THE DENSITY PROCESS OF THE MINIMAL ENTROPY MARTINGALE MEASURE IN A STOCHASTIC VOLATILITY MODEL WITH JUMPS WITH APPLICATIONS TO OPTION PRICING

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ABSTRACT. We derive the density process of the minimal entropy martingale measure in the stochastic volatility model proposed by Barndorff-Nielsen and Shephard [3]. The density is represented by the logarithm of the value function for an investor with exponential utility and no claim issued, and a Feynman-Kac representation of this function is provided. The dynamics of the processes determining the price and volatility are explicitly given under the minimal entropy martingale measure, and we derive a Black & Scholes equation with integral term for the price dynamics of derivatives. It turns out that the price is the solution of a coupled system of two integro-partial differential equations.

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1. Introduction

In this paper we derive the density process of the minimal entropy martingale measure in a Black & Scholes market with a stochastic volatility model given by Barndorff-Nielsen and Shephard [3]. We apply our results to find the minimal entropy price of derivatives in this market, and present a system of integro-partial differential equations (integro-PDEs) that determines the price. The knowledge of the density process also enables us to describe the price dynamics of the market under the minimal entropy measure.

Barndorff-Nielsen and Shephard [3] propose a geometric Brownian motion where the squared volatility is modeled by a non-Gaussian Ornstein-Uhlenbeck process as the price dynamics for a financial asset. In their model, the volatility level will revert toward zero, with random upward shifts given by the jumps of a subordinator process (being an increasing Lévy process). Due to the stochastic volatility, this asset price model leads to an incomplete market, and the arbitrage theory does not provide a unique price for derivatives written on the asset due to the existence of

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a continuum of equivalent martingale measures. Thus, the risk preferences of the market participants need to be included in the price formation of the derivatives.

Utility indifference pricing (see Hodges and Neuberger [14]) gives an alternative to the arbitrage theory to derive the fair premium of derivatives in incomplete markets. One considers an investor trying to maximize his exponential utility by either entering into the market by his own account, or issuing a derivative and investing his incremental wealth after collecting the premium. The indifference price of the claim is then defined as the premium for which the investor becomes indifferent between the two investment alternatives. It is well-known (see e.g. Frittelli [11], Rouge and El-Karoui [20] and Delbaen *et al.* [9]) that the zero risk aversion limit of the indifference price corresponds to the minimal entropy martingale measure price. The zero risk aversion limit is of particular interest, since this is the only price for which the buyer and seller agree on the indifference price.

We state the density of the minimal entropy martingale measure by appealing to general results by Grandits and Rheinländer [12] and verification results by Rheinländer [19]. The density process is introduced via a function H which is related to the solution of the portfolio optimization problem of the investor having an exponential utility function and not issuing any claim. In fact, it arises from the logarithmic transform of the value function in a similar fashion as demonstrated in Musiela and Zariphopoulou [17]. This function is represented as an expectation of the exponential of a ratio between the drift and squared volatility, and shown to be the Feynman-Kac solution of an integro-PDE. It provides us with the scaling of the jumps when considering the minimal entropy dynamics of the stochastic volatility model.

We apply our results to find the minimal entropy price of a class of claims with payoff given by a function of the underlying at maturity of the contract. The price is written as an expected value of the payoff, where we have complete knowledge of the dynamics of the asset and volatility processes. Furthermore, we state the integro-PDE for the pricing equation, which will become a Black & Scholes partial differential equation with an additional integral term arising from the stochastic volatility. This integral term will also include the function H, and thus to solve it we need to consider a coupled system of two integro-PDEs. Related papers studying the minimal entropy martingale measure for stochastic volatility markets are Hobson [13], Becherer [4, 5] and Benth and Karlsen [7].

The paper is organized as follows: In the next section we define our financial market, and in Section 3 we study the density of the minimal entropy martingale measure. Section 4 is devoted to the density process and the analysis of the function H, while in Section 5 we apply our results to the minimal entropy pricing of claims.

