Pricing of unemployment insurance products with doubly stochastic Markov chains

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Abstract. This paper provides a new approach for modeling and calculating premiums for unemployment insurance products. The innovative modeling concept consists of combining the benchmark approach with its real-world pricing formula and Markov chain techniques in a doubly stochastic setting. We describe individual insurance claims based on a special type of unemployment insurance contracts, which are offered on the private insurance market. The pricing formulas are first given in a general setting and then specified under the assumption that the individual employment-unemployment process of an employee follows a time-homogeneous doubly stochastic Markov chain. In this framework, formulas for the premiums are provided depending on the $\mathbb{P}$-numéraire portfolio of the Benchmark approach. Under a simple assumption on the $\mathbb{P}$-numéraire portfolio, the model is tested on its sensitivities to several parameters. With the same specification the model's employment and unemployment intensities are estimated on public data of the Federal Employment Office in Germany.

1. Introduction

In 2005 the unemployment rate in Germany reached its highest level since 1933. In the meantime the situation has improved but currently the unstable economic conditions with Europe's fear of recession have impact on job markets all over Europe. The fear of unemployment with its financial consequences is therefore very present among employees and the demand on financial securities increases.

In addition to social security systems, existing in most European states, private insurance companies have started to offer unemployment insurance products.

The present paper describes a new approach for modeling and calculating fair insurance premiums of unemployment insurance contracts by using martingale pricing techniques. To this end, we consider a special type of unemployment insurance product, which pays a deterministic amount of insurance claim to the insured person as soon as he gets unemployed and fulfills several other claim criteria. For example, one could think of Payment Protection Insurances against unemployment, which are always linked to some payment obligation of the insured person. If an insured event occurs, the insurance company pays the (deterministic) instalments during the respective period.

Pricing of random claims has ever been one of the core subjects in both actuarial and financial mathematics and there exist various approaches for calculating (fair) prices. The actuarial way of pricing

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usually considers the classical premium calculation principles that consist of *net premium* and *safety loading*: if C describes a random claim which the insurance company has to pay (eventually) at time $T > 0$ then a premium $P(C)$ to be charged for the claim is defined by

$$P(C) = \underbrace{\mathbb{E}\left[\frac{C}{D_T}\right]}_{\text{net premium}} + \underbrace{A\left(\frac{C}{D_T}\right)}_{\text{safety loading}}, \tag{1}$$

where D is a discounting process chosen according to actuarial judgement, see also Kull [21] for further remarks.

Note that the net premium is the expected value of $\frac{C}{D_T}$ with respect to the *real-world* (or *objective*) probability measure. Possible safety loadings could be $A(\frac{C}{D_T}) = 0$ (*net premium principle*), $A(\frac{C}{D_T}) = a \cdot \mathbb{E}[\frac{C}{D_T}]$ (*expected value principle*, where $a \geq 0$), $A(\frac{C}{D_T}) = a \cdot \mathbb{V}ar(\frac{C}{D_T})$ (*variance principle*, where $a > 0$) or $A(\frac{C}{D_T}) = a \cdot \sqrt{\mathbb{V}ar(\frac{C}{D_T})}$ (*standard deviation principle*, where $a > 0$), see e.g. Rolski et al. [27]. The existence of a safety loading is justified by ruin arguments and the risk-averseness of the insurance company: the net premium principle with zero safety loading is unfavourable for the insurance company as the ruin probability of an increasing collective tends towards 50% (central limit theorem). In competitive insurance markets however, the safety loading has to decrease.

Widely used pricing approaches in finance base on no-arbitrage assumptions (see e.g. the famous papers of Black and Scholes [7] and Merton [22]). A financial market, consisting of several primary assets, is assumed to be in an economic equilibrium in which riskless gains with positive probability (arbitrage) by trading in the assets are impossible. A fundamental result in this context is then the essential equivalence of absence of arbitrage and the existence of an equivalent (local) martingale measure, i.e. a probability measure, which is equivalent to the real-world measure and according to which all assets, discounted with some numéraire, are (local) martingales. There are different versions of this result which is often called the fundamental theorem of asset pricing (in short FTAP), see. e.g. Delbaen and Schachermayer [9], Delbaen and Schachermayer [10], Föllmer and Schied [13], Harrison and Pliska [15] or Kabanov and Kramkov [18].

Based on the FTAP, it can then be shown that, at any time t, an arbitrage-free price $P_t(C)$ of a (contingent) claim C (paid at time $T \geq t$) is given by

$$P_t(C) := N_t \mathbb{E}_{\mathbb{Q}}\left[\frac{C}{N_T} | \mathcal{F}_t\right] , \tag{2}$$

where $\mathbb{Q}$ is an equivalent (local) martingale measure, $(N_t)_{0 \leq t < \infty}$ the numéraire process and $(\mathcal{F}_t)_{0 \leq t < \infty}$ the filtration that expresses the information, existing in the market. Hence, the (new) discounted price process is assumed to follow a $\left(\mathbb{Q}, (\mathcal{F}_t)_{0 \leq t \leq T}\right)$-martingale.

Approaches, which base on no-arbitrage assumptions are strong tools for the purpose of modeling price structures, because they provide access to the powerful theory of martingales. Other advantages are the dynamic description of price processes and the close connection to hedging.

From an economic point of view both the safety loading in equation (1) and the change to an equivalent (local) martingale measure in equation (2) express the risk-averseness of the insurance company. There exist several works which connect actuarial premium calculation principles with the financial no-arbitrage theory. The papers Delbaen and Haezendonck [8] and Sondermann [30] both describe a competitive and liquid reinsurance market, in which insurance companies can "trade" their risks among each other. Since riskless profits shall be excluded also in this setting, the no-arbitrage theory applies and insurance premiums can be calculated by equation (2). Both papers actually show that under some assumptions[1] there exist risk-neutral[2] equivalent (local) martingale measures, which explain premiums of the form (1), so that these principles provide arbitrage-free prices, too. Further papers connecting actuarial and financial valuation principles are for example Kull [21] and Schweizer [28].

Martingale approaches can therefore be applied also to actuarial applications. Moreover, they get more and more important due to the fact that insurance markets and financial markets may no longer be viewed as some disjoint objects: on the one hand, insurance companies have the possibility to invest in financial markets and therefore hedge against their risks with financial instruments, and on the other hand, they can sell parts of their insurance risk by putting insurance linked products on the financial markets. Hence, one should consider insurance and financial markets as one arbitrage-free market, search for appropriate numéraires and (local) martingale measures and apply equation (2) for pricing all claims. Note that most insurance claims are not replicable by other financial instruments, which implies that the hybrid market of financial and insurance products is incomplete. As a consequence, there usually exist several equivalent (local) martingale measures, corresponding to the same numéraire, that guarantee the absence of arbitrage in the market. By equation (2) it is then clear that *defining* a premium calculation principle in the market is equivalent to *choosing* a numéraire and an equivalent (local) martingale measure. The usual procedure in this context is, to fix some numéraire and then to search for an equivalent (local) martingale measure, satisfying some desired properties. Examples, among others, are the minimal martingale measure and the minimal entropy measure. However, several measure choices seem not to be economically reasonable for hybrid markets. Moreover, it can be shown that for several insurance linked products with random jumps, the density of the minimal martingale measure may become negative and is therefore useless in the context of pricing.

To avoid this problem, we choose the so called benchmark approach for our pricing issue. This method is based on the assumption that the so called $\mathbb{P}$-numéraire portfolio exists in our market. This provides that the non-negative primary assets and also every non-negative portfolio become supermartingales with respect to the *real-world* probability measure $\mathbb{P}$, when discounted (or benchmarked) with this portfolio. The existence and uniqueness of the $\mathbb{P}$-numéraire portfolio could be shown in a sufficiently general setting, see Becherer [1] or Karatzas and Kardaras [20]. In particular it exists if there exists a risk-neutral equivalent (local) martingale measure, one of the most common assumptions in financial applications.

The existence of the $\mathbb{P}$-numéraire portfolio guarantees the absence of arbitrage, which is defined in a stronger way than usual. There could still exist some weak form of arbitrage in the market, which would require for negative portfolios of total wealth, however. In a realistic market model, such portfolios should be impossible due to the law of limited liability.

Moreover, choosing the benchmark approach for pricing of hybrid insurance products presents several

[1] The equivalent martingale measure is required here to be structure preserving, i.e. the claim process remains a compound Poisson process under $\mathbb{Q}$.

[2] For risk-neutral martingale measures, the numéraire is chosen to be a bank account in the domestic currency.

advantages. We discuss this issue thoroughly in Section 3.
To summarize, the main innovations of this paper are

- to analyze the premium determination for a relatively *new* class of insurance products, which is gaining more and more importance on the insurance market,
- to apply an alternative method to the classical actuarial approaches,
- and to provide analytical formulas for the insurance premiums, in a doubly stochastic Markovian specification of the model.

The paper is organized as follows: Section 2 contains the general setting and a short description of the benchmark approach. Section 3 discusses the intrinsic advantages of using the benchmark approach for premium determination of insurance contracts. In Section 4 we describe some details of the unemployment insurance products regarding exclusion clauses of claim payment, which influence the premium calculation. In Section 5 we present a general premium calculation formula without further distribution assumptions. In Section 6 we model the employment and unemployment progress of an insured person with a time-homogeneous doubly stochastic Markov chain and provide general formulas for the insurance premiums, depending on the $\mathbb{P}$-numéraire portfolio. Under a simple assumption on the $\mathbb{P}$-numéraire portfolio, we calculate first premiums and present premium sensitivity results in Section 7. Finally, in Section 8, we present estimation results for employment and unemployment intensities that base on data of the "Federal Employment Office of Germany".

2. The benchmark approach

As stated in the introduction, we adopt the benchmark approach for modeling price structures. All fundamental results of this approach can be found in Platen and Heath [25] for jump diffusion and Itô process driven markets and in Platen [23] for a general semimartingale market. The approach enables us to use the real-world probability measure for determining insurance premiums.
We consider a frictionless financial market model in continuous time, which is set up on a complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$, equipped with some filtration $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$ that is assumed to satisfy $\mathcal{G}_t \subseteq \mathcal{G}\ \forall t \in [0, T]$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, as well as the usual hypotheses, see Protter [26]. $T \in (0, \infty)$ is some arbitrary time horizon.
On the market, we can find $d+1$ nonnegative, adapted tradable (primary) security account processes, denoted by $S^j = (S^j_t)_{0 \leq t \leq T}$, $j \in \{0, 1, ..., d\}$, $d \geq 1$. The security account process S^0 is often taken as a riskless bank account in the domestic currency. We write $S = (S_t)_{0 \leq t \leq T} = (S^0_t, S^1_t, ..., S^d_t)^{tr}_{0 \leq t \leq T}$ for the $d+1$-dimensional random vector process, consisting of the $d+1$ assets, and assume that S is a càdlàg semimartingale.
Because we want our market model to cover both financial and actuarial markets, we interpret the securities S^j to describe, among others, stocks and insurance products.
Market participants can trade in the assets in order to reallocate their wealth. The mean of trading in financial stocks is thereby as usual. However, we may need to explain what we mean by trading insurance products: on the one hand, insurance companies can forward fractions of their claims to other

insurers or reinsurers according to proportional reinsurance contracts, see also Sondermann [30]. Hence one could say that there is the possibility for insurance companies to "buy" (and "sell") insurance claims and to trade them in different fractions. On the other hand, one can also think of trading insurance claims when contracts are concluded or cancelled either by the insurance company or by the insured. Let $L(S)$ denote the space of $\mathbb{R}^{d+1}$-valued, predictable strategies

$$\delta = (\delta_t)_{0 \leq t \leq T} = (\delta_t^0, \delta_t^1, ..., \delta_t^d)_{0 \leq t \leq T}^{tr} ,$$

for which the corresponding gains from trading in the assets, i.e. $\int_0^t \delta_s \cdot dS_s$, exist for all $t \in [0,T]$. Here, δ_t^j represents the units of asset j held at time t by a market participant. The portfolio value S_t^δ at time $t \in [0,T]$ is then given by

$$S_t^\delta = \delta_t \cdot S_t = \sum_{j=0}^{d} \delta_t^j S_t^j .$$

A strategy $\delta \in L(S)$ is called *self-financing* if changes in the portfolio value are only due to changes in the assets and not due to in- or outflow of money, i.e. if

$$S_t^\delta = S_0^\delta + \int_0^t \delta_s \cdot dS_s \ , t \in [0,T] ,$$

or equivalently

$$dS_t^\delta = \delta_t \cdot dS_t .$$

We write $\mathcal{V}_x^+$ ($\mathcal{V}_x$) for the set of all strictly positive (nonnegative), finite and self financing portfolios S^δ with initial capital $S_0^\delta = x$.

