Markovian Integral Equations

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Abstract. We analyze multidimensional Markovian integral equations that are formulated with a progressive time-inhomogeneous Markov process that has Borel measurable transition probabilities. In the case of a path process of a path-dependent diffusion, the solutions to these integral equations lead to the concept of mild solutions to path-dependent partial differential equations (PPDEs). Our goal is to establish uniqueness, stability, existence and non-extendibility of solutions among a certain class of maps. By requiring the Feller continuity of the Markov process, we give weak conditions under which solutions become continuous. Moreover, we provide a multidimensional Feynman-Kac formula and a one-dimensional global existence- and uniqueness result.

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1 Introduction

Markovian integral equations arise when dealing with diffusion processes and mild solutions to semilinear parabolic partial differential equations (PDEs). This fact was utilized by Dynkin [7, 8] to give probabilistic formulas for mild solutions via the log-Laplace functionals of superprocesses. In this context, Schied [17] used Markovian integral equations to solve problems of optimal stochastic control in mathematical finance. By studying path-dependent diffusion processes, the connection of Markovian equations to PDEs can be extended to path-dependent partial differential equations (PPDEs), as verified in [16], [10] and [4]. Inspired by the applications of one-dimensional Markovian equations, the aim of this paper is to construct solutions even in a multidimensional framework.

Let S be a separable metrizable topological space, $T > 0$ and $\mathscr{X} = (X, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a progressive Markov process on some measurable space $(\Omega, \mathscr{F})$ with state space S and Borel measurable transition probabilities $\mathbb{P} = \{P_{r,x} \,|\, (r, x) \in [0, T] \times S\}$. We consider the following *multidimensional Markovian integral equation* coupled with a terminal value condition:

$$E_{r,x}[g(X_T)] = u(r, x) + E_{r,x}\left[\int_r^T f(s, X_s, u(s, X_s))\, \mu(ds)\right] \tag{M}$$

for all $(r, x) \in [0, T] \times S$. Here, we implicitly assume that the dimension is $k \in \mathbb{N}$, the unknown map $u : [0, T] \times S \to \mathbb{R}^k$ takes all its values in $D \in \mathscr{B}(\mathbb{R}^k)$ and $f : [0, T] \times S \times D \to \mathbb{R}^k$ is Borel measurable and may depend on u in a nonlinear way. Further, μ is an atomless Borel measure on $[0, T]$ and the terminal value condition $g : S \to D$ is Borel measurable and bounded.

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We first remark that for $D = \mathbb{R}^k$ a Picard iteration and Banach's fixed-point theorem produce solutions to (M) that are local in time. This can be found, for example, in Pazy [15, Theorem 6.1.4] when $\mathscr{X}$ is a time-homogeneous diffusion process. Regarding existence, we will suppose more generally that D is convex with non-empty interior. By modifying analytical methods from the theory of ordinary differential equations (ODEs), we will derive unique non-extendible solutions to (M) that are admissible in a topological sense. Moreover, weak conditions ensuring the continuity of the derived solutions will be provided. In the particular case when $D = \mathbb{R}^k$ and f is an affine map in the third variable $w \in \mathbb{R}^k$, we will prove a representation for solutions to (M). This gives a multidimensional generalization of the Feynman-Kac formula in Dynkin [9, Theorem 4.1.1].

Let us emphasize that non-negative solutions to one-dimensional Markovian equations are well-studied. Namely, for $k = 1$ and $D = \mathbb{R}_+$, solutions to (M) have been deduced by a Picard iteration approach. For instance, some classical references are Watanabe [18, Proposition 2.2], Fitzsimmons [11, Proposition 2.3] and Iscoe [12, Theorem 1.1]. In these works the existence of solutions to (M) is used for the construction of measure-valued Markov branching processes. Dynkin [5, 6, 9] establishes superprocesses with probabilistic methods by means of branching particle systems, which in turn yields another existence result to our Markovian integral equations.

These treatments of (M) in one dimension require that the function f admits a representation that is related to measure-valued Markov branching processes. To give one of the main examples, the following case is included in [5, 6, 9]:

$$f(t, x, w) = b_1(t, x)w^{\alpha_1} + \cdots + b_n(t, x)w^{\alpha_n} \tag{1.1}$$

for each $(t, x, w) \in [0, T] \times S \times \mathbb{R}_+$, where $n \in \mathbb{N}$, $b_1, \ldots, b_n : [0, T] \times S \to \mathbb{R}_+$ are Borel measurable and bounded and $\alpha_1, \ldots, \alpha_n \in [1, 2]$. Here, the bound $\alpha_i \leq 2$ for all $i \in \{1, \ldots, n\}$ is crucial. However, this paper intends to derive solutions without imposing a specific form of f. Rather, as in the multidimensional case, we will introduce regularity conditions for f with respect to the Borel measure μ like local Lipschitz μ-continuity. This will allow for a more general treatment of (M). In particular, our approach includes the case

$$f(t, x, w) = a(t, x) + b_1(t, x)\varphi_1(w) + \cdots + b_n(t, x)\varphi_n(w)$$

for all $(t, x, w) \in [0, T] \times S \times \mathbb{R}_+$, where $a : [0, T] \times S \to (-\infty, 0]$ is Borel measurable and bounded and $\varphi_1, \ldots, \varphi_n : \mathbb{R}_+ \to \mathbb{R}_+$ are locally Lipschitz continuous with $\varphi_i(0) = 0$ for each $i \in \{1, \ldots, n\}$. Hence, (1.1) is also feasible if $\alpha_i > 2$ for some $i \in \{1, \ldots, n\}$. Note that we will not restrict our attention to the case $D = \mathbb{R}_+$. In fact, the one-dimensional global existence and uniqueness result, which we will establish, is applicable provided D is a non-degenerate interval. In this connection, the same weak conditions as before grant the continuity of solutions to (M).

The paper is based on the doctoral thesis [13] and structured as follows. Section 2 sets up the framework and states the main results. First, in Section 2.1 we consider product spaces endowed with a pseudometric and introduces several map spaces. Section 2.2 presents regularity conditions for multidimensional Borel measurable maps relative to a Borel measure. In Section 2.3 we give an adjusted definition of a Markov process that is in line with the classical notion. In Section 2.4 we introduce the Markovian terminal value problem (M), by defining (approximate) solutions. In Section 2.5 the main results are presented. Section 3 shows our approach to the main results. In Section 3.1 we compare solutions, prove their stability and also investigate their growth behavior, while in Section 3.2 we construct solutions that are local in time. Finally, the main results are proven in Section 4.

2 Preliminaries and main results

Throughout the paper, let S be a separable metrizable topological space, $T > 0$ and μ be an atomless Borel measure on $[0, T]$. We fix $k \in \mathbb{N}$ and let $\mathbb{I}_k$ be the identity matrix in $\mathbb{R}^{k\times k}$. To keep notation simple, we use $|\cdot|$ for the absolute value function, the Euclidean norm on $\mathbb{R}^k$ and the spectral norm on $\mathbb{R}^{k\times k}$.

2.1 Time-space Cartesian products

We endow $[0, T] \times S$ with a pseudometric d_S that ensures its separability and which generates a topology that is coarser than the product topology. Then $\mathscr{B}([0, T] \times S) \subset \mathscr{B}([0, T]) \otimes \mathscr{B}(S)$, since S is separable. For instance, d_S could be a product metric, in which case the Borel σ-field coincides with the product σ-field. However, the presence of a pseudometric allows us to include path processes of path-dependent diffusions as specific continuous strong Markov processes.

In this context, we assume that for each $(t, x), (s, y) \in [0, T] \times S$ we have $d_S((t, x), (s, y)) = 0$ only if $s = t$. Let for the moment I be a non-degenerate interval in $[0, T]$ and E be a separable Banach space with complete norm $\|\cdot\|$. For each Borel set $D \in \mathscr{B}(E)$ we let

$$B(I \times S, D) \quad \text{and} \quad B(S, D) \tag{2.1}$$

denote the sets of all D-valued Borel measurable maps on $I \times S$ and S, respectively. By $B_b(I \times S, D)$ and $B_b(S, D)$ we denote the sets of all bounded maps in $B(I \times S, D)$ and $B(S, D)$, respectively. In the case $D = E = \mathbb{R}$ we omit to highlight the set of all attainable values.

Definition 2.1. Let $u : I \times S \to E$.

(i) u is called *consistent* if we have $u(t, x) = u(t, y)$ for all $t \in I$ and each $x, y \in S$ with $d_S((t, x), (t, y)) = 0$.

(ii) We say that u is *right-continuous* if for each $(r, x) \in I \times S$ and every $\varepsilon > 0$ there is $\delta > 0$ such that $\|u(s, y) - u(r, x)\| < \varepsilon$ for all $(s, y) \in I \times S$ with $s \geq r$ and $d_S((s, y), (r, x)) < \delta$.

By the monotone class theorem, if a map $u : I \times S \to E$ is Borel measurable, then it is consistent. Further, (right-)continuity of u also implies its consistency and it ensures that $u(\cdot, x)$ is (right-)continuous for each $x \in S$ and $u(t, \cdot)$ is continuous for all $t \in I$, which entails product measurability.

We recall that any Borel set in a separable Banach space can be viewed as a separable metrizable topological space. In particular, S could be a countable intersection of open sets in $\mathbb{R}^d$ with $d \in \mathbb{N}$, which serves as state space for diffusion processes. To deal with path processes of path-dependent diffusions, the following framework, as essentially introduced in [2], can be used.

Example 2.2. For $d \in \mathbb{N}$ we consider the separable Banach space $C([0, T], \mathbb{R}^d)$ of all $\mathbb{R}^d$-valued continuous maps on $[0, T]$ equipped with the maximum norm $\|\cdot\|_\infty$ and denote any map $x : [0, T] \to \mathbb{R}^d$ stopped at time $t \in [0, T]$ by x^t. Suppose that $U \subset \mathbb{R}^d$ is a countable intersection of open sets, $S = C([0, T], U)$ and

$$d_S((t, x), (s, y)) = |t - s|^\alpha + \|x^t - y^s\|_\infty$$

for all $(t, x), (s, y) \in [0, T] \times S$ and some $\alpha \in (0, 1]$. Then $[0, T] \times S$ endowed with d_S is indeed a separable pseudometric space whose topology is coarser than the product topology and it is complete if and only if U is closed. Moreover, the following two facts hold:

(i) The map u is consistent if and only if it is *non-anticipative* in the sense that $u(t, x) = u(t, x^t)$ for all $(t, x) \in I \times S$.

(ii) u is Borel measurable if and only if it is non-anticipative and product measurable. In particular, if u is right-continuous, then it is Borel measurable.

2.2 Regularity with respect to Borel measures

Let again I be a non-degenerate interval in $[0,T]$ and E be a separable Banach space with complete norm $\|\cdot\|$. We introduce regularity conditions from [13][Chapter 2] and recall that a measurable function $\overline{a} : I \to \mathbb{R}$ is locally μ-integrable if and only if $\int_r^t |\overline{a}(s)|\,\mu(ds) < \infty$ for all $r,t \in I$ with $r \leq t$.

Definition 2.3. Suppose that $a \in B(I \times S, E)$.

(i) The map a is called *(locally) μ-dominated* if there is a measurable (locally) μ-integrable function $\overline{a} : I \to \mathbb{R}_+$ such that $\|a(\cdot,y)\| \leq \overline{a}$ for all $y \in S$ μ-a.s. on I.

(ii) We say that a is *μ-suitably bounded* if for each $r,t \in I$ with $r \leq t$ there is a μ-null set $N \in \mathscr{B}([0,T])$ such that $\sup_{(s,y)\in(N^c\cap[r,t])\times S} \|a(s,y)\| < \infty$.

For each $D \in \mathscr{B}(E)$ it is readily seen that the set of all D-valued Borel measurable locally μ-dominated maps on $I \times S$, which we denote by

$$B_\mu(I \times S, D), \tag{2.2}$$

contains every D-valued Borel measurable μ-suitably bounded map on $I \times S$. If $D = E = \mathbb{R}$, then we write $B_\mu(I \times S)$ for (2.2), and note that $\mathscr{B}([0,T] \times S) \otimes \mathscr{B}(D)$ is the Borel σ-field of $[0,T] \times S \times D$ in general.

Definition 2.4. Let $f : [0,T] \times S \times D \to \mathbb{R}^k$ be Borel measurable.

(i) We call f *affine μ-bounded* if there are $a,b \in B_\mu([0,T] \times S, \mathbb{R}_+)$ so that $|f(t,x,w)| \leq a(t,x) + b(t,x)\|w\|$ for all $(t,x,w) \in [0,T] \times S \times D$. If one can take $b = 0$, then f is called *μ-bounded.*

(ii) We say that f is *locally μ-bounded* at $\hat{w} \in \overline{D}$ if there is a neighborhood W of $\hat{w}$ in $\overline{D}$ for which the restriction of f to $[0,T] \times S \times (W \cap D)$ is μ-bounded. The map f is called locally μ-bounded if it is locally μ-bounded at each $\hat{w} \in D$.