2. The Market

Given a probability space $(\Omega, \mathcal{F}, P)$ and a time horizon T, consider a financial market consisting of a bond and a risky asset with prices at time $0 \leq t \leq T$ denoted by R_t and S_t, respectively. Assume without loss of generality that the bond yields a risk-free rate of return equal to zero, i.e.,

$$dR_t = 0, \tag{2.1}$$

together with the convention that $R_0 = 1$. In this paper we will consider the stochastic volatility model introduced by Barndorff-Nielsen and Shepard [3], but let us mention that our results can be achieved for more general stochastic volatility models (under appropriate integrability conditions) as long as the volatility driving process L_t is independent from B_t (see (2.2) and (2.3) below). In the Barndorff-Nielsen and Shepard model the price of the risky asset is evolving according to the following dynamics

$$dS_t = \alpha(Y_t)S_t dt + \sigma(Y_t)S_t dB_t, \quad S_0 = s > 0 \tag{2.2}$$

$$dY_t = -\lambda Y_t dt + dL_{\lambda t}, \quad Y_0 = y > 0, \tag{2.3}$$

where B_t is a Brownian motion and L_t a pure jump subordinator (that is, an increasing pure jump Lévy process) with Poisson random measure denoted by $N(dt, dz)$. In this paper we assume B_t and L_t to be independent. The Lévy measure $\nu(dz)$ of L_t satisfies $\int_0^\infty \min(1, z)\, \nu(dz) < \infty$. Further, we denote by $\{\mathcal{F}_t\}_{t \geq 0}$ the completion of the filtration $\sigma(B_s, L_{\lambda s}; s \leq t)$ generated by the Brownian motion and the subordinator such that $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ becomes a complete filtered probability space. In this paper we will assume the following specification of the parameter functions α and σ:

$$\alpha(y) = \mu + \beta y, \quad \sigma(y) = \sqrt{y}, \tag{2.4}$$

with μ and β being constants.

The process Y_t models the squared volatility, and will be an Ornstein-Uhlenbeck process reverting toward zero, and having positive jumps given by the subordinator. An explicit representation of the squared volatility is

$$Y_t = y \exp\left(-\lambda t\right) + \int_0^t \exp\left(-\lambda(t-u)\right)\, dL_{\lambda u}. \tag{2.5}$$

The scaling of time by λ in the subordinator is to decouple the modeling of the marginal distribution of the (log)returns of S and their autocorrelation structure. We note that in [3] it is proposed to use a superposition of processes Y_t with different speeds of mean-reversion. However, in this paper we will stick to only one process Y_t, but remark that there are no essential difficulties in generalizing to the case of a superposition of Y's. The modeling idea is to specify a stationary distribution of Y that implies (at least approximately) a desirable distribution for the returns of S. Given this stationary distribution, one needs to derive a subordinator L. In [3], several examples of such distributions and their associated subordinators are given in the context of financial applications.

We denote by $\psi(\theta)$ the cumulant function of L_t, which is defined as the logarithm of the characteristic function

$$\psi(\theta) = \ln \mathbb{E}\left[\exp\left(i\theta L_1\right)\right], \quad \theta \in \mathbb{R}. \tag{2.6}$$

From the Lévy-Kintchine Formula we have

$$\psi(\theta) = \int_0^\infty \left\{ e^{i\theta z} - 1 \right\} \nu(dz). \tag{2.7}$$

We suppose that the Lévy measure satisfies an exponential integrability condition, that is, there exists a constant $k > 0$ such that

$$\int_1^\infty e^{kz}\, \nu(dz) < \infty. \tag{2.8}$$

Later we will be more precise about the size of k (see Prop. 3.2 and the examples following), and relate it to parameters in the specification of the Lévy measure. Under condition (2.8), the moment generating function is defined for all $|\theta| \leq k$, and

$$\mathbb{E}\left[\exp\left(\theta L_1\right)\right] = \exp\left(\phi(\theta)\right) \tag{2.9}$$

with

$$\phi(\theta) = \psi(-i\theta). \tag{2.10}$$

Note that $L_{\lambda t}$ is also a subordinator, and the cumulant function of this is $\lambda\psi(\theta)$. The process L_t has the decomposition

$$L_t = \int_0^t \int_0^\infty z\ \nu(dz)dt + \int_0^t \int_0^\infty z(N(dz,dt) - \nu(dz)dt), \tag{2.11}$$

where the second integral on the right-hand side is a martingale. The reader is referred to [1], [8], [18] and [22] for more information about Lévy processes and subordinators.