Definition 2.1. *A nonnegative, self-financing portfolio $S^\delta \in \mathcal{V}_x$ permits* arbitrage *if*

$$\mathbb{P}(S_\tau^\delta = 0) = 1$$

and

$$\mathbb{P}(S_\sigma^\delta > 0 \,|\, \mathcal{G}_\tau) > 0$$

for some stopping times $0 \leq \tau \leq \sigma \leq T$ and some initial capital x. A given market model is said to be arbitrage-free *if there exists no nonnegative portfolio $S^\delta \in \mathcal{V}_x$ of the above kind.*

We now introduce the notion of the $\mathbb{P}$-numéraire portfolio.

Definition 2.2. *A portfolio $S^{\delta_*} \in \mathcal{V}_1^+$ is called* $\mathbb{P}$-numéraire portfolio *if every nonnegative portfolio $S^\delta \in \mathcal{V}_x$, discounted (or benchmarked) with S^{δ_*}, forms a $(\mathbb{G}, \mathbb{P})$-supermartingale for every $x \geq 0$. In* *particular, we have*

$$\mathbb{E}\left[\frac{S^{\delta}_{\sigma}}{S^{\delta_*}_{\sigma}}\,\Big|\,\mathcal{G}_{\tau}\right] \leq \frac{S^{\delta}_{\tau}}{S^{\delta_*}_{\tau}} \qquad a.s. \tag{3}$$

for all stopping times $0 \leq \tau \leq \sigma \leq T$.

From now on we call a portfolio, an asset or any payoff, when expressed in units of the $\mathbb{P}$-numéraire portfolio, a "benchmarked" portfolio, asset or payoff.
If a $\mathbb{P}$-numéraire portfolio exists, it is unique as can be easily seen with the help of the supermartingale property and Jensen's inequality, see Becherer [1].
To establish the further modeling framework, we make the following key assumption.

Assumption 2.3. *The* $\mathbb{P}$*-numéraire portfolio* $S^{\delta_*} \in \mathcal{V}_1^+$ *exists in our market.*

If it exists, the $\mathbb{P}$-numéraire portfolio is equal to the "growth optimal portfolio" (in short GOP), which is defined as the portfolio with the maximal growth-rate in the market and which satisfies several other optimality criteria, see Becherer [1], Hulley and Schweizer [16], Platen [23] or Platen and Heath [25].
The assumption on the existence of the $\mathbb{P}$-numéraire portfolio obviously depends on the model specifications of the market. However, it is a rather weak assumption as the existence was proven for most model specifications of nowadays practical interest, see Becherer [1], Karatzas and Kardaras [20] or Platen and Heath [25]. In particular it exists for all models with existing risk-neutral equivalent (local) martingale measure. Regarding estimation and calibration of the $\mathbb{P}$-numéraire portfolio it is shown in Platen and Heath [25] that every sufficiently diversified market portfolio yields a good proxy for it.
With the existence of the $\mathbb{P}$-numéraire portfolio and the corresponding supermartingale property (3), arbitrage opportunities, as defined in Definition 2.1, are excluded, see Platen [23]. There could still exist some weaker forms of arbitrage, which would require to allow for negative portfolios of total wealth, however. Because of the (mostly legally established) principle of limited liability, these portfolios should be excluded in a realistic market model: a market participant generally holds a nonnegative portfolio of total wealth, otherwise he would have to declare bankruptcy. Assumption 2.3 therefore guarantees the absence of a strong form of arbitrage, which is the one, needed in our market model.
Let us now consider two portfolios $S^{\delta} \in \mathcal{V}_x$ and $S^{\delta'} \in \mathcal{V}_y$ with

$$\frac{S^{\delta}_T}{S^{\delta_*}_T} = \frac{S^{\delta'}_T}{S^{\delta_*}_T} \qquad \mathbb{P}\text{- a.s..}$$

Let the benchmarked portfolio process $\left(\frac{S^{\delta}_t}{S^{\delta_*}_t}\right)_{t\in[0,T]}$ be a martingale and the benchmarked portfolio process $\left(\frac{S^{\delta'}_t}{S^{\delta_*}_t}\right)_{t\in[0,T]}$ be a supermartingale. Then

$$\frac{S^{\delta}_t}{S^{\delta_*}_t} = \mathbb{E}\left[\frac{S^{\delta}_T}{S^{\delta_*}_T}\,\Big|\,\mathcal{G}_t\right] = \mathbb{E}\left[\frac{S^{\delta'}_T}{S^{\delta_*}_T}\,\Big|\,\mathcal{G}_t\right] \leq \frac{S^{\delta'}_t}{S^{\delta_*}_t}, \qquad \forall t \in [0,T]\,, \tag{4}$$

and in particular

$$x = S_0^{\delta} \le S_0^{\delta'} = y \ .$$

Hence, a rational (risk-averse) investor would always invest in a benchmarked martingale portfolio (if it exists) and we can give the following definition of "fair" wealth processes, see Platen [23].

Definition 2.4. *A portfolio process* $S^{\delta} = (S_t^{\delta})_{t\in[0,\infty)}$ *is called* fair *if its benchmarked value process*

$$\widehat{S^{\delta}}_t := \frac{S_t^{\delta}}{S_t^{\delta_*}} \ , \quad t \in [0,\infty) \ ,$$

forms a $(\mathbb{G},\mathbb{P})$*-martingale.*

Definition 2.5. *Given some maturity* $T \in (0,\infty)$*, a* T-contingent claim C *is a* $\mathcal{G}_T$*-measurable random variable with* $\mathbb{E}\left[\frac{|C|}{S_T^{\delta_*}}\right] < \infty$*.*
We denote by

$$\widehat{C} := \frac{C}{S_T^{\delta_*}} \tag{5}$$

the benchmarked payoff *of the* T*-contingent claim* C*.*

According to Definition 2.4, it is natural to define the so called *real-world pricing formula* for a T-contingent claim C as follows:

Definition 2.6. *For a* T*-contingent claim* C *the* fair price $P_t(C)$ *of* C *at time* $t \in [0,T]$ *is given by*

$$P_t(C) := S_t^{\delta_*}\,\mathbb{E}\left[\frac{C}{S_T^{\delta_*}}\Big|\mathcal{G}_t\right] = S_t^{\delta_*}\,\mathbb{E}\left[\widehat{C}\,\big|\,\mathcal{G}_t\right] \ . \tag{6}$$

The corresponding benchmarked fair price process $(\widehat{P}_t)_{t\in[0,T]} = \left(\frac{P_t}{S_t^{\delta_*}}\right)_{t\in[0,T]}$ hence forms a $(\mathbb{G},\mathbb{P})$-martingale.
As stated in the introduction, the considered, hybrid market is incomplete. Therefore, not every T-contingent claim, introduced to the market, can be replicated by a self-financing portfolio. Fortunately, the real-world pricing formula (6) yields economically reasonable prices for both the replicable and non-replicable case: relation (4) shows that for non-negative, replicable T-contingent claims and under Assumption 2.3, the real-world pricing formula defines the claim's minimal price for every $t \le T$. For any non-replicable claim, the real-world pricing method is consistent with asymptotic utility indifference pricing in a very general setting, see Platen and Heath [25]. More advantages of the benchmark approach, particularly for actuarial applications, are provided in the next section.

3. The benchmark approach for actuarial applications

In this section we examine some advantages of using the benchmark approach and its real-world pricing formula for actuarial aspects of premium determination and risk classification.

As already stated in the introduction the concepts of no-arbitrage theory often contain standard actuarial premium principles in the sense that there exists some risk-neutral equivalent (local) martingale measure such that the price structures with respect to this measure coincide with the prices obtained by the standard actuarial premium principles, see Delbaen and Haezendonck [8], Kull [21], Schweizer [28] or Sondermann [30]. Moreover, with the possibility to trade on and forward actuarial risks to financial markets, insurance and financial markets may no longer be considered as disjoint objects, but can be viewed as one arbitrage-free market. In this framework, the standard actuarial premium principles are often considered to be rather ad hoc approaches, which don't take into account the insurance company's economic environment. Well established mathematical methods for arbitrage-free pricing of financial contingent claims can be applied as well and may be rather suitable for premium determination, in particular for hybrid insurance products. In the present paper, we choose the benchmark approach, introduced in the literature by several authors, e.g. Fernholz [11], Fernholz and Karatzas [12], Karatzas and Kardaras [20], Platen [23], Platen [24] or Platen and Heath [25]. In most applications, the price of a claim, evaluated with the real-world pricing formula (6) coincides with the risk-neutral price of the claim with respect to the risk-neutral equivalent (local) martingale measure $\mathbb{P}^*$ with density process $\Lambda_t = \frac{\widehat{S}^0_t}{\widehat{S}^0_0}$, if $\mathbb{P}^*$ exists. This means that under some conditions the benchmark approach is equivalent to the usual no-arbitrage pricing theory that, as already stated, contains the standard actuarial premium principles. In this sense, the real-world pricing formula can be interpreted as a premium principle which is in the line to classical actuarial premium principles but offers more advantages for applications.

In particular, we claim that real-world pricing defines a more natural framework for actuarial applications than risk-neutral pricing: First of all it provides a pricing rule under the real-world probability measure $\mathbb{P}$, which is valid for realistic market models, even if no risk-neutral equivalent (local) martingale measure exists. By using this approach, one also benefits from the statistical advantages of working directly under the real-world probability measure.

Another advantage is the fact that we take directly into account the role of investment opportunities in assessing premiums and reserves. The $\mathbb{P}$-numéraire portfolio is a direct and intuitive global indicator of (hybrid) market performance and dependence structure. This is particularly relevant for insurance structures depending heavily on the performance of financial markets and macro-economic factors, as in the case of unemployment insurance products. On the contrary, the choice of a particular (local) martingale measure for actuarial applications appears quite artificial, since it is exclusively determined in relation to the primitive financial assets on the financial market.