(iii) Let $k = 1$, then f is said to be *affine μ-bounded from below* if $f(t,x,w) \geq -a(t,x) - b(t,x)\|w\|$ for all $(t,x,w) \in [0,T] \times S \times D$ and some $a,b \in B_\mu([0,T] \times S, \mathbb{R}_+)$. If $b = 0$ is possible, then f is *μ-bounded from below.* Moreover, f is *(affine) μ-bounded from above* if $-f$ is (affine) μ-bounded from below.

For a Borel measurable map $f : [0,T] \times S \times D \to \mathbb{R}^k$ to be locally μ-bounded, it is sufficient that it is affine μ-bounded. If f is locally μ-bounded, then the Borel measurable map $f(\cdot,\cdot,\hat{w})$ is μ-dominated for each $\hat{w} \in D$. Of course, for $k = 1$ the function f is (affine) μ-bounded if and only if it is (affine) μ-bounded from below and from above.

Definition 2.5. Let $f : [0,T] \times S \times D \to \mathbb{R}^k$ be Borel measurable.

(i) We call f *Lipschitz μ-continuous* if there is $\lambda \in B_\mu([0,T] \times S, \mathbb{R}_+)$ satisfying $|f(t,x,w) - f(t,x,\tilde{w})| \leq \lambda(t,x)\|w - \tilde{w}\|$ for all $(t,x) \in [0,T] \times S$ and each $w, \tilde{w} \in D$.

(ii) We call f *locally Lipschitz μ-continuous* at $\hat{w} \in \overline{D}$ if there is a neighborhood W of $\hat{w}$ in $\overline{D}$ such that f restricted to $[0,T] \times S \times (W \cap D)$ is Lipschitz μ-continuous. The map f is locally Lipschitz μ-continuous if it is locally Lipschitz μ-continuous at every $\hat{w} \in D$.

In what follows, the linear space of all $\mathbb{R}^k$-valued Borel measurable, locally μ-bounded and locally Lipschitz μ-continuous maps on $[0,T] \times S \times D$ is denoted by

$$BC_\mu^{1-}([0,T] \times S \times D, \mathbb{R}^k). \tag{2.3}$$

For $k = 1$ we write $BC_\mu^{1-}([0,T] \times S \times D)$ for this space. Clearly, if $f : [0,T] \times S \times D \to \mathbb{R}^k$ is a Borel measurable map that is locally Lipschitz μ-continuous and $f(\cdot,\cdot,\hat{w})$ is μ-dominated for all $\hat{w} \in D$, then f is locally μ-bounded. If instead f is Lipschitz μ-continuous and $f(\cdot,\cdot,\hat{w})$ is μ-dominated for at least one $\hat{w} \in D$, then f is affine μ-bounded.

Examples 2.6. (i) Let $a \in B_\mu([0,T] \times S, \mathbb{R}^k)$ and $b \in B_\mu([0,T] \times S, \mathbb{R}^{k\times k})$. Assume that $\varphi : D \to \mathbb{R}^k$ is Borel measurable and $f(t,x,w) = a(t,x) + b(t,x)\varphi(w)$ for all $(t,x,w) \in [0,T] \times S \times D$. Then the following two assertions hold:

(1) f is (affine) μ-bounded whenever φ is (affine) bounded. If instead φ is locally bounded, then f is locally μ-bounded. For $k = 1$ and $b \geq 0$, it follows that f is (affine) μ-bounded from below (resp. from above) if φ is (affine) bounded from below (resp. from above).

(2) From the (local) Lipschitz continuity of φ the (local) Lipschitz μ-continuity of f follows. Thus, if φ is locally Lipschitz continuous, then $f \in BC_\mu^{1-}([0,T] \times S \times D, \mathbb{R}^k)$.

(ii) Let $(U, \mathscr{U})$ be a measurable space, η be a kernel from $[0,T] \times S$ to $(U, \mathscr{U})$ and $\varphi : U \times D \to \mathbb{R}^k$ be product measurable such that $\varphi(\cdot, w)$ is $\eta(t,x,\cdot)$-integrable and f is of the form

$$f(t,x,w) = \int_U \varphi(u,w)\, \eta(t,x,du)$$

for each $(t,x,w) \in [0,T] \times S \times D$. Then the subsequent two assertions are valid:

(1) f is locally μ-bounded if for each $\hat{w} \in D$ there are a neighborhood W of $\hat{w}$ in D and an $\mathscr{U}$-measurable function $a : U \to \mathbb{R}_+$ with $|\varphi(u,w)| \leq a(u)$ for all $(u,w) \in U \times W$ such that $\int_U a(u)\, \eta(\cdot,\cdot,du)$ is finite and μ-dominated.

(2) f is locally Lipschitz μ-continuous if to all $\hat{w} \in D$ there are a neighborhood W of $\hat{w}$ in D and an $\mathscr{U}$-measurable function $\lambda : U \to \mathbb{R}_+$ with $|\varphi(u,w) - \varphi(u,\tilde{w})| \leq \lambda(u)\|w - \tilde{w}\|$ for all $u \in U$ and each $w, \tilde{w} \in W$ such that $\int_U \lambda(u)\, \eta(\cdot,\cdot,du)$ is finite and μ-dominated.

2.3 Time-inhomogeneous Markov processes

In the sequel, let $\mathscr{X} = (X, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a *Markov process* on some measurable space $(\Omega, \mathscr{F})$ with state space S and *Borel measurable transition probabilities*, which is composed of a process $X : [0,T] \times \Omega \to S$, a filtration $(\mathscr{F}_t)_{t\in[0,T]}$ to which X is adapted and a set $\mathbb{P} = \{P_{r,x} \,|\, (r,x) \in [0,T] \times S\}$ of probability measures on $(\Omega, \mathscr{F})$ such that the following three conditions hold:

(i) $d_S((r, X_r), (r,y)) = 0$ for all $r \in [0,s]$ $P_{s,y}$-a.s. for each $(s,y) \in [0,T] \times S$.

(ii) The function $[0,t] \times S \to [0,1]$, $(s,y) \mapsto P_{s,y}(X_t \in B)$ is Borel measurable for all $t \in [0,T]$ and each $B \in \mathscr{B}(S)$.

(iii) $P_{r,x}(X_t \in B|\mathscr{F}_s) = P_{s,X_s}(X_t \in B)$ $P_{r,x}$-a.s. for all $r,s,t \in [0,T]$ with $r \leq s \leq t$, each $x \in S$ and every $B \in \mathscr{B}(S)$.

If d_S is a product metric, then (i) reduces to $X_r = y$ for all $r \in [0,s]$ $P_{s,y}$-a.s. for each $(s,y) \in [0,T] \times S$ and we recover the classical definition of a time-inhomogeneous Markov process with Borel measurable transition probabilities. Moreover, we let $\mathscr{X}$ be *progressive*. That is, X is progressively measurable with respect to both its natural filtration and its natural backward filtration. For example, this is the case if X has left- or right-continuous paths.

Whenever necessary, we will require that $\mathscr{X}$ is *Feller (right-)continuous*, which means that the function $[0,t] \times S \to \mathbb{R}$, $(r,x) \mapsto E_{r,x}[\varphi(X_t)]$ is (right-)continuous for all $t \in [0,T]$ and each $\varphi \in C_b(S)$. In this case, it follows that the bounded map

$$[0,t] \times S \to \mathbb{R}^k, \quad (r,x) \mapsto E_{r,x}\bigg[\int_r^t \varphi(s, X_s)\, \mu(ds)\bigg] \tag{2.4}$$

is (right-)continuous for each $t \in [0,T]$ and every $\varphi \in B_\mu([0,t] \times S, \mathbb{R}^k)$ for which $\varphi(s,\cdot)$ is continuous for μ-a.e. $s \in [0,t]$, by dominated convergence. Moreover, if $\mathscr{X}$ is Feller continuous, then it automatically has Borel measurable transition probabilities.

Example 2.7. Let the setting of Example 2.2 hold with $U = \mathbb{R}^d$. On the Polish space $(\Omega, \mathscr{F}) = (S, \mathscr{B}(S))$ we let X be the path process and $(\mathscr{F}_t)_{t\in[0,T]}$ be the natural filtration of the canonical process. That is, $X_t(\omega) = \omega^t$ and $\mathscr{F}_t = \sigma(X_t)$ for all $(t,\omega) \in [0,T] \times \Omega$. Let $b \in B([0,T] \times S, \mathbb{R}^d)$ and $\sigma \in B([0,T] \times S, \mathbb{R}^{d\times d})$ satisfy

$$|b(t,x)| \vee |\sigma(t,x)| \leq c(1 + \|x\|_\infty) \quad \text{and} \quad |b(t,x) - b(t,y)| \vee |\sigma(t,x) - \sigma(t,y)| \leq c\|x - y\|_\infty$$

for all $t \in [0,T]$, each $x, y \in S$ and some $c \in \mathbb{R}_+$, where we also use $|\cdot|$ to denote the Euclidean norm on $\mathbb{R}^d$ and the spectral norm on $\mathbb{R}^{d\times d}$. Choose a filtered probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, (\tilde{\mathscr{F}}_t)_{t\in[0,T]}, \tilde{P})$ on which there is a standard d-dimensional $(\tilde{\mathscr{F}}_t)_{t\in[0,T]}$-Brownian motion W. Then for each $(r,x) \in [0,T] \times S$, pathwise uniqueness holds for the following path-dependent stochastic differential equation (SDE)

$$dY_t = b(t, Y^t)\, dt + \sigma(t, Y^t)\, dW_t \quad \text{for } t \in [r,T] \tag{2.5}$$

and it admits a unique strong solution Y satisfying $Y^r = x^r$ $\tilde{P}$-a.s., see [3] for further details. Hence, we may let $P_{r,x}$ denote the law of Y on S. Then $\mathscr{X} = (X, (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ is indeed a progressive Markov process that is even Feller continuous. This class of path processes of path-dependent diffusions is considered in [14] to study path-dependent PDEs.

2.4 The Markovian terminal value problem

From now on, let $D \in \mathscr{B}(\mathbb{R}^k)$ have non-empty interior and $f : [0,T] \times S \times D \to \mathbb{R}^k$ be Borel measurable. Suppose that $g \in B(S, D)$ is consistent in the sense that $g(x) = g(y)$ for all $x, y \in S$ with $d_S((T,x),(T,y)) = 0$ and

$$E_{r,x}[|g(X_T)|] < \infty \quad \text{for all } (r,x) \in [0,T] \times S.$$

We let $\varepsilon \in B_\mu([0,T] \times S, \mathbb{R}_+)$ and define an interval I in $[0,T]$ to be *admissible* if it is of the form $I = (t,T]$ or $I = [t,T]$ for some $t \in [0,T)$. This allows us to introduce the *Markovian terminal value problem* (M), by defining ε-approximate solutions in a weak sense.

Definition 2.8. An *ε-approximate solution* to (M) on an admissible interval I is a map $u \in B(I \times S, D)$ for which $\int_r^T |f(s, X_s, u(s, X_s))|\, \mu(ds)$ is finite and $P_{r,x}$-integrable such that

$$\left| E_{r,x}[g(X_T)] - u(r,x) - E_{r,x}\left[\int_r^T f(s, X_s, u(s, X_s))\, \mu(ds)\right]\right| \leq E_{r,x}\left[\int_r^T \varepsilon(s, X_s)\, \mu(ds)\right]$$

for all $(r,x) \in I \times S$. Every 0-approximate solution is called a *solution*. If in addition $I = [0,T]$, then we will speak about a *global* solution.

In the framework of Example 2.7, global solutions to (M) are exactly mild solutions to an underlying path-dependent PDE, as can be verified in [14]. Moreover, based on the following fact, mild solutions can be related to viscosity solutions, see [4][Corollary 4.17].

Remark 2.9. By the Markov property of $\mathscr{X}$, a map $u \in B(I \times S, D)$ is a solution to (M) on I if and only if $|u(t, X_t)|$ and $\int_r^t |f(s, X_s, u(s, X_s))|\, \mu(ds)$ are finite and $P_{r,x}$-integrable such that

$$E_{r,x}[u(t, X_t)] = u(r,x) + E_{r,x}\left[\int_r^t f(s, X_s, u(s, X_s))\, \mu(ds)\right] \quad \text{and} \quad u(T,x) = g(x)$$

for all $r, t \in I$ with $r \leq t$ and each $x \in S$. We notice that these conditions can be extended in the sense of above definition to obtain a strong notion of ε-approximate solutions.

For our main results, we introduce admissibility and non-extendibility of solutions.

Definition 2.10. Assume that u is a solution to (M) on an admissible interval I.

(i) We say that u is *(weakly) admissible* if $u([r,T] \times S)$ is relatively compact in D° (resp. D) for each $r \in I$.

(ii) Let u be admissible, then we call u *extendible* if there is an admissible solution $\tilde{u}$ to (M) on an admissible interval $\tilde{I}$ with $I \subsetneq \tilde{I}$ and $u = \tilde{u}$ on $I \times S$. Otherwise, u is *non-extendible* and I is called a *maximal interval of existence.*

2.5 The main results

We begin with non-extendibility and assume until the end of the paper that g is bounded, as this requirement is necessary for an admissible solution to exist.

