# Optimal Portfolio for an Insider in a Market driven by Lévy Processes

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Dedicated to the memory of Axel Grorud

**Abstract**

We consider a financial market driven by a Lévy process with filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$. An *insider* in this market is an agent who has access to more information than an honest trader. Mathematically, this is modelled by allowing a strategy of an insider to be adapted to a bigger filtration $\mathcal{G}_t \supseteq \mathcal{F}_t$. The corresponding anticipating stochastic differential equation of the wealth is interpreted in the sense of *forward integrals*. In this framework, we study the optimal portfolio problem of an insider with logarithmic utility function. Explicit results are given in the case where the jumps are generated by a Poisson process.

*Key words and phrases*: forward integral, Malliavin derivative, insider trading, utility function, enlargement of filtration.

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## 1 Introduction

A trader on the stock market is usually assumed to make his decisions relying on all the information which is generated by the market events. This information is assumed to be free and at everyone's disposal: a dealer who is selecting some portfolio can exploit the knowledge of the whole history of market events up to the time in which his decisions are to be taken. In general the vast number of varied market events (i.e. *full information*) makes it difficult for traders to take advantage of the total information available and, most of the times, decisions are actually based on some *partial information*. In any of these cases it is always assumed that dealers can only read the information needed from the flow of the market events up to their present time.

However it is registered that some people have *more* detailed information than others, in the sense that they act with present time knowledge of some *future* event. This is the so-called *insider information* and those dealers taking advantage of it are the *insiders*. Insider trading is illegal and prosecuted by law.

Nevertheless it is mathematically challenging to model their behavior on the market and some part of the most recent literature in mathematical finance is related to this problem. The aim of the research in this direction is however not to help the insider trading, but to

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give a picture of how much better and with which strategies an insider can perform on the market if he uses optimally the extra information he has at disposal.

The mathematical challenges are all nested in the fact that information from the future is used and thus the usual techniques of stochastic integration cannot be directly applied.

In this line of interests our paper deals with an optimal portfolio problem for an insider in a market driven by a Lévy process. In the sequel we give a detailed description of the market model we are considering and our approach to insider trading modeling. A sketch of the content of this paper is given in the final part of this introduction.

Given a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, $t \in [0,T]$, and a time horizon $T$, consider a financial market with two investment possibilities:

$$\text{(1.1)} \qquad \text{(Bond price)} \quad dS_0(t) = r(t)S_0(t)dt; \quad S_0(0) = 1$$

$$\text{(1.2)} \qquad \text{(Stock price)} \quad dS_1(t) = S_1(t^-)[\mu(t)dt + \int_{\mathbb{R}} \theta(t,z)\widetilde{N}(dt,dz)]; \quad S_1(0) > 0.$$

Here we assume that

(1.3) $r(t) = r(t,\omega),\ \mu(t) = \mu(t,\omega),\ \theta(t,z) = \theta(t,z,\omega)$ are $\mathcal{F}_t$-adapted caglad processes (i.e. left continuous and with right limits), where $t \in [0,T],\ \omega \in \Omega,\ z \in \mathbb{R}\backslash\{0\}$.

(1.4) $-1 < \theta(t,z) \qquad dt \times \nu_{\mathcal{F}}(dz)$ a.e.,

$$\text{(1.5)} \qquad E\left[\int_0^T \left\{|r(t)| + |\mu(t)| + \int_{\mathbb{R}} \theta^2(t,z)\nu_{\mathcal{F}}(dz)\right\} dt\right] < \infty,$$

where $\widetilde{N}(dt,dz) = N(dt,dz) - dt\nu_{\mathcal{F}}(dz)$ is the compensated Poisson random measure associated to a given compensated pure jump $\mathcal{F}_t$-Lévy process $\eta(t) = \eta(t,\omega),\ \omega \in \Omega$ . This means that

$$\text{(1.6)} \qquad \eta(t) = \int_0^t \int_{\mathbb{R}} z\widetilde{N}(dt,dz).$$

See e.g. [A], [B], [P] and [Sa] for more information about Lévy processes.

Note that here the filtration $\{\mathcal{F}_t\}$ ($\mathcal{F}_0$ trivial) represent the increasing flow of information that is generated by the market events according to the above dynamics. This represents the *full information* at disposal to all *honest* traders.

Since a square integrable Lévy process $\Lambda(t)$ can be written in the form

$$\Lambda(t) = \alpha t + \beta B(t) + \int_0^t \int_{\mathbb{R}} z\widetilde{M}(ds,dz),$$

where $B(t)$ is a Brownian motion, $\widetilde{M}(ds,dz)$ a compensated Poisson random measure and $\alpha$, $\beta$ are constants, we see that the model (1.2) may be regarded as the *pure jump part* of the model

$$\text{(1.7)} \qquad dS_1(t) = S_1(t^-)[\mu(t)dt + \gamma(t)d\Lambda(t)]$$

driven by the Lévy process $\Lambda(t)$. In the other extreme, the *continuous part* of (1.7) is the more widely studied model

$$dS_1(t) = S_1(t)[\mu(t)dt + \sigma(t)dB(t)]. \tag{1.8}$$

It has been argued (see e.g. [Ba], [CT], [ER], [S]) that (1.2) represents a better model for stock prices than (1.8).

Now suppose that we use the model (1.1) & (1.2) and that a trader is free to choose at any time $t$ the *fraction* $\pi(t) = \pi(t,\omega)$, $\omega \in \Omega$ of his total wealth invested in the stocks. The corresponding wealth $X(t) = X^{(\pi)}(t)$ will have the dynamics

(1.9)
$$dX(t) = X(t^-)\left[\{r(t)(1-\pi(t)) + \pi(t)\mu(t)\}\,dt + \int_{\mathbb{R}} \pi(t)\theta(t,z)\widetilde{N}(dt,dz)\right]; \quad X(0) = x > 0.$$

We have referred to a dealer whose choice of portfolio $\pi(t)$ at time $t$ is only based on the information $\{\mathcal{F}_t\}$ available from the market up to time $t$ as an *honest* trader. In this case the mathematical modeling deals with $\pi(t)$ as an $\mathcal{F}_t$-adapted stochastic process and the integral on the extreme right-hand side of (1.9) is well defined as an Itô integral.

In this paper we study the situation in which an agent, the *insider*, has access to larger information modeled by a *general* filtration $\{\mathcal{G}_t\}$ larger than the one available to any honest trader, i.e.

$$\mathcal{G}_t \supset \mathcal{F}_t, \quad t \in [0,T],$$

The insider relies on this wider information at decision making time and the corresponding stochastic process $\pi(t)$ is $\mathcal{G}_t$-adapted. This opens new mathematical challenges since it is no longer clear how to interpret the integral

$$'' \int_{\mathbb{R}} \pi(t)X(t^-)\theta(t,z)\widetilde{N}(dt,dz) \; '' \tag{1.10}$$

stemming from the right-hand side of (1.9).

We choose to model the integral above as a *forward integral*, which will be denoted by

$$\int_{\mathbb{R}} \pi(t)X(t^-)\theta(t,z)\widetilde{N}(d^-t,dz). \tag{1.11}$$

See Section 2 for definition and properties.

The reasons for taking this approach into account are

1. The forward integral may be regarded as the limit of the natural Riemann sums coming from the situation we are modelling, see e.g. [BØ].

2. The forward integral provides the natural interpretation of the gains from trade process. Indeed, suppose a trader buys one stock at a random time $\tau_1$ and keeps it until another random time $\tau_2 > \tau_1$, when he sells it, then the gains obtained is $S(\tau_1) - S(\tau_2) = \int \varphi(s)d^-S(s)$ where $\varphi(s) = \chi_{(\tau_1,\tau_2]}(s)$ is the portfolio.

3. If the Lévy process $\eta(t)$ happens to be a semimartingale with respect to $\mathcal{G}_t$, then (1.10) would make sense within semimartingale theory and as such the integral would coincide with the forward integral (1.11).

In this paper we specifically deal with the following problem. For a given $T$ and a filtration $\{\mathcal{G}_t\}$ such that $\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{F}$ for all $t \in [0,T]$, we want to find a function $\Phi(x)$ and a portfolio $\pi^*(t) \in \mathcal{A}$ such that

$$\Phi(x) := \sup_{\pi \in \mathcal{A}} E^x \left[ \ln X^{(\pi)}(T) \right] = E^x \left[ \ln X^{(\pi^*)}(T) \right], \tag{1.12}$$

where $\mathcal{A}$ is the family of admissible portfolios and $E^x$ denotes the expectation with respect to $P$ when $X^{(\pi)}(0) = x > 0$. We call $\pi^*(t) = \pi_i^*(t)$ an *optimal portfolio* for the insider and $\Phi(x) = \Phi_i(x)$ the *value function* for the insider. See Section 3 for a more precise formulation of this problem.

