Evaluating hybrid products: the interplay between financial and insurance markets

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Abstract. A current issue in the theory and practice of insurance and reinsurance markets is to find alternative ways of securitizing risks. Insurance companies have the possibility of investing in financial markets and therefore hedge against their risks with financial instruments. Furthermore they can sell part of their insurance risk by introducing insurance linked products on financial markets. Hence insurance and financial markets may no longer be considered as disjoint objects, but can be viewed as one arbitrage-free market. Here we provide an introduction to how mathematical methods for pricing and hedging financial claims such as the benchmark approach and local risk minimization can be applied to the valuation of hybrid financial insurance products, as well as to premium determination, risk mitigation and claim reserve management.

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1. Introduction

A current issue in the theory and practice of insurance and reinsurance markets is to find alternative ways of securitizing risks. To this purpose, insurance companies have tried to take advantage of the vast potential of capital markets by introducing exchange-traded insurance-linked instruments such as *mortality derivatives* and *catastrophe insurance options*. At the same time, insurance products such as *unit-linked life insurance contracts*, where the insurance benefits depend on the price of some specific traded stocks, offer a combination of traditional life insurance and financial investment. Furthermore, new kinds of insurance instruments, which offer protection against risks connected to macro-economic factors such as unemployment, are recently offered on the market. Hence insurance and financial markets may no longer be viewed as disjoint objects, but can be considered as one arbitrage-free market. Here we provide an introduction to how mathematical methods for pricing and hedging financial claims can be applied to the valuation and

hedging of the hybrid products mentioned above, as well as to premium determination, risk mitigation and claim reserve management. In this paper we propose to use the **benchmark approach** for pricing and the **(local) risk minimization** method for hedging purposes. We motivate these choices as follows.
We have already remarked that insurance markets and financial markets can be seen as one arbitrage-free market. However insurance claims are in general not replicable by other financial instruments, which implies that the hybrid market consisting of financial and insurance products is incomplete. As a consequence, there usually exist several equivalent (local) martingale measures, corresponding to the same numéraire, that guarantee the absence of arbitrage in the market. In incomplete markets a pricing and hedging criterion with a corresponding equivalent (local) martingale measure must then be selected. Rather natural and tractable are quadratic hedging criteria such as *mean-variance hedging* and *local risk minimization*, see [41] and [25] for extensive surveys. The local risk minimization approach provides for a given square-integrable contingent claim H a perfect hedge by using strategies that are not necessarily self-financing, with (discounted) portfolio value given by the gain of trade plus an instantaneous adjustment called the cost. The optimal strategy, when it exists, is determined by the property of having minimal risk, in the sense that the optimal cost is given by a square-integrable martingale strongly orthogonal to the martingale part of the asset price process. This implies that the optimal strategy is "self-financing on average", i.e. remains as close as possible to being self-financing. In this setting, one can hedge a contingent claim H by investing in the primary assets on the market and by compensating other source of risks by using the cost. In particular (local) risk-minimization naturally appears as suitable hedging method when market incompleteness derives by the presence of an additional source of randomness external to the financial market (such as for example mortality risk, catastrophe risk, insurance risks), that is "orthogonal" to the asset price dynamics, but not necessarily independent of them and vice versa. This is the case of market models containing financial insurance-linked instruments, such as mortality derivatives (survival swaps, longevity bonds) recently introduced on the markets to hedge against systematic mortality risk in life insurance contracts, and unit-linked life insurance contracts, i.e. contracts that combine insurance benefits and financial investment. Some references on this topic are for example [1], [4], [5], [8], [9], [15], [16], [31], [32], [37] and [38].
Local risk minimization is mainly an hedging criterion and provides a no-arbitrage price only as a "by-product" of the method, but such a valuation is not its primary objective. Hence for what concerns the pricing issue, we consider here the benchmark approach, introduced in the literature by several authors ([20], [21], [28], [33], [34], [35]). The benchmark approach provides a pricing rule (*real-world pricing*) under the real-world probability measure $\mathbb{P}$ by using a particular discounting factor called *benchmark* or $\mathbb{P}$-*numéraire portfolio*, and does not require at all the existence of an equivalent (local) martingale measure (ELMM). The numéraire portfolio contains information on macro-economic influences and on risks generated by the complex of hybrid products on the market. Hence it can be seen as a general indicator of the market's financial and economic conditions (cost of capital, interest

rates, expected investment returns, macro-economic influences, market dependence structure). Real-world pricing uses the numéraire as a measure of market performance and then results to be more natural than pricing by selecting a particular equivalent martingale measure. In this way we also benefit from the statistical advantages of working directly under the real-world probability measure.
Furthermore, for hedgeable claims, the real-world pricing formula gives their minimal price and for non-hedgeable claims the method is consistent with (asymptotic) utility indifference pricing as defined in [35] in a very general setting. Moreover there is an intrinsic relation between (local) risk minimization approach and real-world pricing, that justifies the use of the benchmark approach for pricing also in incomplete markets. To this extent we refer to the detailed discussion contained in Section 5.
For what concerns the application of the no-arbitrage pricing theory to premium determination for insurance contracts, this topic has been already discussed in the literature by several authors, see [17], [29], [40] and [42], as explained in Section 3. Here we consider the benchmark approach also for actuarial application as more natural pricing method with respect to the martingale methods of the standard no-arbitrage pricing theory, since it keeps a close connection to the classical premium calculation principles, which also use the real-world probability measure $\mathbb{P}$.
Furthermore, in the case of real-world pricing of insurance contracts, we take directly in account the role of investment opportunities in assessing premiums and reserves, since the benchmark is a direct and intuitive global indicator of (hybrid) market performance. This is of course even more relevant for insurance structures depending heavily on the performance of financial markets and macro-economic factors, such as for example unemployment insurance products. On the contrary the choice of a particular martingale measure for actuarial applications appears quite artificial, since it is exclusively determined in relation to the primitive financial assets on the financial market. A detailed discussion on the relation between actuarial premium calculation principles and real-world pricing is contained in Section 5.
The structure of the paper is the following. First of all we introduce shortly the benchmark approach. Then in Section 3.1 we illustrate an application of real-world pricing to premium determination for unemployment insurance products, after having discussed no-arbitrage pricing of insurance claims in Section 3. Afterwards we consider local risk minimization for hybrid markets: in Section 4 we recall the main features of this hedging method and in Section 4.1 we apply it to dynamic hedging with longevity bonds. Finally a discussion on the relation between (local) risk minimization approach, real-world pricing and actuarial premium calculation principles concludes the paper in Section 5.