Theorem 2.11. *Let D be convex, $f \in BC_\mu^{1-}([0,T] \times S \times D, \mathbb{R}^k)$ and g be bounded away from ∂D. Then there is a unique non-extendible admissible solution u_g to* (M) *on a maximal interval of existence I_g that is open in $[0,T]$. With $t_g^- := \inf I_g$, either $I_g = [0,T]$ or*

$$\lim_{t \downarrow t_g^-} \inf_{x \in S} \min\left\{\operatorname{dist}(u_g(t,x), \partial D), \frac{1}{1 + |u_g(t,x)|}\right\} = 0. \tag{2.6}$$

Moreover, if $\mathscr{X}$ is Feller (right-)continuous, $f(s,\cdot,\cdot)$ is continuous for μ-a.e. $s \in [0,T]$ and g is continuous, then u_g is (right-)continuous.

Let us for the moment assume that the hypotheses of the theorem hold. If u_g is bounded away from ∂D, that is, if $\operatorname{dist}(u_g(t,x), \partial D) \geq \varepsilon$ for all $(t,x) \in I_g \times S$ and some $\varepsilon > 0$, and $I_g \neq [0,T]$, then from the boundary and growth condition (2.6) it follows that

$$\lim_{t \downarrow t_g^-} \sup_{x \in S} |u_g(t,x)| = \infty.$$

Let us instead suppose that u_g is bounded. For instance, this occurs whenever f is affine μ-bounded, as justified in Lemma 3.5. Then the theorem says that either u_g is a global solution or

$$\lim_{t \downarrow t_g^-} \inf_{x \in S} \operatorname{dist}(u_g(t,x), \partial D) = 0. \tag{2.7}$$

In particular, if u_g is not only bounded but also its image $u_g(I_g \times S)$ is relatively compact in D°, then $I_g = [0,T]$. For $D = \mathbb{R}^k$ we combine these considerations with a Picard iteration to get the following result.

Proposition 2.12. *Let $D = \mathbb{R}^k$ and f belong to $BC_\mu^{1-}([0,T] \times S \times \mathbb{R}^k, \mathbb{R}^k)$ and be affine μ-bounded. Then $I_g = [0,T]$ and the sequence $(u_n)_{n \in \mathbb{N}_0}$ in $B_b([0,T] \times S, \mathbb{R}^k)$, recursively defined by $u_0(r,x) := E_{r,x}[g(X_T)]$ and*

$$u_n(r,x) := u_0(r,x) - E_{r,x}\left[\int_r^T f(s, X_s, u_{n-1}(s,X_s))\,\mu(ds)\right]$$

for all $n \in \mathbb{N}$, converges uniformly to u_g, the unique global bounded solution to (M)*. Furthermore, if $\mathscr{X}$ is Feller (right-)continuous, $f(s,\cdot,\cdot)$ is continuous for μ-a.e. $s \in [0,T]$ and g is continuous, then u_n is (right-)continuous for each $n \in \mathbb{N}$.*

Let us at this place assume that $D = \mathbb{R}^k$ and f is an affine map in $w \in \mathbb{R}^k$. In other words, there are two maps $a : [0,T] \times S \to \mathbb{R}^k$ and $b : [0,T] \times S \to \mathbb{R}^{k\times k}$ such that

$$f(t,x,w) = a(t,x) + b(t,x)w \quad \text{for all } (t,x,w) \in [0,T] \times S \times \mathbb{R}^k.$$

As a and b are necessarily Borel measurable, we infer from Examples 2.6 that f is affine μ-bounded and Lipschitz μ-continuous as soon as a and b are μ-dominated. Thus, we get a multidimensional Feynman-Kac formula.

Proposition 2.13. *Let $D = \mathbb{R}^k$ and $a \in B_\mu([0,T] \times S, \mathbb{R}^k)$ and $b \in B_\mu([0,T] \times S, \mathbb{R}^{k\times k})$ be such that $f(t,x,w) = a(t,x) + b(t,x)w$ for each $(t,x,w) \in [0,T] \times S \times \mathbb{R}^k$. Then $I_g = [0,T]$ and*

$$u_g(r,x) = E_{r,x}[\mathrm{B}_{r,T}\, g(X_T)] - E_{r,x}\bigg[\int_r^T \mathrm{B}_{r,t}\, a(t,X_t)\,\mu(dt)\bigg] \tag{2.8}$$

for all $(r,x) \in [0,T] \times S$ and some bounded map $\mathrm{B} : [0,T] \times [0,T] \times \Omega \to \mathbb{R}^{k\times k}$, $(r,t,\omega) \mapsto \mathrm{B}_{r,t}(\omega)$ with the following three properties:

(i) $\mathrm{B}_{r,t}$ is $\sigma(X_s : s \in [r,t])$-measurable, $|\mathrm{B}_{r,t}| \le \exp(\int_r^t |b(s,X_s)|\,\mu(ds))$, the map $\mathrm{B}(\omega)$ is continuous and we have $\mathrm{B}_{r,t} = \mathbb{I}_k - \int_r^t b(s,X_s)\mathrm{B}_{s,t}\,\mu(ds)$ for all $r,t \in [0,T]$ with $r \le t$ and each $\omega \in \Omega$.

(ii) $\mathrm{B}_{r,r} = \mathbb{I}_k$ and $\mathrm{B}_{r,s}\mathrm{B}_{s,t} = \mathrm{B}_{r,t}$ for all $r,s,t \in [0,T]$. If in addition $r \le t$ and $\omega \in \Omega$, then $\mathrm{B}_{r,t}(\omega)$ is invertible with $\mathrm{B}_{r,t}(\omega)^{-1} = \mathrm{B}_{t,r}(\omega)$ and $|\mathrm{B}_{t,r}| \le \exp(\int_r^t |b(s,X_s)|\,\mu(ds))$.

(iii) If $b(r,x)b(s,y) = b(s,y)b(r,x)$ for every $(r,x),(s,y) \in [0,T] \times S$, then $\mathrm{B}_{r,t} = \exp(-\int_r^t b(s,X_s)\,\mu(ds))$ for all $r,t \in [0,T]$ with $r \le t$.

Clearly, if there are a function $c \in B_\mu([0,T] \times S)$ and a matrix $A \in \mathbb{R}^{k\times k}$ such that the map b in above proposition satisfies $b = cA$, then the commutation condition in (iii) holds. Hence, we may consider an example involving trigonometric functions.

Example 2.14. Let $k = 2$ and $a = 0$. Suppose that there are $c \in B_\mu([0,T] \times S)$ and $\delta, \varepsilon \in \mathbb{R}\backslash\{0\}$ such that

$$b(t,x) = c(t,x)\begin{pmatrix} 0 & \delta \\ \varepsilon & 0 \end{pmatrix} \quad \text{for all } (t,x) \in [0,T] \times S.$$

We set $\rho := 1$, if $\delta\varepsilon > 0$, and $\rho := i \in \mathbb{C}$, otherwise. Then we can write $u_g(r,x)$ coordinatewise for all $(r,x) \in [0,T] \times S$ in the form

$$\begin{aligned}
(u_g)_1(r,x) &= E_{r,x}\bigg[\cosh\bigg(-\rho\sqrt{|\delta\varepsilon|}\int_r^T c(s,X_s)\,\mu(ds)\bigg)g_1(X_T)\bigg] \\
&\quad + \rho\frac{\sqrt{|\delta\varepsilon|}}{\varepsilon}E_{r,x}\bigg[\sinh\bigg(-\rho\sqrt{|\delta\varepsilon|}\int_r^T c(s,X_s)\,\mu(ds)\bigg)g_2(X_T)\bigg], \\
(u_g)_2(r,x) &= \rho\frac{\sqrt{|\delta\varepsilon|}}{\delta}E_{r,x}\bigg[\sinh\bigg(-\rho\sqrt{|\delta\varepsilon|}\int_r^T c(s,X_s)\,\mu(ds)\bigg)g_1(X_T)\bigg] \\
&\quad + E_{r,x}\bigg[\cosh\bigg(-\rho\sqrt{|\delta\varepsilon|}\int_r^T c(s,X_s)\,\mu(ds)\bigg)g_2(X_T)\bigg].
\end{aligned}$$

Let us now restrict our attention to $k = 1$. While Proposition 2.12 covers the case $D = \mathbb{R}$, we can also derive global solutions if D is a non-degenerate interval.

Theorem 2.15. *Let D be a non-degenerate interval, $\underline{d} := \inf D$ and $\overline{d} := \sup D$. Assume that f belongs to $BC_{\mu}^{1-}([0,T] \times S \times D)$ and the following two conditions hold:*

(i) If $\underline{d} > -\infty$ (resp. $\overline{d} < \infty$), then f is both locally μ-bounded and locally Lipschitz μ-continuous at $\underline{d}$ (resp. $\overline{d}$) so that $\lim_{w\downarrow\underline{d}} f(\cdot,x,w) \leq 0$ (resp. $\lim_{w\uparrow\overline{d}} f(\cdot,x,w) \geq 0$) for all $x \in S$ μ-a.s.

(ii) If $\underline{d} = -\infty$ (resp. $\overline{d} = \infty$), then f is affine μ-bounded from above (resp. from below).

Then there is a unique global bounded solution $\overline{u}_g$ to (M) *that agrees with u_g if g is bounded away from $\{\underline{d},\overline{d}\} \cap \mathbb{R}$. Moreover, if $\mathscr{X}$ is Feller (right-)continuous, $f(s,\cdot,\cdot)$ is continuous for μ-a.e. $s \in [0,T]$ and g is continuous, then $\overline{u}_g$ is (right-)continuous.*

In the case $D = \mathbb{R}_+$, global bounded solutions to (M) can be expressed via the log-Laplace functionals of superprocesses provided f admits the representation required below.

Example 2.16. Let S be complete, $D = \mathbb{R}_+$ and $b, c \in B_b([0,T] \times S, \mathbb{R}_+)$. We let η be a kernel from $[0,T] \times S$ to $(0,\infty)$ for which $\int_0^\infty u \wedge u^2\, \eta(\cdot,\cdot,du)$ is bounded. Assume that f is of the form

$$f(t,x,w) = b(t,x)w + c(t,x)w^2 + \int_0^\infty (e^{-uw} - 1 + uw)\, \eta(t,x,du) \tag{2.9}$$

for all $(t,x,w) \in [0,T] \times S \times \mathbb{R}_+$, then $f \geq 0$ and $f \in BC_{\mu}^{1-}([0,T] \times S \times \mathbb{R}_+)$, due to Examples 2.6. Hence, Theorem 2.15 applies. For instance, let $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_n \in (1,2)$ and $d_1, \ldots, d_n \in B_b([0,T] \times S, \mathbb{R}_+)$, then f could admit the representation

$$f(t,x,w) = b(t,x)w + c(t,x)w^2 + \sum_{i=1}^n d_i(t,x) w^{\alpha_i}$$

for each $(t,x,w) \in [0,T] \times S \times \mathbb{R}_+$. This follows directly from integration by parts and the choice $\eta(t,x,B) = \sum_{i=1}^n d_i(t,x)\alpha_i(\alpha_i - 1)\Gamma(2-\alpha_i)^{-1} \int_B u^{-1-\alpha_i}\, du$ for all $(t,x) \in [0,T] \times S$ and each Borel set B in $(0,\infty)$, where Γ denotes the Gamma function.

In the general case (2.9), Theorem 1.1 in Dynkin [6] yields an $(\mathscr{X},\mu,f)$-superprocess, which is a progressive Markov process $\mathscr{Z} = (Z, (\mathscr{G}_t)_{t\in[0,T]}, \mathbb{Q})$ with state space $\mathscr{M}(S)$, the Polish space of all finite Borel measures on S, so that for each $t \in (0,T]$ and every consistent $\tilde{g} \in B_b(S,\mathbb{R}_+)$, the function

$$[0,t] \times S \to \mathbb{R}_+, \quad (r,x) \mapsto -\log\left(E_{r,\delta_x}^{Q}\left[e^{-\int_S \tilde{g}(y)\, Z_t(dy)}\right]\right)$$

is Borel measurable and a global solution to (M) when T and g are replaced by t and $\tilde{g}$, respectively. Here, $\mathbb{Q}$ is of the form $\mathbb{Q} = \{Q_{r,\nu} \,|\, (r,\nu) \in [0,T] \times \mathscr{M}(S)\}$ and E_{r,δ_x}^{Q} denotes the expectation with respect to Q_{r,δ_x} for all $(r,x) \in [0,T] \times S$. Thus,

$$\overline{u}_g(r,x) = -\log\left(E_{r,\delta_x}^{Q}\left[e^{-\int_S g(y)\, Z_T(dy)}\right]\right) \quad \text{for each } (r,x) \in [0,T] \times S.$$

Finally, a combination of Theorem 2.15 with Proposition 2.13 gives the following result.