Optimal insider portfolio problems of this type were first studied by Karatzas and Pikovsky [KP]. They assumed the following:

**a)** The market is described by the Brownian motion model (1.1) and (1.8).

**b)** The insider filtration is of the form

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L)$$

where $L = L(\omega)$, $\omega \in \Omega$ is some $\mathcal{F}$-measurable random variable.

**c)** $B(t)$ is a semimartingale with respect to $\{\mathcal{G}_t\}$. This case is mostly studied in literature and it is commonly referred to as enlargement of filtration.

In particular, they showed that if

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B(T_0))$$

for some $T_0 > T$, then

$$\pi_i^*(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)} + \frac{1}{\sigma(t)} \frac{B(T_0) - B(t)}{T_0 - t}$$

is an optimal portfolio for the insider. For comparison, the corresponding optimal portfolio $\pi_h^*(t)$ for the *honest* trader is

$$\pi_h^*(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)}.$$

Moreover the difference between the values $\Phi_i(x)$ and $\Phi_h(x)$ for the insider and the honest trader, respectively, is

$$\Phi_i(x) - \Phi_h(x) = \frac{1}{2} \int_0^T \frac{ds}{T_0 - s}$$

In particular, if $T_0 = T$ then $\Phi_i(x) = \infty$.

Subsequently, optimal strategies for an insider have been studied by many researchers in the last years, but to the best of our knowledge most of them assume a priori a semimartingale framework, i.e. the assumption c) above. In [BØ] however, the optimal insider portfolio problem in the Brownian motion model (1.1) & (1.8) was aproached without assuming a priori conditions b) and c) above. Our paper may be regarded as jump diffusion version of the paper [BØ].

The present paper is organized as follows.

In Section 2 we revise the main concepts and results on the anticipative calculus. As announced, the forward integration is our main tool and lays beneath the approach to insider modeling we are considering here. From the technical point of view some results on forward integration depend on the deep relationship between the forward and the Malliavin calculus, see e.g. Proposition 2.6. Hence in this section we recall the definitions and the results on both type of anticipative stochastic calculus which play a crucial role in the solution of our problem. We refer to [DMØP] for a full discussion on these results.

In Section 3, we concentrate on the optimal insider portfolio problem for the *pure jump* model (1.1) & (1.2). We consider a general insider filtration $\mathcal{G}_t \supset \mathcal{F}_t$, without assuming a priori the pure jump conditions corresponding to b) and c) above. In fact, one of our main results is that if there exists an optimal portfolio for the insider, then the driving process

$$\int_0^t \int_{\mathbb{R}} \theta(t,z) \widetilde{N}(dt,dz)$$

is a $\mathcal{G}_t$-semimartingale (see Theorem 3.5). Thus we show that given the existence of an optimal portfolio, we are necessarily in the enlargement of filtration framework. This result is actually a variant of the result of Delbaen and Schachermayer in ([DS], Th 7.2) which is restricted to locally bounded, adapted cadlag processes. We obtain this result by deriving explicitly the special semimartingale decomposition in terms of the optimal portfolio of the process

$$\int_0^t \int_{\mathbb{R}} \frac{\theta(s,z)}{1+\pi^*(s)\theta(s,z)} \widetilde{N}(d^-s,dz),$$

where $\pi^*(s)$ is the optimal portfolio. We also obtain an equation for the optimal insider portfolio, provided that it exists (Theorem 3.3).

In Section 4, we consider the *mixed financial market*

$$\text{(Bond price)} \quad dS_0(t) = r(t)S_0(t)dt \tag{1.13}$$

$$\text{(Stock price)} \quad dS_1(t) = S_1(t^-)[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \theta(t,z)\widetilde{N}(dt,dz)], \tag{1.14}$$

where both a continuous and a pure jump component are taken into account. Here in addition to (1.3)-(1.5) we assume

$$E\left[\int_0^T \sigma^2(t)dt\right] < \infty \qquad \text{and} \qquad \sigma \neq 0. \tag{1.15}$$

We obtain analogous results as in Section 3 for the market (1.13) & (1.14). (For information on forward integrals with respect to Brownian motion see [NP],[RV1]-[RV3] and [BØ]).

Finally, in Section 5, we apply the above results to the special case when the insider has at most knowledge about the value of the underlying driving processes $B(T_0)$ and $\eta(T_0)$ at some time $T_0 \geq T$. This means that the insider filtration $\mathcal{G}_t$ is such that

$$\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{G}'_t,$$

where $\mathcal{G}'_t = \mathcal{F}_t \vee \sigma\left(B(T_0), \eta(T_0)\right)$ corresponds to full information about the underlying processes at time $T_0$. We derive necessary and sufficient conditions for an optimal insider portfolio and give explicit results about the optimal portfolio and the finiteness of the value function $\Phi(x)$ for both the pure jump market (1.1) & (1.2) and the mixed market (1.13) & (1.14) in the special case in which the underlying jump process $\eta(t)$ is a compensated Poisson process. For related works in the context of insider modeling and portfolio optimization see also [EJ], [EGK], [KY1], [KY2], [K].

# 2 Framework

In this section we briefly recall the framework and the results achieved in [DMØP] (see also [DØP], [ØP]) which we are using in Section 3. As presented in the introduction, our object of interest is a square integrable *pure jump* Lévy process with no drift defined on a probability space $(\Omega, \mathcal{F}, P)$ with time horizon $T$:

$$\eta(t) = \int_0^t \int_{\mathbb{R}} z \widetilde{N}(dt, dz), \quad t \in [0, T]. \tag{2.1}$$

where $\widetilde{N}(dt, dz) = (N - \nu_{\mathcal{F}})(dt, dz)$ is the compensated Poisson random measure of $\eta$. Let $\mathcal{F}_t$ be the filtration generated by $\eta(s),\ s \leq t,\ t \in [0, T]$. Note that with abuse of notation we put

$$\nu_{\mathcal{F}}(dt, dz) := dt\, \nu_{\mathcal{F}}(dz).$$

In the first subsection we first define the Malliavin derivative for Lévy processes and in the second subsection we focus on the forward integral and its relation to the Malliavin derivative. The relation between forward integrals and Malliavin calculus (see Proposition 2.6) is used as a technical tool in order to exchange limits and forward integrals in computations in Section 3.

## 2.1 Chaos expansion and Malliavin derivative

Here and in the sequel let $\lambda = \lambda(dt) = dt$ denote the Lebesgue measure on $[0, T]$ and let $L^2\big((\lambda \times \nu_{\mathcal{F}})^n\big)$ be the set of all deterministic functions $f : ([0, T] \times \mathbb{R})^n \to \mathbb{R}$ such that

$$\|f\|^2_{L^2((\lambda \times \nu_{\mathcal{F}})^n)} := \int_{([0,T]\times\mathbb{R})^n} f^2(t_1, z_1, ..., t_n, z_n) dt_1 \nu_{\mathcal{F}}(dz_1) ... dt_n \nu_{\mathcal{F}}(dz_n) < \infty.$$

Let $\tilde{L}^2\big((\lambda \times \nu_{\mathcal{F}})^n\big)$ denote the set of all *symmetric* functions in $L^2\big((\lambda \times \nu_{\mathcal{F}})^n\big)$.
Put

$$G_n = \{(t_1, z_1, ..., t_n, z_n); 0 \leq t_1 \leq ... \leq t_n \leq T \text{ and } z_i \in \mathbb{R}, i = 1, ..., n\}$$

and let $L^2(G_n)$ denote the set of all functions $g : G_n \to \mathbb{R}$ such that

$$\|g\|^2_{L^2(G_n)} := \int_{G_n} g^2(t_1, z_1, ..., t_n, z_n) dt_1 \nu_{\mathcal{F}}(dz_1) ... dt_n \nu_{\mathcal{F}}(dz_n) < \infty.$$

Then

$$\|f\|^2_{L^2((\lambda\times\nu_{\mathcal{F}})^n)} = n! \|g\|^2_{L^2(G_n)}; \qquad f \in \tilde{L}^2\big((\lambda \times \nu_{\mathcal{F}})^n\big).$$

If $f \in L^2(G_n)$, we define its n-fold iterated integral with respect to $\widetilde{N}$ by

$$J_n(f) := \int_0^T \int_{\mathbb{R}} ... \int_0^{t_2} \int_{\mathbb{R}} f(t_1, z_1, ..., t_n, z_n) \widetilde{N}(dt_1, dz_1) ... \widetilde{N}(dt_n, dz_n)$$

and if $f \in \tilde{L}^2\big((\lambda \times \nu_{\mathcal{F}})^n\big)$ we define

$$I_n(f) := n! J_n(f)$$

Then we have the following *chaos expansion theorem*, originally due to Itô ([I2]) (see also [L]).