3. The Minimal Entropy Martingale Measure

In this section we derive the density of the minimal entropy martingale measure of the model (2.2)-(2.3), which we denote by Q_{ME}. In Grandits and Rheinländer [12] the density of Q_{ME} for general stochastic volatility models driven by an independent noise process is shown to be of the form

$$\begin{aligned} Z_T &= c \cdot \exp\left(- \int_0^T \frac{\alpha(Y_t)}{\sigma^2(Y_t)} S_t^{-1} dS_t \right) \\ &= c \cdot \exp\left(- \int_0^T \frac{\alpha(Y_t)}{\sigma(Y_t)} dB_t - \int_0^T \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)} dt \right). \end{aligned} \tag{3.1}$$

Here, c is the normalizing constant given by

$$c^{-1} = \mathbb{E}\left[\exp\left(- \int_0^T \frac{\alpha^2(Y_t)}{2\sigma^2(Y_t)} dt \right) \right].$$

However, their boundedness assumptions on α and σ are not covering our assumptions in model (2.2)-(2.3). Thus, we give a short proof that Z_T in (3.1) is indeed the density we are looking for by appealing to the sufficient conditions developed by Rheinländer [19].

Proposition 3.1. *Suppose we have*

$$\mathbb{E}\left[\exp\left(\int_0^T \frac{\alpha^2(Y_s)}{\sigma^2(Y_s)}ds\right)\right] < \infty. \tag{3.2}$$

Then Z_T *as defined in* (3.1) *is the density of the minimal entropy martingale measure* Q_{ME}.

Proof. Referring to the results in [19], it is enough to verify the following three statements:

i): The expectation $E[Z_T]$ is equal to one.
ii): The measure induced by Z_T, denoted by Q_{ME}, has finite entropy.
iii): We have

$$\int_0^T \left(\frac{\alpha(Y_t)}{\sigma^2(Y_t)}S_t^{-1}\right)^2 d[S]_t \in L_{exp}(P), \tag{3.3}$$

where $[S]_t$ is the quadratic variation process of S_t and $L_{exp}(P)$ is the Orlicz space generated by the Young function $\exp(\cdot)$.

i): Define

$$Z_T' = \exp\left(-\int_0^T \frac{\alpha(Y_t)}{\sigma(Y_t)}dB_t - \int_0^T \frac{1}{2}\frac{\alpha^2(Y_t)}{\sigma^2(Y_t)}dt\right). \tag{3.4}$$

Then by assumption (3.2) and the Novikov condition, we know that Z_t' is a true martingale. We denote its corresponding probability measure by Q' and note that Y_t has the same dynamics under P and Q'. Hence, we get

(3.5)
$$\mathbb{E}[Z_T] = c\mathbb{E}\left[Z_T' \exp\left(-\int_0^T \frac{\alpha^2(Y_t)}{2\sigma^2(Y_t)}dt\right)\right] = c\mathbb{E}_{Q'}\left[\exp\left(-\int_0^T \frac{\alpha^2(Y_t)}{2\sigma^2(Y_t)}dt\right)\right] = 1.$$

ii): Using the same arguments as in *i)*, we see that

(3.6)
$$\begin{aligned}\mathbb{E}\left[Z_T |\ln Z_T|\right] &= \mathbb{E}_{Q'}\left[\exp\left(-\int_0^T \frac{\alpha^2(Y_t)}{2\sigma^2(Y_t)}dt\right) \left| \left(\int_0^T \frac{\alpha(Y_t)}{\sigma(Y_t)}dB_t + \int_0^T \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)}dt\right) \right|\right] \\ &= \mathbb{E}_{Q'}\left[\exp\left(-\int_0^T \frac{\alpha^2(Y_t)}{2\sigma^2(Y_t)}dt\right) \left| \int_0^T \frac{\alpha(Y_t)}{\sigma(Y_t)}d\widetilde{B}_t \right|\right] < \infty,\end{aligned}$$

where $\widetilde{B}_t$ is the Brownian motion under Q'.

iii) Since we have

$$\exp\left(\int_0^T \left(\frac{\alpha(Y_t)}{\sigma^2(Y_t)} S_t^{-1}\right)^2 d[S]_t\right) = \exp\left(\int_0^T \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)} dt\right),$$

assumption (3.2) implies condition (3.3). □

The following Proposition gives a sufficient condition for assumption (3.2) stated in terms of the Lévy measure of L_1. Moreover, it determines an exact constant k in the exponential integrability condition (2.8):

Proposition 3.2. *If*

$$\int_0^\infty \left\{\exp\left(\frac{\beta^2}{\lambda}\left(1-\exp(-\lambda T)\right) z\right) - 1\right\} \nu(dz) < \infty, \tag{3.7}$$

then Z_T *defined in* (3.1) *is the density of the minimal entropy martingale measure* Q_{ME}.