Corollary 2.17. *Let D be a non-degenerate interval, $\underline{d} := \inf D$ and $\overline{d} := \sup D$. Suppose that there are $a, b \in B_{\mu}([0,T] \times S)$ such that*

$$f(t,x,w) = a(t,x) + b(t,x)w \quad \textit{for all } (t,x,w) \in [0,T] \times S \times D.$$

Additionally, for $\underline{d} > -\infty$ *(resp.* $\overline{d} < \infty$*) let* $a(\cdot, x) + b(\cdot, x)\underline{d} \leq 0$ *(resp.* $a(\cdot, x) + b(\cdot, x)\overline{d} \geq 0$*) for all* $x \in S$ μ*-a.s. Then*

$$\overline{u}_g(r, x) = E_{r,x}\Big[e^{-\int_r^T b(s,X_s)\,\mu(ds)} g(X_T)\Big] - E_{r,x}\Big[\int_r^T e^{-\int_r^t b(s,X_s)\,\mu(ds)} a(t, X_t)\,\mu(dt)\Big] \tag{2.10}$$

for every $(r, x) \in [0, T] \times S$*. Furthermore, whenever* $\mathscr{X}$ *is Feller (right-)continuous,* $a(s, \cdot)$ *and* $b(s, \cdot)$ *are continuous for* μ*-a.e.* $s \in [0, T]$ *and* g *is continuous, then* $\overline{u}_g$ *is (right-)continuous.*

3 Approach to the main results

3.1 Comparison, stability and growth behavior of solutions

We let I be an admissible interval and first give a Markovian Gronwall inequality. A well-known result in this direction is provided by Dynkin [5, Lemma 3.2].

Lemma 3.1. *Let* $h \in B(S, \mathbb{R}_+)$ *be so that* $E_{r,x}[h(X_T)] < \infty$ *for all* $(r, x) \in I \times S$ *and* $a, b \in B_\mu(I \times S, \mathbb{R}_+)$*. Assume that* $u \in B(I \times S, \mathbb{R}_+)$ *is* μ*-suitably bounded and satisfies*

$$u(r, x) \leq E_{r,x}[h(X_T)] + E_{r,x}\Big[\int_r^T a(s, X_s) + b(s, X_s)u(s, X_s)\,\mu(ds)\Big]$$

for each $(r, x) \in I \times S$*, then*

$$u(r, x) \leq E_{r,x}\Big[e^{\int_r^T b(s,X_s)\,\mu(ds)}\Big(h(X_T) + \int_r^T a(s, X_s)\,\mu(ds)\Big)\Big]$$

for every $(r, x) \in I \times S$*.*

Proof. It follows inductively from the Markov property of $\mathscr{X}$ and integration by parts that

$$\begin{aligned} u(r, x) \leq{}& \sum_{i=0}^n E_{r,x}\Big[\frac{1}{i!}\Big(\int_r^T b(s, X_s)\,\mu(ds)\Big)^i \Big(h(X_T) + \int_r^T a(s, X_s)\,\mu(ds)\Big)\Big] \\ &+ E_{r,x}\Big[\int_r^T \Big(\int_r^t b(s, X_s)\,\mu(ds)\Big)^n \frac{b(t, X_t)}{n!} u(t, X_t)\,\mu(dt)\Big] \end{aligned}$$

for all $(r, x) \in I \times S$ and each $n \in \mathbb{N}_0$. Since u is μ-suitably bounded, dominated convergence yields that

$$\lim_{n\uparrow\infty} E_{r,x}\Big[\int_r^T \Big(\int_r^t b(s, X_s)\,\mu(ds)\Big)^n \frac{b(t, X_t)}{n!} u(t, X_t)\,\mu(dt)\Big] = 0.$$

Hence, monotone convergence establishes the asserted estimate. □

Let us compare approximate solutions.

Lemma 3.2. *Assume that there is* $W \subset D$ *for which* f *restricted to* $[0, T] \times S \times W$ *is Lipschitz* μ*-continuous. That is, there is* $\lambda \in B_\mu([0, T] \times S, \mathbb{R}_+)$ *such that*

$$|f(t, x, w) - f(t, x, \tilde{w})| \leq \lambda(t, x)|w - \tilde{w}| \quad \text{for all } (t, x) \in [0, T] \times S$$

and each $w, \tilde{w} \in W$*. Let* $\tilde{g} \in B_b(S, D)$ *be consistent and* $\varepsilon, \tilde{\varepsilon} \in B_\mu([0, T] \times S, \mathbb{R}_+)$*. Then every* ε*-approximate solution* u *to* (M) *on* I *and each* $\tilde{\varepsilon}$*-approximate solution* $\tilde{u}$ *to* (M) *on* I*, where* g *is replaced by* $\tilde{g}$*, satisfy*

$$|u - \tilde{u}|(r, x) \leq E_{r,x}\Big[e^{\int_r^T \lambda(s,X_s)\,\mu(ds)}\Big(|g - \tilde{g}|(X_T) + \int_r^T (\varepsilon + \tilde{\varepsilon})(s, X_s)\,\mu(ds)\Big)\Big]$$

for all $(r, x) \in I \times S$*, provided* u *and* $\tilde{u}$ *are* μ*-suitably bounded and take all its values in* W*.*

Proof. The triangle inequality yields that

$$|u-\tilde{u}|(r,x) \leq E_{r,x}[|g-\tilde{g}|(X_T)] + E_{r,x}\bigg[\int_r^T (\varepsilon+\tilde{\varepsilon})(s,X_s) + \lambda(s,X_s)|u-\tilde{u}|(s,X_s)\,\mu(ds)\bigg]$$

for each $(r,x) \in I \times S$, since we have $|f(s,X_s,u(s,X_s)) - f(s,X_s,\tilde{u}(s,X_s))| \leq \lambda(s,X_s)|u-\tilde{u}|(s,X_s)$ for μ-a.e. $s \in [r,T]$. So, Lemma 3.1 leads us to the asserted estimate. □

From the comparison we get an uniqueness result provided f belongs to the linear space (2.3). Note that the procedure of the proof originates from Theorem 6.7 in Amann [1].

Corollary 3.3. *Suppose that $f \in BC_\mu^{1-}([0,T]\times S\times D,\mathbb{R}^k)$. Then there is at most a unique weakly admissible solution to* (M) *on I. In particular, if D is closed, then* (M) *admits at most one global bounded solution.*

Proof. Assume that u and $\tilde{u}$ are two weakly admissible solutions to (M) on I and let $r \in I$. Then there is a compact set K in D such that $u(t,x), \tilde{u}(t,x) \in K$ for all $(t,x) \in [r,T]\times S$. As K is compact, it follows from Proposition 2.8 in [13] that there is a neighborhood W of K in D such that f restricted to $[0,T]\times S\times W$ is Lipschitz μ-continuous. Hence, $u = \tilde{u}$ on $[r,T]\times S$, by Lemma 3.2, and the assertions follow. □

Now we consider stability.

Proposition 3.4. *Let $f \in BC_\mu^{1-}([0,T]\times S\times D,\mathbb{R}^k)$ and for each $n \in \mathbb{N}$ let $g_n \in B_b(S,D)$ be consistent, $\varepsilon_n \in B_\mu([0,T]\times S,\mathbb{R}_+)$ and u_n be an ε_n-approximate solution to* (M) *on I with g replaced by g_n. Assume that the following two conditions hold:*

(i) $(g_n)_{n\in\mathbb{N}}$ and $(\int_0^T \varepsilon_n(t,X_t)\,\mu(dt))_{n\in\mathbb{N}}$ converge uniformly to g and 0, respectively.

(ii) For each $r \in I$ there is a compact set K in D such that $u_n([r,T]\times S) \subset K$ for all $n \in \mathbb{N}$.

Then $(u_n)_{n\in\mathbb{N}}$ converges locally uniformly in $t \in I$ and uniformly in $x \in S$ to the unique weakly admissible solution to (M) *on I.*

Proof. As uniqueness is covered by Corollary 3.3, we directly turn to the convergence claim. Let $r \in I$ and K be a compact set in D so that $u_n(t,x) \in K$ for all $n \in \mathbb{N}$ and each $(t,x) \in [r,T]\times S$. Then there are a neighborhood W of K in D and $\lambda \in B_\mu([r,T]\times S,\mathbb{R}_+)$ with $|f(t,x,w) - f(t,x,\tilde{w})| \leq \lambda(t,x)|w-\tilde{w}|$ for all $(t,x) \in [r,T]\times S$ and each $w,\tilde{w} \in W$. Thus, Lemma 3.2 ensures that

$$|u_m - u_n|(s,x) \leq E_{s,x}\bigg[e^{\int_s^T \lambda(t,X_t)\,\mu(dt)}\bigg(|g_m - g_n|(X_T) + \int_s^T (\varepsilon_m + \varepsilon_n)(t,X_t)\,\mu(dt)\bigg)\bigg]$$

for all $m,n \in \mathbb{N}$ and every $(s,x) \in [r,T]\times S$. From (i) we infer that $(u_n)_{n\in\mathbb{N}}$ is a uniformly Cauchy sequence on $[r,T]\times S$, and as $r \in I$ has been arbitrarily chosen, it converges locally uniformly in $t \in I$ and uniformly in $x \in S$ to some map $u \in B(I\times S,D)$.

We now check that u is a weakly admissible solution to (M) on I. Let as before $r \in I$ and K be a compact set in D with $u_n([r,T]\times S) \subset K$ for all $n \in \mathbb{N}$, which immediately gives $u([r,T]\times S) \subset K$. Further, let us pick $\lambda \in B_\mu([r,T]\times S,\mathbb{R}_+)$ with $|f(t,x,w) - f(t,x,\tilde{w})| \leq \lambda(t,x)|w-\tilde{w}|$ for all $(t,x) \in [r,T]\times S$ and every $w,\tilde{w} \in K$, then

$$\bigg|u_n(s,x) - E_{s,x}[g(X_T)] + E_{s,x}\bigg[\int_s^T f(t,X_t,u(t,X_t))\,\mu(dt)\bigg]\bigg|$$
$$\leq E_{s,x}[|g_n - g|(X_T)] + E_{s,x}\bigg[\int_s^T \varepsilon_n(t,X_t) + \lambda(t,X_t)|u_n - u|(t,X_t)\,\mu(dt)\bigg]$$

for all $n \in \mathbb{N}$ and each $(s,x) \in [r,T] \times S$. This entails that $(u_n)_{n\in\mathbb{N}}$ also converges locally uniformly in $t \in I$ and uniformly in $x \in S$ to the map

$$I \times S \to \mathbb{R}^k, \quad (r,x) \mapsto E_{r,x}[g(X_T)] - E_{r,x}\bigg[\int_r^T f(s,X_s,u(s,X_s))\,\mu(ds)\bigg],$$

which proves the proposition. □

We conclude with a growth estimate.

Lemma 3.5. *Assume that f is affine μ-bounded. In other words, there are $a,b \in B_\mu([0,T] \times S, \mathbb{R}_+)$ such that $|f(t,x,w)| \leq a(t,x) + b(t,x)|w|$ for all $(t,x,w) \in [0,T] \times S \times D$. Then every μ-suitably bounded solution u to* (M) *on I satisfies*

$$|u(r,x)| \leq E_{r,x}\bigg[e^{\int_r^T b(s,X_s)\,\mu(ds)}\bigg(|g(X_T)| + \int_r^T a(s,X_s)\,\mu(ds)\bigg)\bigg] \tag{3.1}$$

for each $(r,x) \in I \times S$.

Proof. We directly see that $|u(r,x)| \leq E_{r,x}[|g(X_T)|] + E_{r,x}[\int_r^T a(s,X_s) + b(s,X_s)|u(s,X_s)|\,\mu(ds)]$ for every $(r,x) \in I \times S$. In consequence, Lemma 3.1 implies the claimed estimate. □

3.2 Local existence in time

We aim to construct an approximate solution that is local in time. Once this is achieved, we apply the stability result of the previous section to deduce a solution as uniform limit of a sequence of approximate solutions.

For each $\beta > 0$ we define $N_{\mathscr{X},\beta}(g)$ to be the set of all $w \in \mathbb{R}^k$ such that $|w - E_{r,x}[g(X_T)]| < \beta$ for some $(r,x) \in [0,T] \times S$. Because we are dealing with the transition probabilities $\mathbb{P}$, the convexity of D should be required, as the lemma below indicates.

