

LINEARIZED FILTERING OF AFFINE PROCESSES USING STOCHASTIC RICCATI EQUATIONS

LUKAS GONON AND JOSEF TEICHMANN

ABSTRACT. We consider an affine process X which is only observed up to an additive white noise, and we ask for the law of X_t , for some $t > 0$, conditional on all observations up to time t . This is a general, possibly high dimensional filtering problem which is not even locally approximately Gaussian, whence essentially only particle filtering methods remain as solution techniques. In this work we present an efficient numerical solution by introducing an approximate filter for which conditional characteristic functions can be calculated by solving a system of generalized Riccati differential equations depending on the observation and the process characteristics of X . The quality of the approximation can be controlled by easily observable quantities in terms of a macro location of the signal in state space. Asymptotic techniques as well as maximization techniques can be directly applied to the solutions of the Riccati equations leading to novel very tractable filtering formulas. The efficiency of the method is illustrated with numerical experiments for Cox–Ingersoll–Ross and Wishart processes, for which Gaussian approximations usually fail.

Key words and phrases: affine process, filtering, conditional law, Riccati equation, Zakai equation.

MSC 2010 Classification: 60G35, 62M20

1. INTRODUCTION

Consider a time-dependent stream of multi-variate signals which can not be observed directly, but only through a stream of noisy measurements. Given the observations made up to a specified moment in time t , what can we optimally say about the signal at time t , i.e. what is the best estimate for the signal’s value? There are various mathematical formulations of this fundamental problem. Research fields such as time-series analysis, signal processing and (frequentist) non-parametric statistics model the signal process as a deterministic function or focus on a discrete-time setting. In *stochastic filtering* the signal and observation processes are modeled as continuous-time stochastic processes, i.e. a (dynamic) Bayesian perspective is adopted.

The mathematical formulation of the stochastic filtering problem is the following: consider a D -valued stochastic process X , a p -dimensional Brownian motion W and an observation function $h: D \rightarrow \mathbb{R}^p$. Define the observation process Y as

$$(1.1) \quad Y_t = \int_0^t h(X_s) ds + W_t, \quad t \geq 0.$$

Note that both X and W are defined on *one* probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $p \in \mathbb{N}$, the state space D is some set and some regularity on h and the sample paths of X needs to be imposed to make (1.1) well-defined. Here X models the signal and Y the observation process. The *filtering problem* is to calculate π_t , the conditional distribution of X_t given $\mathcal{F}_t^Y := \sigma(Y_s: s \in [0, t])$, i.e. the observations up to time t , for each $t \geq 0$.

Starting in the mid-twentieth century, stochastic filtering has received an enormous amount of attention and has influenced many fields of mathematics – we refer to the introductory textbooks [LS01], [BC09], the historical overview in [Cri14] and the handbook [CR11]. Theoretically the filtering problem has been solved: $(\pi_t)_{t \geq 0}$ can be characterized as the unique solution to a

Both authors thank for the generous support of ETH Foundation and of SNF.

measure-valued stochastic differential equation (the Fujisaki-Kallianpur-Kunita or Kushner-Stratonovich equation). For applications, e.g., in mathematical finance [BH98] or geophysics [LSZ15], also a quick numerical calculation of π_t is quintessential – in fact for any application of a continuous-time stochastic model that features latent factors. It has been shown that apart from a few special cases, e.g. when h is affine and X is an Ornstein-Uhlenbeck process or when the state space D consists of finitely many points, the equation for $(\pi_t)_{t \geq 0}$ is truly infinite-dimensional. As a consequence, devising numerical methods to calculate π_t or even just the conditional mean $\mathbb{E}[X_t | \mathcal{F}_t^Y]$ is challenging. In most cases it is in-feasible due to computational constraints. Standard numerical methods ([BC09, Chapters 8-10]) either only work for low-dimensional state spaces or for approximately Gaussian setups.¹ However, post-crisis financial modeling asks for factor processes X which are both high-dimensional and not approximately Gaussian. The lack of numerical filtering methods for such processes has put serious limitations on the modeling flexibility: one has not been able to include latent factors in them.

In the present article, we fill this gap and show that the narrow class of processes for which an efficient numerical solution is possible (see above) also includes *affine processes*. More precisely, we consider the case when h is affine and the signal process X is an *affine process* with state space $D = \mathbb{R}_+^m \times \mathbb{R}^{d-m}$ as characterized in [DFS03]. This class of processes includes for example Lévy processes, Cox-Ingersoll-Ross processes [CIR85] or the Heston model [Hes93] and is very widely used in financial applications (see e.g. [DFS03], [KRM15, Section 3] for a list of references). The filtering problem arises naturally in this context; for example, X could model the short rate and Y the observed yields of bond prices as in [GP99], [CS03], see also [BH98].

Let us briefly summarize the key ideas of our approach. As a first step the distribution of X conditional on \mathcal{F}^Y is rewritten in terms of the pathwise filtering functional as studied by [Dav80], [Cla78]. Although the functional itself is not directly tractable, it can be approximated by a linearized version thereof. This new *linearized filtering functional* (LFF) is numerically tractable, since the Fourier coefficients can be calculated by solving a system of generalized Riccati equations with vector fields depending on the observation Y . This gives rise to Fourier filtering techniques, analogously to the Fourier pricing techniques used for affine (log-price) models, see e.g. [CM99] and [DFS03]. In addition the (approximate) conditional moments can be calculated by solving a system of ordinary differential equations. In contrast to existing numerical methods (e.g. a particle filter), this is very well-suited to parallel computations and thus promising for high-dimensional filtering.

There is also another equally fruitful viewpoint on this approach: the Zakai equation for the (un-normalized) distribution σ_t is a stochastic partial differential equation (SPDE) of the following form (under mild regularity conditions)

$$d\sigma_t(dx) = \mathcal{A}^* \sigma_t(dx) dt + \sum_{i=1}^d h_i(x) \sigma_t(dx) dY_t^i,$$

where \mathcal{A}^* denotes the adjoint of the generator of X . This equation, even though linear, has a quite complicated geometry: essentially only the Kalman filter, which corresponds to an Ornstein-Uhlenbeck process X and linear observation $h(x) = x$ allows for a finite dimensional realization, i.e. a way to write the SPDE's solution via solutions of finite dimensional stochastic differential equations. This is due to the fact that the geometrically relevant Stratonovich formulation of the equation has an additional term of type $h(x)^2 \sigma_t(dx)$ in the drift, which in turn causes the infinite dimensional analogon of hypo-ellipticity, whence no finite dimensional realizations can exist. The only way to cure this phenomenon in the relevant Brownian case is by replacing the Stratonovich correction by a linear expression, which is of course locally

¹In high-dimensional geophysical applications for example, only approximate Gaussian filters are routinely used (see the preface of [LSZ15]).

possible in a well controlled way. Then this *modified Zakai equation* has a completely different solution structure which can often be described by finite dimensional stochastic differential equations. In case of general affine processes X , even beyond the canonical setting used in this article, the modified Zakai equation under linear observation can be considered as an *affine SPDE* with time-dependent affine potential term (for this interpretation one necessarily needs the Stratonovich formulation), whose solution can be described by generalized stochastic Riccati equations. Notice also that the modification of the Zakai equation depends on the nature of the Stratonovich correction, in particular in case of finite variation noises the modification would vanish and we would actually have a solution theory for the classical Zakai equation by Fourier methods.

All of this is explained in detail in Section 3, while Section 2 provides background on affine processes and the filtering problem. The proofs of the statements on the LFF as well as local existence and uniqueness of solutions to the Riccati equations in Section 3 are then given in Section 4. They are based on a change of measure and comparison results for generalized Riccati equations. These are of independent interest and extend results from [KMK10] and [KRM15] to Riccati equations associated to non-conservative time-inhomogeneous affine “processes” that do not necessarily satisfy the admissibility conditions.

This theoretical analysis is complemented by a numerical study. In Section 5 the methodology is applied to the problem of filtering a Cox-Ingersoll-Ross (CIR) process. In numerical examples the filter induced by the linearized filtering functional, the *affine functional filter* (AFF), is compared to the benchmark (a bootstrap particle filter) and two standard Gaussian- and Gamma-approximation approaches (extended Kalman filter and [Bat06]). Not only is the AFF very close to the benchmark (and in particular more accurate than the two approximations), but it can also be calculated more efficiently than a particle filter. In examples in higher dimensions the situation turns out to be even more extreme: In Section 6 the methodology is applied to Wishart processes [Bru91], a matrix-valued extension of CIR processes (and a special case of affine processes taking values in S_d^+ , the set of symmetric positive semi-definite matrices). While in theory particle methods are applicable to this problem, in practice this requires enormous computational resources. Numerical experiments (already) for $d = 3$ show that in order to achieve the same level of accuracy (measured in terms of mean square error) as the AFF an outrageous number of particles would be necessary. Conversely, if one only uses a number of particles yielding similar computing times for the two methods, the mean-square error of a bootstrap particle filter is still by far larger than the error of the AFF. This makes the AFF the first numerically feasible method for filtering Wishart processes.

1.1. Notation. Fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which all random variables are defined.

Fix $p \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $d \in \mathbb{N}$ with $d \geq m$ and set $D = \mathbb{R}_+^m \times \mathbb{R}^{d-m}$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on \mathbb{R}^d and $|\cdot|$ the associated norm. Also write $\langle \cdot, \cdot \rangle$ for the linear extension of the inner product to $\mathbb{R}^d + i\mathbb{R}^d$, but without complex conjugation. Set

$$I := \{1, \dots, m\}, \quad J := \{m+1, \dots, d\}.$$

For $k \in \mathbb{N}$, write

$$\mathbb{C}_-^k = \{u \in \mathbb{C}^k : \operatorname{Re} u_i \leq 0, \forall i\}, \quad \mathbb{C}_{--}^k = \{u \in \mathbb{C}^k : \operatorname{Re} u_i < 0, \forall i\}$$

and define $\mathcal{U} = \mathbb{C}_-^m \times i\mathbb{R}^n$.

Denote by $B(D)$ and $C_b(D)$ the sets of bounded measurable functions and bounded continuous functions on D and by $\mathcal{P}(D)$ the set of probability measures on D . As usually, $\mathcal{P}(D)$ is equipped with the topology of weak convergence. Let $\mathcal{M}^+(D)$ denote the set of finite measures on the Borel σ -algebra $\mathcal{B}(D)$. Given $\mu \in \mathcal{M}^+(D)$ and a measurable, μ -integrable function f on D , write $\mu f := \int_D f(x) \mu(dx)$.

Fix a continuous truncation function $\chi: \mathbb{R}^d \rightarrow [-1, 1]^d$ with $\chi(\xi) = \xi$ in a neighborhood of 0 and bounded away from 0 outside that neighborhood. In fact, in order to be able to rely on a result from [KMK10] for $k = 1, \dots, d$ we choose

$$\chi_k(x) = \begin{cases} 0 & \text{if } x_k = 0, \\ (1 \wedge |x_k|) \frac{x_k}{|x_k|} & \text{otherwise.} \end{cases}$$

Let $\pi_0 \in \mathcal{P}(D)$, $D(\mathcal{L}) \subset C_b(D)$ and $\mathcal{L}: D(\mathcal{L}) \rightarrow C_b(D)$ linear. Recall that a D -valued stochastic process $(X_t)_{t \geq 0}$ defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is called a *solution to the martingale problem for $(D(\mathcal{L}), \mathcal{L}, \pi_0)$* , if $\tilde{\mathbb{P}} \circ X_0^{-1} = \pi_0$ and for each $h \in D(\mathcal{L})$, the process

$$h(X_t) - h(X_0) - \int_0^t \mathcal{L}h(X_s) ds, \quad t \geq 0,$$

is a martingale (in its own filtration). The martingale problem for $(D(\mathcal{L}), \mathcal{L}, \pi_0)$ is said to be *well-posed* if there exists a solution and any two solutions have the same finite-dimensional marginal distributions.