2. The Benchmark Approach

As stated in the introduction, we adopt the benchmark approach for our pricing issue. All fundamental results of this approach can be found in [35] for jump diffusion and Itô process driven markets and in [33] for a general semimartingale market.
Let $T > 0$ be a finite time horizon. We consider a frictionless financial market

model in continuous time, which is set up on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t < T}$ that is assumed to satisfy $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \in [0,T]$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, as well as the usual hypotheses, see [36].
On the market, we can find $d+1$ nonnegative, adapted tradable (primary) security account processes, represented the $(d+1)$-dimensional càdlàg semimartingale $S = (S_t)_{t\in[0,T]} = (S_t^0, S_t^1, ..., S_t^d)^{tr}_{t\in[0,T]}$. Here we interpret S_t^0 as the value of the adapted, strictly positive savings account at time t, $t \in [0,T]$.
Let $L(S)$ denote the space of $\mathbb{R}^{d+1}$-valued, predictable strategies

$$\delta = (\delta_t)_{t\in[0,T]} = (\delta_t^0, \delta_t^1, ..., \delta_t^d)^{tr}_{t\in[0,T]} ,$$

for which the corresponding gain from trading in the assets, i.e. $\int_0^t \delta_s \cdot dS_s$, exists for all $t \in [0,T]$.
Here, δ_t^j represents the units of asset j held at time t by a market participant. The portfolio value S_t^δ at time $t \in [0,T]$ is then given by

$$S_t^\delta = \delta_t \cdot S_t = \sum_{j=0}^{d} \delta_t^j S_t^j .$$

A strategy $\delta \in L(S)$ is called *self-financing* if changes in the portfolio value are only due to changes in the assets and not due to in- or outflow of money, i.e. if

$$S_t^\delta = S_0^\delta + \int_0^t \delta_s \cdot dS_s \ , \ t \in [0,T] ,$$

or equivalently

$$dS_t^\delta = \delta_t \cdot dS_t .$$

In the sequel we won't always request strategies to be self-financing. We write $\mathcal{V}_x^+$ $(\mathcal{V}_x)$ for the set of all strictly positive (nonnegative), finite and self financing portfolios S^δ with initial capital $S_0^\delta = x$. We now introduce the notion of the $\mathbb{P}$-numéraire portfolio.

Definition 2.1. *A portfolio $S^{\delta_*} \in \mathcal{V}_1^+$ is called* $\mathbb{P}$-numéraire portfolio *if every nonnegative portfolio $S^\delta \in \mathcal{V}_1$, discounted (or benchmarked) with S^{δ_*}, forms a $(\mathbb{F}, \mathbb{P})$-supermartingale. In particular, we have*

$$\mathbb{E}\left[\frac{S_\sigma^\delta}{S_\sigma^{\delta_*}} \,\middle|\, \mathcal{F}_\tau \right] \le \frac{S_\tau^\delta}{S_\tau^{\delta_*}} \qquad a.s. \tag{2.1}$$

for all stopping times $0 \le \tau \le \sigma \le T$.

From now on, we choose the $\mathbb{P}$-numraire portfolio as *benchmark*. We call any security, when expressed in units of the numraire portfolio, a *benchmarked security* and refer to this procedure as *benchmarking*. The benchmarked value of a portfolio S^δ is given by

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}}, \ t \in [0,T].$$

If a $\mathbb{P}$-numéraire portfolio exists, it is unique by the supermartingale property and Jensen's inequality, see [3].
To establish the further modeling framework, we make the following (rather weak) assumption, see [3], [28] or [35].

Assumption 2.2. *The $\mathbb{P}$-numéraire portfolio $S^{\delta_*} \in \mathcal{V}_1^+$ exists in our market.*

If it exists, the $\mathbb{P}$-numéraire portfolio is equal to the "growth optimal portfolio" (in short: GOP), which is defined as the portfolio with the maximal growth-rate in the market. It also satisfies several other optimality criteria, see [3], [26], [33] and [35], and can be approximated under fairly weak assumptions by a sequence of well-diversified portfolios (see Theorem 3.6 of [34]). The existence and uniqueness of the $\mathbb{P}$-numéraire portfolio can be shown in a sufficiently general setting, see [3], [28] or [35].

Definition 2.3. *A benchmarked nonnegative self-financing portfolio $\hat{S}^\delta$ is a* strong arbitrage *if it starts with zero initial capital, that is $\hat{S}^\delta_0 = 0$, and generates some strictly positive wealth with strictly positive probability at a later time $t > 0$, that is $\mathbf{P}(\hat{S}^\delta_t > 0) > 0$.*

With the existence of the $\mathbb{P}$-numéraire portfolio and the corresponding supermartingale property (2.1), strong arbitrage opportunities, as defined in Definition 2.3, are excluded, see [33]. There could still exist some weaker forms of arbitrage, which would require to allow for negative portfolios of total wealth, however. Because of the (often legally established) principle of limited liability, these portfolios should be excluded in a realistic market model: a market participant generally holds a nonnegative portfolio of total wealth, otherwise he would have to declare bankruptcy. This holds in particular for insurance companies that must take care of several legal constraints for trading.
Let us now consider two portfolios $S^\delta \in \mathcal{V}_x$ and $S^{\delta'} \in \mathcal{V}_y$ with $\hat{S}^\delta_T = \hat{S}^{\delta'}_T$ $\mathbb{P}-$a.s. Let the benchmarked portfolio process $\hat{S}^\delta_t, t \in [0,T]$, be a martingale and the benchmarked portfolio process $\hat{S}^{\delta'}_t, t \in [0,T]$, be a supermartingale. Then

$$\hat{S}^\delta_t = \mathbb{E}\left[\hat{S}^\delta_T \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\hat{S}^{\delta'}_T \,\middle|\, \mathcal{F}_t\right] \le \hat{S}^{\delta'}_t, \quad \forall t \in [0,T] \, , \tag{2.2}$$

and in particular

$$x = \hat{S}^\delta_0 \le \hat{S}^{\delta'}_0 = y \; .$$

Then S^δ (if it exists) has minimal price among all benchmarked portfolios with the same terminal value. Hence, a rational (risk-averse) investor would always invest in a benchmarked martingale portfolio (if it exists). This justify the following definition of "fair" wealth processes, see [33].