Proof. Since $Y_t \geq y \exp(-\lambda T)$, we have

$$\frac{\alpha^2(Y_t)}{\sigma^2(Y_t)} = \frac{\mu^2}{Y_t} + 2\mu\beta + \beta^2 Y_t \leq C + \beta^2 Y_t$$

for a positive constant C. But this gives

$$\begin{aligned} \mathbb{E}\left[\exp\left(\int_0^T \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)}\, dt\right)\right] &\leq C'\mathbb{E}\left[\exp\left(\beta^2 \int_0^T Y_t\, dt\right)\right] \\ &= C'\mathbb{E}\left[\exp\left(\frac{\beta^2}{\lambda}\left(y(1-\exp(-\lambda T)) + \int_0^T \left(1-\exp(-\lambda(T-t))\right)\, dL_{\lambda t}\right)\right)\right] \\ &= C'' \exp\left(\lambda \int_0^T \int_0^\infty \left(\exp(f(t)z) - 1\right)\, \nu(dz)\, dt\right), \end{aligned} \tag{3.8}$$

where C', C'' are positive constants, and $f(t) = \beta^2\left(1-\exp(-\lambda(T-t)\right)/\lambda$. □

Let us consider some examples of the process L_t that are relevant in finance, and state sufficient conditions for the density of the minimal entropy martingale measure. If we choose the stationary distribution of Y_t to be an inverse Gaussian law with parameters δ and γ, that is $Y_t \sim \mathrm{IG}(\delta, \gamma)$, the Lévy measure of L becomes

$$\nu(dz) = \frac{\delta}{2\sqrt{2\pi}} z^{-3/2}(1+\gamma z)\exp\left(-\frac{1}{2}\gamma z\right)\, dz.$$

Hence, the exponential integrability condition in Prop. 3.2 is satisfied whenever

$$\beta^2\left(1-\exp(-\lambda T)\right) < \frac{1}{2}\gamma\lambda.$$

When $Y_t \sim IG(\delta, \gamma)$, the log-returns of S_t will be approximately normal inverse Gaussian distributed, a family of laws that has been successfully fitted to log-returns of stock prices (see e.g., Barndorff-Nielsen [2] and Rydberg [21]).

Another popular distribution in finance is the variance gamma law (see Madan and Seneta [16]). If the stationary distribution of Y_t is a gamma law with parameters δ and α, that is $Y_t \sim \Gamma(\delta, \alpha)$, the marginal distribution of the log-returns of S_t is approximately following a variance gamma law. The Lévy measure of L becomes

$$\nu(dz) = \delta\alpha \exp(-\alpha z)\, dz,$$

for which the integrability condition in Prop. 3.2 is satisfied whenever

$$\beta^2 (1 - \exp(-\lambda T)) < \alpha\lambda.$$

Note that the case $\beta = \frac{1}{2}$ corresponds to symmetrically distributed log-returns. When the log-returns are symmetric, it is sufficient for the integrability condition in Prop. 3.2 that $\gamma\lambda > 1/2$ (inverse Gaussian law) or $\alpha\lambda > 1/2$ (the variance gamma law).

4. The Density Process

Section 3 determines the minimal entropy martingale measure Q_{ME} for the model (2.2)-(2.3). It is of interest (for example for pricing of derivatives) to know the dynamics of the processes S_t and Y_t under Q_{ME}. In this Section we identify the density process of Q_{ME} as a certain stochastic exponential, which then by means of the Girsanov theorem gives us the dynamics of S_t and Y_t under Q_{ME}.

A key ingredient in the description of the density process of the minimal entropy martingale measure is the function $H(t, y)$ defined as follows:

$$H(t,y) = \mathbb{E}\left[\exp\left(-\frac{1}{2}\int_t^T \frac{\alpha^2(Y_u)}{\sigma^2(Y_u)}\,du\right)\Big|Y_t = y\right], \qquad (t,y) \in [0,T] \times \mathbb{R}_+, \tag{4.1}$$

where $\mathbb{R}_+ = (0, \infty)$. We remark that our motivation for considering the function H comes from portfolio optimization with an exponential utility function. It turns out that the difference between the value function of the utility maximization problem and the utility function itself can be represented as H. This can be seen by, e.g., considering the Hamilton-Jacobi-Bellman of the stochastic control problem. We refer to Musiela and Zariphopoulou [17] for more on this in a different market context than ours.