2. BACKGROUND: AFFINE PROCESSES AND THE FILTERING PROBLEM

2.1. Affine processes.

2.1.1. Definition and characterization. Let us review the definition of an affine process and some consequences thereof. We refer to [DFS03], [KRST11] and [CT13] for further details and references.

Consider a D -valued time-homogeneous Markov process $((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in D})$ defined on (Ω, \mathcal{F}) , see [RW00, Chapter III]. Denote by $(P_t)_{t \geq 0}$ the associated semigroup on $B(D)$ and assume $P_t 1 = 1$ for all $t \geq 0$ (i.e. the process is conservative). $((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in D})$ is called *affine*, if it is stochastically continuous, X has RCLL-paths (\mathbb{P}_x -a.s. for any $x \in D$) and there exist functions $\phi: \mathbb{R}_{\geq 0} \times \mathcal{U} \rightarrow \mathbb{C}$ and $\psi: \mathbb{R}_{\geq 0} \times \mathcal{U} \rightarrow \mathbb{C}^d$ such that for all $x \in D$, $(t, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$:

$$(2.1) \quad \mathbb{E}_x[e^{\langle X_t, u \rangle}] = \exp(\phi(t, u) + \langle x, \psi(t, u) \rangle).$$

Remark 2.1. As shown in [KRST11] this definition implies that for all $u \in \mathcal{U}$,

$$(2.2) \quad F(u) := \left. \frac{\partial \phi}{\partial t}(t, u) \right|_{t=0+}, \quad R(u) := \left. \frac{\partial \psi}{\partial t}(t, u) \right|_{t=0+}$$

exist and are continuous at $u = 0$. Thus, in the terminology of [DFS03] we are considering a conservative, regular affine process.

Remark 2.2. Alternatively, we could only assume that $((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in D})$ is conservative, stochastically continuous and (2.1) holds for $(t, u) \in \mathbb{R}_{\geq 0} \times i\mathbb{R}^d$. Then [KRST11] implies that it is a Feller process and in particular, we may choose an RCLL version of X on D (under \mathbb{P}_x , for any $x \in D$).² Finally [DFS03, Theorem 2.7] implies that (2.1) can be extended to $\mathbb{R}_{\geq 0} \times \mathcal{U}$.

Let us now review some key properties of affine processes. To formulate these, an additional definition is required: A collection of parameters

$$(2.3) \quad (a, \alpha, b, \beta, c, \gamma, \mu^0, \mu)$$

²Since the process is conservative, there is no need to consider the one-point compactification of D .

is called admissible, if it satisfies the following (admissibility) conditions:

$$(2.4) \quad a \in \text{Sem}^d \text{ with } a_{i,j} = 0 \text{ for } i, j \in I$$

$$(2.5) \quad \alpha = (\alpha^1, \dots, \alpha^m) \text{ with } \alpha^i \in \text{Sem}^d \text{ and } \alpha_{k,j}^i = 0 \text{ for } k, j \in I \setminus \{i\}$$

$$(2.6) \quad b \in \mathbb{R}^d \text{ with } b_i - \int_{D \setminus \{0\}} \chi_i(\xi) \mu^0(d\xi) \geq 0 \text{ for } i \in I$$

$$(2.7) \quad \beta \in \mathbb{R}^{d \times d} \text{ with } \beta_{i,j} - \int_{D \setminus \{0\}} \chi_i(\xi) \mu^j(d\xi) \geq 0 \text{ for } i, j \in I \text{ and } i \neq j$$

$$(2.8) \quad \beta_{i,k} = 0 \text{ for } i \in I, k \in J$$

$$(2.9) \quad c \in \mathbb{R}_+$$

$$(2.10) \quad \gamma \in \mathbb{R}_+^m$$

$$(2.11) \quad \mu = (\mu^1, \dots, \mu^m) \text{ and for } i \in I \cup \{0\}, \mu^i \text{ is a Borel measure on } D \setminus \{0\}$$

$$(2.12) \quad \int_{D \setminus \{0\}} \chi_k(\xi) \mu^i(d\xi) < \infty \text{ for } i \in I \cup \{0\}, k \in I \setminus \{i\}$$

$$(2.13) \quad \int_{D \setminus \{0\}} \chi_k(\xi)^2 \mu^i(d\xi) < \infty \text{ for } i \in I \cup \{0\}, k \in (J \cup \{i\}) \setminus \{0\}$$

Remark 2.3. The admissibility conditions are identical with [DFS03, Definition 2.6]. We have only changed notation slightly in order to match the semimartingale notation in [KMK10]. The measure m in [DFS03, Definition 2.6] is denoted μ^0 here, the truncation function is arbitrary (as in [Fil05]) and we denote by b, β the parameters $\tilde{b}, \tilde{\beta}$ from [DFS03, Theorem 2.12]. Our conditions (2.6), (2.7) for these are equivalent to conditions (2.6) and (2.7) in [DFS03, Definition 2.6]. This leads to different expressions below for (2.14) and (2.16) than in [DFS03], see also [DFS03, Remark 2.13].

Suppose $((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in D})$ is an affine process and denote again by $(P_t)_{t \geq 0}$ the restriction of the associated semigroup to $C_0(D)$. Then (see [DFS03, Theorem 2.7, Theorem 2.12 and Proposition 9.1]) there exists a collection of admissible parameters³ (2.3) with $c = 0$ and $\gamma = 0$ such that the following properties hold:

- F and R in (2.2) are given as

$$(2.14) \quad \begin{aligned} F(u) &= \frac{1}{2} \langle u, au \rangle + \langle b, u \rangle + \int_{D \setminus \{0\}} \left(e^{\langle \xi, u \rangle} - 1 - \langle \chi(\xi), u \rangle \right) \mu^0(d\xi) \\ R_i(u) &= \frac{1}{2} \langle u, \alpha^i u \rangle + \langle \beta^i, u \rangle + \int_{D \setminus \{0\}} \left(e^{\langle \xi, u \rangle} - 1 - \langle \chi(\xi), u \rangle \right) \mu^i(d\xi) \end{aligned}$$

for $i = 1, \dots, m$ and $R_i(u) = \langle \beta^i, u \rangle$ for $i = m + 1, \dots, d$. Here $\beta^i \in \mathbb{R}^d$ is defined via

$$\beta_j^i := \beta_{j,i}, \quad \text{for } 1 \leq i, j \leq d.$$

- ϕ and ψ solve the generalized Riccati equations

$$(2.15) \quad \begin{aligned} \partial_t \phi(t, u) &= F(\psi(t, u)), & \phi(0, u) &= 0 \\ \partial_t \psi(t, u) &= R(\psi(t, u)), & \psi(0, u) &= u \end{aligned}$$

for $t \geq 0, u \in \mathcal{U}$.

- $(P_t)_{t \geq 0}$ is a Feller semigroup (in the sense of [RY99, Chapter III]). Denote by $(D(\mathcal{A}), \mathcal{A})$ its infinitesimal generator. Then $C_c^\infty(D)$ is a core for \mathcal{A} , $C_0^2(D) \subset D(\mathcal{A})$ and for any

³Recall that we only consider conservative affine processes here.

$$\begin{aligned}
(2.16) \quad & f \in C_0^2(D), x \in D, \\
& \mathcal{A}f(x) = \frac{1}{2} \sum_{k,l=1}^d \alpha_{kl}(x) \frac{\partial^2 f(x)}{\partial x_k \partial x_l} + \langle \beta(x), \nabla f(x) \rangle \\
& \quad + \int_{D \setminus \{0\}} (f(x + \xi) - f(x) - \langle \chi(\xi), \nabla f(x) \rangle) K(x, d\xi)
\end{aligned}$$

where

$$\begin{aligned}
(2.17) \quad & \alpha(x) = a + \sum_{i=1}^m \alpha^i x_i \\
& \beta(x) = b + \sum_{i=1}^d \beta^i x_i \\
& K(x, d\xi) = \mu^0(d\xi) + \sum_{i=1}^m x_i \mu^i(d\xi).
\end{aligned}$$

- X is a semimartingale (under \mathbb{P}_x , for any $x \in D$) admitting characteristics (B, C, ν) with respect to χ given by

$$(2.18) \quad B_t = \int_0^t \beta(X_s) ds, \quad C_t = \int_0^t \alpha(X_s) ds, \quad \nu(dt, d\xi) = K(X_t, d\xi) dt,$$

where α, β, K are as in (2.17).

Finally, let us put (conservative) affine processes into the framework of [EK86]. This is the purpose of Lemma 2.4 below. It is very close to [DFS03, Lemma 10.2], but considers arbitrary initial laws and establishes uniqueness also within the class of solutions to the martingale problem which are not necessarily RCLL. This extension is required to establish uniqueness for evolution equations (as the Zakai equation in Theorem 2.9 below) associated to \mathcal{A} .

Lemma 2.4. *Fix a collection of admissible parameters (2.3) and define \mathcal{A}_0 as the restriction of \mathcal{A} (see (2.16)) to $C_c^\infty(D)$. Then for any $\pi_0 \in \mathcal{P}(D)$, the martingale problem for $(C_c^\infty(D), \mathcal{A}_0, \pi_0)$ is well-posed and the solution has RCLL-sample paths.*

Proof. The statement of [DFS03, Theorem 2.7] that X is a Feller process means that $(P_t)_{t \geq 0}$ is a strongly continuous, positive contraction semigroup on $C_0(D)$ in the terminology of [EK86]. Furthermore, by [EK86, Chap.4, Cor. 2.8] and since X is conservative, $(D(\mathcal{A}), \mathcal{A})$ is conservative (in the terminology of [EK86]). Thus $(P_t)_{t \geq 0}$ is a Feller semigroup (on $C_0(D)$) also in the terminology of [EK86]. Set $D(\mathcal{A}_0) = C_c^\infty(D)$. By [DFS03, Theorem 2.7], $D(\mathcal{A}_0)$ is a core for $(D(\mathcal{A}), \mathcal{A})$ and so the closure of the operator $(D(\mathcal{A}_0), \mathcal{A}_0)$ is again $(D(\mathcal{A}), \mathcal{A})$. Combining [EK86, Chap.4, Thm. 2.2, 2.7 and 4.1] then yields the statement. \square

In view of Lemma 2.4 the following terminology is sensible: Fix $\pi_0 \in \mathcal{P}(D)$. We call an RCLL stochastic process X on $(\Omega, \mathcal{F}, \mathbb{P})$ an affine process started from π_0 , if it is a solution to the martingale problem for $(C_c^\infty(D), \mathcal{A}_0, \pi_0)$. By Lemma 2.4 this uniquely determines the law of X under \mathbb{P} .