Definition 2.4. *A portfolio process $S^\delta = (S^\delta_t)_{t\ge 0}$ is called* fair *if its benchmarked value process*

$$\hat{S}^\delta_t = \frac{S^\delta_t}{S^{\delta_*}_t} \, , \quad t \in [0,T] \, ,$$

forms a $(\mathbb{F},\mathbb{P})$-martingale.

Definition 2.5. *A* T-contingent claim H *is a* $\mathcal{F}_T$*-measurable random variable with* $\mathbb{E}\left[\frac{|H|}{S_T^{\delta_*}}\right] < \infty$*. We denote by*

$$\widehat{H} := \frac{H}{S_T^{\delta_*}} \tag{2.3}$$

the benchmarked payoff *of the* T*-contingent claim* H*.*

According to Definition 2.4, it is natural to define the so called *real-world pricing formula* for a T-contingent claim H as follows:

Definition 2.6. *For a* T*-contingent claim* H *the* fair price $P_t(H)$ *of* H *at time* $t \in [0,T]$ *is given by*

$$P_t(H) := S_t^{\delta_*} \mathbb{E}\left[\frac{H}{S_T^{\delta_*}} \middle| \mathcal{F}_t\right] = S_t^{\delta_*} \mathbb{E}\left[\hat{H} \mid \mathcal{F}_t\right] . \tag{2.4}$$

Here (2.4) *is addressed as* real-world pricing formula*.*

Hence the corresponding benchmarked fair price process $(\hat{P}_t)_{t\in[0,T]} = \left(\frac{P_t}{S_t^{\delta_*}}\right)_{t\in[0,T]}$ forms a $(\mathbb{F},\mathbb{P})$-martingale.

Definition 2.7. *We say that a nonnegative benchmarked contingent claim* $\hat{H} \in L^1(\mathcal{F}_T,\mathbb{P})$ *is* hedgeable *if there exists a self-financing strategy* $\delta^{\hat{H}} = (\delta_t^{\hat{H}})_{t\in[0,T]} = (\delta_t^{\hat{H},1}, \delta_t^{\hat{H},2}, \dots, \delta_t^{\hat{H},d})^{tr}_{t\in[0,T]}$ *such that*

$$\hat{H} = \hat{H}_0 + \int_0^T \delta_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u. \tag{2.5}$$

In the case of an hedgeable benchmarked payoff $\hat{H}$, the real-world pricing formula (2.4) provides the description for the fair portfolio of minimal price among all replicating self-financing portfolios for $\hat{H}$, since the benchmarked fair portfolio value forms by definition a $\mathbb{P}$-martingale. The benchmark approach allows other self-financing hedge portfolios to exist for $\hat{H}$, see [35]. However, these nonnegative portfolios are not $\mathbb{P}$-martingales and, as supermartingales, therefore more expensive than the $\mathbb{P}$-martingale given by the benchmarked fair portfolio process obtained by (2.4), see (2.2).

Remark 2.8. *If a* T*-contingent claim* H *and the value* $S_T^{\delta_*}$ *at time* T *of the* $\mathbb{P}$*-numéraire portfolio are independent, we get*

$$\begin{aligned} P_t(H) &= S_t^{\delta_*} \mathbb{E}\Big[\frac{1}{S_T^{\delta_*}} |\mathcal{F}_t\Big] \,\mathbb{E}\left[H|\mathcal{F}_t\right] \\ &= P(t,T)\mathbb{E}\left[H|\mathcal{F}_t\right] , \end{aligned} \tag{2.6}$$

where $P(t,T)$ *is the fair price at time* $t \leq T$ *of a zero-coupon bond with nominal value one, paid at time* T*. This formula is often called the* actuarial pricing formula*.*

3. No-arbitrage Pricing of Insurance Claims

Pricing of random claims has ever been one of the core subjects in both actuarial and financial mathematics and there exist various approaches for calculating (fair) prices. The actuarial way of pricing usually considers the classical premium calculation principles that consist of *net premium* and *safety loading*: if H describes a random claim, which the insurance company has to pay (eventually) in the future at time τ, then a premium $P(H)$ to be charged for the claim is defined by

$$P(H) = \underbrace{\mathbb{E}\left[\frac{H}{D_\tau}\right]}_{\text{net premium}} + \underbrace{A\left(\frac{H}{D_\tau}\right)}_{\text{safety loading}}, \tag{3.1}$$

where D is a discounting factor chosen according to actuarial judgement (see also [29] for further remarks). Note that the net premium is the expected value of H with respect to the *real-world* (or *objective*) probability measure. Possible safety loadings could be $A(\frac{H}{D_\tau}) = 0$ (*net premium principle*), $A(\frac{H}{D_\tau}) = a \cdot \mathbb{E}\left[\frac{H}{D_\tau}\right]$ (*expected value principle*, where $a \geq 0$), $A(\frac{H}{D_\tau}) = a \cdot \mathbb{V}ar(\frac{H}{D_\tau})$ (*variance principle*, where $a > 0$) or $A(\frac{H}{D_\tau}) = a \cdot \sqrt{\mathbb{V}ar(\frac{H}{D_\tau})}$ (*standard deviation principle*, where $a > 0$), see e.g. [39]. The existence of a safety loading is justified by ruin arguments and the risk-averseness of the insurance company.