**Theorem 2.1** *Every $\mathcal{F}_T$-measurable random variable $F \in L^2(P)$ admits the representation*

$$F = E[F] + \sum_{n=1}^{\infty} I_n(f_n) \tag{2.2}$$

*for a unique sequence of symmetric functions $f_n \in \tilde{L}^2\big((\lambda \times \nu_{\mathcal{F}})^n\big)$. Moreover,*

$$\|F\|^2_{L^2(P)} = (E[F])^2 + \sum_{n=1}^{\infty} n! \|f_n\|^2_{L^2((\lambda\times\nu_{\mathcal{F}})^n)}.$$

Using this expansion, we define Malliavin differentiation as follows.

**Definition 2.2** *The space $\mathbb{D}_{1,2}$ is the set of all $\mathcal{F}_T$-measurable random variables $F \in L^2(P)$ admitting the chaos expansion: $F = E[F] + \sum_{n=1}^{\infty} I_n(f_n)$, such that*

$$\|F\|^2_{\mathbb{D}_{1,2}} := \sum_{n=1}^{\infty} n \cdot n! \, \|f_n\|^2_{L^2((\lambda\times\nu_{\mathcal{F}})^n)} < \infty. \tag{2.3}$$

*The* Malliavin derivative $D_{t,z}$ *is an operator defined on $\mathbb{D}_{1,2}$ with values in the standard $L^2$-space $L^2(P \times \lambda \times \nu_{\mathcal{F}})$ given by*

$$D_{t,z} F := \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t, z)), \tag{2.4}$$

*where $f_n(\cdot, t, z) = f_n(t_1, z_1, ..., t_{n-1}, z_{n-1}; t, z)$.*

For more information on $D_{t,z}$ we refer to [L]. Note that if (2.3) holds then

$$E\left[\int_0^T\int_{\mathbb{R}}(D_{t,z}F)^2\nu_{\mathcal{F}}(dz)dt\right]=\sum_{n=1}^{\infty}n\cdot n!\,\|f_n\|^2_{L^2((\lambda\times\nu_{\mathcal{F}})^n)}<\infty.$$

If $F\in \mathbb{D}_{1,2}$ we say that $F$ is *Malliavin differentiable.* The operator $D_{t,z}$ is proved to be closed and to coincide with a certain difference operator defined in [Pi], in the sense that

$$D_{t,z}(F\cdot G)=F\cdot D_{t,z}G+G\cdot D_{t,z}F+D_{t,z}F\cdot D_{t,z}G \qquad \lambda\times\nu_{\mathcal{F}}-\text{a.e.} \tag{2.5}$$

if both $F$ and $G$ are Malliavin differentiable. From this we deduce

**Lemma 2.3** *Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and let $F\in\mathbb{D}_{1,2}$. Then*

$$D_{t,z}(f(F))=f\left(F+D_{t,z}(F)\right)-f(F). \tag{2.6}$$

For a proof see [ØS], Lemma 3.6.

## 2.2 Forward integrals and the Itô formula

In this subsection we recall the *forward integral* with respect to a Poisson random measure $\widetilde{N}$, introduced in [DMØP]. The notion of the forward integral has its origin in the works [RV1] and [NP], from which also our notation for the forward integral has been inspired. Then we recall some formulas concerning the forward integral, in particular the Itô formula for forward processes.

**Definition 2.4** *The* forward integral

$$\int_0^T\int_{\mathbb{R}}\psi(t,z)\widetilde{N}(d^-t,dz)$$

*with respect to the Poisson random measure $\widetilde{N}$, of a caglad stochastic function $\psi(t,z)$, $t\in\mathbb{R}_+$, $z\in\mathbb{R}$, with*

$$\psi(t,z):=\psi(t,z,\omega),\qquad \omega\in\Omega,$$

*is defined as*

$$\lim_{m\to\infty}\int_0^T\int_{\mathbb{R}}\psi(t,z)1_{U_m}(z)\widetilde{N}(dt,dz)$$

*if the limit exists in $L^2(P)$. Here the $\{U_m\}$ is an increasing sequence of compact sets $U_m\subseteq\mathbb{R}\setminus\{0\}$ with $\nu_{\mathcal{F}}(U_m)<\infty$, $m=1,2,...,$ such that $\bigcup_{m=1}^{\infty}U_m=\mathbb{R}\setminus\{0\}$.*

Note that for each $m$ the integral above is well-defined as a Lebesgue integral in $t,z$.

**Remark 2.5** *Note that if $G$ is a random variable then*

$$G\cdot\int_0^T\int_{\mathbb{R}}\psi(t,z)\widetilde{N}(d^-t,dz)=\int_0^T\int_{\mathbb{R}}G\cdot\psi(t,z)\widetilde{N}(d^-t,dz). \tag{2.7}$$

For certain forward integrands we have the following *duality formula* (see [DMØP])

**Proposition 2.6** *Let $\psi(t,z)$ be forward integrable and assume*

**i)** $\psi(t,z)=\psi_1(t)\psi_2(t,z)$ *where* $\psi_1(t)$, $\psi_2(t,z)$ *are caglad,* $\psi_1(t)\in \mathbb{D}_{1,2}$ *for a.a.* $t$ *and* $\psi_2(t,z)$ *is* $\mathcal{F}_t$*-adapted such that*

$$E\left[\int_0^T\int_{\mathbb{R}}(\psi_2(t,z))^2\nu_{\mathcal{F}}(dz)dt\right]<\infty$$

**ii)** $D_{t^+,z}\psi(t,z)=\lim_{s\to t^+}D_{s,z}\psi(t,z)$ *exists for a.a.* $(t,z)$ *with*

$$E\left[\int_0^T\int_{\mathbb{R}}|D_{t^+,z}\psi(t,z)|\nu_{\mathcal{F}}(dz)dt\right]<\infty.$$

*Then*

$$E\int_0^T\int_{\mathbb{R}}\psi(t,z)\widetilde{N}(d^-t,dz)=E\int_0^T\int_{\mathbb{R}}D_{t^+,z}\psi(t,z)\nu_{\mathcal{F}}(dz)dt. \tag{2.8}$$

**Definition 2.7** *A* forward process *is a measurable stochastic function* $X(t)=X(t,\omega)$, $t\in\mathbb{R}_+$, $\omega\in\Omega$, *that admits the representation*

$$X(t)=x+\int_0^t\int_{\mathbb{R}}\psi(s,z)\widetilde{N}(d^-s,dz)+\int_0^t\alpha(s)ds, \tag{2.9}$$

*where* $x=X(0)$ *is a constant. A shorthand notation for* (2.9) *is*

$$d^-X(t)=\int_{\mathbb{R}}\psi(t,z)\widetilde{N}(d^-t,dz)+\alpha(t)dt;\qquad X(0)=x. \tag{2.10}$$

*We call* $d^-X(t)$ *the* forward differential *of* $X(t)$, $t\in\mathbb{R}_+$.

We can now state the *Itô formula for forward integrals.*

**Theorem 2.8** **[DMØP]** *Let* $X(t)$, $t\in\mathbb{R}_+$, *be a forward process of the form* (2.9) *and assume* $\psi(\omega,t,z)$ *continuous in* $z$ *around zero for a.a.* $(\omega,t)$ *and* $\int_0^T\int_{\mathbb{R}}\psi(t,z)^2\nu_{\mathcal{F}}(dz)dt<\infty$ *a.e. Let* $f\in C^2(\mathbb{R})$. *Then the forward differential of* $Y(t)=f\big(X(t)\big)$, $t\in\mathbb{R}_+$, *is given by the following formula:*

$$\begin{aligned} d^-Y(t)&=f'\big(X(t)\big)\alpha(t)dt \\ &+\int_{\mathbb{R}}\Big(f\big(X(t^-)+\psi(t,z)\big)-f\big(X(t^-)\big)-f'\big(X(t^-)\big)\psi(t,z)\Big)\nu(dz)dt \\ &+\int_{\mathbb{R}}\Big(f\big(X(t^-)+\psi(t,z)\big)-f\big(X(t^-)\big)\Big)\widetilde{N}(d^-t,dz). \end{aligned} \tag{2.11}$$

Note that this formula has the same form as in the non-anticipating case, see e.g. [A].

# 3 Optimal insider portfolio in a pure jump market

Suppose now that our financial market is as in the introduction of the form (1.1) & (1.2) and in addition we assume $\theta(\omega,t,z),\ \omega\in\Omega,\ t\in[0,T],\ z\in\mathbb{R}\backslash\{0\}$, continuous in z around zero for a.a. $(\omega,t)$.