2.1.2. Exponential moments of affine processes. For the analysis of this article, it will be necessary to extend (2.1) to $U_0 \subset \mathbb{R}_{\geq 0} \times \mathbb{C}^d$, where U_0 is open and $0 \in U_0$. This means that an assumption on exponential moments is required. Suppose that

$$(2.19) \quad \int_{D \setminus \{|z| \leq 1\}} |z| e^{\langle z, u \rangle} \mu^i(dz) < \infty \quad \text{for all } i = 0, \dots, m \text{ and } u \in \mathbb{R}^d.$$

Suppose $((X_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in D})$ is an affine process and define

$$(2.20) \quad E = \{(t, u) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^d : \mathbb{E}_x[e^{\langle X_t, u \rangle}] < \infty \text{ for all } x \in D\}.$$

By definition, this is the maximal domain on which the left hand side of (2.1) is finite. Under assumption (2.19), E is open, $0 \in E$ and ϕ and ψ can be extended to E . This is summarized in the next Lemma, which directly follows from [KRM15] and [FM09]. See also [SV10] and further references in all these articles.

Lemma 2.5. *Suppose (2.19) holds. Then*

- (i) E is open in $\mathbb{R}_{\geq 0} \times \mathbb{R}^d$,
- (ii) for any $(T, u) \in \mathbb{R}_{\geq 0} \times \mathbb{C}^d$ with $(T, \operatorname{Re} u) \in E$, there exists a unique solution to (2.15) on $[0, T]$ and (2.1) holds.

Proof. By [DFS03, Lemma 5.3] and (2.19), F and R are analytic functions. Therefore the same reasoning as in the proof of [FM09, Lemma 2.3] shows that for any $u \in \mathbb{C}^d$, there exists $t_+(u) \in (0, \infty]$ such that (2.15) has a unique solution on $[0, t_+(u))$ and the set

$$D_{\mathbb{R}} := \{(t, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^d : t < t_+(y)\}$$

is open in $\mathbb{R}_{\geq 0} \times \mathbb{R}^d$. Furthermore, by [KRM15, Theorem 2.14(b)], [KRM15, Theorem 2.17(b)] and (2.19), one has $D_{\mathbb{R}} \subset E$ and (2.1) holds for all $(t, u) \in D_{\mathbb{R}}$, $x \in D$. [KRM15, Theorem 2.14(a)] implies $E \subset D_{\mathbb{R}}$ and hence $E = D_{\mathbb{R}}$. This shows (i). $(T, \operatorname{Re} u) \in E$ yields $(T, \operatorname{Re} u) \in D_{\mathbb{R}}$ and so [KRM15, Theorem 2.26] implies (ii). \square

A further consequence of (2.19) is the following:

Lemma 2.6. *Assume (2.19). Then for any $T \geq 0$, $k \in \mathbb{N}$, $x \in D$*

$$(2.21) \quad \mathbb{E}_x[|X_T|^{2k}] < \infty,$$

$$(2.22) \quad \mathbb{E}_x \left[\int_0^T |X_t|^{2k} dt \right] < \infty.$$

Proof. By [DFS03, Lemma 5.3] and (2.19), F and R are analytic functions on \mathbb{C}^d . Thus by [DFS03, Lemma 6.5(i)], ϕ and ψ are in $C^\infty(\mathbb{R}_+ \times \mathcal{U})$. Combining this with $i\mathbb{R}^d \subset \mathcal{U}$ and [DFS03, Theorem 2.16(i)] yields (2.21). By [DFS03, Lemma A.1], for any $t \in [0, T]$, $\mathbb{E}_x[|X_t|^{2k}]$ is a sum of partial derivatives (up to order k) of $\psi(t, \cdot)$ and $\phi(t, \cdot)$ at 0. But all of these are continuous (as argued above) and so $t \mapsto \mathbb{E}_x[|X_t|^{2k}]$ is bounded on $[0, T]$. Hence (2.22) follows. \square

2.1.3. Time-inhomogeneous affine processes. As it turns out, linear filtering of an affine process gives rise to a time-inhomogeneous affine process. This class of time-inhomogeneous Markov processes has been studied in [Fil05]. Similar to the time-homogeneous case (as summarized in Section 2.1.1), [Fil05] has obtained characterizations in terms of a martingale problem or (for conservative processes) semimartingale characteristics. We do not repeat these here; for our purposes it is sufficient to understand the conditions on the parameters that are necessary and sufficient for the existence of such a process. For more details we refer to [Fil05].

A collection of parameters (depending on $t \geq 0$)

$$(2.23) \quad (a(t), \alpha(t), b(t), \beta(t), c(t), \gamma(t), \mu^0(t), \mu(t))$$

is called admissible (or strongly admissible), if the following (admissibility) conditions are satisfied:

- for any $t \geq 0$, (2.23) satisfies conditions (2.4)-(2.13),
- $(a(t), \alpha(t), b(t), \beta(t), c(t), \gamma(t))$ are continuous in $t \in \mathbb{R}_+$,
- the measures $\chi_k(\cdot) \mu^i(t, \cdot)$ (on $D \setminus \{0\}$) are weakly continuous in $t \in \mathbb{R}_+$ for any $i \in I \cup \{0\}$, $k \in I \setminus \{i\}$,
- the measures $\chi_k(\cdot)^2 \mu^i(t, \cdot)$ (on $D \setminus \{0\}$) are weakly continuous in $t \in \mathbb{R}_+$ for any $i \in I \cup \{0\}$, $k \in (J \cup \{i\}) \setminus \{0\}$.

Remark 2.7. As before, b and β here denote $\tilde{b}, \tilde{\beta}$ in [Fil05, Theorem 2.13]. Since χ_k is bounded and continuous, the third continuity condition guarantees that $\tilde{b}, \tilde{\beta}$ in [Fil05, Theorem 2.13] are continuous if and only if b and β in [Fil05, Definition 2.5] are continuous. Together with Remark 2.3 this implies that the present admissibility conditions are identical with [Fil05, Definition 2.5].

Remark 2.8. If $c(t) = 0, \gamma(t) = 0$ for all $t \geq 0$, then the admissibility condition here is equivalent to [KMK10, Definition 2.4].

By [Fil05, Theorem 2.13, Lemma 3.1 and Proposition 4.3] for any collection of parameters satisfying these conditions (and only under these), there exists a strongly regular time-inhomogeneous affine process $(\bar{X}, (\mathbb{P}_{(r,x)})_{(r,x) \in \mathbb{R}_+ \times D})$ (a time-inhomogeneous, stochastically continuous Markov process with an additional regularity condition as (2.2), see [Fil05]) with transition function $(P_{t,T})$ satisfying for any $u \in \mathcal{U}, 0 \leq t \leq T$,

$$(2.24) \quad P_{t,T} \exp(\langle u, \cdot \rangle)(x) = \exp(\Phi(t, T, u) + \langle x, \Psi(t, T, u) \rangle), \quad \forall x \in D,$$

where Φ and Ψ solve the generalized Riccati equations

$$(2.25) \quad \begin{aligned} -\partial_t \Phi(t, T, u) &= F(t, \Psi(t, T, u)), & \Phi(T, T, u) &= 0 \\ \partial_t \Psi(t, T, u) &= R(t, \Psi(t, T, u)), & \Psi(T, T, u) &= u, \quad 0 \leq t \leq T \end{aligned}$$

with vector fields

$$(2.26) \quad \begin{aligned} F(t, u) &= \frac{1}{2} \langle u, a(t)u \rangle + \langle b(t), u \rangle - c(t) + \int_{D \setminus \{0\}} \left(e^{\langle \xi, u \rangle} - 1 - \langle \chi(\xi), u \rangle \right) \mu^0(t, d\xi) \\ R_i(t, u) &= \frac{1}{2} \langle u, \alpha^i(t)u \rangle + \langle \beta^i(t), u \rangle - \gamma_i(t) \\ &\quad + \int_{D \setminus \{0\}} \left(e^{\langle \xi, u \rangle} - 1 - \langle \chi(\xi), u \rangle \right) \mu^i(t, d\xi), \quad i = 1, \dots, m, \\ R_i(t, u) &= \langle \beta^i(t), u \rangle, \quad i = m + 1, \dots, d, \end{aligned}$$

where $\beta_j^i(t) := \beta_{j,i}(t)$.

Finally, fix $(r, x) \in \mathbb{R}_+ \times D$. As noted in [Fil05] one may assume that \bar{X} has RCLL paths, $\mathbb{P}_{(r,x)}$ -a.s. and so the following terminology makes sense: Suppose Y is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ with RCLL paths. We will say that (under \mathbb{P}) Y is a time-inhomogeneous affine process started in (r, x) with admissible parameters (2.23), if the law of Y under \mathbb{P} (on the space of RCLL-paths) is identical to the law of \bar{X} under $\mathbb{P}_{(r,x)}$.

2.2. The filtering problem.

2.2.1. Problem formulation and the Zakai equation. Fix $\pi_0 \in \mathcal{P}(D)$ and suppose X is an affine process started from π_0 (see Section 2.1.1) on $(\Omega, \mathcal{F}, \mathbb{P})$. Further, suppose \mathbb{F} is a right-continuous filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to which X is adapted and such that \mathcal{F}_0 contains all \mathbb{P} -nullsets.

Let us introduce the problem of filtering X given noisy observations Y , as in the standard setup, see [LS01] and [BC09]. The exposition here follows [KO88].

Define Y as

$$(2.27) \quad Y_t = \int_0^t h(X_s) ds + W_t, \quad t \geq 0,$$

where W is a p -dimensional \mathbb{F} -Brownian motion independent of X , $h: D \rightarrow \mathbb{R}^p$ is measurable and

$$(2.28) \quad \mathbb{E} \left[\int_0^T |h(X_s)|^2 ds \right] < \infty,$$

for all $T \geq 0$. Set

$$(2.29) \quad \mathcal{F}_t^Y = \sigma(Y_s : 0 \leq s \leq t) \vee \mathcal{N}, \quad t \geq 0,$$

where \mathcal{N} denotes the collection of \mathbb{P} -nullsets of (Ω, \mathcal{F}) .

The goal of filtering theory is to calculate, for $t \in [0, \infty)$, the conditional distribution of X_t given \mathcal{F}_t^Y . Formally this is described by a measure-valued process as follows: By [BC09, Theorem 2.1] there exists a $\mathcal{P}(D)$ -valued $(\mathcal{F}_t^Y)_{t \geq 0}$ -adapted, RCLL-process $(\pi_t)_{t \geq 0}$ such that for any $f \in B(D)$, $t \geq 0$,

$$\pi_t f = \mathbb{E}[f(X_t) | \mathcal{F}_t^Y] \quad \mathbb{P}\text{-a.s.}$$

It can be shown that π satisfies the *Kushner-Stratonovich equation*. This is a stochastic partial differential equation for the process π , usually written in weak form, i.e. applied to test functions $f \in D(\mathcal{A})$.

Alternatively, one may consider an $\mathcal{M}^+(D)$ -valued (but not $\mathcal{P}(D)$ -valued) process, which leads to the linear Duncan-Mortensen-Zakai equation or shortly *Zakai equation*: Define

$$(2.30) \quad \sigma_t := \exp \left(\int_0^t (\pi_s h)^\top dY_s - \frac{1}{2} \int_0^t |\pi_s h|^2 ds \right) \pi_t$$

which is nonzero \mathbb{P} -a.s., for any $t \geq 0$, because

$$(2.31) \quad \mathbb{E} \left[\int_0^t |\pi_s h|^2 ds \right] < \infty,$$

as can be deduced from (2.28).