There is also an interesting relation between the (local) risk minimization approach[3], a hedging technique used in incomplete markets, and the use of the benchmark approach, see Biagini [2] and Biagini et al. [5] for further references. This special relation provides a strong link between the real-world pricing formula and actuarial risk classification or hedging schemes. Let C be some bounded contingent claim maturing at T. We decompose the benchmarked claim $\widehat{C}$ as the sum of its *hedgeable part* $\widehat{C}^h$

[3]The local risk minimization approach provides for a given square-integrable contingent claim C with maturity T a perfect hedge by using strategies that are not necessarily self-financing, with (discounted) portfolio value given by the gain of trade plus an instantaneous adjustment called the cost. The optimal strategy, when it exists, is determined by the property of having minimal risk, in the sense that the optimal cost is given by a square-integrable martingale strongly orthogonal to the martingale part of the asset price process. This implies that the optimal strategy is "self-financing on average", i.e. remains as close as possible to being self-financing.

and its *unhedgeable part* $\widehat{C}^u$, i.e.

$$\widehat{C} = \widehat{C}^h + \widehat{C}^u \ . \tag{7}$$

The benchmarked hedgeable part $\widehat{C}^h$ can be replicated perfectly by a self-financing trading strategy $(\xi_t)_{t\in[0,T]}$ on the market's primary assets, whose benchmarked value $\widehat{V}_t^\xi$ at time $t \in [0,T]$ is in particular given by

$$\widehat{V}_t^\xi = \mathbb{E}_{\mathbb{P}}\left[\widehat{C}^h \mid \mathcal{G}_t\right] \ . \tag{8}$$

The remaining benchmarked unhedgeable part can be diversified and will be covered by introducing a benchmarked cost process $L^{\widehat{C}}$. This cost process is optimal in the sense that it is chosen to obtain a perfect replication for $\widehat{C}$ with minimal risk in a sense technically specified by the (local) risk minimization method, see Schweizer [29]. Mathematically, the cost process is represented by a (square-integrable) martingale $\left(L_t^{\widehat{C}}\right)_{t\in[0,T]}$ strongly orthogonal to the underlying benchmarked primary assets with $L_0^{\widehat{C}} = 0$. In particular, we have

$$\mathbb{E}_{\mathbb{P}}\left[\widehat{C} \mid \mathcal{G}_t\right] = \mathbb{E}_{\mathbb{P}}\left[\widehat{C}^h \mid \mathcal{G}_t\right] + L_t^{\widehat{C}}, \ t \in [0,T], \tag{9}$$

and for $t = 0$, the initial benchmarked value $\widehat{V}_0^\xi$ of the replicating strategy for $\widehat{C}^h$ coincides with the real-world price of the hedgeable part or the claim $\widehat{C}$, while the benchmarked unhedgeable part remains totally untouched. This is reasonable because any extra trading could only create unnecessary uncertainty and potential additional benchmarked profits or losses. Hence, pricing insurance claims with the benchmark approach provides the possibility to invest in a strategy given by a self-financing component and a cost process (i.e. an instantaneous readjustment of the portfolio) in order to obtain perfect replication of the claim. The cost process is thereby chosen to minimize the risk for the insurance company. For similar results on the relation between actuarial valuation principles and mean-variance hedging we also refer to Schweizer [28]. Also see Fontana and Runggaldier [14] for a discussion on the relation among real-world pricing, upper-hedging pricing and utility indifference pricing.
The close relation of the real-world pricing formula with (local) risk minimization provides also a straightforward insight on the classification of risks to which insurance companies are exposed. The hedgeable part represents the contract's replicable risk and the cost process the residual risk for the insurance company. Since we consider insurance contracts for one individual, we do not take into account the idiosyncratic risk of a contract with respect to a whole collective of insured persons. This, however, can be done by weighting a pool of single insured persons with different risk profiles by using some appropriate weighting measure. This has been done among others in Biagini et al. [4] or Biffis and Millossovich [6]. Therefore, real-world pricing offers also interesting perspectives on risk investigation in actuarial theory.

4. Unemployment Insurance

We now introduce the structure of the considered unemployment insurance products. This is necessary, as we are about to calculate insurance premiums according to individual claim processes. Therefore,

the detailed contract specifications have to be considered.
The product's basic idea is that the insurance company compensates to some extend the financial deficiencies, which an unemployed insured person is exposed to. As already stated in the introduction, we only consider contracts with deterministic, a priori fixed claim payments c_i, which can be interpreted as an annuity during an unemployment period. Obviously, they take place at predefined payment dates T_i, $i = 1, ..., N$. Hence, the randomness of the claims are only due to their occurrence and not to their amount.
As a practical example, one could think of Payment Protection Insurance (in short PPI) products against unemployment, which are linked to some payment obligation of an obligor to its creditor. The claim amount is hereby defined by the (a priori known) instalments, which are paid at predefined payment dates.
The following details of the insurance contract are important for the later model specifications:

- Regarding the method of premium payment, we have to differentiate between single rates, where the whole insurance premium is paid at the beginning of the contract, and periodical rates. For our modeling purpose, we want to focus on calculating single premiums. This is motivated by PPI unemployment products, which are often sold as an add-on directly by the creditor. The insurance company then receives a single rate from the creditor, who in turn allocates this rate to the instalments.

- In order to conclude the insurance contract, the prospective buyer must have been employed at least for a certain period before the beginning of the contract. Therefore, we only consider employed insured persons at time $t = 0$.

We also consider three time periods that belong to the exclusion clauses of the contracts and impact the insurance premium.

- The *waiting period* starts with the beginning of the contract. If an insured person becomes unemployed at any time of this period, he is not entitled to receive any claim payments during the whole unemployment time.

- The *deferment period* starts with the first day of unemployment. An insured person is not entitled to receive claim payments until the end of this period.

- The third period is comparable to the waiting period and is called the *requalification period*. The difference between waiting and requalification period is their beginning. The waiting period starts with the beginning of the contract and the requalification period with the end of any unemployment period that occured during the contract's duration. If an insured person becomes (again) unemployed at any time of the requalification period, he is not entitled to receive any claim payment during the whole time of unemployment.

For existing unemployment insurance contracts, the waiting, deferment and requalification periods currently vary from three to twelve months.

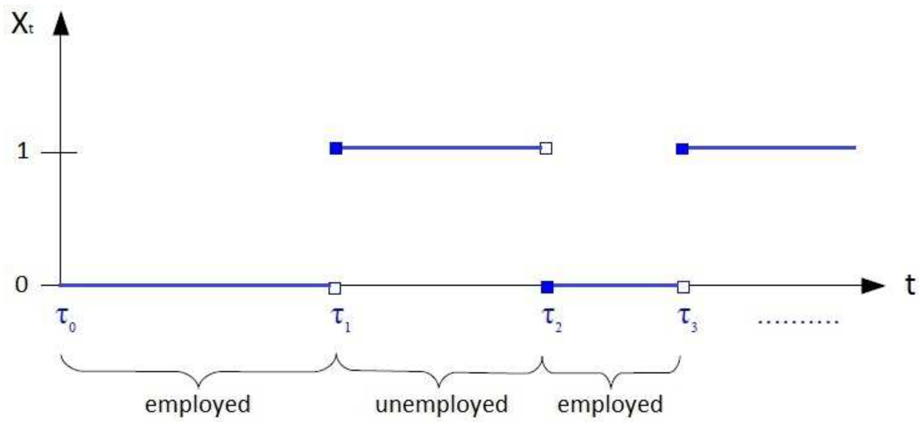

Figure 1: Trajectory of a right-continuous stochastic process X with state space $\{0,1\}$ and with respective jump times $\tau_0, \tau_1, \ldots$. The stochastic process is interpreted as the employment-unemployment progress of a person.

5. The Model

We now present the basic model for pricing unemployment insurance products. The random claims of the contracts can be defined and priced as contingent claims (with respective maturity) according to Definition 2.5. As stated in the previous section, the randomness of the claims only depends on their random occurrence and hence on future unemployment characteristics of the insured person. This implies that the claims are generally not replicable on our hybrid market. We therefore apply the benchmark approach with its real-world pricing formula (6), described in Section 2, because for non-replicable claims this is consistent with asymptotic utility indifference pricing under very general hypothesis, see Platen and Heath [25].

Under the assumptions of Section 2, we fix a maturity date $T = T_N$ which is also the final payment date of the respective insurance contract. The individual progress of a person of being employed and/or unemployed is given as a càdlàg, $\mathbb{G}$-adapted stochastic process $X = (X_t)_{0 \in [0,T]}$ with state space $\{0,1\}$. If $X_t = 0$ for some $t \in [0,T]$, the insured person is assumed to be employed at that time whereas for $X_t = 1$ she is unemployed.

In order to compute the insurance premium, we introduce the random jump times, when X "jumps" from one state to the other. Formally, we define the increasing sequence of random times recursively by

- $\tau_0 := 0$
- $$\tau_n := \inf\{t > \tau_{n\text{-}1} : X_t \neq X_{t\text{-}}\} \qquad n \geq 1\,, \tag{10}$$

where $X_{t\text{-}} := \lim_{s \nearrow t} X_s$. Figure 1 shows one possible "trajectory" of X with its respective jump times. We denote by c_i the deterministic claim amount, to be (eventually) paid at time T_i, $i = 1, \ldots, N$. Moreover, we define $W, D, R \in \mathbb{N}$ to be the waiting, deferment and requalification period, respectively. We think of $t = 0$ as the beginning of the contract. As already stated in Section 4, we only consider insured persons which are employed at the contract's beginning. We therefore set $\{X_0 = 0\} = \Omega$.

How do the random claim costs of the unemployment insurance contract arise? The insurance company has to pay the claim to the amount of c_i at time T_i if the following conditions are satisfied:

- $W < \tau_1 \leq T_i - D$
  The first jump τ_1 to unemployment of the insured person must occur *after* the waiting period W. Moreover, at least the deferment period D must lie between τ_1 and the payment date T_i, i.e. $T_i - \tau_1 \geq D$.
  and

- $T_i < \tau_2$
  The insured person must not have jumped back to employment *before* the payment date T_i.

OR for $j \geq 2$

- $\tau_{2j\text{-}1} - \tau_{2j\text{-}2} > R$
  At least the requalification period R must lie in between a jump $\tau_{2j\text{-}2}$ to employment and the next jump $\tau_{2j\text{-}1}$ back to unemployment.
  and

- $W < \tau_{2j\text{-}1} \leq T_i - D$
  Any jump to unemployment $\tau_{2j\text{-}1}$ must occur after the waiting period W. Moreover, at least the deferment period D must lie between $\tau_{2j\text{-}1}$ and the payment date T_i, i.e. $T_i - \tau_{2j\text{-}1} \geq D$.
  and

- $\tau_{2j} > T_i$
  Before the payment date T_i the insured person must not have jumped back to employment.

Based on this insight, the random insurance claim C_i at the payment date T_i can be defined as

$$C_i(\omega) := c_i \mathbb{1}_{\{W<\tau_1 \leq T_i - D, \tau_2 > T_i\} \cup \bigcup_{j=2}^{\infty} \{\tau_{2j\text{-}1} - \tau_{2j\text{-}2} > R, W < \tau_{2j\text{-}1} \leq T_i - D, \tau_{2j} > T_i\}}(\omega) \tag{11}$$

In the following, we denote

$$A_j^x := \{\tau_j \leq x\}, B_j^x := \{x < \tau_j\}, D_j^x := \{\tau_j - \tau_{j\text{-}1} > x\}.$$

Note that, due to the definition of the jump times, the events $A_{2j\text{-}1}^{T_i - D} \cap B_{2j\text{-}1}^{W} \cap B_{2j}^{T_i}$, $j \geq 1$ are disjoint. Therefore, we can rewrite the random insurance claim C_i in (11) as

$$C_i(\omega) = c_i \left(\mathbb{1}_{B_1^W \cap A_1^{T_i - D} \cap B_2^{T_i}}(\omega) + \sum_{j=2}^{\infty} \mathbb{1}_{D_{2j\text{-}1}^R \cap B_{2j\text{-}1}^W \cap A_{2j\text{-}1}^{T_i - D} \cap A_{2j}^{T_i}}(\omega) \right). \tag{12}$$

As X is assumed to be $\mathbb{G}$-adapted, all jump times τ_j, $j \geq 0$, are $\mathbb{G}$-stopping times. It follows then by (12) that C_i is a T_i-contingent claim according to Definition 2.5. With the real-world pricing formula (6), the price $P_t(C_i)$ of C_i at time $t \in [0, T_i]$ is given as

$$P_t(C_i) = S_t^{\delta_*} c_i \left(\mathbb{E}\left[\hat{\mathbb{1}}_{B_1^W \cap A_1^{T_i - D} \cap B_2^{T_i}} \middle| \mathcal{G}_t \right] + \sum_{j=2}^{\infty} \mathbb{E}\left[\hat{\mathbb{1}}_{D_{2j\text{-}1}^R \cap B_{2j\text{-}1}^W \cap A_{2j\text{-}1}^{T_i - D} \cap B_{2j}^{T_i}} \middle| \mathcal{G}_t \right] \right),$$

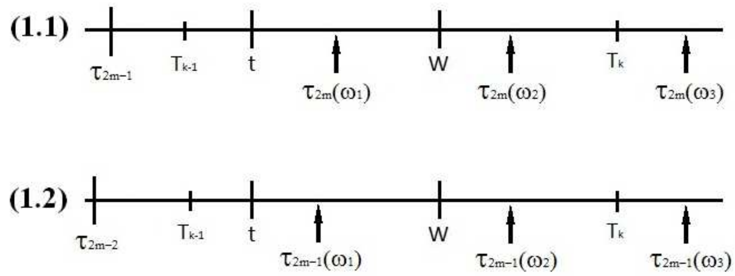

Figure 2: Illustration of different scenarios at time t, given that $t < W$ and $X_0 = 0$. In (1.1) the jumps $\tau_1, ..., \tau_{2m\text{-}2}$ have already occurred and the insured person is employed at time t. In (1.2) the jumps $\tau_1, ..., \tau_{2m\text{-}1}$ have already occurred and the insured person is unemployed at time t

where we used the notation for the benchmarked payoff, introduced in (5).
Now we can sum up over all payment dates T_i to receive the (overall) insurance premium P_t at time $t \in [T_{k\text{-}1}, T_k)$, $k \geq 1$,[4] given by

$$P_t = \sum_{i=k}^{N} S_t^{\delta_*} c_i \left(\mathbb{E}\left[\widehat{\mathbb{1}}_{B_1^W A_1^{T_i - D} B_2^{T_i}} \middle| \mathcal{G}_t \right] + \sum_{j=2}^{\infty} \mathbb{E}\left[\widehat{\mathbb{1}}_{D_{2j\text{-}1}^R B_{2j\text{-}1}^W A_{2j\text{-}1}^{T_i - D} B_{2j}^{T_i}} \middle| \mathcal{G}_t \right] \right) . \tag{13}$$

Note that we suppress the symbol "$\cap$" for notational convenience.
Depending on the time period length $T_k - T_{k\text{-}1}$ and the length of the waiting period W, it may happen that $T_{k\text{-}1} \leq W$ for some $k \in \{1, ..., N\}$. Hence, we have to distinguish between the two cases

(1) $t \in [T_{k\text{-}1}, W)$ and

(2) $t \in [W, T_k)$.