Let an *insider filtration* be given by $\mathcal{G}_t\supset\mathcal{F}_t$ for all $t\in[0,T]$. Let $\pi(t)=\pi(t,\omega)$ denote the portfolio of the insider measured as the fraction of the wealth invested in the stock at time $t$. Then we give the following

**Definition 3.1** *The set $\mathcal{A}$ of admissible portfolios consists of all processes $\pi(t)$ such that*

(3.1) $\pi(t)$ *is a* $\mathcal{G}_t$*-adapted process.*

(3.2) $\pi(t)\theta(t,z)$ *is caglad and forward integrable w.r.t.* $\widetilde{N}(dt,dz)$.

(3.3) $\pi(t)\theta(t,z)>-1+\epsilon$ *for* $\nu_{\mathcal{F}}(dz)dt$*-a.a. (t,z) for some* $\epsilon>0$ *depending on* $\pi$.

(3.4) $E\left[\int_0^T\int_{\mathbb{R}}(\pi(t)\theta(t,z))^2\,\nu_{\mathcal{F}}(dz)dt\right]<\infty.$

(3.5) $\pi$ *is Malliavin differentiable and* $D_{t+,z}\pi(t)=\lim_{s\to t^+}D_{s,z}\pi(t)$ *exists for a.a.* $(t,z)$.

(3.6) $\theta(t,z)(\pi(t)+D_{t+,z}\pi(t))>-1+\epsilon$ *for a.a. (t,z) for some* $\epsilon>0$ *depending on* $\pi$.

(3.7) $E\left[\int_0^T\int_{\mathbb{R}}|\theta(t,z)D_{t+,z}\pi(t)|\nu_{\mathcal{F}}(dz)dt\right]<\infty.$

Note that by Lemma 2.3 we can write , since $D_{t+,z}F=0$ when $F$ is $\mathcal{F}_t$-measurable

$$D_{t+,z}\ln\left(1+\pi(t)\theta(t,z)\right)=\ln\big(1+\theta(t,z)(\pi(t)+D_{t+,z}\pi(t))\big)-\ln\left(1+\pi(t)\theta(t,z)\right),$$

such that together with conditions (3.2) and (3.6) we have

(3.8) $\ln\left(1+\pi(t)\theta(t,z)\right)$ is caglad and forward integrable.

(3.9) $\ln\left(1+\pi(t)\theta(t,z)\right)\in I\!D_{1,2}$ and $D_{t+,z}\ln\left(1+\pi(t)\theta(t,z)\right)$ exists for a.a. $(t,z)$.

(3.10) $E\left[\int_0^T\int_{\mathbb{R}}|D_{t+,z}\ln\left(1+\pi(t)\theta(t,z)\right)|\nu_{\mathcal{F}}(dz)dt\right]<\infty.$

If we now interpret the integral (1.10) as the forward integral, the wealth $X(t)=X^{(\pi)}(t)$ of the insider is described by the equation

(3.11)

$$dX(t)=X(t^-)\left[\{r(t)(1-\pi(t))+\pi(t)\mu(t))\}\,dt+\int_{\mathbb{R}}\pi(t)\theta(t,z)\widetilde{N}(d^-t,dz)\right],\ X(0)=x.$$

This gives a mathematical framework in which we can proceed to solve the optimization problem (1.12) for the insider:

(3.12) $$\Phi(x):=\sup_{\pi\in\mathcal{A}}E^x\left[\ln X^{(\pi)}(T)\right]=E^x\left[\ln X^{(\pi^*)}(T)\right].$$

Note that (3.2), (3.3), (3.5) and (3.6) ensure the forward integrability of $\ln(1+\pi(t)\theta(t,z))$ in $L^2(P)$. By the Itô formula for forward integrals (Theorem 2.8) together with the conditions (3.4), the solution of equation (3.11) is therefore

(3.13)
$$\frac{X(T)}{x} = \exp\Big(\int_0^T (r(s)+(\mu(s)-r(s))\pi(s))\,ds + \int_0^T\int_{\mathbb{R}} \ln(1+\pi(s)\theta(s,z))\,\widetilde{N}(d^-s,dz) - \int_0^T\int_{\mathbb{R}} (\pi(s)\theta(s,z) - \ln(1+\pi(s)\theta(s,z)))\,\nu_{\mathcal{F}}(dz)ds\Big).$$

Note that this is a generalization of the Doleans-Dade exponential. Then we know from Proposition 2.6 that

$$E\Big[\int_0^T\int_{\mathbb{R}} \ln(1+\pi(s)\theta(s,z))\,\widetilde{N}(d^-s,dz)\Big] = E\Big[\int_0^T\int_{\mathbb{R}} D_{s+,z}\ln(1+\pi(s)\theta(s,z))\,\nu_{\mathcal{F}}(dz)ds\Big].$$

Combining this with (3.13), we get

$$\text{(3.14)}\quad E\Big[\ln\frac{X^{(\pi)}(T)}{x}\Big] = E\Big[\int_0^T\Big(r(s)+(\mu(s)-r(s))\pi(s) + \int_{\mathbb{R}}\{\ln(1+\pi(s)\theta(s,z)) - \pi(s)\theta(s,z) + D_{s+,z}\ln(1+\pi(s)\theta(s,z))\}\,\nu_{\mathcal{F}}(dz)\Big)ds\Big].$$

Now suppose $\pi=\pi^*$ is optimal for the problem (3.12). Fix $t\in[0,T)$ and $h>0$ such that $t+h\le T$. Choose $\beta\in\mathcal{A}$ of the form

$$\beta(s) = \chi_{[t,t+h]}(s)\beta_0;\quad 0\le s\le T,$$

where $\beta_0$ is a bounded $\mathcal{G}_t$-measurable random variable such that $D_{t,z}\beta_o$ is bounded a.e. Then it is clear from Definition 3.1 that there exists a $\delta>0$ such that $\pi^*+y\beta\in\mathcal{A}$ for all $y\in(-\delta,\delta)$. Then the function

$$g(y) := E\Big[\ln X^{(\pi+y\beta)}(T)\Big],\quad y\in(-\delta,\delta),$$

is maximal for $y=0$. Hence, by (3.14),

$$\text{(3.15)}\quad 0 = g'(0) = E\Big[\int_0^T\Big((\mu(s)-r(s))\beta(s) + \int_{\mathbb{R}}\Big\{\frac{\theta(s,z)\beta(s)}{1+\pi(s)\theta(s,z)} - \theta(s,z)\beta(s) + D_{s+,z}\Big(\frac{\theta(s,z)\beta(s)}{1+\pi(s)\theta(s,z)}\Big)\Big\}\,\nu_{\mathcal{F}}(dz)\Big)ds\Big]$$

Note that some easy calculations, using conditions (3.3)-(3.7), justify the differentiation inside the integration. Now, by putting

$$\xi(s,z) = \frac{\theta(s,z)}{1+\pi(s)\theta(s,z)}$$

and using again Proposition 2.6, we deduce by an approximation argument that (3.15) becomes

(3.16)
$$0 = E\left[\int_0^T \beta(s)\left\{\mu(s) - r(s) - \int_{\mathbb{R}} \theta(s,z)\pi(s)\xi(s,z)\nu_{\mathcal{F}}(dz)\right\} ds + \int_0^T \int_{\mathbb{R}} \beta(s)\xi(s,z)\widetilde{N}(d^- s, dz)\right].$$

Moreover, from the special form of $\beta(s)$ we get

(3.17)
$$0 = E\left[\beta_0\left\{\int_t^{t+h}\left(\mu(s) - r(s) - \int_{\mathbb{R}} \theta(s,z)\pi(s)\xi(s,z)\nu_{\mathcal{F}}(dz)\right) ds + \int_t^{t+h}\int_{\mathbb{R}} \xi(s,z)\widetilde{N}(d^- s, dz)\right\}\right].$$

Since $\beta_0$ was an arbitrarily chosen $\mathcal{G}_t$-measurable, we deduce that

(3.18)
$$0 = E\left[\int_t^{t+h}\left\{\mu(s) - r(s) + \int_{\mathbb{R}} -\theta(s,z)\pi(s)\xi(s,z)\nu_{\mathcal{F}}(dz)\right\} ds + \int_t^{t+h}\int_{\mathbb{R}} \xi(s,z)\widetilde{N}(d^- s, dz)\middle|\mathcal{G}_t\right].$$

Define

(3.19)
$$M^{(\pi)}(t) = \int_0^t \left\{\mu(s) - r(s) - \int_{\mathbb{R}} \frac{\pi(s)\theta^2(s,z)}{1+\pi(s)\theta(s,z)}\nu_{\mathcal{F}}(dz)\right\} ds + \int_0^t \int_{\mathbb{R}} \frac{\theta(s,z)}{1+\pi(s)\theta(s,z)}\widetilde{N}(d^- s, dz).$$