We are now concerned with the filtering problem on the time interval $[0, T]$, for some $T > 0$ fixed. By (2.28) and independence,

$$(2.32) \quad \frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left(- \int_0^T h(X_s)^\top dW_s - \frac{1}{2} \int_0^T |h(X_s)|^2 ds \right)$$

defines a new probability measure \mathbb{Q} on (Ω, \mathcal{F}) that is equivalent to \mathbb{P} on \mathcal{F}_T .⁴ Furthermore, the law of X under \mathbb{P} is the same as under \mathbb{Q} and, on $[0, T]$ under the measure \mathbb{Q} , Y is a Brownian motion independent of X .

It can be shown (see [BC09, Exercise 3.37]) that $\sigma_t 1$ defined in (2.30) is equal to $\mathbb{E}_{\mathbb{Q}}[\frac{d\mathbb{P}}{d\mathbb{Q}} | \mathcal{F}_t^Y]$. Combining this with the abstract Bayes' rule and the definition (2.30), one obtains (see [BC09, Proposition 3.16]) that for any $t \in [0, T]$, $f \in B(D)$,

$$(2.33) \quad \sigma_t f = \mathbb{E}_{\mathbb{Q}} \left[f(X_t) \exp \left(\int_0^t h(X_s)^\top dY_s - \frac{1}{2} \int_0^t |h(X_s)|^2 ds \right) \middle| \mathcal{F}_t^Y \right],$$

\mathbb{P} -a.s., and the *Kallianpur-Striebel* formula

$$(2.34) \quad \pi_t f = \frac{\sigma_t f}{\sigma_t 1}.$$

Furthermore, σ satisfies the *Zakai equation*

$$(2.35) \quad \sigma_t f = \pi_0 f + \int_0^t \sigma_s(\mathcal{A}f) ds + \int_0^t \sigma_s(hf) dY_s \quad \text{for any } f \in D(\mathcal{A}).$$

By (2.31) and (2.30), h is σ_t -integrable for all $t \leq T$ and $\int_0^T |\sigma_t h|^2 ds < \infty$, \mathbb{P} -a.s. Hence all terms in (2.35) are indeed well-defined.

⁴See [LS01, Example I.6.2.4]. Independence is crucial here, otherwise a Novikov' type assumption would be needed.

2.2.2. *Uniqueness for the Zakai equation.* The following result is a consequence of [KO88, Theorem 4.2]:

Theorem 2.9 (Well-posedness of the Zakai equation). *Let $h \in C(D)$, \mathcal{A} the generator of a (conservative) affine process (see (2.16)) and σ as in (2.35). Assume (2.28).*

Suppose $(\rho_t)_{t \in [0, T]}$ is an $(\mathcal{F}_t^Y)_{t \in [0, T]}$ -adapted RCLL $\mathcal{M}^+(D)$ -valued process such that h is ρ_t -integrable for all $t \leq T$, $\int_0^T |\rho_t h|^2 ds < \infty$, \mathbb{P} -a.s. and satisfying

$$(2.36) \quad \rho_t f = \pi_0 f + \int_0^t \rho_s (\mathcal{A}f) ds + \int_0^t \rho_s (hf) dY_s, \quad \text{for any } f \in C_c^\infty(D)$$

and for $f = 1$ (with $\mathcal{A}1 := 0$). Then $\rho_t = \sigma_t$ for all $t < T$, \mathbb{P} -a.s.

Proof. Define $D(\mathcal{A}_0) := C_c^\infty(D)$ and \mathcal{A}_0 the restriction of \mathcal{A} to $D(\mathcal{A}_0)$. Then by Lemma 2.4, for any $\pi_0 \in \mathcal{P}(D)$, the martingale problem for $(D(\mathcal{A}_0), \mathcal{A}_0, \pi_0)$ is well-posed. Furthermore, for any $f \in D(\mathcal{A}_0)$, $h_i f \in C_0(D)$ for $i = 1, \dots, p$ and so the assumptions of [KO88, Theorem 4.2] are indeed satisfied. \square

Since the assumptions for [KO88, Theorem 4.1] are the same as for [KO88, Theorem 4.2], as a corollary one also obtains a uniqueness result for the Kushner-Stratonovich equation.

Let us point out that Theorem 2.9 holds in the setting considered in Section 3. Taking $h(x) = x$, $\pi_0 = \delta_x$ for some $x \in D$ and assuming that the jump-measures of the affine process satisfy (2.19), one obtains from Lemma 2.6 that (2.28) is indeed satisfied.

2.2.3. *Robust filtering.* Thanks to the uniqueness result for the Zakai equation in Theorem 2.9, theoretically the filtering problem is settled: One finds a solution to the Zakai equation and uses the Kallianpur-Striebel formula (2.34) to calculate the filter. However, in practice one is given a fixed $y \in C([0, T], \mathbb{R}^p)$ (of *finite variation*), whereas (2.33) only specifies the filter \mathbb{P} -a.s. Thus a definition of (2.33) for *all* $y \in C([0, T], \mathbb{R}^p)$ is needed.

Let us briefly review the main result of [Dav80]. See [BC09, Chapter 5] and [vH07, Section 1.4] for further references on robust filtering. Suppose $h \in D(\mathcal{A})$ so that $h(X)$ is a semimartingale. Since X and Y are independent, one can integrate by parts

$$(2.37) \quad \int_0^t h(X_s)^\top dY_s = Y_t^\top h(X_t) - \int_0^t Y_s^\top dh(X_s)$$

and rewrite σ in (2.33) as

$$\sigma_t f = \mathbb{E}_{\mathbb{Q}} \left[f(X_t) \exp \left(Y_t^\top h(X_t) - \int_0^t Y_s^\top dh(X_s) - \frac{1}{2} \int_0^t |h(X_s)|^2 ds \right) \middle| \mathcal{F}_t^Y \right].$$

Recalling that X and Y are independent under \mathbb{Q} and X has the same distribution under \mathbb{P} as under \mathbb{Q} , the conditional expectation is actually given as $\mathbb{E}[F(X, y)]|_{y=Y}$ for a suitable function $F: D \times \mathbb{R}^p \rightarrow \mathbb{R}$. In fact, the following *robustness* property has been established in [Cla78], [CC05]: Define the *pathwise filtering functional* $\sigma_t: B(D) \times C([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}$ by

$$(2.38) \quad \sigma_t(f, y) = \mathbb{E} \left[f(X_t) \exp \left(y_t^\top h(X_t) - \int_0^t y_s^\top dh(X_s) - \frac{1}{2} \int_0^t |h(X_s)|^2 ds \right) \right],$$

then $\sigma_t(f, \cdot)/\sigma_t(1, \cdot)$ is locally Lipschitz continuous and

$$(2.39) \quad \frac{\sigma_t(f, Y)}{\sigma_t(1, Y)} = \mathbb{E}[f(X_t) | \mathcal{F}_t^Y], \quad \mathbb{P}\text{-a.s.}$$

See also [CDFO13] for an extension to multidimensional observation and correlated noise.

In [Dav80] the following observation is made: Fix $y \in C[0, T]$ and define a two-parameter semigroup of operators on $B(D)$ by

$$(2.40) \quad T_{s,t}^y f(x) = \mathbb{E}_x \left[f(X_{t-s}) \exp \left(- \int_s^t y_u^\top dh(X_{u-s}) - \frac{1}{2} \int_s^t |h(X_{u-s})|^2 du \right) \right],$$

for $t \geq s \geq 0$, $x \in D$. Then

$$(2.41) \quad \sigma_t(f, y) = \int_D T_{0,t}^y(e^{y(t)h} f)(x) \pi_0(dx)$$

and, this is the main result of [Dav80], the (extended) generator \mathcal{A}_t^y of the semigroup $T_{s,t}^y$ is given by

$$\mathcal{A}_t^y f = e^{y(t)h} \left(\mathcal{A} - \frac{1}{2} h^2 \right) (e^{-y(t)h} f).$$

This is closely related to applying a Doss-Sussmann method (see e.g. [RW00, Theorem 28.2]) to the Zakai equation, as explained in [Dav11].

3. THE LINEARIZED FILTERING FUNCTIONAL

In this section we introduce and study a computationally tractable approximation of the pathwise filtering functional (2.38) when both X and h are affine. Throughout this section X is an affine process on $(\Omega, \mathcal{F}, \mathbb{P})$ started from $\pi_0 \in \mathcal{P}(D)$ with admissible parameters (2.3) and F, R are as in (2.14). If $\pi_0 = \delta_x$ for $x \in D$, we write \mathbb{P}_x for \mathbb{P} .

3.1. Definition and main results.

3.1.1. *Definition of the approximate filter.* Fix an observation $y \in C([0, \infty), \mathbb{R}^d)$ with $y(0) = 0$ and functions $\gamma \in C([0, \infty), \mathbb{R}^d)$, $c \in C([0, \infty), \mathbb{R})$. The *linearized filtering functional* (LFF) ρ is defined as

$$(3.1) \quad \rho_t(f, y) = \mathbb{E} \left[f(X_t) \exp \left(y_t^\top X_t - \int_0^t y_s^\top dX_s - \int_0^t \gamma_s^\top X_s - c_s ds \right) \right]$$

for any $t \geq 0$ and $f: D \rightarrow \mathbb{R}$ measurable such that the right hand side of (3.1) is well-defined (e.g. $f \geq 0$). If $\rho_t(1, y)$ is finite, define the approximate pathwise filter (the *affine functional filter* or AFF) by

$$(3.2) \quad \bar{\pi}_t(f, y) = \frac{\rho_t(f, y)}{\rho_t(1, y)}.$$

If $\pi_0 = \delta_x$ for $x \in D$, we write $\rho_t^x(f, y)$ for $\rho_t(f, y)$ and $\bar{\pi}_t^x(f, y)$ for $\bar{\pi}_t(f, y)$.

3.1.2. *Heuristic motivation.* The linearized filtering functional (3.1) is the same as the pathwise filtering functional (2.38) for $h(x) = x$, but with $\frac{1}{2}|x|^2$ approximated by the affine function $\gamma_s x + c_s$. The motivation for studying ρ_t is the following: if for some $t > 0$, $x_0 \in D$ and (small) $\varepsilon > 0$, $\mathbb{P}(\{X_s \in B_\varepsilon(x_0) \forall s \in [0, t]\})$ is almost 1, then (3.1) and (2.38) (with $\gamma_s = x_0$ and $c_s = \frac{x_0^2}{2}$) are very close. Consequently, (2.39) implies that also the approximate filter $\bar{\pi}_t(f, Y)$ should be close to $\pi_t(f)$.

3.1.3. *Fourier filtering.* The key point is that (3.1) is computationally tractable, since one can calculate the Fourier coefficients of (3.1) by solving a system of generalized Riccati equations:

Theorem 3.1. *Assume (2.19) holds. Let $u \in \mathbb{C}^d$ and $T \in \mathbb{R}_+$. Suppose $\Phi \in C^1([0, T], \mathbb{R})$ and $\Psi \in C^1([0, T], \mathbb{R}^d)$ solve*

$$(3.3) \quad \begin{aligned} -\partial_t \Phi(t, T, u) &= F(\Psi(t, T, u) - y_t) - c_t, & \Phi(T, T, u) &= 0 \\ -\partial_t \Psi(t, T, u) &= R(\Psi(t, T, u) - y_t) - \gamma_t, & \Psi(T, T, u) &= u + y_T, \quad 0 \leq t \leq T. \end{aligned}$$

Then for any $x \in D$, the Fourier coefficient of $\rho_T^x(\cdot, y)$ is well-defined and given as

$$(3.4) \quad \rho_T^x(\exp(\langle u, \cdot \rangle), y) = \exp(\Phi(0, T, u) + \langle x, \Psi(0, T, u) \rangle).$$

Furthermore, there exists $T_0 > 0$ such that for all $u \in i\mathbb{R}^d$ and $T \leq T_0$, the system (3.3) has a unique solution on $[0, T]$.