Note that (2) also contains the cases, where $W \leq T_{k\text{-}1} \leq t < T_k$, $k = 1, ..., N$.
We begin by considering situation (1) and evaluate the insurance premium for this case. First, however, we introduce two illustrative examples which give some intuition about how to compute the terms appearing in (13). The prove of the subsequent proposition is then straightforward and therefore omitted.

Example 5.1. *We assume that the jumps $\tau_1, ..., \tau_{2m\text{-}2}$ have already occurred up to time t for some $m \geq 2$. Hence, the insured person is employed at time t. Figure 2 illustrates this scenario.*
Since in this situation

- *$\tau_{2m\text{-}2}$ is known at time t and*

- *the restriction $\tau_{2j\text{-}1} > W$ is impossible to be satisfied for $j \leq m-1$,*

we obtain the insurance premium P_t as

$$P_t = \sum_{i=k}^{N} S_t^{\delta_*} c_i \left(\mathbb{E}\left[\widehat{\mathbb{1}}_{B_{2m\text{-}1}^{W \vee (R+\kappa)} A_{2m\text{-}1}^{T_i - D} B_{2m}^{T_i}} \middle| \mathcal{G}_t \right] \Bigg|_{\kappa = \tau_{2m\text{-}2}} + \sum_{j=m+1}^{\infty} \mathbb{E}\left[\widehat{\mathbb{1}}_{D_{2j\text{-}1}^R B_{2j\text{-}1}^W A_{2j\text{-}1}^{T_i - D} B_{2j}^{T_i}} \middle| \mathcal{G}_t \right] \right) .$$

[4]We set $T_0 := 0$ here.

Example 5.2. *Assume that for some $m \geq 2$, only the jumps $\tau_1, ..., \tau_{2m\text{-}1}$ have occurred up to time t. Figure 2 illustrates this scenario.*
In this case, the restriction $\tau_{2j\text{-}1} > W$ is impossible to be satisfied for $j \leq m$ and the calculation of the insurance premium P_t boils down to

$$P_t = \sum_{i=k}^{N} S_t^{\delta_*} c_i \sum_{j=m+1}^{\infty} \mathbb{E}\left[\widehat{\mathbb{1}}_{D^R_{2j\text{-}1} B^W_{2j\text{-}1} A^{T_i-D}_{2j\text{-}1} B^{T_i}_{2j}} \middle| \mathcal{G}_t\right] .$$

We obtain the following proposition.

Proposition 5.3. *If $t \in [T_{k\text{-}1}, W)$ for some $k \geq 1$, the insurance premium P_t is given by*

$$\begin{aligned} P_t = \sum_{i=k}^{N} S_t^{\delta_*} c_i \cdot \Bigg(& \mathbb{1}_{\{t<\tau_1\}} \left(\mathbb{E}\left[\widehat{\mathbb{1}}_{B^W_1 A^{T_i-D}_1 B^{T_i}_2} \middle| \mathcal{G}_t\right] + \sum_{j=2}^{\infty} \mathbb{E}\left[\widehat{\mathbb{1}}_{D^R_{2j\text{-}1} B^W_{2j\text{-}1} A^{T_i-D}_{2j\text{-}1} B^{T_i}_{2j}} \middle| \mathcal{G}_t\right] \right) \\ & + \mathbb{1}_{\{\tau_1 \leq t < \tau_2\}} \sum_{j=2}^{\infty} \mathbb{E}\left[\widehat{\mathbb{1}}_{D^R_{2j\text{-}1} B^W_{2j\text{-}1} A^{T_i-D}_{2j\text{-}1} B^{T_i}_{2j}} \middle| \mathcal{G}_t\right] \\ & + \sum_{m=2}^{\infty} \mathbb{1}_{\{\tau_{2m\text{-}2} \leq t < \tau_{2m\text{-}1}\}} \left(\mathbb{E}\left[\widehat{\mathbb{1}}_{B^{W \vee (R+\kappa)}_{2m\text{-}1} A^{T_i-D}_{2m\text{-}1} B^{T_i}_{2m}} \middle| \mathcal{G}_t\right]\Bigg|_{\kappa=\tau_{2m\text{-}2}} + \sum_{j=m+1}^{\infty} \mathbb{E}\left[\widehat{\mathbb{1}}_{D^R_{2j\text{-}1} B^W_{2j\text{-}1} A^{T_i-D}_{2j\text{-}1} B^{T_i}_{2j}} \middle| \mathcal{G}_t\right] \right) \\ & + \sum_{m=2}^{\infty} \mathbb{1}_{\{\tau_{2m\text{-}1} \leq t < \tau_{2m}\}} \sum_{j=m+1}^{\infty} \mathbb{E}\left[\widehat{\mathbb{1}}_{D^R_{2j\text{-}1} B^W_{2j\text{-}1} A^{T_i-D}_{2j\text{-}1} B^{T_i}_{2j}} \middle| \mathcal{G}_t\right] \Bigg) . \end{aligned} \tag{14}$$

We now consider situation (2), where $t \in [W, T_k)$. In this case, we have

$$\{\tau_j > t\} \subseteq \{\tau_j > W\} .$$

Hence, the latter restriction is automatically fulfilled for every $\omega \in \{\tau_j > t\}$. By taking this into account and with similar arguments as in Examples 5.1 and 5.2, we obtain the following proposition.

Proposition 5.4. *If $T_{k\text{-}1} \leq W \leq t < T_k$ or $W \leq T_{k\text{-}1} \leq t < T_k$ for some $k \geq 1$, the insurance premium P_t is given by*

$$\begin{aligned} P_t = \sum_{i=k}^{N} S_t^{\delta_*} c_i \cdot \Bigg(& \mathbb{1}_{\{t<\tau_1\}} \left(\mathbb{E}\left[\widehat{\mathbb{1}}_{A^{T_i-D}_1 B^{T_i}_2} \middle| \mathcal{G}_t\right] + \sum_{j=2}^{\infty} \mathbb{E}\left[\widehat{\mathbb{1}}_{D^R_{2j\text{-}1} A^{T_i-D}_{2j\text{-}1} B^{T_i}_{2j}} \middle| \mathcal{G}_t\right] \right) \\ & + \mathbb{1}_{\{\tau_1 \leq t < \tau_2\}} \left(\mathbb{1}_{B^W_1 A^{T_i-D}_1} \mathbb{E}\left[\widehat{\mathbb{1}}_{B^{T_i}_2} \middle| \mathcal{G}_t\right] + \sum_{j=2}^{\infty} \mathbb{E}\left[\widehat{\mathbb{1}}_{D^R_{2j\text{-}1} A^{T_i-D}_{2j\text{-}1} B^{T_i}_{2j}} \middle| \mathcal{G}_t\right] \right) \\ & + \sum_{m=2}^{\infty} \mathbb{1}_{\{\tau_{2m\text{-}2} \leq t < \tau_{2m\text{-}1}\}} \left(\mathbb{E}\left[\widehat{\mathbb{1}}_{B^{R+\kappa}_{2m\text{-}1} A^{T_i-D}_{2m\text{-}1} B^{T_i}_{2m}} \middle| \mathcal{G}_t\right]\Bigg|_{\kappa=\tau_{2m\text{-}2}} + \sum_{j=m+1}^{\infty} \mathbb{E}\left[\widehat{\mathbb{1}}_{D^R_{2j\text{-}1} A^{T_i-D}_{2j\text{-}1} B^{T_i}_{2j}} \middle| \mathcal{G}_t\right] \right) \\ & + \sum_{m=2}^{\infty} \mathbb{1}_{\{\tau_{2m\text{-}1} \leq t < \tau_{2m}\}} \left(\mathbb{1}_{D^R_{2m\text{-}1} B^W_{2m\text{-}1} A^{T_i-D}_{2m\text{-}1}} \mathbb{E}\left[\widehat{\mathbb{1}}_{B^{T_i}_{2m}} \middle| \mathcal{G}_t\right] + \sum_{j=m+1}^{\infty} \mathbb{E}\left[\widehat{\mathbb{1}}_{D^R_{2j\text{-}1} A^{T_i-D}_{2j\text{-}1} B^{T_i}_{2j}} \middle| \mathcal{G}_t\right] \right) \Bigg) . \end{aligned} \tag{15}$$

Formulas (14) and (15) show that we have to investigate the joint (conditional) distributions of the $\mathbb{P}$-numéraire portfolio and the different *sojourn-times* $\tau_j - \tau_{j\text{-}1}$, $j \geq 1$, in the respective states in order to calculate the fair insurance premium.

Equations (14) and (15) may look quite nasty, not least because they contain infinite sums. These emerge due to the fact that we have to consider all possible jumps between the states of unemployment and employment, which may occur during the contract's duration. However in practice, in particular for simulations, we don't have to consider all jumps. The employment-unemployment process X is supposed to be càdlàg and it is a well known fact that such processes have only finitely many jumps with absolute jump size bigger than $\varepsilon > 0$ on every compact interval $[0, T]$. Therefore, every (simulated) path of X has only finitely many jumps on the interval $[0, T_N]$ of our contracts.

Furthermore in the following section, we provide an example of modeling joint distributions by assuming X to follow a time-homogeneous doubly stochastic Markov chain. In this case, we get rid of the infinite sums, as they converge to some well-known functions, and obtain analytic solutions for the premiums.

6. Doubly stochastic Markov setting

In this section we present a first way of modeling the joint conditional distributions of the jump times and the $\mathbb{P}$-numéraire portfolio, which are necessary to calculate insurance premiums according to the pricing formulas (14) and (15).

To do so, we assume the general filtration $\mathbb{G}$ to satisfy $\mathbb{G} = \mathbb{F} \vee \mathbb{F}^X$, where $\mathbb{F}^X$ is the natural filtration generated by X and $\mathbb{F}$ is some arbitrary reference filtration. The process X is taken not to be adapted to the filtration $\mathbb{F}$ such that in particular the jump times (10) are obviously $\mathbb{G}$- but not $\mathbb{F}$-stopping times. In addition, the reference filtration $\mathbb{F}$ is assumed to collect among others the information generated by all security price processes $S^0, ..., S^d$, presented in Section 2. In particular, the $\mathbb{P}$-numéraire portfolio S^{δ_*} is $\mathbb{F}$-adapted. Note that all assumptions, made in Sections 2 and 5 are compatible with this specification.

The following definitions are based on the ideas given in Jakubowski and Niewęgłowski [17].