Lemma 3.6. *Let D be convex and g be bounded away from ∂D. That is, there is $\varepsilon > 0$ such that* $\mathrm{dist}(g(x), \partial D) \geq \varepsilon$ *for all $x \in S$. Then there exists $\beta > 0$ such that*

$$N_{\mathscr{X},\beta}(g) \text{ is relatively compact in } D^\circ. \tag{3.2}$$

Proof. Let K be a compact set in D° such that $g(S) \subset K$, then $\int_S g(x)\,P(dx)$ belongs to the convex hull of K for each probability measure P on $(S, \mathscr{B}(S))$. As the convexity of D entails that of D°, it follows from Carathéodory's Convex Hull Theorem that along with K the convex hull of K is a compact set in D°. Hence, there is $\beta > 0$ so that $\inf_{(r,x)\in[0,T]\times S} \mathrm{dist}(E_{r,x}[g(X_T)], \partial D) > \beta$. Since $N_{\mathscr{X},\beta}(g)$ is simply the β-neighborhood of $\{E_{r,x}[g(X_T)]\,|\,(r,x) \in [0,T] \times S\}$, the asserted condition (3.2) follows. □

Until the end of this section, let D be convex, f be locally μ-bounded and g be bounded away from ∂D. Due to Lemma 3.6, we can choose $\beta > 0$ satisfying (3.2). Let $a \in B_\mu([0,T] \times S, \mathbb{R}_+)$ be such that $|f(t,x,w)| \leq a(t,x)$ for all $(t,x) \in [0,T] \times S$ and each $w \in \overline{N}_{\mathscr{X},\beta}(g)$, the closure of $N_{\mathscr{X},\beta}(g)$. Then

$$E_{r,x}\bigg[\int_r^T a(s,X_s)\,\mu(ds)\bigg] \leq \beta \tag{3.3}$$

for all $(r,x) \in [T-\alpha,T] \times S$ and some $\alpha \in (0,T]$. The choices of β and α such that (3.2) and (3.3) hold, respectively, are used to construct an $\overline{N}_{\mathscr{X},\beta}(g)$-valued solution to (M) on $[T-\alpha,T]$.

Proposition 3.7. *Let $\varepsilon \in B_\mu([0,T] \times S, \mathbb{R}_+)$ and $\delta > 0$ be so that $|f(t,x,w) - f(t,x,\tilde{w})| \leq \varepsilon(t,x)$ for all $(t,x) \in [0,T]\times S$ and each $w, \tilde{w} \in \overline{N}_{\mathscr{X},\beta}(g)$ with $|w-\tilde{w}| < \delta$. Then there is an $\overline{N}_{\mathscr{X},\beta}(g)$-valued ε-approximate solution u to* (M) *on $[T-\alpha, T]$. In addition, if $\mathscr{X}$ is Feller (right-)continuous, $f(s,\cdot,\cdot)$ is continuous for μ-a.e. $s \in [0,T]$ and g is continuous, then u is (right-)continuous.*

Proof. At first, since a is μ-dominated, there is $\eta \in (0,\alpha]$ such that $E_{r,x}[\int_r^t a(s,X_s)\,\mu(ds)] < \delta$ for all $r,t \in [T-\alpha,T]$ with $r \leq t < r+\eta$ and each $x \in S$. Given η, we choose $n \in \mathbb{N}$ and $t_0,\dots,t_n \in [T-\alpha,T]$ such that

$$T - \alpha = t_n < \cdots < t_0 = T \quad \text{and} \quad \max_{i\in\{1,\dots,n\}}(t_{i-1} - t_i) < \eta.$$

Starting with $u_0 : [T-\alpha,T]\times S \to \overline{N}_{\mathscr{X},\beta}(g)$ given by $u_0(r,x) := E_{r,x}[g(X_T)]$, we recursively introduce a sequence $(u_i)_{i\in\{1,\dots,n\}}$ of Borel measurable maps, by letting $u_{i+1} : [t_{i+1}, t_i] \times S \to \overline{N}_{\mathscr{X},\beta}(g)$ be defined via

$$u_{i+1}(r,x) := E_{r,x}[u_i(t_i, X_{t_i})] - E_{r,x}\bigg[\int_r^{t_i} f(s, X_s, E_{s,X_s}[u_i(t_i, X_{t_i})])\,\mu(ds)\bigg]$$

for each $i \in \{0,\dots,n-1\}$. It follows by induction over $i \in \{1,\dots,n\}$ that u_i is indeed a well-defined Borel measurable map taking all its values in $\overline{N}_{\mathscr{X},\beta}(g)$ such that

$$|E_{r,x}[u_i(t,X_t)] - u_i(r,x)| \leq E_{r,x}\bigg[\int_r^t a(s,X_s)\,\mu(ds)\bigg] \tag{3.4}$$

for all $r,t \in [t_i, t_{i-1}]$ with $r \leq t$ and each $x \in S$. This is an immediate consequence of the facts that $u_{i-1}(t_{i-1},x) = u_i(t_{i-1},x)$ and $|E_{r,x}[u_i(t,X_t)] - u_0(r,x)| \leq E_{r,x}[\int_t^{t_0} a(t',X_{t'})\,\mu(dt')]$ for each $r,t \in [t_i,t_{i-1}]$ with $r \leq t$ and every $x \in S$.

The crucial outcome of this construction is that the map $u : [T-\alpha,T]\times S \to \overline{N}_{\mathscr{X},\beta}(g)$ defined by $u(r,x) := u_i(r,x)$ with $i \in \{1,\dots,n\}$ so that $r \in [t_i,t_{i-1}]$ is an ε-approximate solution to (M) on $[T-\alpha,T]$. To see this, let $i \in \{1,\dots,n\}$, then

$$\begin{aligned}
&\bigg|E_{r,x}[u(t,X_t)] - u(r,x) - E_{r,x}\bigg[\int_r^t f(s,X_s,u(s,X_s))\,\mu(ds)\bigg]\bigg| \\
&\quad = \bigg|E_{r,x}\bigg[\int_r^t f(s,X_s,E_{s,X_s}[u_{i-1}(t_{i-1},X_{t_{i-1}})]) - f(s,X_s,u_i(s,X_s))\,\mu(ds)\bigg]\bigg| \\
&\quad \leq E_{r,x}\bigg[\int_r^t \varepsilon(s,X_s)\,\mu(ds)\bigg]
\end{aligned}$$

for every $r,t \in [t_i,t_{i-1}]$ with $r \leq t$ and each $x \in S$, since $u_{i-1}(t_{i-1},X_{t_{i-1}}) = u_i(t_{i-1},X_{t_{i-1}})$ and from $t_{i-1} - t_i < \eta$ in combination with (3.4) we infer that $|E_{s,X_s}[u_i(t_{i-1},X_{t_{i-1}})] - u_i(s,X_s)| < \delta$ for all $s \in [t_i,t_{i-1}]$. Hence, the first assertion follows.

Now let $\mathscr{X}$ be Feller (right-)continuous, $f(s,\cdot,\cdot)$ be continuous for μ-a.e. $s \in [0,T]$ and g be continuous. Then for each non-degenerate interval I in $[0,T]$ and every (right-)continuous map $\tilde{u} \in B(I\times S, D)$, we readily see that $f(s,\cdot,\tilde{u}(s,\cdot))$ is continuous for μ-a.e. $s \in I$. In combination with the assertion made at (2.4), it follows inductively that $u_1,\dots,u_n$ are (right-)continuous, which yields the second claim. □

By constructing a suitable sequence of approximate solutions, a local existence result can be derived.

Proposition 3.8. *Let $f \in BC_\mu^{1-}([0,T]\times S\times D, \mathbb{R}^k)$, then there is a unique admissible solution u to* (M) *on $[T-\alpha,T]$, which is $\overline{N}_{\mathscr{X},\beta}(g)$-valued. Moreover, if $\mathscr{X}$ is Feller (right-)continuous, $f(s,\cdot,\cdot)$ is continuous for μ-a.e. $s \in [0,T]$ and g is continuous, then u is (right-)continuous.*

Proof. The uniqueness assertion is implied by Corollary 3.3. To establish existence, we note that, as $\overline{N}_{\mathscr{X},\beta}(g)$ is compact, there exists $\lambda \in B_\mu([T-\alpha,T]\times S,\mathbb{R}_+)$ such that

$$|f(t,x,w) - f(t,x,\tilde{w})| \leq \lambda(t,x)|w-\tilde{w}| \quad \text{for all } (t,x) \in [T-\alpha,T]\times S$$

and each $w,\tilde{w} \in \overline{N}_{\mathscr{X},\beta}(g)$. Thus, Proposition 3.7 provides an $\overline{N}_{\mathscr{X},\beta}(g)$-valued (λ/n)-approximate solution u_n to (M) on $[T-\alpha,T]$ for each $n \in \mathbb{N}$. Additionally, if $\mathscr{X}$ is Feller (right-)continuous, $f(s,\cdot,\cdot)$ is continuous for μ-a.e. $s \in [0,T]$ and g is continuous, then u_n is (right-)continuous.

Proposition 3.4 entails that $(u_n)_{n\in\mathbb{N}}$ converges uniformly to an $\overline{N}_{\mathscr{X},\beta}(g)$-valued solution u to (M) on $[T-\alpha,T]$, which proves the first claim. Since the uniform limit of a sequence of $\mathbb{R}^k$-valued (right-)continuous maps on $[T-\alpha,T]\times S$ is again (right-)continuous, the second assertion follows directly from what we have just shown. □

Now we prove a fixed-point result, which we need later on.

Lemma 3.9. *Let I be a compact admissible interval, $\mathscr{U} \subset B_b(I\times S,\mathbb{R}^k)$ be closed under uniform convergence and $\Psi : \mathscr{U} \to \mathscr{U}$ be a map for which there is $\lambda \in B_\mu(I\times S,\mathbb{R}_+)$ such that*

$$|\Psi(u) - \Psi(\tilde{u})|(r,x) \leq E_{r,x}\bigg[\int_r^T \lambda(s,X_s)|u-\tilde{u}|(s,X_s)\,\mu(ds)\bigg] \tag{3.5}$$

for all $u,\tilde{u} \in \mathscr{U}$ and each $(r,x) \in I\times S$. Then for every $u_0 \in \mathscr{U}$, the sequence $(u_n)_{n\in\mathbb{N}_0}$, recursively given by $u_n := \Psi(u_{n-1})$ for all $n \in \mathbb{N}$, converges uniformly to the unique fixed-point of Ψ.

Proof. Because the uniqueness assertion can be easily inferred from Lemma 3.1, we just show that $(u_n)_{n\in\mathbb{N}_0}$ converges uniformly to some fixed-point of Ψ. By induction,

$$|u_{n+1}-u_n|(r,x) \leq E_{r,x}\bigg[\int_r^T \bigg(\int_r^t \lambda(s,X_s)\,\mu(ds)\bigg)^{n-1}\frac{\lambda(t,X_t)}{(n-1)!}\Delta(t,X_t)\,\mu(dt)\bigg]$$

for each $n \in \mathbb{N}$ and every $(r,x) \in I\times S$, where $\Delta := |\Psi(u_0)-u_0|$. From the triangle inequality and integration by parts we obtain that

$$|u_m - u_n|(r,x) \leq \sum_{i=n}^{m-1}\frac{1}{i!}E_{r,x}\bigg[\bigg(\int_r^T \lambda(s,X_s)\,\mu(ds)\bigg)^i\bigg]\sup_{(t,y)\in[r,T]\times S}\Delta(t,y)$$

for all $m,n \in \mathbb{N}$ with $m > n$ and each $(r,x) \in I\times S$. This shows that $(u_n)_{n\in\mathbb{N}_0}$ is a uniformly Cauchy sequence. Hence, it converges uniformly to some $u \in \mathscr{U}$. As $(u_{n+1})_{n\in\mathbb{N}_0}$ converges uniformly to $\Psi(u)$, we conclude that $u = \Psi(u)$. □

Let us indicate another local existence approach.

Remark 3.10. The set $\mathscr{U} := B_b([T-\alpha,T]\times S,\overline{N}_{\mathscr{X},\beta}(g))$ is closed under uniform convergence and (3.3) guarantees that the map $\Psi : \mathscr{U} \to B([T-\alpha,T]\times S,\mathbb{R}^k)$ defined via

$$\Psi(u)(r,x) := E_{r,x}[g(X_T)] - E_{r,x}\bigg[\int_r^T f(s,X_s,u(s,X_s))\,\mu(ds)\bigg]$$

maps $\mathscr{U}$ into itself. So, let f be locally Lipschitz μ-continuous, then there is $\lambda \in B_\mu([T-\alpha,T]\times S,\mathbb{R}_+)$ satisfying (3.5) for all $u,\tilde{u} \in \mathscr{U}$ and each $(r,x) \in [T-\alpha,T]\times S$. For this reason, Lemma 3.9 implies that Ψ has a unique fixed-point u, which is exactly the unique admissible solution to (M) on $[T-\alpha,T]$ that takes all its values in $\overline{N}_{\mathscr{X},\beta}(g)$.

Moreover, if $\mathscr{X}$ is Feller (right-)continuous, $f(s,\cdot,\cdot)$ is continuous for μ-a.e. $s \in [0,T]$ and g is continuous, then from (2.4) we see that $\Psi(\tilde{u})$ is (right-)continuous whenever $\tilde{u} \in \mathscr{U}$ is. In this case, u is (right-)continuous as uniform limit of a sequence of (right-)continuous maps in $\mathscr{U}$.

4 Proofs of the main results

4.1 Proof of Theorem 2.11

After having constructed solutions that are local in time, we derive unique non-extendible solutions. In this regard, the proof of Theorem 7.6 in [1] has been a good source for ideas.