Then we have proved

**Theorem 3.2** *Suppose $\pi(s) = \pi^*(s)$ is optimal for problem (3.12). Then $M^{(\pi)}(t)$ is a martingale with respect to the filtration $\mathcal{G}_t$. Further, the process*

$$R_t := \int_0^t \int_{\mathbb{R}} \frac{\theta(s,z)}{1+\pi(s)\theta(s,z)}\widetilde{N}(d^- s, dz)$$

*is a special $\mathcal{G}_t$-semimartingale with decomposition given by* (3.19) *(for the definition of a special semimartingale see e.g. [P] p. 129).*

Because $\mathcal{F}_t \subset \mathcal{G}_t$ the random measure $N(dt,dz)$ has a unique predictable compensator w.r.t. $\mathcal{G}_t$, say $\nu_{\mathcal{G}}(dt,dz)$ (see [JS], p.66). Note, however, that this alone would not imply that $R_t$ is a $\mathcal{G}_t$-semimartingale, because $\nu_{\mathcal{G}}(dt,dz)$ need not integrate to a process of finite variation. We may write

(3.20)
$$\begin{aligned} M^{(\pi)}(t) = \int_0^t \int_{\mathbb{R}} \frac{\theta(s,z)}{1+\pi(s)\theta(s,z)}(N - \nu_{\mathcal{G}})(ds,dz) + \int_0^t \int_{\mathbb{R}} \frac{\theta(s,z)}{1+\pi(s)\theta(s,z)}(\nu_{\mathcal{G}} - \nu_{\mathcal{F}})(ds,dz) \\ + \int_0^t \left\{\mu(s) - r(s) - \int_{\mathbb{R}} \frac{\pi(s)\theta^2(s,z)}{1+\pi(s)\theta(s,z)}\nu_{\mathcal{F}}(dz)\right\} ds. \end{aligned}$$

Hence by uniqueness of the semimartingale decomposition of the $\mathcal{G}_t$-semimartingale $M^{(\pi)}(t)$ (see e.g. [P], Th.30, Ch.7) we conclude that the finite variation part above must be 0. Therefore we get the following result.

**Theorem 3.3** *Suppose $\pi \in \mathcal{A}$ is optimal for problem* (3.12). *Then $\pi$ solves the equation*
(3.21)
$$\int_0^t \left\{ \mu(s) - r(s) - \int_{\mathbb{R}} \frac{\pi(s)\theta^2(s,z)}{1+\pi(s)\theta(s,z)} \nu_{\mathcal{F}}(dz) \right\} ds = \int_0^t \int_{\mathbb{R}} \frac{\theta(s,z)}{1+\pi(s)\theta(s,z)} (\nu_{\mathcal{F}} - \nu_{\mathcal{G}})(ds,dz).$$

And in particular we get

**Corollary 3.4** *Suppose $\mathcal{F}_t = \mathcal{G}_t$, $t \in [0,T]$. Then a necessary condition for $\pi$ to be optimal is that for a.a. $s$*

$$\mu(s) - r(s) - \int_{\mathbb{R}} \frac{\pi(s)\theta^2(s,z)}{1+\pi(s)\theta(s,z)} \nu_{\mathcal{F}}(dz) = 0. \tag{3.22}$$

*(This could also be seen by direct computation.)*

The following result may be regarded as a jump diffusion version of the result in [BØ] in the Brownian motion context. It may also be regarded as a variant (in the Malliavin calculus setting) of the result of [DS], stating that if $S(t)$ is a given locally bounded, adapted cadlag price process with filtration $\mathcal{G}_t$ and there is no arbitrage by simple strategies on $S(t)$, then $S(t)$ is a $\mathcal{G}_t$-semimartingale.

**Theorem 3.5** *Suppose there exists an optimal portfolio for problem* (3.12). *Then the process*

$$\int_0^t \int_{\mathbb{R}} \theta(s,z) \widetilde{N}(ds,dz), \qquad 0 \le t \le T,$$

*is a $\mathcal{G}_t$-semimartingale.*

**Proof.** We only need that $\int_0^t \int_{\mathbb{R}} \theta(s,z)(\nu_{\mathcal{F}} - \nu_{\mathcal{G}})(ds,dz)$ exists and is of finite variation. Because then the $\mathcal{G}_t$-martingale

$$\int_0^t \int_{\mathbb{R}} \theta(s,z)(N - \nu_{\mathcal{G}})(ds,dz) = \int_0^t \int_{\mathbb{R}} \theta(s,z)(N - \nu_{\mathcal{F}})(ds,dz) + \int_0^t \int_{\mathbb{R}} \theta(s,z)(\nu_{\mathcal{F}} - \nu_{\mathcal{G}})(ds,dz)$$

exists and $\int_{\mathbb{R}} \theta(t,z)\widetilde{N}(dt,dz)$ is a $\mathcal{G}_t$-semimartingale. By Theorem 3.2 we know that

$$\int_0^t \int_{\mathbb{R}} \frac{\theta(s,z)}{1+\pi(s)\theta(s,z)} (\nu_{\mathcal{F}} - \nu_{\mathcal{G}})(ds,dz)$$

is of finite variation. So by our assumption (3.3) it follows that

$$\int_0^t \int_{\mathbb{R}} \theta(s,z)(\nu_{\mathcal{F}} - \nu_{\mathcal{G}})(ds,dz)$$

is of finite variation also. ■

# 4 Optimal portfolio in a mixed market

In this section we treat the more general situation of the financial market given by (1.13) & (1.14), i.e. the risky asset is driven by a Lévy-Itô diffusion which in addition to the jump part contains Brownian motion as source of uncertainty. The reasoning in this situation is completely analogous to Section 3 and will therefore not be carried out in detail. For the analogous definitions and results of Section 2 concerning the forward integral for Brownian motion we refer to [BØ], for general information on forward integrals see [NP] and [RV1]-[RV3].

Again we assume $\mathcal{G}_t \supset \mathcal{F}_t$ to be an *insider filtration* and $\pi(t) = \pi(t, \omega)$ to be the portfolio of the insider measured as the fraction of the wealth invested in the stock at time $t$. The set $\mathcal{A}$ of *admissible* portfolios now consists of all processes $\pi(t)$ as in Definition 3.1 which in addition are such that $\pi(t)\sigma(t)$ is forward integrable w.r.t $B(t)$. Then the wealth $X(t) = X^{(\pi)}(t)$ of the insider is described by the equation

(4.1)

$$dX(t) = X(t^-)\left[\{r(t)(1-\pi(t)) + \pi(t)\mu(t)\}\,dt + \pi(t)\sigma(t)d^-B(t) + \int_{\mathbb{R}} \pi(t)\theta(t,z)\widetilde{N}(d^-t, dz)\right],$$

where $d^-B(t)$ denotes the forward integral w.r.t. Brownian motion. The optimization problem for the insider in which we are interested is as before

$$\Phi(x) := \sup_{\pi \in \mathcal{A}} E^x\left[\ln X^{(\pi)}(T)\right] = E^x\left[\ln X^{(\pi^*)}(T)\right]. \tag{4.2}$$

Combining the Itô formula for Brownian motion and jump measure forward integrals (see [BØ] and Theorem 2.8), the solution of equation (4.1) is

(4.3)

$$\frac{X(T)}{x} = \exp\left(\int_0^T \left\{r(s) + (\mu(s) - r(s))\pi(s) - \frac{1}{2}\sigma^2(s)\pi^2(s)\right\} ds + \int_0^T \sigma(s)\pi(s)d^-B(s)\right.$$
$$\left. - \int_0^T \int_{\mathbb{R}} \{\pi(s)\theta(s,z) - \ln(1+\pi(s)\theta(s,z))\}\,\nu_{\mathcal{F}}(dz)ds + \int_0^T \int_{\mathbb{R}} \ln(1+\pi(s)\theta(s,z))\,\widetilde{N}(d^-s, dz)\right).$$

Plugging in $\pi(s) + y\beta(s)$ as portfolio and using Proposition 2.6 yields

$$E\left[\ln\frac{X^{(\pi+y\beta)}(T)}{x}\right] = E\left[\int_0^T \left\{r + (\mu - r)(\pi + y\beta) - \frac{1}{2}(\pi+y\beta)^2\sigma^2\right\}ds\right. \tag{4.4}$$
$$+ \int_0^T \int_{\mathbb{R}} \left\{\ln(1+(\pi+y\beta)\theta) - (\pi+y\beta)\theta + D_{s^+,z}\ln(1+(\pi+y\beta)\theta)\right\}\nu_{\mathcal{F}}(dz)ds$$
$$\left. + \int_0^T (\pi+y\beta)\sigma d^-B(s)\right],$$

where we have ommitted the arguments of the integrands for the sake of notational simplicity. Differentiating expression (4.4) w.r.t. $y$, setting $y = 0$ and using the same arguments as in Section 3 results in the following