Widely used pricing approaches in finance base on no-arbitrage assumptions (see e.g. the famous papers of Black and Scholes [11] and Merton [30]). A financial market, consisting of several primary assets, is assumed to be in an economic equilibrium, in which riskless gains with positive probability (arbitrage) by trading in the assets are impossible. A fundamental result in this context is that absence of arbitrage is implied by the existence of an equivalent (local) martingale measure, i.e. a probability measure, which is equivalent to the real-world measure and according to which all assets, discounted with some numéraire, are (local) martingales. There are different versions of this result which is often called the fundamental theorem of asset pricing (in short: FTAP), see. e.g. [18], [19], [22], [24] or [27].

Based on the FTAP, it can then be shown that, at any time t, an arbitrage-free price $P_t(H)$ of a (contingent) claim H (paid at time $T \geq t$) can be defined by

$$P_t(H) := N_t \mathbb{E}_{\mathbb{Q}}\left[\frac{H}{N_T} | \mathcal{F}_t\right], \tag{3.2}$$

where $\mathbb{Q}$ is an equivalent (local) martingale measure and $(N_t)_{t\in[0,T]}$ a discounting factor process.

From an economic point of view both the safety loading in equation (3.1) and the change to an equivalent (local) martingale measure in equation (3.2) express the risk-averseness of the insurance company. Hence, there have been several attempts to connect actuarial premium calculation principles with the financial no-arbitrage theory. The papers [17] and [42] both describe a competitive and liquid reinsurance market, in which insurance companies can "trade" their risks among each other. Since riskless profits shall be excluded also in this setting, the no-arbitrage theory applies and insurance premiums can be calculated by equation (3.2). Both papers

actually show that under some assumptions there exist equivalent martingale measures, which explain premiums of the form (3.1), so that these principles provide arbitrage-free prices, too.
Therefore no-arbitrage pricing theory can be applied also to actuarial premium determination. To this purpose, in this paper we choose the benchmark approach, as we have already thoroughly explained in the Introduction. In Section 5 we comment extensively on the relation between actuarial premium calculation principles and real-world pricing. We now illustrate an application of real-world pricing to premium determination of unemployment insurance products.

3.1. Real-world Pricing for Unemployment Insurance Products

We first introduce the structure of the considered unemployment insurance products. The product's basic idea is that the insurance company compensates to some extend the financial deficiencies, which an unemployed insured person is exposed to. Here we only consider contracts with deterministic, a priori fixed claim payments c_i, which can be interpreted as an annuity during an unemployment period, and predefined payment dates T_i, $i = 1, ..., N$. An example for this kind of contracts is given by Payment Protection Insurance (in short: PPI) products against unemployment, which are linked to some payment obligation of an obligor to its creditor.
The following details of the insurance contract are important for the later model specifications:

- Regarding the method of premium payment, we have to differentiate between single rates, where the whole insurance premium is paid at the beginning of the contract, and periodical rates. For our modeling purpose, we want to focus on calculating single premiums. This is again motivated by PPI unemployment products, which are often sold as an add-on directly by the creditor. The insurance company then receives a single rate from the creditor, who in turn allocates this rate to the instalments.
- The obligor must have been employed at least for a certain period before the contract's conclusion. Hence we assume that she is employed at the beginning of the contract.

We also consider three time periods that belong to the exclusion clauses of the contracts and impact the insurance premium.

- The *waiting period* W starts with the beginning of the contract. If an insured person becomes unemployed at any time of this period, he is not entitled to receive any claim payments during the whole unemployment time.
- The *deferment period* D starts with the first day of unemployment. An insured person is not entitled to receive claim payments until the end of this period.
- The third period is comparable to the waiting period and is called the *requalification period* and denoted by R. The difference between waiting and requalification period is their beginning. The waiting period starts with the beginning of the contract and the requalification period with the end of any unemployment period that occurred during the contract's duration. If an insured person becomes (again) unemployed at any time of the requalification

period, he is not entitled to receive any claim payment during the whole time of unemployment.

For existing unemployment insurance contracts, the waiting, deferment and requalification periods currently vary from three to twelve months.
According to the contract structure defined above, the random insurance claim H_i at the payment date T_i can be defined as

$$H_i(\omega) := c_i I_{\{W<\tau_1\leq T_i-D,\tau_2>T_i\}\cup\bigcup_{j=2}^{\infty}\{\tau_{2j\text{-}1}-\tau_{2j\text{-}2}>R,W<\tau_{2j\text{-}1}\leq T_i-D,\tau_{2j}>T_i\}}(\omega)\,, \qquad (3.3)$$

where $(\tau_j)_{j\in\mathbb{N}}$ with $\tau_0 := 0$ are the random jump times of the employment-unemployment process $X := (X_t)_{t\in[0,T]}$ that describes at time t if the insured person is employed ($X_t = 0$) or not ($X_t = 1$).

Assumption 3.1. *Every (random) insurance claim H_i of the unemployment insurance contract, paid at time T_i, is independent of the respective value $S_{T_i}^{\delta_*}$ of the $\mathbb{P}$-numéraire portfolio at time T_i.*

Under this Assumption we can apply the actuarial pricing formula (2.6), that requires in this case only the conditional joint distributions of the jump times $\tau_j, j \in \mathbb{N}$. However we note that this assumption may be too strong for a realistic model. The insurance claims obviously depend on macroeconomic unemployment factors, which in turn may have interdependencies with financial markets, represented by the $\mathbb{P}$-numéraire portfolio (or the GOP). For the study of dependence effects between the insurance claims and the $\mathbb{P}$-numéraire portfolio, we refer to [10].
Furthermore, we assume that there is the possibility of putting money on a bank account with constant interest rate $r > 0$, and that the employment-unemployment process X follows a time-homogeneous strong Markov chain with respect to $\mathbb{P}$ and

$$\mathcal{F}_t = \mathcal{F}_t^X = \sigma(X_u, u \leq t),\ t \in [0, T].$$

This assumptions may again be too strong for a realistic model. Actually, the probability of an insured person of getting unemployed or employed may depend on his past employment-unemployment development. An extension of this model can be found in [10].
Under these hypotheses the sojourn times $\tau_j - \tau_{j-1}$, $j \geq 1$, given $X_0 = i_0$, $i_0 \in \{0, 1\}$, are conditionally independent and exponentially distributed, with parameters given by the intensity matrix