Proof of Theorem 2.11. We begin with the first claim and define I_g to be the set consisting of $\{T\}$ and of all $t \in [0,T)$ for which (M) admits an admissible solution on $[t,T]$. By Proposition 3.8, we have $\{T\} \subsetneq I_g$ and hence, $t_g^- = \inf I_g < T$. Let $t \in (t_g^-, T]$, then there is $s \in I_g$ with $s < t$, which means that there is an admissible solution u to (M) on $[s,T]$. As u restricted to $[t,T] \times S$ is an admissible solution to (M) on $[t,T]$, we get that $t \in I_g$. Thus, I_g is an admissible interval.

To verify that I_g is open in $[0,T]$, we have to show that if $I_g \neq [0,T]$, then $t_g^- \notin I_g$. On the contrary, assume that $I_g \neq [0,T]$ but $t_g^- \in I_g$. Then $t_g^- > 0$ and there is an admissible solution u to (M) on $[t_g^-, T]$. Since $u(t_g^-, \cdot)$ is bounded and bounded away from ∂D, Proposition 3.8 entails that the Markovian terminal value problem (M) with T and g replaced by t_g^- and $u(t_g^-, \cdot)$, respectively, has an admissible solution v on $[t_g^- - \alpha, t_g^-]$ for some $\alpha \in (0, t_g^-]$. Consequently, the map $w : [t_g^- - \alpha, T] \times S \to D^\circ$ given by $w(r,x) := u(r,x)$, if $r \geq t_g^-$, and $w(r,x) := v(r,x)$, otherwise, is another admissible solution to (M) on $[t_g^- - \alpha, T]$ extending u and v. We conclude that $t_g^- - \alpha \in I_g$, which contradicts the definition of t_g^-.

Let us now introduce the unique non-extendible admissible solution to (M). We recall that if $r, t \in I_g$ satisfy $r \leq t$ and u, v are two admissible solutions to (M) on $[r,T]$ and $[t,T]$, respectively, then $u = v$ on $[t,T] \times S$, by Corollary 3.3. So, for each $r \in I_g$ we can mark the unique admissible solution to (M) on $[r,T]$ by $u_{g,r}$. Then

$$u_g : I_g \times S \to D^\circ, \quad u_g(r,x) := u_{g,r}(r,x)$$

is the unique non-extendible admissible solution to (M). In fact, if $t_g^- \in I_g$, which occurs if and only if $t_g^- = 0$ and $I_g = [0,T]$, then $u_g = u_{g,0}$. This in turn implies that u_g is well-defined and a global admissible solution. Let instead $t_g^- \notin I_g$, then $I_g = (t_g^-, T]$. In this case, we pick a strictly decreasing sequence $(t_n)_{n\in\mathbb{N}}$ in I_g with $\lim_{n\uparrow\infty} t_n = t_g^-$, then

$$u_g^{-1}(B) = \bigcup_{n\in\mathbb{N}} u_{t_n}^{-1}(B) \in \mathscr{B}(I_g \times S)$$

for all $B \in \mathscr{B}(D)$, since $u_{t_n}^{-1}(B) \in \mathscr{B}([t_n, T] \times S)$ for each $n \in \mathbb{N}$. Thus, u_g is Borel measurable and an admissible solution to (M) on I_g, since it coincides with $u_{g,r}$ on $[r,T] \times S$ for each $r \in I_g$. Suppose that I is an admissible interval with $I_g \subsetneq I$ and u is an admissible solution to (M) on I, then there is $t \in I$ with $t \leq t_g^-$. By the definition of I_g, we obtain that $t \in I_g$, which is a contradiction to $I_g = (t_g^-, T]$. This justifies that u_g is non-extendible.

We turn to the second claim. By way of contradiction, assume that $I_g \neq [0,T]$ but (2.6) fails. Then $I_g = (t_g^-, T]$ and there are $\varepsilon \in (0, 1/\sqrt{2})$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in I_g with $\lim_{n\uparrow\infty} t_n = t_g^-$ such that

$$\inf_{x\in S} \min\left\{ \operatorname{dist}(u_g(t_n,x), \partial D), \frac{1}{1 + |u_g(t_n,x)|} \right\} \geq 2\varepsilon \quad \text{for all } n \in \mathbb{N}.$$

As Corollary A.16 in [13] ensures that $D_\eta := \{w \in D \,|\, \operatorname{dist}(w, \partial D) \geq \eta \text{ and } |w| \leq 1/\eta\}$ is a convex compact set in D° for each $\eta \in (0, 2\varepsilon]$, it holds that $E_{r,x}[u_g(t_n, X_{t_n})] \in D_{2\varepsilon}$ for all $n \in \mathbb{N}$ and every $(r,x) \in [0,t_n] \times S$. Let $a \in B_\mu([t_g^-, T] \times S, \mathbb{R}_+)$ satisfy

$$|f(t,x,w)| \leq a(t,x)$$

for all $(t,x,w) \in [t_g^-, T] \times S \times D_\varepsilon$, then there is $\delta \in (0, T - t_g^-]$ so that $\sup_{x\in S} E_{r,x}[\int_r^t a(s, X_s)\,\mu(ds)] < \varepsilon$ for each $r, t \in [t_g^-, T]$ with $r \leq t < r + \delta$. This entails that

$$u_g(t, S) \text{ is relatively compact in } D_\varepsilon^\circ \tag{4.1}$$

for every $n \in \mathbb{N}$ and each $t \in (t_n - \delta_n, t_n]$, where $\delta_n := \delta \wedge (t_n - t_g^-)$. Indeed, suppose this is false, then there is $n \in \mathbb{N}$ for which $u_g(t, S)$ fails to be relatively compact in D_ε° for at least one $t \in (t_n - \delta_n, t_n]$. We set

$$s_n := \sup\{t \in (t_n - \delta_n, t_n] \,|\, u_g(t, S) \text{ is not relatively compact in } D_\varepsilon^\circ\},$$

then another application of Proposition 3.8 shows that $u_g(s_n, S)$ cannot be relatively compact in D_ε°. In particular, $s_n < t_n$, as $u_g(t_n, S) \subset D_{2\varepsilon}$. These considerations imply that

$$|E_{s_n,x}[u_g(t_n, X_{t_n})] - u_g(s_n, x)| \leq E_{s_n,x}\bigg[\int_{s_n}^{t_n} a(s, X_s)\,\mu(ds)\bigg] < \varepsilon$$

for every $x \in S$, since $t_n - s_n < \delta_n \leq \delta$. From $E_{s_n,x}[u_g(t_n, X_{t_n})] \in D_{2\varepsilon}$ and $\varepsilon^2 < 1/2$ it follows that $|u_g(s_n, x)| < |E_{s_n,x}[u_g(t_n, X_{t_n})]| + \varepsilon \leq 1/(2\varepsilon) + \varepsilon < 1/\varepsilon$ for each $x \in S$. Moreover,

$$\begin{aligned}\operatorname{dist}(u_g(s_n, x), \partial D) &\geq \operatorname{dist}(E_{s_n,x}[u_g(t_n, X_{t_n})], \partial D) - |E_{s_n,x}[u_g(t_n, X_{t_n})] - u_g(s_n, x)| \\ &\geq 2\varepsilon - |E_{s_n,x}[u_g(t_n, X_{t_n})] - u_g(s_n, x)| > \varepsilon\end{aligned}$$

for all $x \in S$. In consequence, it follows that $u_g(s_n, S)$ is relatively compact in D_ε°, which is a contradiction. Therefore, condition (4.1) is valid.

Next, since $\lim_{n\uparrow\infty} t_n = t_g^-$, there is $n_0 \in \mathbb{N}$ such that $t_n - t_g^- \leq \delta$ and hence, $t_n - \delta_n = t_g^-$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Thus, (4.1) leads us to

$$|E_{t_g^-,x}[u_g(t, X_t)] - E_{t_g^-,x}[u_g(r, X_r)]| \leq E_{t_g^-,x}\bigg[\int_r^t a(s, X_s)\,\mu(ds)\bigg] < \varepsilon$$

for every $r, t \in (t_g^-, t_{n_0}]$ with $r \leq t$ and each $x \in S$. So, the map $(t_g^-, T] \times S \to D^\circ$, $(t, x) \mapsto E_{t_g^-,x}[u_g(t, X_t)]$ is uniformly continuous in $t \in (t_g^-, T]$, uniformly in $x \in S$. By Proposition A.12 in [13], there exists a unique map $\hat{w} \in B_b(S, D_\varepsilon)$ such that

$$\lim_{t\downarrow t_g^-} E_{t_g^-,x}[u_g(t, X_t)] = \hat{w}(x), \quad \text{uniformly in } x \in S.$$

At the same time, it follows from (4.1) together with dominated convergence that

$$\lim_{r\downarrow t_g^-} E_{t_g^-,x}\bigg[\int_r^T f(s, X_s, u_g(s, X_s))\,\mu(ds)\bigg] = E_{t_g^-,x}\bigg[\int_{(t_g^-,T]} f(s, X_s, u_g(s, X_s))\,\mu(ds)\bigg] \tag{4.2}$$

for every $x \in S$. Since the map $(t_g^-, T] \times S \to \mathbb{R}^k$, $(r, x) \mapsto E_{t_g^-,x}[\int_r^T f(s, X_s, u_g(s, X_s))\,\mu(ds)]$ is uniformly continuous in $r \in (t_g^-, T]$, uniformly in $x \in S$, the limit (4.2) holds even uniformly in $x \in S$. Thus, we define $u : [t_g^-, T] \times S \to D^\circ$ by

$$u(t, x) := u_g(t, x), \quad \text{if } t > t_g^-, \quad \text{and} \quad u(t, x) := \hat{w}(x), \quad \text{otherwise},$$

then it is immediate to see that u is another admissible solution to (M) on $[t_g^-, T]$. Hence, $t_g^- \in I_g$, which contradicts that I_g is open in $[0, T]$. This concludes the verification of the second claim.

At last, let $\mathscr{X}$ be Feller (right-)continuous, $f(s, \cdot, \cdot)$ be continuous for μ-a.e. $s \in [0, T]$ and g be continuous. We define $\tilde{I}_g$ to be the set consisting of $\{T\}$ and of all $t \in [0, T)$ for which there is a (right-)continuous admissible solution to (M) on $[t, T]$, and set $\tilde{t}_g^- := \inf \tilde{I}_g$. Then Proposition 3.8 ensures that $\{T\} \subsetneq \tilde{I}_g$ and thus, $\tilde{t}_g^- < T$. Similar arguments as before yield that $\tilde{I}_g$ is an admissible interval that is open in $[0, T]$.

By Corollary 3.3, the proof is complete, once we have shown that $\tilde{t}_g^- = t_g^-$. Since $\tilde{t}_g^- \geq t_g^-$, let us suppose that $\tilde{t}_g^- > t_g^-$. Then $\tilde{I}_g \neq [0, T]$ and hence, $\tilde{I}_g = (\tilde{t}_g^-, T]$. As u_g must be (right-)continuous on $\tilde{I}_g \times S$ and

$$u_g(r, x) = E_{r,x}[g(X_T)] - E_{r,x}\bigg[\int_r^T f(s, X_s, u_g(s, X_s))\,\mu(ds)\bigg]$$

for all $(r, x) \in [\tilde{t}_g^-, T] \times S$, we infer from (2.4) that u_g is actually (right-)continuous on $[\tilde{t}_g^-, T] \times S$. For this reason, we must face the contradiction that $\tilde{t}_g^- \in \tilde{I}_g$. $\square$

4.2 Proofs of Propositions 2.12 and 2.13

Proof of Proposition 2.12. To establish the first claim, we will invoke Lemma 3.9. First, since f is affine μ-bounded, Lemma 3.5 implies that u_g is bounded, and as (2.7) cannot hold, we get that $I_g = [0,T]$. Hence, u_g is the unique global bounded solution to (M), by Theorem 2.11.