Definition 6.1. *A stochastic process $X = (X_t)_{t\in[0,T]}$ is called time-homogeneous, doubly stochastic Markov chain with state space $\{0,1\}$ if there exists a matrix-valued stochastic process $P = \Big(P(t)\Big)_{t\in[0,T]} = \Big(\big[p_{i,j}(t)\big]_{i,j\in\{1,2\}}\Big)_{t\in[0,T]}$, such that for every $0 \leq s \leq t \leq T$ and every $i, j \in \{1, 2\}$*

(i) $P(t-s)$ is a stochastic matrix, i.e. the sum of all row entries is one.

(ii) $P(t-s)$ is $\mathcal{F}_T$-measurable.

(ii) $\mathbb{1}_{\{X_s=i-1\}}\mathbb{P}\left(X_t = j-1 \,\middle|\, \mathcal{F}_T \vee \mathcal{F}_s^X\right) = \mathbb{1}_{\{X_s=i-1\}} p_{i,j}(t-s)$.

Remark 6.2. *There are two basic differences between Definition 6.1 and Definition* 2.1 *in Jakubowski and Niewęgłowski [17] besides the fact that we already assume a state space* $\{0,1\}$. *While the authors in Jakubowski and Niewęgłowski [17] assume $\mathcal{F}_t$-measurability of the stochastic matrix $P(s,t)$, we in addition assume time-homogeneity and $\mathcal{F}_T$-measurability. This yields in our case a $\mathcal{F}_T$-measurable intensity matrix, suitable for our present application, whereas the authors in Jakubowski and Niewęgłowski [17] create an $\mathbb{F}$-adapted intensity matrix, which is more general. However, for the proofs of the next statements about doubly stochastic Markov chains we refer to Jakubowski and Niewęgłowski [17] as they are, despite the aforementioned differences, one-to-one.*

Note that due to the definition of a doubly stochastic Markov chain and Lemma 3.1 in Jakubowski and Niewęgłowski [17], we have that the σ-fields $\mathcal{Z}_t^X := \sigma(X_u : u > t)$, $t \in [0,T]$, and $\mathcal{V}_t^X := \sigma(X_u : u < t)$, $t \in [0,T]$, are conditionally independent given $\mathcal{F}_T \vee \sigma(X_t)$.
The following hypothesis is a well known concept in applications of doubly stochastic models.

HYPOTHESIS (H) *For every bounded, $\mathcal{F}_T$ measurable random variable Y and every $t \in [0,T]$ we have*

$$\mathbb{E}\left[Y \mid \mathcal{G}_t\right] = \mathbb{E}\left[Y \mid \mathcal{F}_t\right] \; .$$

In proposition 3.4 of Jakubowski and Niewęgłowski [17] it is shown that hypothesis (H) holds for all doubly stochastic Markov chains X.

Definition 6.3. *We say that a time-homogeneous doubly stochastic Markov chain X has an intensity matrix $\Lambda = [\lambda_{i,j}]_{i,j\in\{1,2\}}$ if Λ is $\mathcal{F}_T$-measurable and satisfies the following conditions*

(i) $0 \leq \lambda_{i,j} < \infty$ *a.s. for all* $i,j \in \{1,2\}$ *with* $i \neq j$,

(ii) $\lambda_{i,i} = -\lambda_{i,j}$ *a.s. for all* $i,j \in \{1,2\}$ *with* $i \neq j$,

(iii) Λ *solves* $\frac{dP(t)}{dt} = \Lambda P(t)$, $P(0) = Id_2$ *(Kolmogorov backward equation) and*
Λ *solves* $\frac{dP(t)}{dt} = P(t)\Lambda$, $P(0) = Id_2$ *(Kolmogorov forward equation)*

Most doubly stochastic Markov chains have an intensity, for example if the transition probability matrix process $\left(P(t)\right)_{t\in[0,T]}$ is standard, i.e. $\lim_{t\searrow 0} P(t) = Id_2$ a.s. and the derivative matrix $\Lambda = \lim_{h\searrow 0} \frac{P(h)-Id_2}{h}$ exists with entries $\lambda_{i,j} < \infty$, see Jakubowski and Niewęgłowski [17]. Moreover, for every given $\mathcal{F}_T$-adapted matrix Λ satisfying conditions (i) and (ii) there is a doubly stochastic Markov chain with intensity matrix Λ. This allows us to make the following assumption:

Assumption 6.4. *We assume that the employment unemployment process X follows a time-homogeneous doubly stochastic Markov chain with an intensity matrix of the form*

$$\Lambda^* = \begin{pmatrix} \lambda_0^* & -\lambda_0^* \\ -\lambda_1^* & \lambda_1^* \end{pmatrix} := \begin{pmatrix} \lambda_0(S_T^{\delta_*}) & -\lambda_0(S_T^{\delta_*}) \\ -\lambda_1(S_T^{\delta_*}) & \lambda_1(S_T^{\delta_*}) \end{pmatrix} , \tag{16}$$

where $\lambda_{0,1} : (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)) \mapsto (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ *are some measurable functions.*

An important property of the sojourn times $\tau_j - \tau_{j\text{-}1}$, $j \geq 1$ of a time-homogeneous doubly stochastic Markov chain X is then

$$\mathbb{P}\left(\tau_j - \tau_{j\text{-}1} > t \middle| \mathcal{F}_T \vee \mathcal{F}^X_{\tau_{j\text{-}1}}\right) = e^{t\lambda^*_{X_{\tau_{j\text{-}1}}}} = e^{t\lambda^*_{(j\text{-}1) \bmod 2}} \; ,$$

see Jakubowski and Niewęgłowski [17]. Note that λ_0^* and λ_1^* both depend on the value of S^{δ_*} at time T and are therefore $\mathcal{F}_T$ measurable. Note also that the second equality holds for our specific two state model under the assumption $\{X_0 = 0\} = \Omega$. This directly implies

$$\mathbb{P}\left(\tau_j - \tau_{j\text{-}1} > t \middle| \mathcal{F}_T \vee \mathcal{F}^X_{\tau_{j\text{-}1}}\right) = \mathbb{P}\left(\tau_j - \tau_{j\text{-}1} > t \middle| \sigma(S^{\delta_*}_T)\right) ,$$

which shows that for every $j \geq 1$, $\tau_j - \tau_{j-1}$ is conditionally independent of $\mathcal{F}^X_{\tau_{j\text{-}1}}$ given $\mathcal{F}_T$. In particular, the family $(\tau_j - \tau_{j-1})_{j\geq 1}$ is conditionally independent given $\mathcal{F}_T$.
Because of $\tau_0 = 0$, every jump time τ_j can be written as a telescoping sum

$$\tau_j = \sum_{l=1}^{j} (\tau_l - \tau_{l\text{-}1}) . \tag{17}$$

Hence, we obtain that given $\mathcal{F}_T$ every jump time τ_j, $j \geq 2$, is the sum of two conditionally independent, gamma distributed random variables with parameters depending on whether j is odd or even.
Let's for the moment assume that $t \in [T_{k\text{-}1}, W)$ and that the first jump to unemployment τ_1 has not occurred up to time t ($t < \tau_1$). According to equation (14) and the above facts, we then get

$$\begin{aligned}
P_t = &\sum_{i=k}^{N} S^{\delta_*}_t c_i \Bigg(\mathbb{E}\left[\widehat{\mathbb{1}}_{\{W<\tau_1\leq T_i - D, T_i < \tau_2\}} | \mathcal{G}_t\right] \\
&\quad + \sum_{j=2}^{\infty} \mathbb{E}\left[\widehat{\mathbb{1}}_{\{\tau_{2j\text{-}1}-\tau_{2j\text{-}2}>R, W<\tau_{2j\text{-}1}\leq T_i-D, T_i<\tau_{2j}\}} | \mathcal{G}_t\right]\Bigg) \\
= &\sum_{i=k}^{N} S^{\delta_*}_t c_i \Bigg(\mathbb{E}\left[\mathbb{E}\left[\frac{1}{S^{\delta_*}_{T_i}} \mathbb{1}_{\{W<\tau_1\leq T_i - D, T_i<\tau_2\}} \middle| \mathcal{F}_T \vee \mathcal{F}^X_t\right] \middle| \mathcal{G}_t\right] \\
&\quad + \sum_{j=2}^{\infty} \mathbb{E}\left[\mathbb{E}\left[\frac{1}{S^{\delta_*}_{T_i}} \mathbb{1}_{\{\tau_{2j\text{-}1}-\tau_{2j\text{-}2}>R, W<\tau_{2j\text{-}1}\leq T_i-D, T_i<\tau_{2j}\}} \middle| \mathcal{F}_T \vee \mathcal{F}^X_t\right] \middle| \mathcal{G}_t\right]\Bigg) \\
= &\sum_{i=k}^{N} S^{\delta_*}_t c_i \Bigg(\mathbb{E}\left[\frac{1}{S^{\delta_*}_{T_i}} \mathbb{P}(W < \tau_1 \leq T_i - D, T_i < \tau_2 | \mathcal{F}_T \vee \mathcal{F}^X_t) \middle| \mathcal{G}_t\right] \\
&\quad + \sum_{j=2}^{\infty} \mathbb{E}\left[\frac{1}{S^{\delta_*}_{T_i}} \mathbb{P}(\tau_{2j\text{-}1} - \tau_{2j\text{-}2} > R, W < \tau_{2j\text{-}1} \leq T_i - D, T_i < \tau_{2j} | \mathcal{F}_T \vee \mathcal{F}^X_t) \middle| \mathcal{G}_t\right]\Bigg) \\
= &\sum_{i=k}^{N} c_i S^{\delta_*}_t \Bigg(\mathbb{E}\left[\frac{1}{S_{T_i}} \mathbb{P}(W < \tau_1 \leq T_i - D, T_i < \tau_2 | \mathcal{F}_T \vee \sigma(X_t)) \middle| \mathcal{G}_t\right] \\
&\quad + \sum_{j=2}^{\infty} \mathbb{E}\left[\frac{1}{S^{\delta_*}_{T_i}} \mathbb{P}(\tau_{2j\text{-}1} - \tau_{2j\text{-}2} > R, W < \tau_{2j\text{-}1} \leq T_i - D, T_i < \tau_{2j} | \mathcal{F}_T \vee \sigma(X_t)) \middle| \mathcal{G}_t\right]\Bigg) \\
= &\sum_{i=k}^{N} c_i S^{\delta_*}_t \Bigg(\mathbb{E}\Bigg[\frac{1}{S^{\delta_*}_{T_i}} \underbrace{\mathbb{P}(W - t < \tau_1 \leq T_i - D - t, T_i - t < \tau_2 | \mathcal{F}_T \vee \sigma(X_0))}_{\text{(A)}} \Bigg| \mathcal{G}_t\Bigg] \\
&\quad + \sum_{j=2}^{\infty} \mathbb{E}\Bigg[\frac{1}{S^{\delta_*}_{T_i}} \underbrace{\mathbb{P}(\tau_{2j\text{-}1} - \tau_{2j\text{-}2} > R, W - t < \tau_{2j\text{-}1} \leq T_i - D - t, T_i - t < \tau_{2j} | \mathcal{F}_T \vee \sigma(X_0))}_{(B)} \Bigg| \mathcal{G}_t\Bigg]\Bigg)
\end{aligned} \tag{18}$$

Let $X := \tau_1 - \tau_0$ and $Y := \tau_2 - \tau_1$. Then X and Y are conditionally independent with $X \sim \mathrm{Exp}(\lambda_0^*)$ and $Y \sim \mathrm{Exp}(\lambda_1^*)$, given $\mathcal{F}_T$.

Similarly, for $j \geq 2$, we denote $X_j := \tau_{2j\text{-}1} - \tau_{2j\text{-}2}$, $Y_j := \sum_{l=1}^{2j\text{-}2} (\tau_l - \tau_{l\text{-}1})$ and $Z_j := \tau_{2j} - \tau_{2j\text{-}1}$. Then X_j, Y_j and Z_j are conditionally independent with $X_j \sim \mathrm{Exp}(\lambda_0^*)$, $Y_j \sim \mathrm{Ga}(j-1, \lambda_0^*) * \mathrm{Ga}(j-1, \lambda_1^*)$ and $Z_j \sim \mathrm{Exp}(\lambda_1^*)$ given $\mathcal{F}_T$.