Using the time-homogeneity of the Lévy process, we can rewrite $H(t, y)$ as

$$H(t,y) = \mathbb{E}\left[\exp\left(-\frac{1}{2}\int_0^{T-t} \frac{\alpha^2(Y_u)}{\sigma^2(Y_u)}\,du\right)\Big|Y_0 = y\right], \qquad (t,y) \in [0,T] \times \mathbb{R}_+. \tag{4.2}$$

This function will describe the change of the jump measure of the Lévy process under the minimal entropy martingale measure. However, before considering this in more detail, we study some simple but useful properties of $H(t, y)$:

The function satisfies the following bounds:

Lemma 4.1. *For all* $(t, y) \in [0, T] \times \mathbb{R}_+$ *it holds that*

$$\exp\left(a(t)y^{-1} + b(t)y + c(t)\right) \le H(t,y) \le 1, \tag{4.3}$$

where

$$a(t) = -\frac{\mu}{2\lambda}\left(\exp(\lambda(T-t)) - 1\right),$$
$$b(t) = -\frac{\beta^2}{2\lambda}\left(1 - \exp(-\lambda(T-t))\right),$$
$$c(t) = -\mu\beta(T-t) + \lambda\int_t^T \phi\left(b(u)\right)\, du.$$

Proof. The upper bound of 1 is clear (which is reached for $t = T$). Denote

$$g(y) := -\frac{1}{2}\frac{\alpha^2(y)}{\sigma^2(y)} = -\frac{1}{2}\left(\frac{\mu^2}{y} + 2\mu\beta + \beta^2 y\right). \tag{4.4}$$

Using the explicit representation of Y_u in (2.5), its lower bound $Y_u \geq y\exp(-\lambda(u-t))$ and the fact that

$$-\lambda\int_t^T Y_u\, du = Y_T - Y_t - (L_{\lambda T} - L_{\lambda t}),$$

it is straightforward to derive

$$\begin{aligned} H(t,y) \geq \exp\big(a(t)y^{-1} + b(t)y - \mu\beta(T-t)\big) \\ \times\, \mathbb{E}\left[\exp\left(-\frac{\beta^2}{2\lambda}\int_t^T \left(1 - \exp(-\lambda(T-u))\right) dL_{\lambda u}\right)\right]. \end{aligned} \tag{4.5}$$

The lower bound follows. □

Later we shall make explicit use of the differentiability of $H(t,y)$ with respect to t and y, proven in the following proposition:

Proposition 4.2. *The function $H(t,y)$ is continuously differentiable in t and y, i.e. $H \in C^{1,1}\left([0,T]\times\mathbb{R}_+\right)$.*

Proof. The random variable

$$X_{t,T} := \exp\left(\int_0^{T-t} g(Y_u)\, du\right)$$

with $g(y)$ as in (4.4) and $Y_0 = y$, is obviously differentiable with respect to y. The derivative is given by

$$\frac{1}{2}\int_0^{T-t}\left(\frac{\mu^2}{Y_u^2} - \beta^2\right)\mathrm{e}^{-\lambda u}\, du \cdot X_{t,T}$$

which can be bounded by $C + C'/y^2$ for two positive constants C, C' after appealing to the inequality $Y_u \geq y\exp(-\lambda u)$. Hence, the dominated convergence theorem implies that $H(t,y)$ is differentiable with respect to y. Moreover, by similar arguments we find that $\partial H/\partial y$ is continuous, which proves the first part of the Proposition.

Concerning the differentiation with respect to t, we rewrite $H(t, y)$ as

$$H(t,y) = \mathbb{E}\left[\int_0^{T-t} \exp\left(\int_0^{T-s} g(Y_u)du\right) g(Y_s)ds\right] + 1$$

$$(4.6) \qquad = \int_0^{T-t} \mathbb{E}\left[\exp\left(\int_0^{T-s} g(Y_u)du\right) g(Y_s)\right] ds + 1,$$

where we have appealed to the Fubini-Tonelli theorem together with the exponential integrability conditions on L to interchange integration and expectation. Because Y_u is a Lévy diffusion and the compensating measure of a Lévy process is diffuse with respect to time, the integrand in (4.6) is continuous in s. Hence $H(t, y)$ is continuously differentiable in t.