We choose $a, b \in B_\mu([0,T]\times S, \mathbb{R}_+)$ so that $|f(t,x,w)| \leq a(t,x) + b(t,x)|w|$ for all $(t,x,w) \in [0,T]\times S\times\mathbb{R}^k$ and let $\mathscr{U}$ denote the set of all $u \in B([0,T]\times S, \mathbb{R}^k)$ satisfying (3.1) for each $(r,x) \in [0,T]\times S$. Then $\mathscr{U}$ is closed under uniform convergence and $u_0, u_g \in \mathscr{U}$. Further, we pick two measurable μ-integrable functions $\overline{a}, \overline{b} : [0,T] \to \mathbb{R}_+$ with $a(\cdot, y) \leq \overline{a}$ and $b(\cdot, y) \leq \overline{b}$ for all $y \in S$ μ-a.s., and set

$$c := e^{\int_0^T \overline{b}(s)\,\mu(ds)}\Big(\sup_{y\in S}|g(y)| + \int_0^T \overline{a}(s)\,\mu(ds)\Big).$$

Then each map $u \in \mathscr{U}$ satisfies $|u(r,x)| \leq c$ for each $(t,x) \in [0,T]\times S$. In addition, we introduce the map $\Psi : \mathscr{U} \to B_b([0,T]\times S, \mathbb{R}^k)$ defined via

$$\Psi(u)(r,x) := u_0(r,x) - E_{r,x}\Big[\int_r^T f(s, X_s, u(s,X_s))\,\mu(ds)\Big].$$

Clearly, a map $u \in \mathscr{U}$ is a global solution to (M) if and only if it coincides with u_g, the unique fixed-point of Ψ. From the Markov property of $\mathscr{X}$ and integration by parts we infer that Ψ maps $\mathscr{U}$ into itself. Finally, let $\lambda \in B_\mu([0,T]\times S, \mathbb{R}_+)$ be such that

$$|f(t,x,w) - f(t,x,\tilde{w})| \leq \lambda(t,x)|w - \tilde{w}| \quad \text{for all } (t,x) \in [0,T]\times S$$

and every $w, \tilde{w} \in \mathbb{R}^k$ with $|w| \vee |\tilde{w}| \leq c$. This guarantees that (3.5) is valid for all $u, \tilde{u} \in \mathscr{U}$ and each $(r,x) \in [0,T]\times S$. As this was the last condition we had to check, the first claim follows.

Regarding the second claim, we merely have to note that if $\mathscr{X}$ is Feller (right-)continuous, $f(s,\cdot,\cdot)$ is continuous for μ-a.e. $s \in [0,T]$ and g is continuous, then $\Psi(\tilde{u})$ is (right-)continuous whenever $\tilde{u} \in \mathscr{U}$ is. $\square$

For the proof of Proposition 2.13 we consider a sequence of $\mathbb{R}^{k\times k}$-valued integral maps. To this end, we use the conventions that $[r,t] := [t,r]$ and $\int_r^t \overline{b}(s)\,\mu(ds) := -\int_t^r \overline{b}(s)\,\mu(ds)$ for all $r,t \in [0,T]$ with $t < r$, each $d \in \mathbb{N}$ and every measurable μ-integrable map $\overline{b} : [0,T] \to \mathbb{R}^{d\times d}$.

Lemma 4.1. *For $b \in B_\mu([0,T]\times S, \mathbb{R}^{k\times k})$ let the sequence $({}_n\mathrm{B})_{n\in\mathbb{N}_0}$ of $\mathbb{R}^{k\times k}$-valued bounded maps on $[0,T]\times[0,T]\times\Omega$ be recursively given by ${}_0\mathrm{B}_{r,t}(\omega) := \mathbb{I}_k$ and*

$${}_n\mathrm{B}_{r,t}(\omega) := \int_r^t b(s, X_s(\omega))\,{}_{n-1}\mathrm{B}_{s,t}(\omega)\,\mu(ds) \quad \text{for all } n \in \mathbb{N}.$$

Then ${}_n\mathrm{B}_{r,t}$ is $\sigma(X_s : s \in [r,t])$-measurable, $|{}_n\mathrm{B}_{r,t}| \leq (1/n!)(|\int_r^t |b(s,X_s)|\,\mu(ds)|)^n$ and the map ${}_n\mathrm{B}(\omega)$ is continuous for all $n \in \mathbb{N}_0$, each $r,t \in [0,T]$ and every $\omega \in \Omega$.

Proof. We prove the lemma inductively. For $n = 0$ the assignment ${}_0\mathrm{B} = \mathbb{I}_k$ gives all the results. Let us suppose that the claims are true for some $n \in \mathbb{N}_0$ and pick $r,t \in [0,T]$. Since $\mathscr{X}$ is progressive, the map $[r,t]\times\Omega \to \mathbb{R}^{k\times k}$, $(s,\omega) \mapsto b(s, X_s(\omega))\,{}_n\mathrm{B}_{s,t}(\omega)$ is $\mathscr{B}([r,t])\otimes\sigma(X_s : s \in [r,t])$-measurable, and as the spectral norm on $\mathbb{R}^{k\times k}$ is submultiplicative,

$$\begin{aligned}\Big|\int_r^t |b(s,X_s)\,{}_n\mathrm{B}_{s,t}|\,\mu(ds)\Big| &\leq \Big|\int_r^t \frac{|b(s,X_s)|}{n!}\Big(\Big|\int_s^t |b(s',X_{s'})|\,\mu(ds')\Big|\Big)^n\,\mu(ds)\Big| \\ &= \frac{1}{(n+1)!}\Big(\Big|\int_r^t |b(s,X_s)|\,\mu(ds)\Big|\Big)^{n+1}.\end{aligned}$$

Thus, ${}_{n+1}\mathrm{B}_{r,t}$ is well-defined and the required estimate holds. In addition, an application of Fubini's theorem to each coordinate ensures that ${}_{n+1}\mathrm{B}_{r,t}$ is $\sigma(X_s : s \in [r,t])$-measurable.

To show that the map $[0,T] \times [0,T] \to \mathbb{R}^{k\times k}$, $(r,t) \mapsto {}_{n+1}\mathrm{B}_{r,t}(\omega)$ is continuous for each $\omega \in \Omega$, let $r,t \in [0,T]$ and $(r_m,t_m)_{m\in\mathbb{N}}$ be a sequence in $[0,T] \times [0,T]$ that converges to (r,t). Then

$$\lim_{m\uparrow\infty} \mathbb{1}_{[r_m,t_m]}(s)\, {}_n\mathrm{B}_{s,t_m}(\omega) = \mathbb{1}_{[r,t]}(s)\, {}_n\mathrm{B}_{s,t}(\omega) \quad \text{for } \mu\text{-a.e. } s \in [0,T].$$

Therefore, $\lim_{m\uparrow\infty} {}_{n+1}\mathrm{B}_{r_m,t_m}(\omega) = {}_{n+1}\mathrm{B}_{r,t}(\omega)$, by dominated convergence. □

Proof of Proposition 2.13. Since f is affine μ-bounded and Lipschitz μ-continuous, Proposition 2.12 entails that the sequence $(u_n)_{n\in\mathbb{N}_0}$ in $B_b([0,T] \times S, \mathbb{R}^k)$, recursively given by $u_0(r,x) := E_{r,x}[g(X_T)]$ and

$$u_n(r,x) := u_0(r,x) - E_{r,x}\bigg[\int_r^T a(s,X_s) + b(s,X_s)u_{n-1}(s,X_s)\,\mu(ds)\bigg]$$

for all $n \in \mathbb{N}$, converges uniformly to u_g, the unique global bounded solution to (M). With the notation of Lemma 4.1, an induction proof shows that u_n is of the form

$$u_n(r,x) = E_{r,x}\bigg[\sum_{i=0}^{n} (-1)^i {}_i\mathrm{B}_{r,T}\, g(X_T)\bigg] - E_{r,x}\bigg[\int_r^T \sum_{i=0}^{n-1} (-1)^i {}_i\mathrm{B}_{r,t}\, a(t,X_t)\,\mu(dt)\bigg]$$

for all $n \in \mathbb{N}$ and each $(r,x) \in [0,T] \times S$. Since $\sum_{n=0}^{\infty} |{}_n\mathrm{B}_{r,t}| \leq \exp(|\int_r^t |b(s,X_s)|\,\mu(ds)|)$ for all $r,t \in [0,T]$, the series map $\sum_{n=0}^{\infty}(-1)^n {}_n\mathrm{B}$ converges absolutely, uniformly in $(r,t,\omega) \in [0,T] \times [0,T] \times \Omega$. Hence, Lemma 4.1 together with dominated convergence imply that the bounded limit map $\mathrm{B} := \sum_{n=0}^{\infty}(-1)^n {}_n\mathrm{B}$ satisfies (i) and the representation formula (2.8) is valid.

Let us verify that (ii) holds as well. From ${}_0\mathrm{B} = \mathbb{I}_k$ and ${}_n\mathrm{B}_{r,r} = 0$ for all $n \in \mathbb{N}$ we get that $\mathrm{B}_{r,r} = \mathbb{I}_k$ for each $r \in [0,T]$. By the Cauchy product for absolutely convergent matrix series, to verify that $\mathrm{B}_{r,s}\mathrm{B}_{s,t} = \mathrm{B}_{r,t}$ for every $r,s,t \in [0,T]$, it is enough to show that

$$\sum_{i=0}^{n} {}_i\mathrm{B}_{r,s}\, {}_{n-i}\mathrm{B}_{s,t} = {}_n\mathrm{B}_{r,t} \quad \text{for all } n \in \mathbb{N}_0,$$

which follows inductively. Furthermore, from $\mathrm{B}_{r,t}\mathrm{B}_{t,r} = \mathrm{B}_{r,r} = \mathbb{I}_k$ we conclude that $\mathrm{B}_{r,t}(\omega)$ is invertible and $\mathrm{B}_{r,t}(\omega)^{-1} = \mathrm{B}_{t,r}(\omega)$ for all $r,t \in [0,T]$ and each $\omega \in \Omega$.

Regarding (iii), let b satisfy $b(r,x)b(s,y) = b(s,y)b(r,x)$ for every $(r,x),(s,y) \in [0,T] \times S$. Then the proposition follows as soon as we have proven that

$${}_n\mathrm{B}_{r,t} = \frac{1}{n!}\bigg(\int_r^t b(s,X_s)\,\mu(ds)\bigg)^n \tag{4.3}$$

for every $n \in \mathbb{N}$ and each $r,t \in [0,T]$ with $r \leq t$. Hence, we write S_n for the set of all permutations of $\{1,\dots,n\}$ and set $C_n^\sigma(r,t) := \{(s_1,\dots,s_n) \in [r,t]^n \,|\, s_{\sigma(1)} \leq \cdots \leq s_{\sigma(n)}\}$ for each $\sigma \in S_n$. From the measure transformation formula we obtain that

$$\int_{C_n^\sigma(r,t)} b(s_1,X_{s_1})\cdots b(s_n,X_{s_n})\,d\mu^n(s_1,\dots,s_n)$$
$$= \int_{C_n(r,t)} b(s_1,X_{s_1})\cdots b(s_n,X_{s_n})\,d\mu^n(s_1,\dots,s_n) = {}_n\mathrm{B}_{r,t},$$

where $C_n(r,t) := \{(s_1,\dots,s_n) \in [r,t]^n \,|\, s_1 \leq \cdots \leq s_n\}$. In the end, we utilize that $[r,t]^n = \bigcup_{\sigma\in S_n} C_n^\sigma(r,t)$. Then the hypothesis that μ is atomless and Fubini's theorem lead to

$$\bigg(\int_r^t b(s,X_s)\,\mu(ds)\bigg)^n = \sum_{\sigma\in S_n} \int_{C_n^\sigma(r,t)} b(s_1,X_{s_1})\cdots b(s_n,X_{s_n})\,d\mu^n(s_1,\dots,s_n) = n!\, {}_n\mathrm{B}_{r,t}.$$

That is, (4.3) is justified and the proposition is proven. □

4.3 Proof of Theorem 2.15

We let $k = 1$ and first use the Feynman-Kac formula (2.8) to represent the difference of two solutions. This idea is essentially based on Proposition 3.1 in [17].

Lemma 4.2. *Let $f, \tilde{f} \in BC_{\mu}^{1-}([0,T] \times S \times D)$, $\tilde{g} \in B_b(S,D)$ be consistent, I be an admissible interval, u be a solution to* (M) *on I and $\tilde{u}$ be a solution to* (M) *on I with $\tilde{f}$ and $\tilde{g}$ instead of f and g, respectively. Assume that $u, \tilde{u}$ are weakly admissible and define $a, b \in B(I \times S)$ by*

$$a(r,x) := (f - \tilde{f})(r,x,\tilde{u}(r,x)), \quad \text{and} \quad b(r,x) := \frac{f(r,x,u(r,x)) - f(r,x,\tilde{u}(r,x))}{(u-\tilde{u})(r,x)},$$

if $u(r,x) \neq \tilde{u}(r,x)$, and $b(r,x) := 0$, otherwise. Then a and b are locally μ-dominated and

$$(u-\tilde{u})(r,x) = E_{r,x}\Big[e^{-\int_r^T b(s,X_s)\,\mu(ds)}(g-\tilde{g})(X_T)\Big] - E_{r,x}\Big[\int_r^T e^{-\int_r^t b(s,X_s)\,\mu(ds)} a(t,X_t)\,\mu(dt)\Big]$$

for each $(r,x) \in I \times S$. In particular, if $f \leq \tilde{f}$ and $g \geq \tilde{g}$, then $u \geq \tilde{u}$.