**Theorem 4.1** *Define*

$$M^{(\pi)}(t) = \int_0^t \left\{ \mu(s) - r(s) - \sigma^2(s)\pi(s) - \int_{\mathbb{R}} \frac{\pi(s)\theta^2(s,z)}{1+\pi(s)\theta(s,z)} \nu_{\mathcal{F}}(dz) \right\} ds + \int_0^t \sigma(s)dB(s) + \int_0^t \int_{\mathbb{R}} \frac{\theta(s,z)}{1+\pi(s)\theta(s,z)} \widetilde{N}(d^- s, dz). \tag{4.5}$$

*Suppose $\pi(s) = \pi^*(s)$ is optimal for problem (4.2). Then $M^{(\pi)}(t)$ is a martingale with respect to the filtration $\mathcal{G}_t$.*

Further, we see that the orthogonal decomposition of $M^{(\pi)}(t)$ into a continuous part $M_c^{(\pi)}(t)$ and a discontinuous part $M_d^{(\pi)}(t)$ is given by

$$M_c^{(\pi)}(t) = \int_0^t \sigma(s)dB(s) + \int_0^t \sigma(s)\alpha(s)ds \tag{4.6}$$

$$M_d^{(\pi)}(t) = \int_0^t \int_{\mathbb{R}} \frac{\theta(s,z)}{1+\pi(s)\theta(s,z)} \widetilde{N}(d^- s, dz) + \int_0^t \gamma(s)ds \tag{4.7}$$

where $\alpha(s)$ and $\gamma(s)$ are unique $\mathcal{G}_s$-adapted processes such that

$$\int_0^t \sigma(s)\alpha(s)ds + \int_0^t \gamma(s)ds = \int_0^t \left\{ \mu - r - \sigma^2\pi - \int_{\mathbb{R}} \frac{\pi(s)\theta^2(s,z)}{1+\pi(s)\theta(s,z)} \nu_{\mathcal{F}}(dz) \right\} ds.$$

So the proof of Theorem 3.5, together with the fact that $\int_0^t \frac{1}{\sigma(s)} dM_c^{(\pi)}(s) = B(t) + \int_0^t \alpha(s)ds$ also is a $\mathcal{G}_t$-martingale, gives

**Theorem 4.2** *Suppose there exists an optimal portfolio for problem (4.2). Then the underlying processes*

$$\int_0^t \int_{\mathbb{R}} \theta(s,z)\widetilde{N}(ds,dz) \quad \text{and} \quad B(t), \qquad 0 \le t \le T,$$

*are $\mathcal{G}_t$-semimartingales.*

Finally, we get as an analog to Theorem 3.3

**Theorem 4.3** *Suppose $\pi \in \mathcal{A}$ is optimal for problem* (4.2). *Then $\pi$ solves the equation*

$$\int_0^t \left\{ \mu(s) - r(s) - \sigma^2(s)\pi(s) - \int_{\mathbb{R}} \frac{\pi(s)\theta^2(s,z)}{1+\pi(s)\theta(s,z)} \nu_{\mathcal{F}}(dz) \right\} ds = \int_0^t \sigma(s)\alpha(s)ds + \int_0^t \int_{\mathbb{R}} \frac{\theta(s,z)}{1+\pi(s)\theta(s,z)} (\nu_{\mathcal{F}} - \nu_{\mathcal{G}})(ds,dz), \tag{4.8}$$

*where $\alpha(s)$ is the process from* (4.6) *and $\nu_{\mathcal{G}}$ is the $\mathcal{G}_t$ compensator of $N$.*

And in particular for the honest trader:

**Corollary 4.4** *Suppose $\mathcal{F}_t = \mathcal{G}_t$, $t \in [0,T]$. Then a necessary condition for $\pi$ to be optimal is that for a.a. $s$*

$$\mu(s) - r(s) - \sigma^2(s)\pi(s) - \int_{\mathbb{R}} \frac{\pi(s)\theta^2(s,z)}{1+\pi(s)\theta(s,z)} \nu_{\mathcal{F}}(dz) = 0. \tag{4.9}$$

## 5 Example: enlargement of filtration

Now let the underlying driving jump process of the risky asset in the mixed financial market (1.13) & (1.14) be a pure jump Lévy process $\eta(t)$, i.e.

$$\int_0^t \int_{\mathbb{R}} \theta(t,z)\widetilde{N}(dt,dz) = \int_0^t \int_{\mathbb{R}} z\widetilde{N}(dt,dz) =: \eta(t).$$

In this Section we want to analyze the optimization problem in which the insider has at most knowledge about the value of the underlying driving processes $B(T_0)$ and $\eta(T_0)$ at some time $T_0 \geq T$. This means that the insider filtration $\mathcal{G}_t$ is such that $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{G}'_t$, where $\mathcal{G}'_t = \mathcal{F}_t \vee \sigma\left(B(T_0), \eta(T_0)\right)$ corresponds to full information about the underlying processes at time $T_0$.

**Proposition 5.1** *Let $\mathcal{G}_t$ be an insider filtration such that $\mathcal{F}_t \subseteq \mathcal{G}_t \subseteq \mathcal{G}'_t$. Then*

$$B(t) - \int_0^t \frac{E\left[B(T_0)|\mathcal{G}_s\right] - B(s)}{T_0 - s} ds \tag{5.1}$$

*and*

$$\eta(t) - \int_0^t \frac{E\left[\eta(T_0)|\mathcal{G}_s\right] - \eta(s)}{T_0 - s} ds = \eta(t) - \int_0^t E\left[\int_s^{T_0} \int_{\mathbb{R}} \frac{z}{T_0 - s} \widetilde{N}(dr,dz) \middle| \mathcal{G}_s\right] ds \tag{5.2}$$

*are $\mathcal{G}_t$-martingales for $0 \leq t \leq T$.*

**Proof.** We know by an extension of a result of Itô [I3] (see also [P] p.356) that for a general Lévy process $\Lambda(t)$ with filtration $\widehat{\mathcal{F}}_t$, the process

$$\Lambda(t) - \int_0^t \frac{\Lambda(T_0) - \Lambda(s)}{T_0 - s} ds$$

is a $\left\{\widehat{\mathcal{F}}_t \vee \sigma\left(\Lambda(T_0)\right)\right\}$-martingale for $t \leq T_0$. Using this result and the fact that $B(t)$ and $\eta(t)$ are independent we get that

$$B(t) - \int_0^t \frac{B(T_0) - B(s)}{T_0 - s} ds \qquad \text{and} \qquad \eta(t) - \int_0^t \frac{\eta(T_0) - \eta(s)}{T_0 - s} ds$$

are $\mathcal{G}'_t$-martingales for $0 \leq t \leq T$. So we have

$$\begin{aligned}
&E\left[B(t) - \int_0^t \frac{E\left[B(T_0)|\mathcal{G}_s\right] - B(s)}{T_0 - s} ds \middle| \mathcal{G}_r\right] \\
&= E\left[B(t) - B(r) - \int_r^t \frac{E\left[B(T_0)|\mathcal{G}_s\right] - B(s)}{T_0 - s} ds \middle| \mathcal{G}_r\right] + B(r) - \int_0^r \frac{E\left[B(T_0)|\mathcal{G}_s\right] - B(s)}{T_0 - s} ds \\
&= E\left[E\left[B(t) - B(r) - \int_r^t \frac{B(T_0) - B(s)}{T_0 - s} ds \middle| \mathcal{G}'_r\right] \middle| \mathcal{G}_r\right] + B(r) - \int_0^r \frac{E\left[B(T_0)|\mathcal{G}_s\right] - B(s)}{T_0 - s} ds \\
&= 0 + B(r) - \int_0^r \frac{E\left[B(T_0)|\mathcal{G}_s\right] - B(s)}{T_0 - s} ds.
\end{aligned}$$

For $\eta(t) - \int_0^t \frac{E[\eta(T_0)|\mathcal{G}_s] - \eta(s)}{T_0 - s} ds$ the reasoning is analogous. ■

Proposition 5.1 tells us that in the present situation of enlargement of filtration the process $\alpha(s)$ from (4.6) is of the form $-\frac{E[B(T_0)|\mathcal{G}_s] - B(s)}{T_0 - s}$. Moreover, we can easily deduce the $\mathcal{G}_t$ compensator $\nu_{\mathcal{G}}$ of $N$ from Proposition 5.1.