The proof of Theorem 3.1 is postponed to Section 4.3 below. Let us briefly discuss how to use Theorem 3.1 in practice, relate it to the literature and discuss its assumptions.

Remark 3.2. Suppose $f: D \rightarrow \mathbb{C}$ is given as

$$f(y) = \int_{\mathbb{R}^d} e^{i\langle v, y \rangle} \hat{f}(v) \, dv, \quad y \in D$$

for some $\hat{f}: \mathbb{R}^d \rightarrow \mathbb{C}$ integrable. Then for any $T > 0$ small enough, by Theorem 3.1, definition (3.1) and Fubini's theorem

$$\rho_T^x(f, y) = \int_{\mathbb{R}^d} \rho_T^x(\exp(\langle iv, \cdot \rangle), y) \hat{f}(v) \, dv = \int_{\mathbb{R}^d} e^{\Phi(0, T, iv) + \langle x, \Psi(0, T, iv) \rangle} \hat{f}(v) \, dv.$$

This is analogous to the Fourier method used in option pricing in the framework of affine models.

Remark 3.3. Expressions of type (3.4) are called affine transform formulas in the literature, see e.g. [KRM15] and the references therein. Note that the present result is not covered in the literature, since the Riccati equations (3.3) are time-inhomogeneous and correspond to a non-conservative affine “process” for which the admissibility conditions (2.9) and (2.10) are not necessarily satisfied.

Remark 3.4. In general, it does not hold that $\rho_T(1, y) < \infty$ for all $T > 0$ and so the statement of Theorem 3.1 really just holds up to a finite T_0 (depending on y). To see this, let $u \in \mathbb{R}^d$ and consider $y_s = us$, $c_s = \gamma_s = 0$ for all $s \geq 0$. Then the product rule (as in (2.37)) and $\dot{y}_s = u$ show

$$\rho_T^x(1, y) = \mathbb{E}_x \left[\exp \left(u^\top \int_0^T X_s \, ds \right) \right]$$

which is not necessarily finite. For example, if $d = m = 1$ and X is a CIR process (see Section 5) with parameters $\beta > 0$, $b \geq 0$ and $\sigma > 0$, then for $u < \beta^2/(2\sigma^2)$ and T large enough (satisfying $\tanh(\gamma T/2) \geq \gamma/\beta$ with $\gamma = \sqrt{\beta^2 - 2\sigma^2 u}$) the expectation is not finite, see [FKR10] or [Duf01].

Finally, note that (2.19) could be weakened to the following assumption: there exists $V \subset \mathbb{R}^d$ open with $0 \in V$ such that (2.19) holds for $u \in V$ (instead of all $u \in \mathbb{R}^d$).

3.1.4. The smoothing distribution. Our approximation (3.1) and (3.2) also gives rise to an approximation of the smoothing distribution, i.e. the distribution of $X_{[0, t]}$ conditional on \mathcal{F}_t^Y .

Fix $t > 0$ and denote by $D[0, t]$ the set of RCLL-mappings $[0, t] \rightarrow D$. Consider $G: D[0, t] \rightarrow \mathbb{R}$ bounded, measurable⁵ and, analogously to (3.1) and (3.2) define

$$(3.5) \quad \begin{aligned} \rho_t(G, y) &= \mathbb{E} \left[G(X_{[0, t]}) \exp \left(y_t^\top X_t - \int_0^t y_s^\top dX_s - \int_0^t \gamma_s^\top X_s - c_s \, ds \right) \right] \\ \bar{\pi}_t(G, y) &= \frac{\rho_t(G, y)}{\rho_t(1, y)} \end{aligned}$$

for any $t \geq 0$ such that $\rho_t(1, y) < \infty$. Then $\bar{\pi}_t(\cdot, y)$ is a probability measure on $D[0, t]$ and an approximation to the smoothing distribution. Again, if $\pi_0 = \delta_x$ for $x \in D$, we write $\rho_t^x(G, y)$ for $\rho_t(G, y)$ and $\bar{\pi}_t^x(G, y)$ for $\bar{\pi}_t(G, y)$.

The following result shows that $\bar{\pi}_t(\cdot, y)$ coincides with the the distribution on $D[0, t]$ of a time-inhomogeneous affine process. It will be used for the calculation of (approximate) conditional moments in Section 5 and 6 below. To formulate it, define

$$\widehat{\pi}_0(z) = \int_D e^{\langle x, z \rangle} \pi_0(dx), \quad \text{for } z \in D_{\pi_0} = \{z \in \mathbb{C}^d : |\exp(\langle \cdot, z \rangle)| \in L^1(D, \pi_0)\}.$$

⁵More precisely, for $s \in [0, T]$ define $Y_s: D[0, t] \rightarrow \mathbb{R}$ by $Y_s(\omega) := \omega(s)$ and equip $D[0, t]$ with the σ -algebra generated by $(Y_s)_{s \in [0, t]}$.

Theorem 3.5. *Let $T_0 > 0$, $t \in (0, T_0]$ and $\Psi(\cdot, t, 0)$ as in Theorem 3.1. Suppose $\Psi(0, t, 0) \in D_{\pi_0}$. Then for any $G \in B(D[0, t])$,*

$$\bar{\pi}_t(G, y) = \frac{\int_D e^{\langle x, \Psi(0, t, 0) \rangle} \mathbb{E}_{\mathbb{Q}_x^{y, t}}[G(X_{[0, t]})] \pi_0(dx)}{\widehat{\pi}_0(\Psi(0, t, 0))},$$

where under $\mathbb{Q}_x^{y, t}$, X is a time-inhomogeneous affine process started from $(0, x)$ with admissible parameters

$$(a(s), \alpha(s), b(s), \beta(s), 0, 0, \mu^0(s), \mu(s))_{s \geq 0}$$

defined by (4.1) below with $g(s) := \Psi(s \wedge t, t, 0) - y_{s \wedge t}$ for $s \geq 0$.

Remark 3.6. If $\pi_0 = \delta_x$ for $x \in D$, then $D_{\pi_0} = \mathbb{C}^d$, $\widehat{\pi}_0(z) = e^{\langle x, z \rangle}$ and so Theorem 3.5 implies

$$(3.6) \quad \bar{\pi}_t^x(G, y) = \mathbb{E}_{\mathbb{Q}_x^{y, t}}[G(X_{[0, t]})].$$

Remark 3.7. As a simple example, consider a CIR process (see Section 5) started in $x > 0$. Theorem 3.5 implies that for $t \leq T_0$ the approximate smoothing distribution is given by (3.6). Under $\mathbb{Q}_x^{y, t}$ the process X is the unique solution to

$$(3.7) \quad dX_s = b + \beta X_s + u(s, X_s) ds + \sigma \sqrt{X_s} dB_s, \quad X_0 = x,$$

where $u(s, x) := \sigma^2(\Psi_{s \wedge t} - y_{s \wedge t})x$, $\Psi_s := \Psi(s, t, 0)$ solves (the second part of) (3.3) and B is a Brownian motion under $\mathbb{Q}_x^{y, t}$. Thus, the approximate smoothing distribution is the distribution (on path space) of a new process, which is obtained by inserting the additional drift term $u(s, X_s)$ in the original SDE (5.1).

From [vH07, Chapters 1.4.3 and 4.2] one obtains formally a representation analogous to (3.6) for the exact smoothing distribution, the only difference being the choice of u in (3.7). However, calculating the function u in this case requires solving a PDE. For the approximate filter u can be obtained by solving an ODE, which is an enormous reduction of complexity.

3.1.5. An alternative point of view. To clarify further the relation to [Dav80], let $x_0 \in \mathbb{R}_{++}^m \times \mathbb{R}^n$, $c_0 > 0$ and define $H(x) = (x_0)_I^\top x_I + c_0$ and

$$(3.8) \quad \bar{T}_t^y f(x) = \mathbb{E}_x \left[f(X_t) \exp \left(- \int_0^t y_u^\top dX_u - \int_0^t H(X_u) du \right) \right],$$

so that, in analogy to (2.41) it holds that (with $\gamma^\top = ((x_0)_I^\top, 0)$ and $c = c_0$ in (3.1))

$$\rho_t^x(f, y) = \bar{T}_t^y(\exp(\langle y_t, \cdot \rangle) f)(x).$$

Thus we have approximated $T_{0, t}^y$ in (2.40) by \bar{T}_t^y . Now if $\beta^i = 0$ for $i \in J$, then (3.8) corresponds to a non-conservative, time-inhomogeneous affine process:

Proposition 3.8. *Assume (2.19) holds and $\beta^i = 0$ for $i \in J$. Then there exists $c_0 > 0, T > 0$ such that for all $t \in [0, T]$, \bar{T}_t^y in (3.8) satisfies $\bar{T}_t^y = P_{0, t}^y$, where $(P_{s, t}^y)$ is the transition semigroup of a time-inhomogeneous affine process with (admissible) parameters (2.23) defined for all $t \geq 0$ by (4.1) below with $g(t) = -y_{t \wedge T}$ and*

$$(3.9) \quad \begin{aligned} c(t) &= c_0 - F(-y_t) \\ \gamma^i(t) &= x_0^i - R_i(-y_t), \quad i \in I. \end{aligned}$$

3.1.6. Discussion.

Remark 3.9. The ordinary differential equation (3.3) is formulated backwards in time, which appears to lead to a non-recursive filter. This can easily be resolved and we now explain how a recursive procedure can be obtained: Fix $T_0 > 0$ sufficiently small. Theorem 3.1 guarantees that for any $v \in \mathbb{R}^d$ and $T \leq T_0$ there exists a unique $u_0 \in \mathbb{C}^d$ such that the ODE

$$(3.10) \quad \begin{aligned} -\partial_t \bar{\Psi}(t, u_0) &= R(\bar{\Psi}(t, u_0) - y_t) - \gamma_t \\ \bar{\Psi}(0, u_0) &= u_0 \end{aligned}$$

has a unique solution on $[0, T]$ with $\bar{\Psi}(T, u_0) = iv + y_T$. More specifically, one chooses $u_0 := \Psi(0, T, iv)$ and $\bar{\Psi}(t, u_0) := \Psi(t, T, iv)$. This gives the following recursive procedure to calculate the approximate filter at time $T \leq T_0$:

- solve for all $u_0 \in \mathbb{C}^d$ (for which a solution exists) the ODE (3.10) up to time T .
- for $v \in \mathbb{R}^d$, find the unique solution $u_0 \in \mathbb{C}^d$ to $\bar{\Psi}(T, u_0) = iv + y_T$ and evaluate

$$\rho_T^x(\exp(\langle iv, \cdot \rangle), y) = \exp\left(\int_0^T F(\bar{\Psi}(s, u_0) - y_s) - c_s ds + \langle x, u_0 \rangle\right).$$

In order to calculate the approximate filter at time $\tilde{T} \in [T, T_0]$ one only needs to continue solving (3.10) on $[T, \tilde{T}]$ (and then repeat the second step for $v \in \mathbb{R}^d$), hence the procedure is indeed recursive.