$$\Lambda = \begin{pmatrix} \lambda_0 & -\lambda_0 \\ -\lambda_1 & \lambda_1 \end{pmatrix} \qquad (3.4)$$

of X. In particular, we have

$$\mathbb{P}\left(\tau_1 - \tau_0 > t_1, ..., \tau_n - \tau_{n-1} > t_n \middle|\, X_0 = i_0\right) = e^{-\lambda_{i_0} t_1} \cdot ... \cdot e^{-\lambda_{i_{n-1}+1} t_n}\,, \qquad (3.5)$$

where $i_0, i_1, ..., i_{n-1} \in \{0, 1\}$ with $i_k = 1 - i_{k-1}$, $t_1, ..., t_n \in [0, \infty)$, and $\lambda_{i_0}, ..., \lambda_{i_{n-1}}$ are defined by (3.4), see [43].
For the sake of simplicity, we now assume that $t \in [T_{k-1}, W)$, that the insured person was employed at the actual beginning of the contract ($X_0 = 0$) and that the first jump to unemployment τ_1 has not occurred up to time t ($t < \tau_1$).

Proposition 3.2. *Under Assumptions* 3.1 *we obtain the insurance premiums* P_t *for* $X_0 = 0$ *and* $t \in [T_{k-1}, W)$ *as follows:*

$$P_t = \sum_{i=k}^{N} S_t^{\delta_*} \mathbb{E}\left[\hat{H}_i \middle| \mathcal{F}_t\right] = \sum_{i=k}^{N} e^{-r(T_i - t)} \mathbb{E}\left[\hat{H}_i \middle| \mathcal{F}_t\right] \tag{3.6}$$

$$= \sum_{i=k}^{N} c_i e^{-(r+\lambda_1)(T_i - t)} \Big(\frac{\lambda_0}{\lambda_0 - \lambda_1} \big(e^{-(\lambda_0-\lambda_1)(W-t)} - e^{-(\lambda_0-\lambda_1)(T_i - D - t)} \big)$$

$$+ \lambda_0^2 \lambda_1 \int_{\max\{W-t,R\}}^{T_i - D - t} \int_R^y e^{-(\lambda_0-\lambda_1)x} \int_0^{y-x} e^{-(\lambda_0-\lambda_1)u} I_0(2\sqrt{\lambda_0\lambda_1 u(y-x-u)}) dudxdy \Big), \tag{3.7}$$

where $I_0(x)$ *is the modified first kind Bessel function of order* 0*. In general, the modified first order Bessel function* $I_\alpha(x)$ *of order* $\alpha \in \mathbb{R}$ *is given by*

$$I_\alpha(x) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}. \tag{3.8}$$

Proof. Pricing formula (3.6) derives by applying (2.6) and (3.7) by the assumptions on the employment-unemployment process X. For further details on the proof, we refer to [10]. □

Note that, due to the "loss of memory" property of X, it is sufficient to calculate the insurance premiums for $t \leq \tau_1$. Analogous computations deliver the price for all the other cases, see [10].

4. (Local) Risk Minimization for Hybrid Markets

We now turn to the hedging issue. To avoid technicalities, we focus on the case where the asset prices discounted with the saving account S^0 are given by local martingales under $\mathbb{P}$. We denote by $\bar{S}$ the vector of the $d+1$ discounted primary security accounts $\bar{S} := \left(\frac{S_t}{S_t^0}\right)_{t\in[0,T]} = (1, \bar{S}_t^1, \ldots, \bar{S}_t^d)^{tr}_{t\in[0,T]}$.

Remark 4.1. *This assumption on the underlying asset price processes may appear quite restrictive. However if we choose as discounting factor the* $\mathbb{P}$*-numéraire portfolio, by Assumption* 2.2 *and Theorem 2.4 of* [26] *it follows that the vector process* $\hat{S}$ *of benchmarked primary security accounts is always a* $\mathbb{P}$*-local martingale, if* S *is given by a continuous semimartingale and also for a wide class of jump-diffusion models.*

Under these assumptions on the discounted financial markets, we can apply the risk-minimization method as originally introduced in [23]. For further details, we also refer to [41].

Definition 4.2. *An* L^2-admissible strategy *is any* $\mathbb{R}^{d+1}$*-valued predictable vector process* $\delta = (\delta)_{t\in[0,T]} = (\delta_t^0, \delta_t^1, \ldots, \delta_t^d)^{tr}_{t\in[0,T]}$ *such that*

(i) *the associated discounted portfolio* $\bar{S}^\delta$ *is a square-integrable stochastic process whose left-limit is equal to* $\bar{S}^\delta_{t-} = \delta_t \cdot \bar{S}_t$, $t \in [0,T]$,

(ii) *the stochastic integral* $\int \delta \cdot \mathrm{d}\bar{S}$ *is such that*

$$\mathbb{E}\left[\int_0^T \delta_u^\top \mathrm{d}[\bar{S}]_u \delta_u\right] < \infty. \tag{4.1}$$

Here $[\bar{S}] = ([\bar{S}^i, \bar{S}^j])_{i,j=1,\dots,d}$ *denotes the matrix-valued optional covariance process of* $\bar{S}$.

Since the market is not complete, we also admit strategies here that are not self-financing and may generate profits or losses over time as defined below.

Definition 4.3. *For any* L^2*-admissible strategy* δ*, the* cost *process* $\bar{C}^\delta$ *is defined by*

$$\bar{C}^\delta_t := \bar{S}^\delta_t - \int_0^t \delta_u \cdot \mathrm{d}\bar{S}_u - \bar{S}^\delta_0, \quad t \in [0,T]. \tag{4.2}$$

Here $\bar{C}^\delta_t$ describes the total costs incurred by δ over the interval $[0,t]$.