With transformation arguments for the conditional densities we then get that the joint conditional density $f_{\tau_1,\tau_2}\big|_{\mathcal{F}_T}$ of $(X, X+Y) = (\tau_1, \tau_2)$ given $\mathcal{F}_T$ is of the form

$$f_{\tau_1,\tau_2}\big|_{\mathcal{F}_T}(u,v) = \lambda_0^* e^{-\lambda_0^* u} \mathbb{1}_{(0,\infty)}(u) \lambda_1^* e^{-\lambda_1^*(v-u)} \mathbb{1}_{(0,\infty)}(v-u) \;.$$

Moreover, for $j \geq 2$, the joint conditional densities $f_{\tau_{2j\text{-}1}-\tau_{2j\text{-}2},\tau_{2j\text{-}1},\tau_{2j}}\big|_{\mathcal{F}_T}$ of $(X_j, X_j+Y_j, X_j+Y_j+Z_j) = (\tau_{2j\text{-}1} - \tau_{2j\text{-}2}, \tau_{2j\text{-}1}, \tau_{2j})$ given $\mathcal{F}_T$ are of the form

$$f_{\tau_{2j\text{-}1}-\tau_{2j\text{-}2},\tau_{2j\text{-}1},\tau_{2j}}\big|_{\mathcal{F}_T}(u,v,w) = \lambda_0^* e^{-\lambda_0^* u} \mathbb{1}_{(0,\infty)}(u) g_{2j\text{-}2}(v-u) \mathbb{1}_{(0,\infty)}(v-u) \lambda_1^* e^{-\lambda_1^*(w-v)} \mathbb{1}_{(0,\infty)}(w-v) \;,$$

where $g_{2j\text{-}2}$ is given by

$$g_{2j\text{-}2}(x) = \frac{(\lambda_0^*)^{j-1}(\lambda_1^*)^{j-1}}{(j-2)!(j-2)!} \int_0^x u^{j-2}(x-u)^{j-2} e^{-\lambda_0^* u} e^{-\lambda_1^*(x-u)} du. \tag{19}$$

Hence we have that (A) and (B) in (18) are given as

$$(A) = \int_{W\text{-}t}^{T_i\text{-}D\text{-}t} \int_{T_i\text{-}t}^{\infty} f_{\tau_1,\tau_2}\big|_{\mathcal{F}_T}(u,v) du dv = \frac{\lambda_0^*}{S_{T_i}^{\delta_*}(\lambda_0^* - \lambda_1^*)} e^{-\lambda_1^*(T_i-t)} \left(e^{-(\lambda_0^*-\lambda_1^*)(W-t)} - e^{-(\lambda_0^*-\lambda_1^*)(T_i-D-t)}\right)$$

and

$$\begin{aligned}(B) &= \int_R^{\infty} \int_{W\text{-}t}^{T_i\text{-}D\text{-}t} \int_{T_i\text{-}t}^{\infty} f_{\tau_{2j\text{-}1}\text{-}\tau_{2j\text{-}2},\tau_{2j\text{-}1},\tau_{2j}}\big|_{\mathcal{F}_T}(u,v,w) du dv dw \\ &= \lambda_0^* e^{-\lambda_1^*(T_i-t)} \int_{(W\text{-}t)\vee R}^{T_i\text{-}D\text{-}t} e^{\lambda_1^* v} \int_R^v e^{-\lambda_0^* u} g_{2j\text{-}2}(v-u) du dv \;.\end{aligned}$$

Therefore

$$\begin{aligned} P_t = \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \Bigg(&\mathbb{E}\left[\frac{\lambda_0^*}{S_{T_i}^{\delta_*}(\lambda_0^*-\lambda_1^*)} e^{-\lambda_1^*(T_i-t)} \left(e^{-(\lambda_0^*-\lambda_1^*)(W-t)} - e^{-(\lambda_0^*-\lambda_1^*)(T_i-D-t)}\right) \Big| \mathcal{G}_t \right] \\ &+ \sum_{j=2}^{\infty} \mathbb{E}\left[\frac{1}{S_{T_i}^{\delta_*}} \lambda_0^* e^{-\lambda_1^*(T_i-t)} \int_{\max\{W-t,R\}}^{T_i-D-t} e^{\lambda_1^* y} \int_R^y e^{-\lambda_0^* x} g_{2j-2}(y-x) dx dy \Big| \mathcal{G}_t \right] \Bigg) \;, \end{aligned} \tag{20}$$

where k^* is chosen to be the first $i > k$ such that $T_i - D \geq W$. By monotone convergence this yields

$$P_t = \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \left(\mathbb{E}\left[\underbrace{\frac{\lambda_0^*}{S_{T_i}^{\delta_*}(\lambda_0^* - \lambda_1^*)} e^{-\lambda_1^*(T_i-t)} \left(e^{-(\lambda_0^*-\lambda_1^*)(W-t)} - e^{-(\lambda_0^*-\lambda_1^*)(T_i-D-t)} \right) \Big| \mathcal{G}_t}_{(*)} \right] \right.$$
$$\left. + \underbrace{\mathbb{E}\left[\frac{(\lambda_0^*)^2 \lambda_1^* e^{-\lambda_1^*(T_i-t)}}{S_{T_i}^{\delta_*}} \int_{(W\text{-}t)\vee R}^{T_i\text{-}D\text{-}t} \int_R^v e^{-(\lambda_0^*-\lambda_1^*)u} \int_0^{v-u} e^{-(\lambda_0^*-\lambda_1^*)x} I_0\left(2\sqrt{\lambda_0^* \lambda_1^* x(v-u-x)} \right) dx du dv \Big| \mathcal{G}_t \right]}_{(**)} \right), \tag{21}$$

where I_0 is the modified first kind Bessel function of order 0. In general, the modified first order Bessel function I_α of order $\alpha \in \mathbb{R}$ is given by

$$I_\alpha(y) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{y}{2}\right)^{2m+\alpha}. \tag{22}$$

Recall that λ_0^* and λ_1^* and therefore also $(*)$ and $(**)$ are $\mathcal{F}_T$-measurable and non-negative. Since hypothesis (H) holds[5] we finally have

$$P_t = \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \left(\mathbb{E}\left[\frac{\lambda_0^*}{S_{T_i}^{\delta_*}(\lambda_0^* - \lambda_1^*)} e^{-\lambda_1^*(T_i-t)} \left(e^{-(\lambda_0^*-\lambda_1^*)(W-t)} - e^{-(\lambda_0^*-\lambda_1^*)(T_i-D-t)} \right) \Big| \mathcal{F}_t \right] \right.$$
$$\left. + \mathbb{E}\left[\frac{(\lambda_0^*)^2 \lambda_1^* e^{-\lambda_1^*(T_i-t)}}{S_{T_i}^{\delta_*}} \int_{(W\text{-}t)\vee R}^{T_i\text{-}D\text{-}t} \int_R^v e^{-(\lambda_0^*-\lambda_1^*)u} \int_0^{v-u} e^{-(\lambda_0^*-\lambda_1^*)x} I_0\left(2\sqrt{\lambda_0^* \lambda_1^* x(v-u-x)} \right) dx du dv \Big| \mathcal{F}_t \right] \right), \tag{23}$$

Note that, due to the "loss of memory" property of X, it is sufficient to calculate the insurance premiums for $t \leq \tau_1$. Analogous computations deliver the price for all the other cases, appearing in the pricing equations (14) and (15).
In the following proposition, we give the insurance premium for the most important cases, where $t < \tau_1$. In Appendix A, we compute P_t in the particular case when $\tau_{2m-2} \leq t < \tau_{2m-1}$, for the reader's convenience.

Proposition 6.5. *For $t < \tau_1$ (for example $t = 0$) we obtain the insurance premiums P_t as follows. If*

- *$t \in [T_{k\text{-}1}, W)$, P_t is given in (23).*

[5] Note that $(*)$ and $(**)$ need not to be bounded. However, one can truncate by n and use monotone convergence.

- *$T_{k\text{-}1} \leq W \leq t < T_k$ or $W \leq T_{k\text{-}1} \leq t < T_k$, P_t is given by*

$$P_t = \sum_{i=k}^{N} c_i S_t^{\delta_*} \Bigg(\mathbb{E}\left[\frac{\lambda_0^*}{S_{T_i}^{\delta_*}(\lambda_0^* - \lambda_1^*)} e^{-\lambda_1^*(T_i - t)} \big(1 - e^{-(\lambda_0^* - \lambda_1^*)(T_i - D - t)}\big) \Big| \mathcal{F}_t \right]$$
$$+ \mathbb{E}\left[\frac{(\lambda_0^*)^2 \lambda_1^* e^{-\lambda_1^*(T_i - t)}}{S_{T_i}^{\delta_*}} \int_R^{T_i - D - t} \int_R^{v} e^{-(\lambda_0^* - \lambda_1^*)u} \int_0^{v-u} e^{-(\lambda_0^* - \lambda_1^*)x} I_0\left(2\sqrt{\lambda_0^* \lambda_1^* x (v - u - x)}\right) dx du dv \Big| \mathcal{F}_t \right] \Bigg) . \tag{24}$$

The evaluation of the conditional expectations in equations (23) and (24) can be handled in different ways. Analytical solutions, like inverse Fourier methods may face some difficulties as the functions in the conditional expectations are rather hard to invert, not least because they depend on the value of $S_t^{\delta_*}$ at the three different times t, T_i and T. Of course, one could assume some dynamics for $(S_t^{\delta_*})_{t \in [0,T]}$ and then run Monte Carlo simulations for the case $t = 0$. In the following example we show how to evaluate P_t, depending only on the present value $S_t^{\delta_*}$, if we assume the $\mathbb{P}$-numéraire portfolio to be a Lévy process.

Example 6.6. *Let the process $\left(S_t^{\delta_*}\right)_{t \in [0,T]}$ be a Lévy process with Levy Khintchine characteristics (a, b, ν) and such that its distribution at time $t \in [0, T]$ provides a density h_t^* with respect to the Lebesgue measure, given by*

$$h_t^*(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iyx} \varphi_{S_t^{\delta_*}}(y) dy \ ,$$

where $\varphi_{S_t^{\delta_*}}$ is the characteristic function of $S_t^{\delta_*}$ at time $t \in [0, T]$, equal to

$$\varphi_{S_t^{\delta_*}}(y) = \exp\bigg(-t\Big(\frac{b^2}{2} y^2 - iay + \int_{\{|x| \geq 1\}} (1 - e^{iyx}) \nu(dx) + \int_{\{|x| < 1\}} (1 - e^{iyx} + iyx) \nu(dx) \Big) \bigg) .$$

Moreover, we assume $\mathcal{F}_t = \sigma\left(S_u^{\delta_*} : u \leq t\right)$. For notational convenience we rewrite the expressions $(*)$ and $(**)$ of equation (21) as $\frac{\phi_1(S_T^{\delta_*})}{S_{T_i}^{\delta_*}}$ and $\frac{\phi_2(S_T^{\delta_*})}{S_{T_i}^{\delta_*}}$ for some measurable functions ϕ_1 and ϕ_2.