Remark 3.10. Consider a p -dimensional Brownian motion W , $C \in \mathbb{R}^{p \times d}$, $\Gamma \in \mathbb{R}^{p \times p}$ invertible and an observation process given as

$$(3.11) \quad \bar{Y}_t = \int_0^t CX_s ds + \Gamma W_t, \quad t \geq 0.$$

The present methodology also provides an approximation for this setup: Since \bar{Y} and $Y = \Gamma^{-1}\bar{Y}$ generate the same filtration, the filtering distribution is given by (2.38) and (2.39) with $h(x) = \Gamma^{-1}Cx$. The pathwise functional $\sigma_t(f, y)$ in (2.38) is approximated naturally by $\rho_t(f, (\Gamma^{-1}C)^\top y)$ (see (3.1)) with $\gamma_s = (\Gamma^{-1}C)^\top \Gamma^{-1}Cx_0$ (corresponding to the linearization of h around x_0) and $x_0 = \int_D x\pi_0(dx)$. We do not specify c here, since it cancels out in the normalization (3.2).

In fact, this choice of γ has been used in the examples in Section 5.

Remark 3.11. The choice of the functions γ and c is of course essential for how close ρ_t and $\bar{\pi}_t$ are to σ_t and π_t . In the examples we have always made the choice specified in the previous remark. Let us examine the approximation quality in this setting. Using the product rule (2.37), the definition of \mathbb{Q} and applying the change of measure (2.32) in (3.1) yields \mathbb{P} -a.s.

$$\begin{aligned} \rho_t(f, Y) &= \mathbb{E}_{\mathbb{Q}} \left[f(X_t) \frac{d\mathbb{P}}{d\mathbb{Q}} \exp\left(\frac{1}{2} \int_0^t |h(X_s)|^2 ds - \int_0^t \gamma_s^\top X_s - c_s ds\right) \middle| \mathcal{F}_t^Y \right] \\ &= \mathbb{E} \left[f(X_t) \exp\left(\frac{1}{2} \int_0^t |h(X_s)|^2 ds - \int_0^t \gamma_s^\top X_s - c_s ds\right) \middle| \mathcal{F}_t^Y \right] \sigma_t 1 \end{aligned}$$

and so (in the setting of the previous remark)

$$\bar{\pi}_t(f, Y) = \frac{\mathbb{E} [f(X_t) A_t | \mathcal{F}_t^Y]}{\mathbb{E} [A_t | \mathcal{F}_t^Y]}, \quad A_t = \exp\left(\frac{1}{2} \int_0^t |\Gamma^{-1}C(X_s - x_0)|^2 ds\right).$$

This gives an indication about the approximation quality:

If (with high probability) $\log A_t$ is very small, then the approximation quality is good. This happens for example if $\Gamma = \varepsilon I$ for large $\varepsilon > 0$. If $\varepsilon > 0$ is very small on the other hand, then the approximation quality decreases. However, in this regime there is no need for filtering, since $\int_0^t X_s ds$ can be almost read off from (3.11). For intermediate values of ε this is more difficult to judge and from numerical experiments it appears that there is a range of ε for which the filtering problem is not easy, and nevertheless the approximation is not very good.

Remark 3.12. If the observations arrive only at discrete-time points (as opposed to the continuous-time setting considered here) a similar approximation can be defined. In this case the ordinary differential equations (3.3) are replaced by difference equations.

4. PROOFS

4.1. Proof of auxiliary results. In this section we prepare for the proof of the main results. To this end, we study a change of measure, estimates for the function R in (2.14) and properties of \bar{T}^y in (3.8).

4.1.1. *Change of measure.* One of the key tools in the proofs is a change of measure, which turns the original (time-homogeneous) affine process into a time-inhomogeneous affine process. The next Lemma 4.1 verifies that the associated parameters satisfy the admissibility conditions. Based on this, Proposition 4.2 below will then provide the ingredients for the change of measure.

Lemma 4.1. *Suppose $g: \mathbb{R}_+ \rightarrow \mathbb{R}^d$ is continuous, (2.3) are admissible with $c = 0$, $\gamma = 0$ and (2.19) holds. For $t \geq 0$, define parameters (2.23) by $c(t) = 0$, $\gamma(t) = 0$ and*

$$(4.1) \quad \begin{aligned} a(t) &= a \\ \alpha(t) &= \alpha \\ b(t) &= b + ag_t + \int_{D \setminus \{0\}} \chi(\xi)(e^{\langle g_t, \xi \rangle} - 1) \mu^0(d\xi) \\ \beta_{i,j}(t) &= \beta_{i,j} + (\alpha^j g_t)_i + \int_{D \setminus \{0\}} \chi_i(\xi)(e^{\langle g_t, \xi \rangle} - 1) \mu^j(d\xi), \quad i \in I \cup J, j \in I \\ \beta_{i,j}(t) &= \beta_{i,j}, \quad i \in I \cup J, j \in J \\ \mu^i(t, d\xi) &= e^{\langle g_t, \xi \rangle} \mu^i(d\xi), \quad i \in I \cup \{0\}. \end{aligned}$$

Then (2.23) is strongly admissible and for all $T \geq 0$,

$$(4.2) \quad \sup_{t \in [0, T]} \int_{\{\xi_k > 1\}} \xi_k \mu^i(t, d\xi) < \infty, \quad \text{for } i, k \in I.$$

Proof. Admissibility for fixed $t \geq 0$: Firstly, (2.4) implies $a_{i,j} = 0$ for all $i \in I, j \in I \cup J$ (see (2.4) in [DFS03]). Thus, for $i \in I$, definition (4.1), the assumed integrability (2.12) and the non-negativity condition (2.6) yield

$$b_i(t) - \int_{D \setminus \{0\}} \chi_i(\xi) \mu^0(t, d\xi) = b_i - \int_{D \setminus \{0\}} \chi_i(\xi) \mu^0(d\xi) \geq 0.$$

Similarly, for $i, j \in I$ with $i \neq j$, (2.5) implies $\alpha_{i,k}^j = 0$ for all $k \in I \cup J$. If this was not the case, i.e. if $\alpha_{i,k}^j \neq 0$ for some $k \in J \cup \{j\}$, then defining $v \in \mathbb{R}^d$ by $v_l = \delta_{lk}$ for $l \in J \cup \{j\}$, $v_l = C \delta_{li}$ for $l \in I \setminus \{j\}$ and using (2.5) would yield

$$0 \leq v^\top \alpha^j v = 2C \alpha_{i,k}^j + \alpha_{k,k}^j$$

for all $C \in \mathbb{R}$ and hence a contradiction. Consequently $(\alpha^j g_t)_i = 0$ and as above one uses (2.7) and (2.12) to obtain

$$\beta_{i,j}(t) - \int_{D \setminus \{0\}} \chi_i(\xi) \mu^j(t, d\xi) = \beta_{i,j} - \int_{D \setminus \{0\}} \chi_i(\xi) \mu^j(d\xi) \geq 0.$$

Finally, for $i \in I \cup \{0\}$ and any non-negative $f \in B(D)$ one uses $|e^{\langle g_t, \xi \rangle}| \leq e^{|g_t|}$ on $\{|\xi| \leq 1\}$ to estimate

$$(4.3) \quad \int_{D \setminus \{0\}} f(\xi) \mu^i(t, d\xi) \leq e^{|g_t|} \int_{\{|\xi| \leq 1\} \setminus \{0\}} f(\xi) \mu^i(d\xi) + \|f\|_\infty \int_{D \setminus \{|\xi| \leq 1\}} |\xi| e^{\langle g_t, \xi \rangle} \mu^i(d\xi).$$

Inserting $f = \chi_k$ for $k \in I \setminus \{i\}$ and $f = \chi_k^2$ for $k \in (J \cup \{i\}) \setminus \{0\}$ in (4.3), the integrability conditions for $\mu^i(t, \cdot)$ follow from (2.12), (2.13) and (2.19).

Altogether, it has been verified that (2.23) satisfy for each $t \geq 0$ conditions (2.4)-(2.13).

Continuity in t : Let us first verify the third and fourth admissibility conditions. To do so, note that for any $f: D \setminus \{0\} \rightarrow \mathbb{R}$ which is μ^i -integrable, dominated convergence and continuity of g yield that

$$(4.4) \quad t \mapsto \int_{\{|\xi| \leq 1\} \setminus \{0\}} f(\xi) e^{\langle g_t, \xi \rangle} \mu^i(d\xi) \quad \text{is continuous.}$$

Suppose the following is established: For any $f \in C_b(D)$,

$$(4.5) \quad t \mapsto \int_{D \setminus \{|\xi| \leq 1\}} f(\xi) e^{\langle g_t, \xi \rangle} \mu^i(d\xi) \quad \text{is continuous.}$$

Then for $k \in I \setminus \{i\}$ and any $h \in C_b(D)$, one defines $f := \chi_k h$, notes that $f \in C_b(D)$ (since $\chi \in C_b(D)$) and μ^i -integrable by (2.12) and concludes that

$$t \mapsto \int_{D \setminus \{0\}} h(\xi) \chi_k(\xi) \mu^i(t, d\xi) \quad \text{is continuous,}$$

by (4.4) and (4.5). Thus $\chi_k(\cdot) \mu^i(t, \cdot)$ is weakly continuous and the last strong admissibility condition follows analogously with $f := \chi_k^2 h$ and (2.13).

To verify (4.5), note that (2.19) and [DFS03, Lemma A.2] yield that the function $G_0: \mathbb{R}^d \rightarrow \mathbb{R}$ defined via

$$(4.6) \quad G_0(u) := \int_{D \setminus \{|\xi| \leq 1\}} f(\xi) e^{\langle u, \xi \rangle} \mu^i(d\xi)$$

is analytic. In particular, composing it with the continuous function y preserves continuity and hence (4.5) holds.

Finally, it remains to argue that $b(\cdot)$ and $\beta(\cdot)$ are continuous. To show this, for any $i \in I \cup \{0\}$, $k \in I \cup J$ one uses $\mu^i(D \setminus \{|\xi| \leq 1\}) < \infty$ (since χ is bounded away from 0 on $D \setminus \{|\xi| \leq 1\}$ and by (2.12) and (2.13)) to decompose

$$(4.7) \quad \begin{aligned} \int_{D \setminus \{0\}} \chi_k(\xi) (e^{\langle g_t, \xi \rangle} - 1) \mu^i(d\xi) &= \int_{\{|\xi| \leq 1\} \setminus \{0\}} \chi_k(\xi) (e^{\langle g_t, \xi \rangle} - 1) \mu^i(d\xi) \\ &\quad + \int_{D \setminus \{|\xi| \leq 1\}} \chi_k(\xi) e^{\langle g_t, \xi \rangle} \mu^i(d\xi) - \int_{D \setminus \{|\xi| \leq 1\}} \chi_k(\xi) \mu^i(d\xi). \end{aligned}$$

The second term is continuous in t by (4.5) and so it remains to show that the first integral is continuous in t . But this follows from dominated convergence: for any $T > 0$ one may use Lipschitz continuity of \exp , continuity of g , the Cauchy-Schwarz inequality and the properties of χ to find $C_0, C_1, C_2 > 0$ such that for all $t \in [0, T], \xi \in \{|\xi| \leq 1\} \setminus \{0\}$,

$$|\chi_k(\xi) (e^{\langle g_t, \xi \rangle} - 1)| \leq C_0 |\chi_k(\xi)| |\langle g_t, \xi \rangle| \leq C_1 |\chi(\xi)|^2 \frac{|\xi|}{|\chi(\xi)|} \leq C_2 |\chi(\xi)|^2.$$

But (2.12) and (2.13) imply $\int_{\{|\xi| \leq 1\} \setminus \{0\}} |\chi(\xi)|^2 \mu^i(d\xi) < \infty$ and thus the claim.