Definition 4.4. *For an* L^2*-admissible strategy* δ*, the corresponding* risk *at time* t *is defined by*

$$\bar{R}^\delta_t := \mathbb{E}\left[\left(\bar{C}^\delta_T - \bar{C}^\delta_t\right)^2 \middle| \mathcal{F}_t\right], \quad t \in [0,T],$$

where the cost process $\bar{C}^\delta$*, given in* (4.2)*, is assumed to be square-integrable.*

We now wish to find an L^2-admissible strategy δ which minimizes the associated risk measured by the fluctuations of its cost process in a suitable sense.

Definition 4.5. *Given a discounted contingent claim* $\bar{H} \in L^2(\mathcal{F}_T, \mathbb{P})$*, an* L^2*-admissible strategy* δ *is said to be* risk-minimizing *if the following conditions hold:*

(i) $\bar{S}^\delta_T = \bar{H}$, $\mathbb{P}$-a.s.*;*

(ii) *for any* L^2*-admissible strategy* $\tilde{\delta}$ *such that* $\bar{S}^{\tilde{\delta}}_T = \bar{S}^\delta_T$ $\mathbb{P}$*-a.s., we have*

$$\bar{R}^\delta_t \le \bar{R}^{\tilde{\delta}}_t \quad \mathbb{P}-\text{a.s. for every t} \in [0,\text{T}].$$

Lemma 4.6. *The cost process* $\bar{C}^\delta$ *associated to a risk-minimizing strategy* δ *is a* $\mathbb{P}$*-martingale for all* $t \in [0,T]$.

Proof. For the proof of Lemma 4.6, we refer to [41] and [6]. □

The martingale property of the cost process characterizes the real-world mean-self-financing property of the strategy δ, i.e. L^2-admissible strategies that somehow are kept “self-financing on average”.
The next result shows how to provide a risk-minimizing strategy for a given claim. Let $\mathcal{M}^2_0(\mathbb{P})$ be the space of all square-integrable martingales starting at null at the initial time.

Proposition 4.7. *Every discounted contingent claim* $\bar{H} \in L^2(\mathcal{F}_T, \mathbb{P})$ *admits a unique risk-minimizing strategy* δ *with portfolio value* $\bar{S}^\delta$ *and cost process* $\bar{C}^\delta$*, given respectively by*

$$\delta = \delta^{\bar{H}}, \quad \bar{S}^\delta_t = \mathbb{E}\left[\bar{H} \middle| \mathcal{F}_t\right], \quad \bar{C}^\delta_t = L^{\bar{H}}_t,$$

*for $t \in [0,T]$, where $\delta^{\bar{H}}$ and $L^{\bar{H}}$ are provided by the Galtchouk-Kunita-Watanabe (*GKW*) decomposition of $\bar{H}$, i.e.*

$$\bar{H} = \bar{H}_0 + \int_0^T \delta_u^{\bar{H}} \cdot \mathrm{d}\bar{S}_u + L_T^{\bar{H}}, \quad \mathbb{P}-\text{a.s.}, \tag{4.3}$$

where $\bar{H}_0 \in \mathbb{R}$, $\delta^{\bar{H}}$ is an $\mathbb{F}$-predictable vector process satisfying the integrability condition (4.1) *and $L^{\bar{H}} \in \mathcal{M}_0^2(\mathbb{P})$ is strongly orthogonal to each component of $\bar{S}$.*

Proof. The proof follows from Theorem 2.4 of [41] and Lemma 4.6. □

Thus, the problem of minimizing risk is reduced to finding the representation (4.3). Decomposition (4.3) is often addressed in the literature as the *Föllmer-Schweizer decomposition.*
We now illustrate an application of the local risk minimization approach to hedging of mortality derivatives.

4.1. Application to Mortality Risk: Dynamic Hedging with Longevity Bonds

A large number of life insurance and pensions products have mortality and longevity as a primary source of risk. Life and pension insurance companies typically uses deterministic mortality intensities when determining premiums and reserves. However empirical evidence (see [14] for a literature overview on this topic) shows that this assumption is not realistic, so companies are exposed also to changes in the mortality intensity, i.e. to *systematic mortality risk*. This risk cannot be diversified away by pooling (i.e. by using sufficiently large portfolios) as in the case of *unsystematic mortality risk*, i.e. the risk associated with the status of individual life, but on the contrary its impact increases for larger portfolios of insured persons. Here we use the terminology of *(systematic) mortality risk* to denote all forms of deviations in aggregate mortality rates from those anticipated. More precisely, it can be differentiate in *longevity risk*, i.e. the risk that aggregate survival rates for given cohorts are higher than anticipated, and *short-term, catastrophic mortality risk*, i.e. the risk, that over short period of time, mortality rates are very much higher than would be normally experienced (such as for example in the case of a pandemic influenza or a natural catastrophe). Although mortality and longevity risk can be re-insured, traditional reinsurance is becoming inadequate to offer sufficient protection against these risks. Furthermore the new regulatory regime Solvency II proposal, due to be adopted in 2012, will require insurance companies to hold significant additional capital to guarantee their annuity liabilities if longevity risk cannot be controlled effectively. Since existing markets provide no effective hedge for longevity and mortality risk, recent studies ([2], [12], [13] and [14]) have highlighted the need of encouraging the introduction of a life market in order to address the problem of an extremely fast aging population and the risk of long retirement periods that cannot be afforded anymore by a shrinking (younger) labor force. Hence to this purpose, new forms of investment in mortality derivatives have been recently introduced in alternative or as a complement to traditional reinsurance. Some examples are the followings (for an exhaustive discussion on mortality products, we refer to [14]):

- *Longevity bonds*, where coupon payments are linked to the number of survivors in a given cohort. The first example of longevity bond in the history is represented by Tontine bonds issued by some European governments in the 17^{th} and 18^{th} centuries. The first modern longevity bonds were introduced in 2004 by the European Investment Bank and BNP Paribas.
- *Short-dated, mortality securities*: market traded securities, whose payments are linked to a mortality index. They allow the issuer to reduce its exposure to short-term catastrophic mortality risk. The first bond of type was issued with great success by Swiss Re in 2004.
- *Survivor swaps*, where counterparties swap a fixed series of payments for a series of payments linked to the number of survivors in a given cohort. Until now a small number of survivor swaps have been traded only on a over-the-counter basis.