□

We get from the theory of Markov processes that $H(t, y)$ is the Feynman-Kac representation of the solution of the following integro-PDE:

$$(4.7) \qquad \frac{\partial H}{\partial t} - \frac{\alpha^2(y)}{2\sigma^2(y)} H + \mathcal{L}_Y H = 0, \qquad (t, y) \in [0, T) \times \mathbb{R}_+,$$

with terminal data

$$(4.8) \qquad H(T, y) = 1, \qquad y \in \mathbb{R}_+.$$

Here

$$(4.9) \qquad \mathcal{L}_Y H = -\lambda y \frac{\partial H}{\partial y} + \lambda \int_0^\infty \{H(t, y+z) - H(t, y)\} \nu(dz).$$

The function H in (4.1) solving (4.7)-(4.8) plays a crucial role in the derivation of the density of the minimal entropy martingale measure. In general, (4.1) is rather difficult to calculate explicitly. However, if we consider the special case $\alpha(y) = \beta y$, i.e $\mu = 0$ in (2.2), a direct calculation using the moment generating function of L_1 gives the following explicit solution of the integro-PDE (4.7)-(4.8):

Corollary 4.3. *Suppose $\alpha(y) = \beta y$. Then the solution of* (4.7)-(4.8) *is given as*

$$(4.10) \qquad H(t, y) = \exp\left(b(t)y + c(t)\right),$$

where b and c are defined as

$$b(t) = -\frac{\beta^2}{2\lambda}\left(1 - \exp(-\lambda(T-t))\right), \quad c(t) = \lambda \int_t^T \phi\left(b(u)\right) du,$$

We recall that ϕ is the log moment generating function of L_1 defined in (2.10).

Setting $\mu = 0$ in (2.2) corresponds to an expected log-return of $(\beta - \frac{1}{2})y$ of the risky asset S_t. If we, for instance, specify the stationary distribution of Y to be inverse Gaussian, then the log-returns will be approximately normal inverse Gaussian distributed (see Barndorff-Nielsen and Shephard [3]), and choosing this to be symmetric corresponds to $\beta = \frac{1}{2}$, that is, with $\mu = 0$ we have zero expected log-return.

Now we introduce the notation

$$\delta(y,z,t) := \frac{H(t,y+z)}{H(t,y)} \tag{4.11}$$

and define the following stochastic exponentials

$$Z'_t := \exp\left(-\int_0^t \frac{\alpha(Y_s)}{\sigma(Y_s)}dB_s - \int_0^t \frac{1}{2}\frac{\alpha^2(Y_s)}{\sigma^2(Y_s)}ds\right) \tag{4.12}$$

(4.13)

$$Z''_t = \exp\left(\int_0^t\int_0^\infty \ln\delta(Y_s,z,s)N(dz,ds) + \int_0^t\int_0^\infty (1-\delta(Y_s,z,s))\,\nu(dz)ds\right).$$

We identify the density process in question as follows.

Theorem 4.4. *Suppose condition* (3.2) *is fulfilled. Then*

$$Z_t := Z'_t Z''_t$$

is the density process of the minimal entropy martingale measure Q_{ME}.

Proof. We want to show that

$$Z'_T Z''_T = c\cdot\exp\left(-\int_0^T \frac{\alpha(Y_t)}{\sigma(Y_t)}dB_t - \int_0^T \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)}dt\right),$$

where the right hand side of the above equation is the density in (3.1). Since we have

$$\frac{dS_t}{S_t} = \alpha(Y_t)dt + \sigma(Y_t)dB_t,$$

we get

$$\ln(Z'_T) = -\int_0^T \frac{\alpha(Y_t)}{\sigma^2(Y_t)}S_t^{-1}dS_t + \frac{1}{2}\int_0^T \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)}dt. \tag{4.14}$$

Now, substituting in (4.14) for $\frac{1}{2}\frac{\alpha^2(Y_t)}{\sigma^2(Y_t)}$ the expression we get from the integro-PDE (4.7), we end up with

(4.15)

$$\begin{aligned}\ln(Z'_T) + \ln(Z''_T) = &-\int_0^T \frac{\alpha(Y_t)}{\sigma(Y_t)}S_t^{-1}dS_t + \int_0^T\left(\frac{\partial_t H(t,Y_t)}{H(t,Y_t)} - \lambda Y_t\frac{\partial_y H(t,Y_t)}{H(t,Y_t)}\right)dt\\ &+\int_0^T\int_0^\infty (\ln H(t,Y_t+z) - \ln H(t,Y_t))\,N(dz,dt).\end{aligned}$$