Verification of (4.2): Finally, again (2.19) and [DFS03, Lemma A.2] applied to the measure $|\xi| \mu^i(d\xi)$ on $D \setminus \{|\xi| > 1\}$ (which is finite by (2.19)) shows that the function on $G: \mathbb{R}^d \rightarrow \mathbb{R}$ defined via

$$G(u) := \int_{D \setminus \{|\xi| \leq 1\}} |\xi| e^{\langle u, \xi \rangle} \mu^i(d\xi)$$

is analytic and thus for $i, k \in I$,

$$\sup_{t \in [0, T]} \int_{\{\xi_k > 1\}} \xi_k e^{g_t^\top \xi} \mu^i(d\xi) \leq \sup_{t \in [0, T]} \int_{D \setminus \{|\xi| \leq 1\}} |\xi| e^{g_t^\top \xi} \mu^i(d\xi) = \sup_{u \in K} G(u) < \infty,$$

since $K := \{g_s : s \in [0, T]\}$ is a compact set by continuity of g . \square

Based on Lemma 4.1 and a result from [KMK10] (alternatively, one could use [CFY05]) we can now prove the following key tool:

Proposition 4.2. *Suppose $g: \mathbb{R}_+ \rightarrow \mathbb{R}^d$ is continuous and (2.19) holds. Then for any $x \in D$,*

(i) *the process*

$$(4.8) \quad E_t := \exp \left(\int_0^t g_u^\top dX_u - \int_0^t F(g_u) + \langle X_u, R(g_u) \rangle du \right), \quad t \geq 0,$$

- is a \mathbb{P}_x -martingale,
- (ii) if for some $t \geq 0$ and all $s \geq 0$, $g(s) = g(s \wedge t)$, then $(E_{s \wedge t})_{s \geq 0}$ is the density process (w.r.t. \mathbb{P}_x) of a measure \mathbb{Q} on (Ω, \mathcal{F}) such that, under \mathbb{Q} , X is a time-inhomogeneous affine process started from $(0, x)$ with parameters as in Lemma 4.1.

Proof. The proof of Proposition 4.2 is structured as follows: In Step 1, E in (4.8) is rewritten as $\mathcal{E}(M)$ for a suitable local martingale M . In Step 2 it is verified that Lemma 4.1 implies conditions (4.12), (4.13) and (4.14) below. Finally, in Step 3 we combine Step 1 and 2 with [KMK10] and obtain (i) and (ii).

Step 1: We follow the notation and definitions of [JS03].

Denote by μ^X the jump-measure and by X^c the continuous martingale part of X , respectively. By (2.19), $|e^{g^\top x} - 1 - g^\top \chi(x)| * \nu$ is an adapted, continuous, increasing \mathbb{R} -valued process and thus (combining [JS03, Lemma I.3.10 and Proposition II.1.28]) $e^{g^\top x} - 1 + g^\top \chi(x) \in G_{loc}(\mu^X)$. By linearity and [JS03, Theorem II.2.34], $g^\top \chi(x) \in G_{loc}(\mu^X)$ and so also $e^{g^\top x} - 1 \in G_{loc}(\mu^X)$. Thus by [JS03, Theorem II.1.8(ii)], the process

$$(4.9) \quad M_t = \int_0^t g_s^\top dX_s^c + (e^{g^\top x} - 1) * (\mu^X - \nu)_t, \quad t \geq 0$$

is a local martingale. By an argument as above and [JS03, Corollary II.2.38], $g^\top x \in G_{loc}(\mu^X)$ and $W := e^{g^\top x} - 1 - g^\top x \in G_{loc}(\mu^X)$ and thus, using $\Delta M_t = e^{g_t^\top \Delta X_t} - 1 > -1$ one has

$$(4.10) \quad \begin{aligned} (\log(1+x) - x) * \mu^M &= (-g^\top x + e^{g^\top x} - 1) * \mu^X \\ &= W * (\mu^X - \nu) + W * \nu \\ &\stackrel{(4.9)}{=} (-g^\top x) * (\mu^X - \nu) + M + \int_0^\cdot -g_s^\top dX_s^c + W * \nu \\ &= \int_0^\cdot -g_s^\top dX_s + M + W * \nu + \int_0^\cdot g_s^\top \beta(X_s) ds + g^\top (x - \chi(x)) * \nu \\ &= - \int_0^\cdot g_s^\top dX_s + M + (e^{g^\top x} - 1 - g^\top \chi(x)) * \nu + \int_0^\cdot g_s^\top \beta(X_s) ds. \end{aligned}$$

Denoting by \mathcal{E} the stochastic exponential, the definition (see also [JS03, Theorem 8.10]) and (2.18) yields

$$(4.11) \quad \begin{aligned} \mathcal{E}(M)_t &= \exp \left(M_t - \frac{1}{2} \int_0^t g_s^\top \alpha(X_s) g_s ds - (\log(1+x) - x) * \mu_t^M \right) \\ &\stackrel{(4.10)}{=} \exp \left(\int_0^t g_u^\top dX_u - \int_0^t g_u^\top \beta(X_u) du - \frac{1}{2} \int_0^t g_s^\top \alpha(X_s) g_s ds \right. \\ &\quad \left. + (g^\top \chi(x) - e^{g^\top x} + 1) * \nu_t \right) \\ &= \exp \left(\int_0^t g_u^\top dX_u - \int_0^t F(g_u) + \langle X_u, R(g_u) \rangle du \right), \end{aligned}$$

where the last step follows by definition (2.14).

Step 2: Define $W: \mathbb{R}_+ \times \mathbb{R}^d \rightarrow [0, \infty)$ by $W(t, x) := e^{\langle g_t, x \rangle}$. We now show that for all $j \in I \cup \{0\}$, $t \geq 0$,

$$(4.12) \quad \int_0^t \int_{D \setminus \{0\}} (1 - \sqrt{W(s, x)})^2 \mu^j(dx) ds < \infty,$$

$$(4.13) \quad \int_{D \setminus \{0\}} |\chi(x)(W(t, x) - 1)| \mu^j(dx) < \infty,$$

$$(4.14) \quad \text{the measure } \chi_k(W(t, x) - 1)(W(t, x) - 1) \mu^j(dx) \text{ is weakly continuous in } t \in \mathbb{R}_+.$$

It remains to argue that (4.12)-(4.14) are indeed satisfied. Since \exp is Lipschitz continuous and g is continuous, there exists $C \geq 0$ such that for all $s \in [0, t]$, $|x| \leq 1$,

$$|1 - \sqrt{W(s, x)}| = |1 - e^{-\frac{1}{2}\langle g_s, x \rangle}| \leq C|\langle g_s, x \rangle| \leq C|g_s||x|.$$

Taking $K \subset \mathbb{R}^d$ compact with $g_s, \frac{1}{2}g_s \in K$ for all $s \in [0, t]$ and splitting the integral in $\{|x| \leq 1\}$ and $\{|x| \geq 1\}$, we obtain (for G_0 as in (4.6) with $f = 1$)

$$\begin{aligned} \int_0^t \int_{D \setminus \{0\}} (1 - e^{-\frac{1}{2}\langle g_s, x \rangle})^2 \mu^j(dx) ds &\leq C \int_0^t |g_s|^2 ds \int_{|x| \leq 1} |x|^2 \mu^j(dx) \\ &\quad + 2 \sup_{u \in K} G_0(u) + \int_{D \setminus \{|x| \leq 1\}} \mu^j(dx), \end{aligned}$$

which is finite by the integrability properties of the Lévy-measures (2.12), (2.13) and since G_0 is continuous. Thus (4.12) indeed holds and an analogous reasoning gives (4.13).

To establish (4.14), denote $\tilde{\mu}(t, dx) := \chi_k(W(t, x) - 1)(W(t, x) - 1) \mu^j(dx)$ and again consider $D \setminus \{|\xi| \leq 1\}$ and $\{|\xi| \leq 1\} \setminus \{0\}$ separately, i.e. for $f \in C_b(D)$ write

$$(4.15) \quad \int_{D \setminus \{0\}} f(\xi) \tilde{\mu}(t, d\xi) = \int_{D \setminus \{|\xi| \leq 1\}} f(\xi) \tilde{\mu}(t, d\xi) + \int_{\{|\xi| \leq 1\} \setminus \{0\}} f(\xi) \tilde{\mu}(t, d\xi).$$

The second term is continuous in t by dominated convergence and the same argument used to show that b and β are continuous. The first term in (4.15) is the composition of $F_0: \mathbb{R}^d \rightarrow \mathbb{R}_+$ defined by

$$F_0(u) := \int_{D \setminus \{|\xi| \leq 1\}} f(\xi) \chi_k(e^{\langle u, \xi \rangle} - 1)(e^{\langle u, \xi \rangle} - 1) \mu^j(d\xi)$$

and g . To establish (4.14) it thus suffices to show that F_0 is continuous. To see this, assume $f \geq 0$ (for general f apply the subsequent argument to the positive and negative parts of f separately), define $h: [-1, \infty) \rightarrow \mathbb{R}$ by $h(z) := z^2 - z \chi_k(z)$ and write

$$F_0(u) = G_0(2u) - 2G_0(u) + G_0(0) - \int_{D \setminus \{|\xi| \leq 1\}} f(\xi) h(e^{\langle u, \xi \rangle} - 1) \mu^j(d\xi)$$

with G_0 as in (4.6). For the truncation function χ chosen in [KMK10], $h(z) = \max(0, z^2 - z)$ for all $z \in [-1, \infty)$ and so h is non-decreasing and convex. In particular for any $\xi \in D \setminus \{|\xi| \leq 1\}$, the function on \mathbb{R}^d defined by $u \mapsto h(e^{\langle u, \xi \rangle} - 1)$ is convex and so

$$u \mapsto \int_{D \setminus \{|\xi| \leq 1\}} f(\xi) h(e^{\langle u, \xi \rangle} - 1) \mu^j(d\xi)$$

is a (\mathbb{R}_+ -valued) convex function on \mathbb{R}^d . [Roc70, Corollary 10.1.1] implies that it is continuous and so the proof is complete.