Proof. The second claim is a direct consequence of the first, since $a \leq 0$ whenever $f \leq \tilde{f}$. Thus, we merely have to prove the first assertion. To check that a and b are locally μ-dominated, it suffices to show that for each $r \in I$ there is a measurable μ-integrable function $\overline{c} : [r,T] \to \mathbb{R}_+$ such that

$$|a(\cdot,y)| \vee |b(\cdot,y)| \leq \overline{c} \quad \text{for each } y \in S \quad \mu\text{-a.s. on } [r,T].$$

This follows readily from the local Lipschitz μ-continuity of f, the local μ-boundedness of $\tilde{f}$ and the weak admissibility of u and $\tilde{u}$. By definition, $a(t,x) + b(t,x)(u-\tilde{u})(t,x) = f(t,x,u(t,x)) - \tilde{f}(t,x,\tilde{u}(t,x))$ for each $(t,x) \in I \times S$. Hence, we let $r \in I$ and define $a_r, b_r \in B_{\mu}([0,T] \times S)$ by $a_r(t,x) := a(t,x)$ and $b_r(t,x) := b(t,x)$, if $t \geq r$, and $a_r(t,x) := 0$ and $b_r(t,x) := 0$, otherwise. Then $f_r : [0,T] \times S \times \mathbb{R} \to \mathbb{R}$ given by

$$f_r(t,x,w) := a_r(t,x) + b_r(t,x)w$$

is Borel measurable, affine μ-bounded and Lipschitz μ-continuous. In addition, the restriction of $u - \tilde{u}$ to $[r,T] \times S$ is an admissible solution to (M) with f and g replaced by f_r and $g - \tilde{g}$, respectively. Thus, from Proposition 2.13 and Corollary 3.3 we infer the assertion. ☐

We suppose in the sequel that D is an interval, and set $\underline{d} := \inf D$ and $\overline{d} := \sup D$.

Lemma 4.3. *Let $\underline{d} > -\infty$ and f be affine μ-bounded from below. That is, $f(t,x,w) \geq -a(t,x) - b(t,x)|w|$ for all $(t,x,w) \in [0,T] \times S \times D$ and some $a, b \in B_{\mu}([0,T] \times S, \mathbb{R}_+)$. Then every μ-suitably bounded solution u to* (M) *on an admissible interval I satisfies*

$$u(r,x) - \underline{d} \leq E_{r,x}\Big[e^{\int_r^T b(s,X_s)\,\mu(ds)}\Big(g(X_T) - \underline{d} + \int_r^T (a + b|\underline{d}|)(s,X_s)\,\mu(ds)\Big)\Big]$$

for every $(r,x) \in I \times S$.

Proof. It holds that

$$u(r,x) \leq E_{r,x}[g(X_T)] + E_{r,x}\Big[\int_r^T (a + b|\underline{d}|)(s,X_s) + b(s,X_s)(u(s,X_s) - \underline{d})\,\mu(ds)\Big]$$

for each $(r,x) \in I \times S$, because $|u(s,X_s)| \leq (u(s,X_s) - \underline{d}) + |\underline{d}|$ for all $s \in [r,T]$. By Lemma 3.1, the asserted estimate follows. ☐

Remark 4.4. Suppose instead that $\overline{d} < \infty$ and f is affine μ-bounded from above. To obtain a similar estimate in this case, we replace D by $-D = \{-w \,|\, w \in D\}$ and f by the function $[0,T] \times S \times (-D) \to \mathbb{R}$, $(t,x,w) \mapsto -f(t,x,-w)$, respectively, and apply the above lemma.

Next, we study the boundary behavior of solutions. To this end, we only consider the case $\underline{d} > -\infty$, as the case $\overline{d} < \infty$ can be treated similarly, by considering above remark.

Proposition 4.5. *Let $\underline{d} > -\infty$ and $f \in BC_\mu^{1-}([0,T] \times S \times D)$. Suppose that f is both locally μ-bounded and locally Lipschitz μ-continuous at $\underline{d}$ with $\lim_{w \downarrow \underline{d}} f(\cdot, x, w) \leq 0$ for all $x \in S$ μ-a.s., and let one of the following two conditions hold:*

(i) f is μ-bounded from above.

(ii) $\overline{d} = \infty$ and f is affine μ-bounded from below.

Then there is $c \in (0,1]$ such that each weakly admissible solution u to (M) *on an admissible interval I satisfies $u(r,x) - \underline{d} \geq c(E_{r,x}[g(X_T)] - \underline{d})$ for all $(r,x) \in I \times S$.*

Proof. If $\underline{d} \notin D$, then we define the extension $\overline{f}$ of f to $[0,T] \times S \times (D \cup \{\underline{d}\})$ by $\overline{f}(t,x,\underline{d}) := \lim_{w \downarrow \underline{d}} f(t,x,w)$ for all $(t,x) \in [0,T] \times S$. Otherwise, we simply set $\overline{f} := f$, which gives $\overline{f} \in BC_\mu^{1-}([0,T] \times S \times (D \cup \{\underline{d}\}))$ in either case.

Let u be a weakly admissible solution to (M) on an admissible interval I. Then Lemma 4.2 implies that $b_u \in B(I \times S)$ defined via $b_u(r,x) := (\overline{f}(r,x,u(r,x)) - \overline{f}(r,x,\underline{d}))/(u(r,x) - \underline{d})$, if $u(r,x) > \underline{d}$, and $b_u(r,x) := 0$, otherwise, is locally μ-dominated and satisfies

$$u(r,x) - \underline{d} \geq E_{r,x}\left[e^{-\int_r^T b_u(s,X_s)\,\mu(ds)}(g(X_T) - \underline{d})\right] \quad \text{for each } (r,x) \in I \times S,$$

since $\overline{f}(t, X_t, \underline{d}) \leq 0$ for μ-a.e. $t \in [r,T]$. Thus, we derive some $\eta \in B_\mu([0,T] \times S, \mathbb{R}_+)$ such that every weakly admissible solution u to (M) on an admissible interval I satisfies $b_u(t,x) \leq \eta(t,x)$ for each $(t,x) \in I \times S$. Once this is shown, the claim follows.

Let us at first assume that (i) holds. Then there is $a \in B_\mu([0,T] \times S, \mathbb{R}_+)$ with $\overline{f}(t,x,w) \leq a(t,x)$ for each $(t,x,w) \in [0,T] \times S \times D$. As f is locally Lipschitz μ-continuous at $\underline{d}$, there are $\delta > 0$ and $\lambda \in B_\mu([0,T] \times S, \mathbb{R}_+)$ satisfying $|\overline{f}(t,x,w) - \overline{f}(t,x,\tilde{w})| \leq \lambda(t,x)|w - \tilde{w}|$ for every $(t,x) \in [0,T] \times S$ and all $w, \tilde{w} \in [\underline{d}, \underline{d} + \delta) \cap D$. Hence,

$$b_u(t,x) \leq \lambda(t,x)\mathbb{1}_{[\underline{d},\underline{d}+\delta)}(u(t,x)) + \frac{a(t,x) - \overline{f}(t,x,\underline{d})}{\delta}\mathbb{1}_{[\underline{d}+\delta,\infty)}(u(t,x)) \leq \eta(t,x)$$

for every weakly admissible solution u to (M) on an admissible interval I and each $(t,x) \in I \times S$, where we have set $\eta := \max\{\lambda, (a - \overline{f}(\cdot,\cdot,\underline{d}))/\delta\}$. Since f is locally μ-bounded at $\underline{d}$, we easily see that η is μ-dominated, as desired.

In place of assuming that f is μ-bounded from above, let (ii) be true. Then Lemma 4.3 yields $c \in (\underline{d}, \infty)$ such that $u(I \times S) \subset [\underline{d}, c] \cap D$ for each weakly admissible solution u to (M) on an admissible interval I. As $[\underline{d}, c]$ is compact, there is $\lambda \in B_\mu([0,T] \times S, \mathbb{R}_+)$ so that $|\overline{f}(t,x,w) - \overline{f}(t,x,\tilde{w})| \leq \lambda(t,x)|w - \tilde{w}|$ for all $(t,x) \in [0,T] \times S$ and each $w, \tilde{w} \in [\underline{d}, c] \cap D$. Hence, we set $\eta := \lambda$ and obtain that $|b_u(t,x)| \leq \eta(t,x)$ for every $(t,x) \in I \times S$. □

Finally, we are ready to establish the one-dimensional global existence- and uniqueness result.

Proof of Theorem 2.15. Let us verify the first claim. We begin with the case $\underline{d} > -\infty$ and $\overline{d} < \infty$. By using the function $[0,T] \times S \times (-D) \to \mathbb{R}$, $(t,x,w) \mapsto -f(t,x,-w)$, Proposition 4.5 yields that $I_{\tilde{g}} = [0,T]$ for every $\tilde{g} \in B_b(S,(\underline{d},\overline{d}))$ that is bounded away from $\{\underline{d},\overline{d}\}$. Thus, for all $n \in \mathbb{N}$ we define

$$g_n := (g \vee (\underline{d} + (\overline{d} - \underline{d})2^{-n})) \wedge (\overline{d} - (\overline{d} - \underline{d})2^{-n}), \tag{4.4}$$

then $g_n \in B_b(S,(\underline{d},\overline{d}))$ and $\mathrm{dist}(g_n,\{\underline{d},\overline{d}\}) \geq (\overline{d} - \underline{d})2^{-n}$, which guarantees that $I_{g_n} = [0,T]$. Because $|g_n - g| \leq (\overline{d} - \underline{d})2^{-n}$ for all $n \in \mathbb{N}$, the sequence $(g_n)_{n\in\mathbb{N}}$ converges uniformly to g. If $D \subsetneq [\underline{d},\overline{d}]$, then we let $\overline{f}$ denote the unique extension of f to $[0,T] \times S \times [\underline{d},\overline{d}]$ such that

$$\overline{f} \in BC_\mu^{1-}([0,T] \times S \times [\underline{d},\overline{d}]).$$

Otherwise, we just set $\overline{f} := f$. According to Proposition 3.4, the sequence $(u_{g_n})_{n\in\mathbb{N}}$ converges uniformly to the unique global bounded solution to (M) with $\overline{f}$ instead of f, which we denote by $\overline{u}_g$. By uniqueness, $\overline{u}_g = u_g$ whenever g is bounded away from $\{\underline{d},\overline{d}\}$. Since Proposition 4.5 also shows that $\overline{u}_g$ does not attain the value $\underline{d}$ (resp. $\overline{d}$) if the same is true for g, the function $\overline{u}_g$ is D-valued. Hence, $\overline{u}_g$ is the unique global bounded solution to (M).

Let us turn to the case $\underline{d} > -\infty$ and $\overline{d} = \infty$. Lemma 4.3 and Proposition 4.5 entail that $I_{\tilde{g}} = [0,T]$ for every $\tilde{g} \in B_b(S,(\underline{d},\infty))$ that is bounded away from $\underline{d}$. For each $n \in \mathbb{N}$ we set

$$g_n := g \vee (\underline{d} + 2^{-n}), \tag{4.5}$$

then $g_n \in B_b(S,(\underline{d},\infty))$ and $\mathrm{dist}(g_n,\underline{d}) \geq 2^{-n}$, which implies that $I_{g_n} = [0,T]$. In addition, $|g_n - g| \leq 2^{-n}$ and $g_n(x) - \underline{d} \leq (g(x) - \underline{d}) \vee 2^{-1}$ for all $n \in \mathbb{N}$ and each $x \in S$. We can now infer from Lemma 4.3 and Proposition 3.4 that $(u_{g_n})_{n\in\mathbb{N}}$ converges uniformly to the unique global bounded solution to (M), denoted by $\overline{u}_g$. Once again, uniqueness forces $\overline{u}_g = u_g$ if g is bounded away from $\underline{d}$. From Proposition 4.5 we see that $\overline{u}_g$ cannot attain the value $\underline{d}$ if $g(x) > \underline{d}$ for all $x \in S$. For this reason, $\overline{u}_g$ is D-valued, which concludes the case $\underline{d} > -\infty$ and $\overline{d} = \infty$. The case $\underline{d} = -\infty$ and $\overline{d} < \infty$ is a consequence of the last one, by using the familiar function $[0,T] \times S \times (-D) \to \mathbb{R}$, $(t,x,w) \mapsto -f(t,x,-w)$.

In the end, we note that for each $n \in \mathbb{N}$ the function g_n given either by (4.4) or (4.5), depending on which case occurs, is continuous if g is. Hence, as the uniform limit of a sequence of real-valued (right-)continuous functions on $[0,T] \times S$ is (right-)continuous, Theorem 2.11 implies the second assertion. $\square$

Proof of Corollary 2.17. At first, Theorem 2.15 entails that (M) admits the unique global bounded solution $\overline{u}_g$, which is (right-)continuous whenever $\mathscr{X}$ is Feller (right-)continuous, $a(s,\cdot)$ and $b(s,\cdot)$ are continuous for μ-a.e. $s \in [0,T]$ and g is continuous. Let us set

$$\overline{f}(t,x,w) := a(t,x) + b(t,x)w \quad \text{for all } (t,x,w) \in [0,T] \times S \times \mathbb{R},$$

then Proposition 2.13 implies that the unique global bounded solution $\tilde{u}_g$ to (M) with f replaced by $\overline{f}$ admits the required representation (2.10). However, $\overline{u}_g$ is also a global bounded solution to (M) when f is replaced by $\overline{f}$. Uniqueness gives $\overline{u}_g = \tilde{u}_g$. $\square$

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