Note that we have used the short-hand notation $\partial_t H$ for $\partial H/\partial t$ and $\partial_y H$ for $\partial H/\partial y$. Since $H\in C^{1,1}$ from Prop. 4.2, we can apply Itô's formula on $h(t,Y_t) = \ln H(t,Y_t)$

to derive

$$
\begin{aligned}
h(T,Y_T) =& h(0,Y_0) + \int_0^T \left(\frac{\partial_t H(t,Y_t)}{H(t,Y_t)} - \lambda Y_t \frac{\partial_y H(t,Y_t)}{H(t,Y_t)} \right) dt \\
&+ \int_0^T \int_0^\infty \left(\ln H(t,Y_t+z) - \ln H(t,Y_t) \right) N(dz,dt).
\end{aligned} \tag{4.16}
$$

Finally, substitution of (4.16) in (4.15) yields

$$
\begin{aligned}
Z'_T Z''_T &= \exp\left(- \ln H(0,y) - \int_0^T \frac{\alpha(Y_t)}{\sigma^2(Y_t)} S_t^{-1} dS_t \right) \\
&= c \cdot \exp\left(- \int_0^T \frac{\alpha(Y_t)}{\sigma(Y_t)} dB_t - \int_0^T \frac{\alpha^2(Y_t)}{\sigma^2(Y_t)} dt \right),
\end{aligned} \tag{4.17}
$$

such that $Z'_T Z''_T$ is indeed the density of Q_{ME}. Finally, the orthogonality of Z'_t and Z''_t together with the fact $\mathbb{E}[Z'_T Z''_T] = 1$ (point *i)* in the proof of Prop. 3.1) yields that $Z_t = Z'_t Z''_t$ is a martingale. □

5. THE ENTROPY PRICE OF DERIVATIVES AND INTEGRO-PARTIAL DIFFERENTIAL EQUATIONS

As an application of our results, we consider the price of derivatives written on the asset S under the minimal entropy martingale measure. We derive the corresponding integro-PDE for claims having a payoff given by the asset price S_T at maturity of the contract, a typical example being a European call option on S.

Consider a contingent claim with payoff $f(S_T)$ at maturity time T, where we suppose that f is of linear growth and $f(S_T) \in L^1(Q_{ME})$. Then the entropy price of the claim at time t given $S_t = s$ and $Y_t = y$, denoted by $\Lambda(t,y,s)$, is defined as the conditional expectation under the minimal entropy martingale measure. We thus get

$$
\Lambda(t,y,s) = \mathbb{E}_{Q_{ME}} \left[f\left(S_T\right) \,\middle|\, Y_t = y, S_t = s \right]. \tag{5.1}
$$

Knowing the density process of Q_{ME} we are now able to determine the dynamics of S_t and Y_t under Q_{ME}. Define the two processes $\widetilde{S}_t$ and $\widetilde{Y}_t$ by

$$
d\widetilde{S}_t = \sigma\left(\widetilde{Y}_t\right) \widetilde{S}_t \, d\widetilde{B}_t, \tag{5.2}
$$

$$
d\widetilde{Y}_t = -\lambda \widetilde{Y}_t dt + d\widetilde{L}_{\lambda t}, \tag{5.3}
$$

where $\widetilde{B}_t$ is Brownian motion and $\widetilde{L}_t$ is a pure jump Markov process with the predictable compensating measure

$$
\widetilde{\nu}(\omega,dz,dt) = \frac{H(t,\widetilde{Y}_t(\omega)+z)}{H(t,\widetilde{Y}_t(\omega))} \nu(dz)dt. \tag{5.4}
$$

Observe that the state-dependent jump measure $\widetilde{\nu}(dz)$ becomes deterministic when $\mu = 0$: Indeed, from Cor. 4.3 we find that

$$\widetilde{\nu}(\omega, dz, dt) = \mathrm{e}^{b(t)z}\nu(dz)\,dt,$$

where $b(t)$ is given in Cor. 4.3. Hence, for $\mu = 0$, $\widetilde{L}$ is an independent increment process (see e.g. Sato [22]).

By using the independence of $\widetilde{B}_t$ and $\widetilde{L}_t$ and the Girsanov theorem for Brownian motion and random measures (see Jacod and Shiryaev [15]), respectively, we see that

$$\begin{aligned}\Lambda(t, y, s) &= \mathbb{E}_{Q_{ME}}\left[f(S_T) \,\middle|\, Y_t = y, S_t = s\right] \\ &= \mathbb{E}\left[f\left(\widetilde{S}_T\right) \,\middle|\, \widetilde{Y}_t = y, \widetilde{S}_t = s\right].\end{aligned}$$