Other kind of products such as *mortality options*, i.e. financial contracts with mortality rate as underlying, have been discussed only at theoretical level in the literature. The AFPEN (the association of French Pension Funds) has suggested to introduce also an *annuity futures* market. Some of these new investment products such as some longevity bonds have encountered the favor of the public. However the establishment of a life market is still at the beginning.
As a contribution to the ongoing discussion on the introduction of longevity markets, we now consider an application of risk minimization to dynamic hedging with longevity bonds. For further details on this issue, we refer also to [9]. The mathematical setting is the following. The time of death $\tau > 0$ of a person is modeled as a random variable with $P(\tau > t) > 0$ for any $t \in [0, T]$, and we denote by $H_t = \mathbb{I}_{\{\tau \leq t\}}$ the counting process of death. Let $\mathbb{H} := (\mathcal{H}_t)_{t\in[0,T]}$ be the filtration, generated by H. We assume that the overall information is represented by the filtration $\mathbb{G} := \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ is the augmented natural filtration of some Brownian motion W. To avoid technical difficulties, we suppose that the hypothesis (H) holds, i.e. every $\mathbb{F}$-martingale remains a martingale in the larger filtration $\mathbb{G}$. In particular, W is a $\mathbb{G}$-martingale, and then by Lévy's characterization a $\mathbb{G}$-Brownian motion. The survival probability process G associated to τ is supposed to fulfill

$$G_t := \mathbb{P}\left(\tau > t \middle| \mathcal{F}_t\right) = \exp\left(-\int_0^t \mu_u \, du\right) =: \exp\left(-\Gamma_t\right), \; t \in [0, T],$$

where the *stochastic mortality intensity* μ is given by an $\mathbb{F}$-progressively measurable process driven by W. The counting process martingale M associated with the one-jump process H is given as

$$M_t = H_t - \int_0^t (1 - H_u) \, \mu_u du, \; t \in [0, T]. \tag{4.4}$$

For simplicity we assume here to work with a fixed constant short rate r. We now suppose that it is possible to trade on the financial market in an instrument called a longevity bond which has present value

$$B_t = \int_0^t e^{-ru} G_u \, du, \; t \in [0, T].$$

The payment generated by this bond has the form of an annuity, where the declining rate is given by the survival probability for the age cohort of the insured person. The (discounted) value process associated with the longevity bond is thus given by the conditional expectation

$$V_t = \mathbb{E}\left[\int_0^T e^{-ru} G_u \, du \middle| \mathcal{G}_t\right], \; t \in [0,T]. \tag{4.5}$$

Remark 4.8. *If we consider a benchmarked financial market $\hat{S}_t, \; t \in [0,T]$, then the pricing formula for the benchmarked value process of the longevity bond is given by*

$$\hat{V}_t = \mathbb{E}\left[\int_0^T \frac{G_u}{S_u^{\delta^*}} \, du \middle| \mathcal{G}_t\right], \; t \in [0,T]. \tag{4.6}$$

In the case of dividends paying assets, the benchmark approach presents the disadvantage that we need to know the joint conditional distribution of (S^{δ^*}, G) to compute (4.6)*. Note however that when the interest rate is supposed to be constant and the discounted asset prices to be local martingales, then the pricing formulas* (4.5) *and* (4.6) *coincide, since the $\mathbb{P}$-numéraire portfolio is given in this case by the saving account S^0.*

We assume the existence on the market of a *gratification annuity* with increasing, continuous rate payments equal to $1 - G_t$ as long as the insured person is alive, up to maturity T. As G_t can be inferred from the longevity index which itself bases on realized mortality of some representative group, such an instrument rewards longevity relative to the policyholder's own age cohort. The present value of a gratification annuity is given by

$$C^a = \int_0^T e^{-ru} (1 - H_u)(1 - G_u) \, du.$$

Our goal is now to hedge the risk exposure from having sold the gratification annuity by trading dynamically in the longevity bond with value process V. For this sake we need some technical assumptions. First we assume $e^{\Gamma_T} \in L^2(P)$, and introduce the spaces $L^2(W)$, $L^2(M)$ consisting of all predictable θ, ψ such that

$$\mathbb{E}\left[\int_0^T \theta_s^2 \, ds\right] < \infty, \quad \mathbb{E}\left[\int_0^T \psi_s^2 \, d\Gamma_s\right] < \infty.$$

The space Θ of admissible strategies consists of all predictable ϑ such that

$$\mathbb{E}\left[\int_0^T \vartheta_s^2 \, d\langle V\rangle_s\right] < \infty.$$

In this setting the risk minimizing strategy for the gratification annuity can be found by first computing the GKW decompositions of V and $\mathbb{E}\left[C^a | \mathcal{G}_t\right], t \in [0,T]$, with respect to the $\mathbb{G}$-martingales W and M. By comparing them, one can then deduce the Föllmer-Schweizer decomposition

$$E\left[C^a | \mathcal{G}_t\right] = c + \int_0^t \vartheta_s^* \, dV_s + V_t^{\perp}, \tag{4.7}$$

where $\vartheta \in \Theta$ and $V^{\perp}$ is a square integrable martingale strongly orthogonal to V (i.e. $VV^{\perp}$ is a local martingale). For further details we refer to [9].

