_0^T \phi_y^*(t)dB(t) = a \quad \text{a.s. for all } y \in \mathbb{R}. \tag{5.66}$$

To see this, assume that (5.66) holds. Then by (5.9) we have, with $\hat{\phi}_y(t) = \frac{a}{T_0}(y - 2B(t))$,

$$\int_0^{T_0} \hat{\phi}_y(t)dB(t) = a = \int_0^T \phi_y^*(t)dB(t). \tag{5.67}$$

Hence

$$\int_0^T (\phi_y^*(t) - \hat{\phi}_y(t))dB(t) = \int_T^{T_0} \hat{\phi}_y(t)dB(t), \tag{5.68}$$

which implies that

$$\begin{aligned}\int_0^T (\phi_y^*(t) - \hat{\phi}_y(t))dB(t) &= E\left[\int_0^T (\phi_y^*(t) - \hat{\phi}_y(t))dB(t)|\mathcal{F}_T\right] \\ &= E\left[\int_T^{T_0} \hat{\phi}_y(t)dB(t)|\mathcal{F}_T\right] = 0.\end{aligned}$$

But then

$$\int_0^T \phi_y^*(t)dB(t) = \int_0^T \frac{a}{T_0}(y - 2B(t))dB(t) = \frac{a}{T_0}[B(T)(y - B(T)) + T] \neq a. \tag{5.69}$$

This contradiction shows that (5.66) cannot hold. This completes the proof of Theorem 5.1. □

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References

[1] J. Amendinger, D. Becherer and M. Schweizer, A monetary value for initial information in portfolio optimization, *Finance and Stochastics* **7** (2003) 29–46.
[2] J. Amendinger, P. Imkeller and M. Schweizer, Additional logarithmic utility of an insider, *Stochastic Processes and Their Applications* **75** (1998) 263–286.
[3] J. Asch and J. Potthoff, Itô lemma without non-anticipatory conditions, *Probability Theory and Related Fields* **88** (1991) 17–46.

[4] M. A. Berger and V. J. Mizel, An extension of the stochastic integral, *Annals of Probability* **10** (1982) 235–450.

[5] F. Biagini, P. Guasoni and M. Pratelli, Mean-variance hedging for stochastic volatility models, *Mathematical Finance* **10** (2000) 109–123.

[6] F. Biagini and B. Øksendal, A general stochastic calculus approach to insider trading, *Applied Mathematics and Optimization* **52** (2005) 167–181.

[7] L. Campi, Some results on quadratic hedging with insider trading, *Stochastics* **77** (2005) 327–348.

[8] J. M. Corcuera, P. Imkeller, A. Kohatsu-Higa and D. Nualart, Additional utility of insiders with imperfect dynamical information, *Finance Stochast.* **8** (2004) 437–450.

[9] G. Di Nunno, T. Meyer-Brandis, B. Øksendal and F. Proske, Optimal portfolio for an insider in a market driven by Lévy processes, *Quantitative Finance* **6** (2006) 83–94.

[10] G. Di Nunno, A. Kohatsu-Higa, T. Meyer-Brandis, B. Øksendal, F. Proske and A. Sulem, Optimal portfolio for a "large" insider in a market driven by Lévy processes, Preprint Series in Pure Mathematics 30, University of Oslo (2005).

[11] H. T. Davis, Introduction to Nonlinear Differential and Integral Equations, US Atomic Energy Commission, 1960.

[12] A. Grorud and M. Pontier, Insider trading in a continuous time market model, *International Journal of Theoretical and Applied Finance* **1** (1998) 331–347.

[13] A. Grorud and M. Pontier, Asymmetrical information and incomplete market, *International Journal of Theoretical and Applied Finance* **4** (2001) 285–302.

[14] Y. Hu and B. Øksendal, Optimal smooth portfolio selection for an insider, Preprint Series in Pure Mathematics 12, University of Oslo, 2003.

[15] C. F. Huang and R. Litzenberger, Foundations for Financial Economics, North-Holland, 1988.