We again show the following calculations only for the case $t \in [T_{k\text{-}1}, W)$. We get

$$
\begin{aligned}
P_t &= \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \left(\mathbb{E}\left[\frac{\phi_1(S_T^{\delta_*})}{S_{T_i}^{\delta_*}} \Big| \mathcal{F}_t \right] + \mathbb{E}\left[\frac{\phi_2(S_T^{\delta_*})}{S_{T_i}^{\delta_*}} \Big| \mathcal{F}_t \right] \right) \\
&= \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \left(\mathbb{E}\left[\mathbb{E}\left[\frac{\phi_1(S_T^{\delta_*} - S_{T_i}^{\delta_*} + S_{T_i}^{\delta_*})}{S_{T_i}^{\delta_*}} \big| \mathcal{F}_{T_i} \right] \Big| \mathcal{F}_t \right] + \mathbb{E}\left[\mathbb{E}\left[\frac{\phi_2(S_T^{\delta_*} - S_{T_i}^{\delta_*} + S_{T_i}^{\delta_*})}{S_{T_i}^{\delta_*}} \big| \mathcal{F}_{T_i} \right] \Big| \mathcal{F}_t \right] \right) \\
&= \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \left(\mathbb{E}\left[\mathbb{E}\left[\frac{\phi_1(S_T^{\delta_*} - S_{T_i}^{\delta_*} + x)}{x} \right] \Big|_{x=S_{T_i}^{\delta_*}} \Big| \mathcal{F}_t \right] + \mathbb{E}\left[\mathbb{E}\left[\frac{\phi_2(S_T^{\delta_*} - S_{T_i}^{\delta_*} + x)}{x} \right] \Big|_{x=S_{T_i}^{\delta_*}} \Big| \mathcal{F}_t \right] \right) \\
&= \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \left(\mathbb{E}\left[\int_{\mathbb{R}} \frac{\phi_1(y+x)}{x} h^*_{T\text{-}T_i}(y) dy \ \Big|_{x=S_{T_i}^{\delta_*}} \Big| \mathcal{F}_t \right] + \mathbb{E}\left[\int_{\mathbb{R}} \frac{\phi_2(y+x)}{x} h^*_{T\text{-}T_i}(y) dy \ \Big|_{x=S_{T_i}^{\delta_*}} \Big| \mathcal{F}_t \right] \right) \\
&= \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \left(\mathbb{E}\Big[\underbrace{\int_{\mathbb{R}} \frac{\phi_1(y + S_{T_i}^{\delta_*})}{S_{T_i}^{\delta_*}} h^*_{T\text{-}T_i}(y) dy}_{=:\psi_1(S_{T_i}^{\delta_*})} \Big| \mathcal{F}_t \Big] + \mathbb{E}\Big[\underbrace{\int_{\mathbb{R}} \frac{\phi_2(y + S_{T_i}^{\delta_*})}{S_{T_i}^{\delta_*}} h^*_{T\text{-}T_i}(y) dy}_{=:\psi_2(S_{T_i}^{\delta_*})} \Big| \mathcal{F}_t \Big] \right) \\
&= \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \left(\mathbb{E}\left[\psi_1(S_{T_i}^{\delta_*} - S_t^{\delta_*} + S_t^{\delta_*}) \Big| \mathcal{F}_t \right] + \mathbb{E}\left[\psi_2(S_{T_i}^{\delta_*} - S_t^{\delta_*} + S_t^{\delta_*}) \Big| \mathcal{F}_t \right] \right) \\
&= \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \left(\mathbb{E}\left[\psi_1(S_{T_i}^{\delta_*} - S_t^{\delta_*} + x) \right] \Big|_{x=S_t^{\delta_*}} + \mathbb{E}\left[\psi_2(S_{T_i}^{\delta_*} - S_t^{\delta_*} + x) \right] \Big|_{x=S_t^{\delta_*}} \right) \\
&= \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \left(\int_{\mathbb{R}} \psi_1(z+x) h^*_{T_i - t}(z) dz \ \Big|_{x=S_t^{\delta_*}} + \int_{\mathbb{R}} \psi_2(z+x) h^*_{T_i - t}(z) dz \ \Big|_{x=S_t^{\delta_*}} \right) \\
&= \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \left(\int_{\mathbb{R}} \psi_1(z + S_t^{\delta_*}) h^*_{T_i - t}(z) dz + \int_{\mathbb{R}} \psi_2(z + S_t^{\delta_*}) h^*_{T_i - t}(z) dz \right)
\end{aligned} \tag{25}
$$

For the reader's convenience, we provide the full expressions of (25) *in Appendix B.*
Yet, we already recognize that in this special setting we are able to compute the insurance premium P_t by simulating the value of the $\mathbb{P}$-numéraire portfolio $S_t^{\delta_*}$ at time $t \in [0, T]$.

7. Sensitivities of the premium

In this section, we provide first calculations for the insurance premium P_0 at time $t = 0$ for a purely Markovian setting. To this end we assume that $\mathcal{F}_t = \{\emptyset, \Omega\}$ for all $t \in [0, T]$ and hence that the $\mathbb{P}$-numéraire portfolio is constant. Due to arbitrage arguments we have $S_t^{\delta_*} = e^{-rt}$ for some constant interest rate $r \geq 0$. In this case the expressions in the conditional expectation of formula (23) are

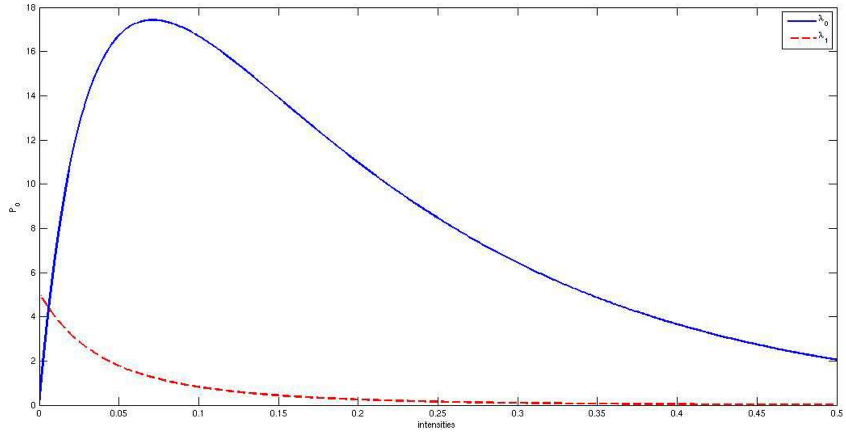

Figure 3: Sensitivity of the insurance premium P_0 to the intensity parameters λ_0^* and λ_1^*.

deterministic and we obtain the premium P_0 as

$$P_0 = \sum_{i=1}^{N} c_i e^{-rT_i} \left(\frac{\lambda_0^*}{(\lambda_0^* - \lambda_1^*)} e^{-\lambda_1^*(T_i-t)} \left(e^{-(\lambda_0^*-\lambda_1^*)(W-t)} - e^{-(\lambda_0^*-\lambda_1^*)(T_i-D-t)} \right) \right.$$
$$\left. + (\lambda_0^*)^2 \lambda_1^* e^{-\lambda_1^*(T_i-t)} \int\limits_{(W\text{-}t)\vee R}^{T_i\text{-}D\text{-}t} \int\limits_{R}^{v} e^{-(\lambda_0^*-\lambda_1^*)u} \int\limits_{0}^{v-u} e^{-(\lambda_0^*-\lambda_1^*)x} I_0 \left(2\sqrt{\lambda_0^*\lambda_1^* x(v-u-x)} \right) dxdudv \right. . \quad (26)$$

.

In order to test the model on its accuracy, we evaluate this premium numerically and investigate its sensitivity with respect to several parameters. For this purpose, we consider an insurance contract, where the insured person is employed at the contract's actual beginning ($X_0 = 0$). Moreover, we assume the insurance company to pay a constant, monthly amount $c_i = 1$ if a claim occurs. The sensitivity of the premium is then tested by the parameter of interest, while fixing the other free variables of formula (21) to one of the following levels:

$$T_N = 72(\text{months}) \ , \ c_i = 1 \ , \ \lambda_0^* = 0,36\% \ , \ \lambda_1^* = 2,7\% \ ,$$
$$W = 6 \text{ (months)} \ , \ D = 6 \text{ (months)} \ , \ R = 6 \text{ (months)} \ , \ r = 4\% \text{ p.a.}$$

- **The intensities λ_0^* and λ_1^*:**
  Figure 3 shows the characteristics of the insurance premium due to changes in the intensities λ_0^* and λ_1^*. The results reflect the natural demands on the insurance premium: because of the exponential distributions we have the relationships $\mathbb{E}_{\mathbb{P}}[\tau_j - \tau_{j-1}] = \frac{1}{\lambda_0}$ for odd $j \in \mathbb{N}$ and $\mathbb{E}_{\mathbb{P}}[\tau_j - \tau_{j-1}] = \frac{1}{\lambda_1}$ for even $j \in \mathbb{N}$. Therefore, an increase in λ_0^* is equal to a decrease in the expected time of employment. As a consequence, the insurance premium has first to increase. In the (unrealistic) case, where the expected time of employment reaches about 14,3 months ($\lambda_0^* \approx 0,07$), more and more claims are not paid because the employment time does often not exceed the waiting or the requalification time. Consequently, the insurance premium decreases. Analogously, an increase in λ_1^* is equal to a decrease in the expected time of unemployment and

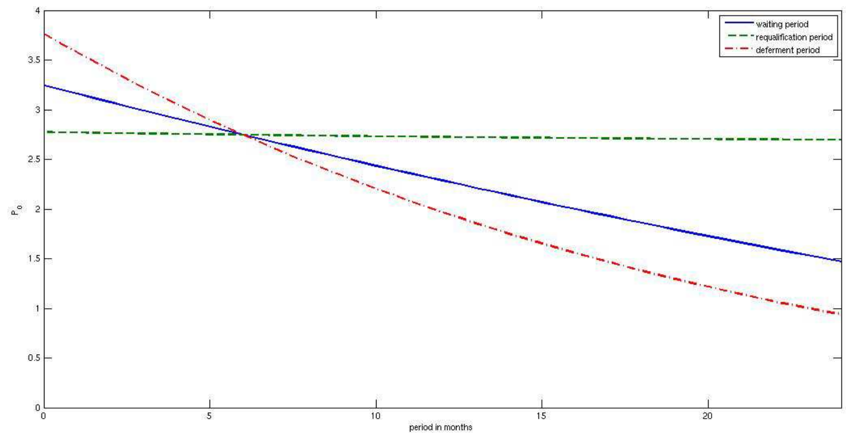

Figure 4: Sensitivity of the insurance premium P_0 to the waiting, deferment and requalification periods.

the insurance premium has to decrease.
Moreover, Figure 3 shows that the insurance premium reacts more sensitive on changes in the employment intensity λ_0^* than in the unemployment intensity λ_1^*.

- **Waiting, deferment and requalification period:**
  Figure 4 shows the changes in the insurance premium due to variations in the waiting, deferment and requalification periods. Again, the results reflect the natural demand on the insurance premium: an increase in all three periods decreases the probability of claim payments and should therefore result in lower insurance premiums.
  Moreover, the insurance premium reacts more sensitive to changes in the waiting and the deferment period as to changes in the requalification period.

8. Estimation of the intensities

We again assume the simple model specifications of Section 7. The insurance premium according to equation (26) is then a function of several parameters. Most of them are fixed in the terms of contract like the waiting, deferment and requalification period, the maturity, the payments c_i or their respective dates T_i. A priori unknown parameters are the intensities λ_0^* and λ_1^* as well as the risk-free interest rate. In this section, we present estimation results for the intensities λ_0^* and λ_1^*, which are based on real data, published by the "Federal Employment Office" of Germany[6].
Because X in this simple setting is supposed to follow a time-homogeneous Markov chain, we can apply the estimation methods presented in Kalbfleisch and Lawless [19]. A constant quantity of persons is here observed over a certain period. Each person is supposed to be in a certain observable state at any time of the period, where the set of states is finite (in our case the states are: "employed" and "unemployed"). At some fixed time points of the period the respective state of each person in the group is listed.