This representation allows us to set up an integro-PDE for the entropy price. Like in Bensoussan and Lions [6], Ch. 3-Thm. 8.1, using the bounds of $H(t, y)$, we get that $\Lambda(t, y, s)$ is the Feynman-Kac representation of the solution of the following integro-PDE

(5.5)

$$\begin{aligned}&\frac{\partial\Lambda}{\partial t} + \frac{1}{2}\sigma^2(y)s^2\frac{\partial^2\Lambda}{\partial s^2} - \lambda y\frac{\partial\Lambda}{\partial y} \\ &\quad + \lambda\int_0^\infty \left(\Lambda(t, y+z, s) - \Lambda(t, y, s)\right)\frac{H(t, y+z)}{H(t, y)}\nu(dz) = 0, \;\; (t, y, s) \in [0, T) \times \mathbb{R}_+^2,\end{aligned}$$

with terminal condition

$$\Lambda(T, y, s) = f(s), \;\; (y, s) \in \mathbb{R}_+^2. \tag{5.6}$$

Note that in order to solve this integro-PDE, we need to consider the equation (4.7)-(4.8) for H as well. Thus, the minimal entropy price of a claim in the Barndorff-Nielsen and Shephard model is given as the solution of a coupled system of two integro-PDEs.

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REFERENCES

[1] Applebaum, D. (2004). *Lévy Processes and Stochastic Calculus.* Cambridge Univ. Press 2004.

[2] Barndorff-Nielsen, O.E. (1998). Processes of normal inverse Gaussian type. *Finan. Stochast.*, **2**, pp. 41-68.

[3] Barndorff-Nielsen, O.E. and Shephard, N. (2001). Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. R. Statist. Soc.*, Ser. B, **63**, pp. 167-241 (with discussion).

[4] Becherer, D. (2003). Rational hedging and valuation of integrated risks under constant absolute risk aversion. *Insurance Math. Econom.*, **33**(1), 1-28.

[5] Becherer, D. (2004). Utility-indifference hedging and valuation via reaction-diffusion systems. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci*, **460**, No. 2041, 27-51.

[6] Bensoussan, A. and Lions, J.-L. (1982). *Impulse control and Quasi-Variational Inequalities.* Gauthiers-Villars.

[7] Benth, F.E. and Karlsen, K.H. (2003). *A pde representation of the density of the minimal entropy martingale measure in stochastic volatility markets.* Preprint, University of Oslo.

[8] Bertoin, J.: Lévy Processes. Cambridge University Press, Cambridge 1996.

[9] Delbaen, F., Grandits, P., Rheinländer, T., Samperi, D., Schweizer, M., and Stricker, C. (2002). Exponential hedging and entropic penalties. *Math. Finance*, **12**(2), 99-123.

[10] Fleming, W. H. and Soner, H. M. (1993). *Controlled Markov processes and viscosity solutions.* Springer-Verlag, New York.

[11] Fritelli, M. (2000). The minimal entropy martingale measure and the valuation problem in incomplete markets. *Math. Finance*, **10**, 39-52.

[12] Grandits, P., Rheinländer, T. (2002). On the minimal entropy martingale measure. *Annals of Probability*, **30**, 1003-1038.

[13] Hobson, D. (2004). Stochastic volatility models, correlation and the q-optimal measure. *Math. Finance*, **14**(4), 537-556.

[14] Hodges, S. D. and Neuberger, A. (1989). *Optimal replication of contingent claims under transaction costs. Rev. Future Markets*, **8**, 222-239.

[15] Jacod, J. and Shiryaev, A. N. (1987). *Limit Theorems for Stochastic Processes.* Springer Verlag, Second Edition

[16] Madan, D. B: and Seneta, E. (1990). The variance gamma model for share market returns. *J. Business* **63**, 511-524.

[17] Musiela, M. and Zariphopoulou, T. (2003). An example of indifference prices under exponential preferences. *Finan. Stochast.*, **8**(2), 229-239.

[18] Protter, P. (2003). *Stochastic Integration and Differential Equations.* Springer-Verlag, Second Edition.

[19] Rheinländer, T. (2003). An entropy approach to the stein/stein model with correlation, *Finan. Stochast.*, to appear.

[20] Rouge, R. and El Karoui, N. (2000). Pricing via utility maximization and entropy. *Math. Finance*, **10**, 259–276.

[21] Rydberg, T. (1999). Generalized hyperbolic diffusion processes with applications towards finance. *Math. Finance*, **9**, 183-201.

[22] Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Studies in Advanced Mathematics, Vol. 68, Cambridge University Press, Cambridge.

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