[16] P. Imkeller, Enlargement of the Wiener filtration by an absolutely continuous random variable via Malliavin calculus, *Prob. Theory Relat. Fields* **106** (1996) 105–135.

[17] P. Imkeller, Random times at which insiders can have free lunches, *Stoch. Stoch. Rep.* **74** (2002) 465–487.

[18] P. Imkeller, Malliavin's calculus in insider models: Additional utility and free lunches *Mathematical Finance* **13** (2003) 153–169.

[19] P. Imkeller, M. Pontier and F. Weiss, Free lunch and arbitrage possibilities in a financial market model with an insider, *Stochastic Processes and Their Applications* **92** (2001) 103–130.

[20] K. Itô, Extension of stochastic integrals, in *Proceedings of International Symposium on Stochastic Differential Equations*, Wiley, 1978, pp. 95–109.

[21] T. Jeulin and M. Yor (eds.), Grossissiments de Filtrations: Exemples et applications, Lectures Notes in Mathematics **1118** (1985), Springer-Verlag.

[22] A. Kohatsu-Higa and A. Sulem, Utility maximization in an insider influenced market, *Mathematical Finance* **16** (2006) 153–179.

[23] A. Kohatsu-Higa and A. Sulem, A large trader-insider model, in *Stochastic Processes and Applications to Mathematical Finance — Proceedings of the 5th Ritsumeikan International Symposium* (World Scientific, 2006).

[24] H. H. Kuo and A. Russek, White noise approach to stochastic integration, *Journal of Multivariate Analysis* **24** (1988) 218–236.

[25] J. P. Laurent and H. Pham, Dynamic programming and mean-variance hedging *Finance and Stochastics* **3** (1999) 83–110.

[26] J. A. León, R. Navarro and D. Nualart, An anticipating calculus approach to the utility maximisation of an insider, *Mathematical Finance* **13** (2003) 171–185.

[27] X. Li, E. B. Lim and Y. Zhou, Dynamic mean-variance portfolio selection with no-shorting constraints, *SIAM Journal of Control and Optimization* **40** (2001) 1540–1555.

[28] D. Nualart, *The Malliavin Calculus and Related Topics* (Springer-Verlag, 1995).

[29] D. Nualart and E. Pardoux, Stochastic calculus with anticipating integrands, *Prob. Th. Rel. Fields* **78** (1988) 535–581.

[30] B. Øksendal, An Introduction to Malliavin Calculus with Applications to Economics, Lecture Notes, Norwegian School of Economics and Business Administration, 1996.

[31] B. Øksendal, *Stochastic Differential Equations*, 6th Edn. (Springer-Verlag, 2003).

[32] B. Øksendal, A universal optimal consumption rate for an insider, *Mathematical Finance* **16** (2006) 119–129.

[33] B. Øksendal and A. Sulem, Partial observation control in an anticipating environment, *Russian Math. Surveys* **50** (2004) 355–375.

[34] H. Pham, T. Rheinländer and M. Schweizer, Mean-variance hedging for continuous processes: New proofs and examples, *Finance and Stochastics* **2** (1998) 173–198.

[35] R. Rishel, Optimal portfolio management with partial observations and power utility function. *Stochastic Analysis, Control, Optimization and Applications, Systems Control Found. Appl.*, Birkhäuser Boston, 1999, pp. 605–619.

[36] F. Russo and P. Vallois, Forward, backward and symmetric stochastic integration, *Prob. Th. Rel. Fields* **97** (1993) 403–421.

[37] F. Russo and P. Vallois, The generalised covariation process and Itô formula, *Stochastic Processes and Their Applications* **59** (1995) 81–104.

[38] F. Russo and P. Vallois, Stochastic calculus with respect to continuous finite quadratic variation processes, *Stochastics and Stochastics Reports* **70** (2000) 1–40.

[39] M. Schweizer, A guided tour through quadratic hedging approaches, in E. Jouini, J. Cvitanic and M. Musiela, eds. *Option Pricing, Interest Rates and Risk Management* (Cambridge University Press, Cambridge, UK, 2001), pp. 538–574.