[6]All data that was used for the estimations can be found on the web page of the Federal Employment Office of Germany, see e.g. http://www.pub.arbeitsagentur.de/hst/services/statistik/detail/a.html

For every person, the random process of jumping from one state to another is supposed to follow a time-homogeneous Markov chain and all processes are supposed to be iid. Under these assumptions, it is then possible to apply a Maximum Likelihood estimation method to derive estimators for the components of the (unique) intensity matrix Λ^*.
In order to perform the estimations, we need

- the number n_{00}^i of persons that are in state 0 at time t_{i-1} and at time t_i,
- the number n_{01}^i of persons that moved from state 0 at time t_{i-1} to state 1 at time t_i,
- the number n_{11}^i of persons that are in state 1 at time t_{i-1} and at time t_i,
- the number n_{10}^i of persons that moved from state 1 at time t_{i-1} to state 0 at time t_i, $i = 1, ..., n$,

for a sequence $t_0, ..., t_n$ of points in time. Unfortunately, there is no public data available, containing the exact numbers. Therefore, we proceed in the following way:
We set the time lag $t_i - t_{i-1}$ equal to one month and extract approximations to the required numbers out of the monthly job market reports of the Federal Employment Office of Germany.
To this end, we interpret the labor force in Germany to consist of all employed and unemployed persons. The total number of unemployed persons as well as the unemployment rate are given in the reports. The unemployment rate however is the fraction of all unemployed persons with respect to the labor force. Hence by using this information, we can derive the total numbers of unemployed and employed persons, respectively, every month. These values are interpreted to be the number of persons that stay in the state "unemployed" or "employed", respectively. The monthly changes of the total values are interpreted to form the number of persons that are employed at time t_{i-1} and become unemployed until t_i and vice versa. Moreover, since the labor force of Germany is identified once a year (at the beginning of May), the observed samples remain constant over a time period of one year. This is important for our estimation method as the sample of observed people must remain constant over the observation period.
This way of extracting the numbers contains some inaccuracies, however. First of all, the labor force is not constant over a time period of one year and may not only consist of employed and unemployed persons. Moreover, the number of changes from one state to another represent cumulated values and not the actual total values.
Hence, our analysis requires further investigation. First refinements will be provided in [3]. However, the sample of available data, although too small, provides already some valuable hints to the real dimensions of the intensities.
We performed the estimation procedure on the monthly data from May to April of the years $'98/'99$ - $'07/'08$, respectively. The results can be found in Table 1.

	$\overline{\boldsymbol{\lambda}_0^*}$	$\overline{\boldsymbol{\lambda}_1^*}$
'98/'99	0,0038552007	0,0282108984
'99/'00	0,0036361118	0,0281011314
'00/'01	0,0034410489	0,0284704359
'01/'02	0,0036146433	0,0265663626
'02/'03	0,0040511845	0,0251325837
'03/'04	0,0042670366	0,0263048480
'04/'05	0,0041309233	0,0244147955
'05/'06	0,0035819277	0,0243732156
'06/'07	0,0027738128	0,0247146137
'07/'08	0,0030655450	0,0325999498

Table 1: Maximum-Likelihood estimators of the intensities λ_0^* and λ_1^* for the periods $'98/'99$ - $'07/'08$.

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Role of the Funding Source

The opinions expressed in this article are those of the authors and do not necessarily reflect the views of Swiss Life Insurance Solutions AG. Moreover, presented simulation concepts are not necessarily used by Swiss Life Insurance Solutions AG or any affiliates.

Swiss Life Insurance Solutions AG provides the general research framework of investigating unemployment insurance products but is neither incorporated in the collection, analysis and interpretation of data, nor in the writing of the working paper. Moreover, submission for publication of the working paper is gratefully accepted.

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Under the assumptions of Section 6, we now compute the insurance premium P_t, when $t \in [T_{k-1}, W)$, and $\tau_{2m-2} \leq t < \tau_{2m-1}$ for some $m \geq 2$. By equation (14) we get

$$
\begin{aligned}
P_t =& \sum_{i=k}^{N} c_i S_t^{\delta_*} \Bigg(\mathbb{E}\Big[\widehat{\mathbb{1}}_{\{W \vee (R+\kappa) < \tau_{2m\text{-}1} \leq T_i - D, \tau_{2m} > T_i\}} \Big| \mathcal{G}_t \Big] \Big|_{\kappa = \tau_{2m\text{-}2}} \\
&\quad + \sum_{j=m+1}^{\infty} \mathbb{E}\Big[\widehat{\mathbb{1}}_{\{\tau_{2j\text{-}1} - \tau_{2j-2} > R, W < \tau_{2j\text{-}1} \leq T_i - D, \tau_{2j} > T_i\}} \Big| \mathcal{G}_t \Big] \Bigg) \\
=& \sum_{i=k}^{N} c_i S_t^{\delta_*} \Bigg(\mathbb{E}\Big[\mathbb{E}\Big[\frac{1}{S_{T_i}^{\delta_*}} \mathbb{1}_{\{W \vee (R+\kappa) < \tau_{2m\text{-}1} \leq T_i - D, \tau_{2m} > T_i\}} \Big| \mathcal{F}_T \vee \mathcal{F}_t^X \Big] \Big| \mathcal{G}_t \Big] \Big|_{\kappa = \tau_{2m\text{-}2}} \\
&\quad + \sum_{j=m+1}^{\infty} \mathbb{E}\Big[\mathbb{E}\Big[\frac{1}{S_{T_i}^{\delta}} \mathbb{1}_{\{\tau_{2j\text{-}1} - \tau_{2j-2} > R, W < \tau_{2j\text{-}1} \leq T_i - D, \tau_{2j} > T_i\}} \Big| \mathcal{F}_T \vee \mathcal{F}_t^X \Big] \Big| \mathcal{G}_t \Big] \Bigg) \\
=& \sum_{i=k}^{N} c_i S_t^{\delta_*} \Bigg(\mathbb{E}\Big[\frac{1}{S_{T_i}^{\delta_*}} \mathbb{P}\Big(W \vee (R+\kappa) < \tau_{2m\text{-}1} \leq T_i - D, \tau_{2m} > T_i \Big| \mathcal{F}_T \vee \mathcal{F}_t^X \Big) \Big| \mathcal{G}_t \Big] \Big|_{\kappa = \tau_{2m\text{-}2}} \\
&\quad + \sum_{j=m+1}^{\infty} \mathbb{E}\Big[\frac{1}{S_{T_i}^{\delta}} \mathbb{P}\Big(\tau_{2j\text{-}1} - \tau_{2j-2} > R, W < \tau_{2j\text{-}1} \leq T_i - D, \tau_{2j} > T_i \Big| \mathcal{F}_T \vee \mathcal{F}_t^X \Big) \Big| \mathcal{G}_t \Big] \Bigg) \\
=& \sum_{i=k}^{N} c_i S_t^{\delta_*} \Bigg(\mathbb{E}\Big[\frac{1}{S_{T_i}^{\delta_*}} \mathbb{P}\Big(W \vee (R+\kappa) < \tau_{2m\text{-}1} \leq T_i - D, \tau_{2m} > T_i \Big| \mathcal{F}_T \vee \sigma(X_t) \Big) \Big| \mathcal{G}_t \Big] \Big|_{\kappa = \tau_{2m\text{-}2}} \\
&\quad + \sum_{j=m+1}^{\infty} \mathbb{E}\Big[\frac{1}{S_{T_i}^{\delta}} \mathbb{P}\Big(\tau_{2j\text{-}1} - \tau_{2j-2} > R, W < \tau_{2j\text{-}1} \leq T_i - D, \tau_{2j} > T_i \Big| \mathcal{F}_T \vee \sigma(X_t) \Big) \Big| \mathcal{G}_t \Big] \Bigg) \\
=& \sum_{i=k}^{N} c_i S_t^{\delta_*} \Bigg(\mathbb{E}\Big[\frac{1}{S_{T_i}^{\delta_*}} \mathbb{P}\Big(W \vee (R+\kappa) < \tau_1 \leq T_i - D, \tau_2 > T_i \Big| \mathcal{F}_T \vee \sigma(X_0) \Big) \Big| \mathcal{G}_t \Big] \Big|_{\kappa = \tau_{2m\text{-}2}} \\
&\quad + \sum_{j=2}^{\infty} \mathbb{E}\Big[\frac{1}{S_{T_i}^{\delta}} \mathbb{P}\Big(\tau_{2j\text{-}1} - \tau_{2j-2} > R, W < \tau_{2j\text{-}1} \leq T_i - D, \tau_{2j} > T_i \Big| \mathcal{F}_T \vee \sigma(X_0) \Big) \Big| \mathcal{G}_t \Big] \Bigg) \qquad (27) \\
\overset{\text{see (21)}}{=}& \sum_{i=k}^{N} c_i S_t^{\delta_*} \Bigg(\mathbb{E}\left[\frac{\lambda_0^*}{S_{T_i}^{\delta_*}(\lambda_0^* - \lambda_1^*)} e^{-\lambda_1^*(T_i - t)} \Big(e^{-(\lambda_0^* - \lambda_1^*)(\max\{W, R+\tau_{2m-2}\} - t)} - e^{-(\lambda_0^* - \lambda_1^*)(T_i - D - t)} \Big) \Big| \mathcal{G}_t \right] \\
&+ \mathbb{E}\left[(\lambda_0^*)^2 \lambda_1^* e^{-\lambda_1^*(T_i - t)} \int_{\max\{W-t, R\}}^{T_i - D - t} \int_{R}^{y} e^{-(\lambda_0^* - \lambda_1^*)x} \int_{0}^{y-x} e^{-(\lambda_0^* - \lambda_1^*)u} I_0\big(2\sqrt{\lambda_0^* \lambda_1^* u(y-x-u)}\big) du\,dx\,dy \Big| \mathcal{G}_t \right] \Bigg). \qquad (28)
\end{aligned}
$$

Note that $\tau_{2m\text{-}1}$ is by assumption the first jump time after time t. Hence, by exploiting the time-homogeneity of X and executing a "time-reduction" by t, $\tau_{2m\text{-}1}$ becomes the first jump time of the "renewed" Markov chain and equality (27) holds. The last equality (28) follows by the same argument as for formula (21) in Section 6.

We provide here the full expressions of Example 6.6 for the insurance premium P_t at time $t \in [0,T]$. According to (25), we have

$$
\begin{aligned}
P_t =& \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \left(\int_{\mathbb{R}} \psi_1(z+S_t^{\delta_*}) h^*_{T_i-t}(z)dz + \int_{\mathbb{R}} \psi_2(z+S_t^{\delta_*}) h^*_{T_i-t}(z)dz \right) \\
=& \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\phi_1(y+z+S_t^{\delta_*})}{z+S_t^{\delta_*}} h^*_{T\text{-}T_i}(y)dy h^*_{T_i-t}(z)dz + \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\phi_2(y+z+S_t^{\delta_*})}{z+S_t^{\delta_*}} h^*_{T\text{-}T_i}(y)dy h^*_{T_i-t}(z)dz \right) \\
=& \sum_{i=k^*}^{N} c_i S_t^{\delta_*} \left(\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\lambda_0(y+z+S_t^{\delta_*})}{(z+S_t^{\delta_*})(\lambda_0(y+z+S_t^{\delta_*})-\lambda_1(y+z+S_t^{\delta_*}))} e^{-\lambda_1(y+z+S_t^{\delta_*})(T_i-t)} \right. \\
& \cdot \left(e^{-(\lambda_0(y+z+S_t^{\delta_*})-\lambda_1(y+z+S_t^{\delta_*}))(W-t)} - e^{-(\lambda_0(y+z+S_t^{\delta_*})-\lambda_1(y+z+S_t^{\delta_*}))(T_i-D-t)} \right) h^*_{T\text{-}T_i}(y) h^*_{T_i-t}(z)dydz \\
& + \int_{\mathbb{R}}\int_{\mathbb{R}} \left[\frac{(\lambda_0(y+z+S_t^{\delta_*}))^2 \lambda_1(y+z+S_t^{\delta_*}) e^{-\lambda_1(y+z+S_t^{\delta_*})(T_i-t)}}{z+S_t^{\delta_*}} \int_{(W\text{-}t)\vee R}^{T_i\text{-}D\text{-}t} \int_{R}^{v} e^{-(\lambda_0(y+z+S_t^{\delta_*})-\lambda_1(y+z+S_t^{\delta_*}))u} \right. \\
& \cdot \left. \int_0^{v-u} e^{-(\lambda_0(y+z+S_t^{\delta_*})-\lambda_1(y+z+S_t^{\delta_*}))x} I_0\left(2\sqrt{\lambda_0(y+z+S_t^{\delta_*})\lambda_1(y+z+S_t^{\delta_*})x(v-u-x)} \right) dxdudv \right] \\
& \cdot \left. h^*_{T\text{-}T_i}(y) h^*_{T_i-t}(z)dydz \right),
\end{aligned}
$$

where

$$
h^*_t(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left(-iyz - t\left(\frac{b^2}{2}y^2 - iay + \int_{\{|x|\geq 1\}} (1-e^{iyx})\nu(dx) + \int_{\{|x|<1\}} (1-e^{iyx}+iyx)\nu(dx) \right) \right) dy \, .
$$

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