Theorem 4.9. *Under the hypotheses above, by martingale representation, for each $u \in [0,T]$ there exists a constant c_u and a predictable process $(\theta_{u,s})_{s\in[0,T]} \in L^2(W)$, with $\theta_{u,s} = 0$ if $s > u$, such that*

$$\mathbb{E}\left[e^{-ru}\,(1-G_u)\,e^{-\Gamma_u}\middle|\,\mathcal{F}_t\right] = c_u + \int_0^{t\wedge u} \theta_{u,s}\,dW_s$$
$$= c_u + \int_0^{t} \theta_{u,s}\mathbb{I}_{[0,u]}(s)\,dW_s, \tag{4.8}$$

for $t \in [0,T]$. We set $c := \int_0^T c_u\,du < \infty$. Then the Föllmer-Schweizer decompositions of the gratification annuity C^a with respect of the longevity bond V is given by

$$C^a = c + \int_0^T \eta_s\,dV_s + V_T^{\perp}\,, \tag{4.9}$$

where $V_T^{\perp} = \int_{0+}^T \gamma_s^M\,dM_s$, the predictable integrand $\gamma^M \in L^2(M)$ is equal to

$$\gamma_s^M = -(1-H_{s-})e^{\Gamma_s}\int_s^T \left(c_u + \int_0^u \theta_{u,v}\,dW_v\right)du,\ \ s \in [0,T], \tag{4.10}$$

and $\eta \in \Theta$ is uniquely determined by the equation

$$\eta_s\xi_s = (1-H_{s-})e^{\Gamma_s}\int_s^T \theta_{u,s}\ du,\ \ s \in [0,T]. \tag{4.11}$$

Here the predictable integrand $\xi \in L^2(W)$ derives by the predictable martingale representation for the longevity bond

$$V_t = E\left[\int_0^T e^{-ru}G_u\middle|\,\mathcal{F}_t\right] = V_0 + \int_0^t \xi_s\,dW_s,\ \ t \in [0,T]. \tag{4.12}$$

Proof. For the proof we refer to [9]. □

5. Relation between (Local) Risk Minimization Approach and Real-world Pricing

We now discuss the relation between (local) risk minimization approach and real-world pricing. For an exhaustive discussion of the connection between risk minimization approach and real-world pricing, we also refer to [6]. For further details on the relation between the existence of the numéraire portfolio and the minimal martingale density, see [26].

For the sake of simplicity, we assume that the underlying financial market contains only continuous asset prices. Then by Theorem 2.4 of [26] follows that the benchmarked asset price process $\hat{S}$ is given by a local martingale. Given a benchmarked contingent claim $\hat{H} \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, by Proposition 4.7 there exists a unique risk-minimizing strategy $\xi^{\hat{H}}$, that can be obtained by the Galtchouk-Kunita-Watanabe decomposition of $\hat{H}$ with respect to $\hat{S}$ given by

$$\hat{H} = E[\hat{H}] + \int_0^T \xi_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u + L_T^{\hat{H}}, \quad \mathbb{P} - \text{a.s.}, \tag{5.1}$$

where $\xi^{\hat{H}}$ is an $\mathbb{F}$-predictable vector process with $\mathbb{E}\left[\int_0^T {\xi_u^{\hat{H}}}^\top \mathrm{d}[\hat{S}]_u \xi_u^{\hat{H}}\right] < \infty$ and $L^{\hat{H}} = (L_t^{\hat{H}})_{t\in[0,T]}$ is a square-integrable martingale with $L_0^{\hat{H}} = 0$, strongly orthogonal to each component of $\hat{S}$. The benchmarked portfolio's value process associated to $\xi^{\hat{H}}$ is then $\mathbb{E}\left[\hat{H}\middle|\mathcal{F}_t\right]$, $t \in [0,T]$, with initial value $\mathbb{E}\left[\hat{H}\right]$ and benchmarked cost process $L^{\hat{H}}$. Hence the real-world pricing formula (2.4) coincides at any time $t \in [0,T]$ with the portfolio's value of the risk-minimizing strategy for $\hat{H}$ in incomplete markets where the benchmarked underlyings are local martingales. This is the case not only for continuous asset price models, but also for a large class of jump-diffusion models, see for example [35], Chapter 14, pages 513 - 549. Moreover we also remark that the risk-minimizing strategy is independent of the choice of the discounting factor in market models driven by continuous asset price processes or where the orthogonal martingale structure is generated by continuous martingales. For further details on this, we refer to [6] and [7].
Furthermore decomposition (5.1) allows us to decompose every square-integrable benchmarked contingent claim as the sum of its *hedgeable part* $\hat{H}^h$ and its *unhedgable part* $\hat{H}^u$ such that we can write

$$\hat{H} = \hat{H}^h + \hat{H}^u, \tag{5.2}$$

where

$$\hat{H}^h := \hat{H}_0 + \int_0^T \xi_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u \tag{5.3}$$

and

$$\hat{H}^u := L_T^{\hat{H}}. \tag{5.4}$$

Here the benchmarked hedgeable part $\hat{H}^h$ can be replicated perfectly, i.e.

$$\hat{U}_{H^h}(t) = \mathbb{E}\left[\hat{H}^h\middle|\mathcal{F}_t\right] = \hat{H}_0 + \int_0^t \xi_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u\,,\ t \in [0,T], \tag{5.5}$$

and $\xi^{\hat{H}}$ yields the fair strategy for the self-financing replication of the hedgeable part of $\hat{H}$. The remaining benchmarked unhedgeable part can be diversified and will be covered through the benchmarked cost process $L^{\hat{H}}$. In particular at $t = 0$ the initial value of the risk-minimizing strategy coincides with the real world price for the hedgeable part at $t = 0$, while the benchmarked unhedgeable part remains totally untouched. This is reasonable because any extra trading could only create unnecessary uncertainty and potential additional benchmarked profits or losses. However for $t > 0$ the cost $L^{\hat{H}}$ will be different from 0 and

$$\mathbb{E}\left[\hat{H}\middle|\mathcal{F}_t\right] = \mathbb{E}\left[\hat{H}^h\middle|\mathcal{F}_t\right] + L_t^{\hat{H}},\ t \in [0,T], \tag{5.6}$$

can be interpreted as an actuarial valuation formula, with the difference that the expectation term involves only the hedgeable part of the claim. The safety loading is given here by the benchmarked cost process. For similar results on the relation between actuarial valuation principles and mean-variance hedging, we also refer to [40].

The connection between risk-minimization and real-world pricing is then an important insight which both gives a clear reasoning for pricing and hedging of contingent claims via real-world pricing also in incomplete markets, and contributes to justify the use of the benchmark approach also for actuarial applications.

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