**Proposition 5.2** *The $\mathcal{G}_t$ compensating measure $\nu_{\mathcal{G}}$ of the jump measure $N$ is given by*

$$\nu_{\mathcal{G}}(ds, dz) = \nu_{\mathcal{F}}(dz)ds + E\left[\frac{1}{T_0 - s}\int_s^{T_0} \widetilde{N}(dr, dz)\middle|\mathcal{G}_s\right] ds \tag{5.3}$$

$$= E\left[\frac{1}{T_0 - s}\int_s^{T_0} N(dr, dz)\middle|\mathcal{G}_s\right] ds. \tag{5.4}$$

**Proof.** We know (see [JS] p.80) that it is sufficient to show that if $\widehat{\nu}_{\mathcal{G}}$ is the right-hand side of (5.3) then

$$\int_0^t \int_{\mathbb{R}} f(z)(N - \widehat{\nu}_{\mathcal{G}})(ds, dz)$$

is a $\mathcal{G}_t$-martingale for all $f \in G$, where $G$ is a family of bounded deterministic functions on $\mathbb{R}$, zero around zero, which determines a measure on $\mathbb{R}$ with weight zero in zero. The same argument holds if we take $G$ to be the family of invertible functions $f(z)$ which are integrable w.r.t. $\widehat{\nu}_{\mathcal{G}}$ (which implies that it is also integrable w.r.t. $\nu_{\mathcal{F}}$). Let $f(z)$ be such a function. Then $\bar{\eta}(t)$ given by

$$\bar{\eta}(t) := \int_0^t \int_{\mathbb{R}} f(z)\widetilde{N}(ds, dz)$$

is a pure jump Lévy process (see e.g. Sec. 2.3.2 in [A]). Consider the Lévy process

$$B(t) + \bar{\eta}(t),$$

whose filtration is denoted by $\bar{\mathcal{F}}_t$. Since $f(z)$ is invertible we have $\bar{\mathcal{F}}_t = \mathcal{F}_t$ and $\bar{\mathcal{G}'}_t = \mathcal{G}'_t$, where $\bar{\mathcal{G}'}_t = \bar{\mathcal{F}}_t \vee \sigma\left(B(T_0), \bar{\eta}(T_0)\right)$. From Proposition 5.1 we then get that

$$\bar{M}(t) := \int_0^t \int_{\mathbb{R}} f(z)\widetilde{N}(ds, dz) - \int_0^t E\left[\int_s^{T_0} \int_{\mathbb{R}} \frac{f(z)}{T_0 - s}\widetilde{N}(dr, dz)\middle|\mathcal{G}_s\right] ds$$

is a $\mathcal{G}_t$-martingale. Equation (5.4) is a straight forward algebraic transformation. ■

Using the measure given by (5.3), we see that the necessary condition for an optimal portfolio given by equation (4.8) in the situation of this section becomes (note that $\theta(t, z) = z$)

$$\int_0^t \left\{\mu(s) - r(s) - \sigma^2(s)\pi(s) - \int_{\mathbb{R}} \frac{\pi(s)z^2}{1 + \pi(s)z}\nu_{\mathcal{F}}(dz)\right\} ds \tag{5.5}$$
$$= \int_0^t \left\{-\sigma(s)\frac{E\left[B(T_0)|\mathcal{G}_s\right] - B(s)}{T_0 - s} - E\left[\int_s^{T_0} \int_{\mathbb{R}} \frac{z}{(1 + \pi(s)z)(T_0 - s)}\widetilde{N}(dr, dz)\middle|\mathcal{G}_s\right]\right\} ds$$

When $\eta(t)$ is of finite variation this can be rewritten as

$$\begin{aligned}(5.6)\quad &\int_0^t \left\{\mu(s) - r(s) - \sigma^2(s)\pi(s) - \int_{\mathbb{R}} z\nu_{\mathcal{F}}(dz)\right\} ds \\ &= \int_0^t \left\{-\sigma(s)\frac{E\left[B(T_0)|\mathcal{G}_s\right] - B(s)}{T_0 - s} - E\left[\int_s^{T_0}\int_{\mathbb{R}} \frac{z}{(1+\pi(s)z)(T_0-s)} N(dr,dz)\Big|\mathcal{G}_s\right]\right\} ds\end{aligned}$$

Given some additional assumptions, this is also a *sufficient* condition for a portfolio $\pi \in \mathcal{A}$ to be optimal:

**Theorem 5.3** *Assume that $\eta(t)$ is of finite variation. The portfolio $\pi = \pi(s,\omega)$, $\omega \in \Omega$, $s \in [0,T]$, is optimal for the insider if and only if $\pi \in \mathcal{A}$ and for a.a. $(s,\omega)$ $\pi$ solves the equation*

$$\begin{aligned}(5.7)\quad &\mu(s) - r(s) - \sigma^2(s)\pi(s) - \int_{\mathbb{R}} z\nu_{\mathcal{F}}(dz) \\ &= -\sigma(s)\frac{E\left[B(T_0)|\mathcal{G}_s\right]^- - B(s)}{T_0 - s} - E\left[\int_s^{T_0}\int_{\mathbb{R}} \frac{z}{(1+\pi(s)z)(T_0-s)} N(dr,dz)\Big|\mathcal{G}_s\right]^-,\end{aligned}$$

*where the notation $E[...]^-$ denotes the left limit in $s$.*

**Proof.** By Proposition 5.1 and Proposition 5.2 equation (4.3) becomes

$$\begin{aligned}E&\left[\ln\frac{X^\pi(T)}{x}\right] \\ &= E\Bigg[\int_0^T \left\{r(s) + (\mu(s) - r(s))\pi(s) - \frac{1}{2}\sigma(s)^2\pi^2(s) - \int_{\mathbb{R}} \pi(s)z\nu_{\mathcal{F}}(dz)\right\}ds \\ &\quad + \int_0^T \sigma(s)\pi(s)d(B(s) + \alpha(s)ds) - \int_0^T \sigma(s)\pi(s)\alpha(s)ds \\ &\quad + \int_0^T\int_{\mathbb{R}} \ln(1+\pi(s)z)(N - \nu_{\mathcal{G}})(ds,dz) + \int_0^T\int_{\mathbb{R}} \ln(1+\pi(s)z)\nu_{\mathcal{G}}(ds,dz)\Bigg] \\ &= E\Bigg[\int_0^T \left\{r(s) + (\mu(s) - r(s))\pi(s) - \frac{1}{2}\sigma(s)^2\pi^2(s) - \int_{\mathbb{R}} \pi(s)z\nu_{\mathcal{F}}(dz)\right\}ds \\ (5.8)\quad &\quad + \int_0^T \left\{\sigma(s)\pi(s)\frac{E\left[B(T_0)|\mathcal{G}_s\right] - B(s)}{T_0 - s} + E\left[\int_s^{T_0}\int_{\mathbb{R}} \frac{\ln(1+\pi(s)z)}{(T_0-s)} N(dr,dz)\Big|\mathcal{G}_s\right]\right\} ds\Bigg].\end{aligned}$$

We can maximize this pointwise for each fixed $(s,\omega)$. Define

$$\begin{aligned}H(\pi) =& r(s) + (\mu(s) - r(s))\pi - \frac{1}{2}\sigma(s)^2\pi^2 - \int_{\mathbb{R}} \pi z\nu_{\mathcal{F}}(dz) \\ &+ \sigma(s)\pi\frac{E\left[B(T_0)|\mathcal{G}_s\right] - B(s)}{T_0 - s} + E\left[\int_s^{T_0}\int_{\mathbb{R}} \frac{\ln(1+\pi z)}{(T_0-s)} N(dr,dz)\Big|\mathcal{G}_s\right]\end{aligned}$$

.

Then a stationary point $\pi$ of $H$ is given by

$$0 = H'(\pi) = \mu(s) - r(s) - \sigma^2(s)\pi(s) - \int_{\mathbb{R}} z\nu_{\mathcal{F}}(dz)$$
$$+ \sigma(s)\frac{E\left[B(T_0)|\mathcal{G}_s\right] - B(s)}{T_0 - s} + E\left[\int_s^{T_0}\int_{\mathbb{R}} \frac{z}{(1+\pi(s)z)(T_0-s)} N(dr,dz)\bigg|\mathcal{G}_s\right].$$

Since $H$ is concave, a stationary point of $H$ is a maximum point of $H$. But, since for a given omega the set of dicontinuities has Lebesgue measure zero, the equation

$$\mu(s) - r(s) - \sigma^2(s)\pi(s) - \int_{\mathbb{R}} z\nu_{\mathcal{F}}(dz)$$
$$= -\sigma(s)\frac{E\left[B(T_0)|\mathcal{G}_s\right]^- - B(s)}{T_0 - s} - E\left[\int_s^{T_0}\int_{\mathbb{R}} \frac{z}{(1+\pi(s)z)(T_0-s)} N(dr,dz)\bigg|\mathcal{G}_s\right]^-,$$

also describes an optimal portfolio. ■

In order to get explicit expressions for $\pi$, we now apply this to the case when $\eta(t)$ is a compensated Poisson process of intensity $\rho > 0$. In this case the corresponding Lévy measure is

$$\nu_{\mathcal{F}}(dz)ds = \rho\delta_1(dz)ds,$$

where $\delta_1(dz)$ is the unit point mass at 1, and we can write

$$\eta(t) = Q(t) - \rho t,$$

$Q$ being a Poisson process of intensity $\rho$. Since in this case

$$K(\pi) = \ln(1+\pi)\frac{E\left[Q(T_0)|\mathcal{G}_s\right] - Q(s)}{T_0 - s}$$

is concave in $\pi$, we get by Theorem 5.3 that a necessary and sufficient condition for an optimal insider portfolio $\pi$ for a.a $(s,\omega)$ is given by the equation

(5.9)
$$0 \;=\; \mu(s) - r(s) - \rho - \sigma^2(s)\pi(s) + \sigma(s)\frac{E\left[B(T_0)|\mathcal{G}_s\right]^- - B(s)}{T_0 - s} + \frac{E\left[Q(T_0) - Q(s)|\mathcal{G}_s\right]^-}{(1+\pi(s))(T_0 - s)}.$$

**1) The pure jump case:**

If we deal with market (1.1) & (1.2), i.e. $\sigma = 0$, we have the following

**Theorem 5.4** *Assume that $r(s)$ and $\mu(s)$ are bounded and $\rho + r(s) - \mu(s) > 0$ and bounded away from 0. Then*

**i)** *There exists an optimal insider portfolio if and only if*

$$E\left[Q(T_0)|\mathcal{G}_s\right] - Q(s) > 0$$

*for a.a $(s,\omega)$. In this case*

(5.10)
$$\pi^*(s) = \frac{E\left[Q(T_0) - Q(s)|\mathcal{G}_s\right]^-}{(T_0 - s)(\rho + r(s) - \mu(s))} - 1$$

*is the optimal portfolio for the insider.*

**ii)** *Assume there exists an optimal insider portfolio. Then the value function $\Phi(x)$ for the insider is finite for all $T_0 \geq T$.*

**Proof.** Part (i) follows from equation (5.9) setting $\sigma(s) = 0$. It remains to prove (ii). We substitute the value (5.10) for $\pi^*$ into the expression (5.8) and get

$$(5.11)\quad E\left[\ln \frac{X^{\pi}(T)}{x}\right] = E\left[\int_0^T \left\{2r(s) - \mu(s) + \rho + \left(\frac{E\left[Q(T_0)|\mathcal{G}_s\right] - Q(s)}{(T_0 - s)}\right)\right.\right. \\ \left.\left. + \ln\left(\frac{E\left[Q(T_0)|\mathcal{G}_s\right] - Q(s)}{(T_0 - s)(\rho + r(s) - \mu(s))}\right)\left(\frac{E\left[Q(T_0)|\mathcal{G}_s\right] - Q(s)}{(T_0 - s)}\right)\right\} ds\right].$$

By means of the value of the moments of the Poisson distribution and of the Jensen inequality in its conditional form, we obtain

$$(5.12)\qquad E\left[\int_0^T \frac{E\left[Q(T_0)|\mathcal{G}_s\right] - Q(s)}{T_0 - s} ds\right] = \int_0^T \frac{E[Q(T_0) - Q(s)]}{T_0 - s} ds = \rho T < \infty$$

and

$$(5.13)\quad E\left[\int_0^T \ln\left(E\left[Q(T_0)|\mathcal{G}_s\right] - Q(s)\right) \frac{E\left[Q(T_0)|\mathcal{G}_s\right] - Q(s)}{T_0 - s} ds\right] \\ \leq E\left[\int_0^T \frac{\left(E[Q_{T_0}|\mathcal{G}_s] - Q(s)\right)^2}{T_0 - s} ds\right] \\ \leq \int_0^T \frac{E\left[E[(Q(T_0) - Q(s))^2|\mathcal{G}_s]\right]}{T_0 - s} ds \\ \leq \int_0^T \frac{E[(Q(T_0) - Q(s))^2]}{T_0 - s} ds \\ = \int_0^T \left(\rho^2(T_0 - s) + \rho\right) ds < \infty$$

and also

$$(5.14)\qquad E\left[\int_0^T \ln\left(\frac{1}{T_0 - s}\right) \frac{E\left[Q(T_0)|\mathcal{G}_s\right] - Q(s)}{T_0 - s} ds\right] = \rho \int_0^T \ln\left(\frac{1}{T_0 - s}\right) ds < \infty.$$

Using (5.12)-(5.14), we see that (5.11) is finite. ■

**Remark.** In the pure Poisson jump case Theorem 5.4 shows that if the insider filtration is $\mathcal{G}' = \mathcal{F} \vee \sigma(Q(T_0))$, then there is no optimal portfolio since $E\left[Q(T_0)|\mathcal{G}_s\right] - Q(s) = 0$ for all $\omega$ such that $Q(T_0) = 0$. This is contrary to the pure Brownian motion case with the enlargement of filtration $\mathcal{G}' = \mathcal{F} \vee \sigma(B(T_0))$, where we have an optimal portfolio (see [KP]). The reason is that the insider has an arbitrage opportunity as soon as he knows where $Q(t)$ does not jump. On the other hand, as soon as there exists an optimal portfolio in the Poisson pure jump market for an insider filtration $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{G}'$, then the value function $\Phi(x)$ for the insider is finite also for $T_0 = T$, which again is contrary to the pure Brownian motion case.

**2) The mixed case:**

If we deal with the mixed market (1.13) & (1.14), we get the following

**Theorem 5.5** *Set*

$$\alpha(s) = -\frac{E\left[B(T_0) - B(s)|\mathcal{G}_s\right]^-}{T_0 - s} \qquad \text{and} \qquad \gamma(s) = -\frac{E\left[Q(T_0) - Q(s)|\mathcal{G}_s\right]^-}{T_0 - s}.$$

*Then*

**i)** *For all insider filtrations $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{G}'$ and $T_0 > T$ there exists an optimal insider portfolio given by*

$$\pi^*(s) = \frac{1}{2\sigma^2(s)} \Big(\mu(s) - r(s) - \rho - \sigma(s)\alpha(s) - \sigma^2(s) + \sqrt{(\mu(s) - r(s) - \rho - \sigma(s)\alpha(s) + \sigma^2(s))^2 - 4\sigma^2(s)\gamma(s)}\Big) \tag{5.15}$$

**ii)** *The value function $\Phi(x)$ for the insider is finite for all $T_0 > T$.*

**Proof.** Part i) follows by solving equation (5.9) for $\pi$. Here, the conditio $\pi(s) > -1$ is not fulfilled $\mu_{\mathcal{F}}(dz)dt$ a.e. but $N(dz, dt)$ a.e. which is sufficient in the our situation of the Poisson process. Concerning part ii), it is sufficient to consider the largest insider filtration $\mathcal{G}'_t = \mathcal{F}_t \vee \sigma\left(B(T_0), Q(T_0)\right)$. Then

$$\alpha(s) = -\frac{B(T_0) - B(s)}{T_0 - s} \qquad \text{and} \qquad \gamma(s) = -\frac{Q(T_0) - Q(s)}{T_0 - s}.$$

Using the fact that

$$E[\int_0^T \alpha^2(s)ds] = \int_0^T \frac{1}{T_0 - s} ds = \ln\left(\frac{T_0}{T_0 - T}\right)$$

in addition to (5.12)-(5.14) and Jensen inequality, one can show the finiteness of $\Phi(x)$ with the same techniques as in the proof of Theorem 5.4. ■

**Remark.** Contrary to the pure Poisson jump case, the mixed case gives rise to an optimal insider portfolio for all insider filtrations $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{G}'$. The reason is that while it is possible to introduce arbitrage possibilities for the insider through an enlargement of filtration in the pure jump case (see e.g. [G]), this is no longer the case in the mixed market. But in contrast to the pure jump case, the finiteness of the value function $\Phi(x)$ is only ensured for $T_0$ stricly bigger than $T$. However, choosing the filtration "small enough with respect to the information $B(T_0)$", one can generate a finite value function also for $T_0 = T$. The most obvious example would be $\mathcal{G}_t = \mathcal{F}_t \vee \sigma\left(Q(T_0)\right)$, in which case $\alpha(s) \equiv 0$. This example is treated in [EJ].


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