Step 3: Recall that (2.3) with $c = 0$ and $\gamma = 0$ is strongly admissible in the sense of [KMK10, Definition 2.4] and by Lemma 4.1 the same holds for (4.1). Furthermore, recall the definition of M in (4.9). Since $g: \mathbb{R}_+ \rightarrow \mathbb{R}^d$ and $W: \mathbb{R}_+ \times \mathbb{R}^d \rightarrow [0, \infty)$ (defined above) are continuous, satisfy (by Step 2) conditions (4.12), (4.13) and (4.14) and since (4.2) holds, [KMK10, Theorem 4.1] and its proof show that $\mathcal{E}(M)$ is a martingale and that $\mathcal{E}(M)$ can be used as the density process

of a probability measure \mathbb{Q} that is locally absolutely continuous w.r.t. \mathbb{P}_x and has the properties stated in (ii). But $E_t = \mathcal{E}(M)_t$ (by Step 1) and hence the claim. \square

4.1.2. Estimates for R .

Lemma 4.3. *There exists a function $g \in C(\mathbb{R}^d, \mathbb{R}_+)$ such that $g(x) = g((x_I^+, x_J))$ for all $x \in \mathbb{R}^d$ (with $x_I^+ = (x_1^+, \dots, x_m^+)$) and for any $u \in \mathbb{C}^d$,*

$$(4.16) \quad \operatorname{Re} \langle \bar{u}_I, R_I(u) \rangle \leq g(\operatorname{Re} u)(1 + |u_J|^2)(1 + |u_I|^2).$$

Proof. Inequality (4.16) is derived in [KRM15, Lemma 5.5] with

$$(4.17) \quad g(x) := c_0(1 + x_I^+) + c_1 e^{x^+} + \sum_{i=1}^m \int_{D \cap \{|\xi| \geq 1\}} e^{\langle \xi, x \rangle} \mu^i(d\xi) + \int_{D \cap \{|\xi| \leq 1\}} \xi_i (e^{\xi_i x_i^+} - 1) \mu^i(d\xi)$$

for some $c_0, c_1 > 0$. Since $\xi_k \geq 0$ for all $k \in I$, $e^{\langle \xi, x \rangle} \leq e^{\langle \xi, (x_I^+, x_J) \rangle}$ and so (4.16) remains valid if instead of g one uses $g((x_I^+, x_J))$. Continuity of this function follows by (2.19) and so the lemma is proved. \square

Lemma 4.4. *Let $r > 0$ and $S_r := \{u \in \mathbb{C}^d : \forall \in I \operatorname{Re} u_i \leq r, |u_J| \leq r\}$. Then there exists $C > 0$ such that for all $u \in S_r$,*

$$|R_I(u)| \leq C(1 + |u|^2).$$

Proof. By the triangle inequality it suffices to find for each $i \in I$ a constant $C_i > 0$ such that $|R_i(u)| \leq C_i(1 + |u|^2)$ for all $u \in S_r$. For $i \in I$,

$$|R_i(u)| \leq \left(\frac{1}{2} |\alpha^i| + |\beta^i| \right) (1 + |u|^2) + \int_{D \setminus \{0\}} \left| e^{\langle \xi, u \rangle} - 1 - \langle \chi(\xi), u \rangle \right| \mu^i(d\xi)$$

and so we only need to analyze the μ^i -integral. Set $B := \{z \in \mathbb{C} : \operatorname{Re} z \leq (d+1)r\}$, then for all $z \in B$,

$$|\exp(z) - 1 - z| \leq |z| \sup_{t \in (0,1)} |e^{tz} - 1| = |z| \sup_{t \in (0,1)} |z| \int_0^t e^{sz} ds \leq |z|^2 e^{(d+1)r}.$$

Furthermore, for any $\xi \in D$ with $|\xi| \leq 1$ and $u \in S_r$,

$$\operatorname{Re} \langle \xi, u \rangle = \langle \xi, \operatorname{Re} u \rangle \leq \left(\sum_{i \in I} \xi_i + |\xi_J| \right) r \leq (d+1)r$$

implies $\langle \xi, u \rangle \in B$. Combining these two observations with the Cauchy-Schwarz inequality and $\chi(\xi) = \xi$ on $\{|\xi| \leq 1\}$ one obtains

$$\int_{D \cap \{|\xi| \leq 1\}} \left| e^{\langle \xi, u \rangle} - 1 - \langle \chi(\xi), u \rangle \right| \mu^i(d\xi) \leq |u|^2 e^{(d+1)r} \int_{D \cap \{|\xi| \leq 1\}} |\xi|^2 \mu^i(d\xi) =: |u|^2 C_i^0,$$

where C_i^0 is finite because of (2.12) and (2.13).

By (2.19) and [DFS03, Lemma A.2], the function

$$\tilde{R}_i(u) := \int_{D \setminus \{|\xi| \leq 1\}} e^{\langle \xi, u \rangle} \mu^i(d\xi), \quad u \in \mathbb{C}^d$$

is analytic. In particular, $C_0 := \sup_{|u| \leq r} |\tilde{R}_i(u)|$ is finite.

Since χ is bounded away from 0 on $D \setminus \{|\xi| \leq 1\}$, $C := \mu^i(D \setminus \{|\xi| \leq 1\})$ is finite (by (2.12) and (2.13)). Combining this with $|\chi(\xi)| \leq d$ one obtains for $u \in S_r$

$$\begin{aligned} \int_{D \setminus \{|\xi| \leq 1\}} \left| e^{\langle \xi, u \rangle} - 1 - \langle \chi(\xi), u \rangle \right| \mu^i(d\xi) &\leq \int_{D \setminus \{|\xi| \leq 1\}} e^{\langle \xi, \operatorname{Re} u \rangle} \mu^i(d\xi) + C(1 + d|u|) \\ &\leq C_0 + C(1 + |u|), \end{aligned}$$

where we have used $\xi_I \in \mathbb{R}_+^m$ for the last estimate. Combining all the estimates yields the desired statement. \square

4.1.3. *Properties of \bar{T}^y .* To prepare the proof of Proposition 3.8 we provide two additional Lemmas. The first is an application of Itô's lemma and essentially identifies the extended generator of \bar{T}^y in (3.8). The second Lemma rephrases a result from [Fil05].

Recall that $H(x) = (x_0)_I^\top x_I + c_0$ and for $f \in C_0^{1,2}(\mathbb{R}_+ \times D)$, $t \geq 0$, set

$$(4.18) \quad \begin{aligned} \mathcal{A}_t^y f(t, x) &= \mathcal{A}f(t, x) + f(t, x)[F(-y_t) + \langle x, R(-y_t) \rangle - H(x)] \\ &\quad - \langle \alpha(x)y_t, \nabla_x f(t, x) \rangle + \int_{D \setminus \{0\}} [f(t, x + \xi) - f(t, x)](e^{-\langle y_t, \xi \rangle} - 1)K(x, d\xi). \end{aligned}$$

Proposition 4.5. *Suppose $y: \mathbb{R}_+ \rightarrow \mathbb{R}^d$ is continuous, (2.3) are admissible with $c = 0$, $\gamma = 0$ and (2.19) holds. Define*

$$(4.19) \quad U_t := \exp\left(-\int_0^t y_u^\top dX_u - \int_0^t H(X_u) du\right)$$

and \mathcal{A}^y as in (4.18). Then for any $f \in C_0^{1,2}(\mathbb{R}_+ \times D)$, the process

$$(4.20) \quad U_t f(t, X_t) - f(0, X_0) - \int_0^t U_u (\partial_u + \mathcal{A}_u^y) f(u, X_u) du, \quad t \geq 0,$$

is a local martingale.

Proof. Define E by (4.8) with $g := -y$. Then $E_t = \mathcal{E}(M)_t$ for M as in (4.9). Furthermore, $U_t = E_t \exp(V_t)$, where

$$V_t = \int_0^t F(-y_u) + \langle X_u, R(-y_u) \rangle du - \int_0^t H(X_u) du$$

is continuous and of bounded variation, $[E, V] = 0$ and thus

$$dU_t = d(\mathcal{E}(M)_t \exp(V_t)) = U_{t-} dM_t + U_t dV_t.$$

For $f \in C_0^{1,2}(\mathbb{R}_+ \times D)$, Itô's formula shows that

$$f(t, X_t) = f(0, X_0) + \int_0^t (\partial_s + \mathcal{A})f(s, X_s) ds + N_t, \quad t \geq 0,$$

where N is a local martingale and the continuous part of N is $\int \nabla_x f(s, X_{s-})^\top dX_s^c$. Combining this with (2.18), the definition (4.9), the fact that f is bounded and $e^{-y^\top x} - 1 \in G_{loc}(\mu^X)$, we obtain

$$\begin{aligned} [M, N]_t &= \langle M^c, N^c \rangle_t + \sum_{s \leq t} (f(s, X_s) - f(s, X_{s-})) (e^{-y_s^\top \Delta X_s} - 1) \\ &\doteq - \int_0^t y_s^\top \alpha(X_s) \nabla_x f(s, X_s) ds \\ &\quad + \int_0^t (f(s, X_{s-} + \xi) - f(s, X_{s-})) (e^{-y_u^\top \xi} - 1) K(X_{u-}, d\xi) du, \end{aligned}$$

where $U \doteq V$ means that $U - V$ is a local martingale.

Putting everything together, Itô's formula written in differential form gives

$$\begin{aligned} dU_t f(t, X_t) &\doteq U_t (\partial_t + \mathcal{A})f(t, X_t) dt + f(t, X_{t-}) U_t dV_t + d[U, N]_t \\ &\doteq U_t \mathcal{A}_t^y f(t, X_t) dt, \end{aligned}$$

which shows that (4.20) is a local martingale. \square

In the following Lemma, we allow the function spaces (defined before) to contain complex valued functions.

Lemma 4.6. *There exists a dense subset $L \subset C_0(D)$ with the following property: for any $T > 0$, $h \in L$ there exists $u \in C^{1,2}([0, T] \times D)$ bounded, satisfying*

$$(4.21) \quad \begin{aligned} \partial_t u(t, x) + \mathcal{A}_t^y u(t, x) &= 0 & (t, x) \in [0, T] \times D, \\ u(T, x) &= h(x) & x \in D. \end{aligned}$$

Proof. Denote by $\Theta_0 \subset C_0(D)$ the set of \mathbb{C} -valued functions from [DFS03, Proposition 8.2]. Any $h \in \Theta_0$ is of the form

$$h(x) = \int_{\mathbb{R}^n} e^{(v, iq)^\top x} g(q) \, dq, \quad x \in D,$$

for some $g \in C_c^\infty(\mathbb{R}^n)$ and $v \in \mathbb{C}_-^m$. Denote by L the complex linear span of Θ_0 . In [DFS03, Lemma 8.4] it is shown that L dense in $C_0(D)$.

Fix $T > 0$ and $h \in \Theta_0$. For $(t, x) \in [0, T] \times D$, define $u(t, x) := P_{t,T} h(x)$. Then u is bounded, satisfies $u(T, \cdot) = h$ and, as established in the proof of [Fil05, Proposition 6.3], $u \in C^{1,2}([0, T] \times D)$ and (4.21) indeed holds. \square

4.2. Proof of Proposition 3.8.

Proof of Proposition 3.8. Since $x_0^i > 0$ for $i \in I$, $y_0 = 0$, $R(0) = 0$ and R, y are continuous, there exists $T > 0$ such that $x_0^i - R_i(-y_t) \geq 0$ for $t \in [0, T]$. Taking $c_0 = \sup_{t \in [0, T]} F(-y_t)$ it follows that $c(t) \geq 0$ and $\gamma(t) \in \mathbb{R}_+^m$ for all $t \in [0, T]$. Combining this with Lemma 4.1, it follows that the parameters are indeed admissible.

To prove the proposition, it suffices to show $\bar{T}_t^y = P_{0,t}^y$ and for this it is sufficient to show $\bar{T}_t^y h = P_{0,t}^y h$ for all h in a dense subset of $C_0(D)$. Taking L from Lemma 4.6, for any $h \in L$ we find $f \in C_0^{1,2}([0, t] \times D)$ such that $f(t, \cdot) = h$ and the du -integral in (4.20) vanishes. Hence

$$N_s := U_{s \wedge t} f(s \wedge t, X_{s \wedge t}), \quad s \geq 0,$$

is a local martingale by Proposition 4.5, where U is as in (4.19). On the other hand,

$$U_t = \exp(V_t) E_t,$$

where E is defined in (4.8) (with $g = -y$) and

$$V_t := \int_0^t F(-y_u) + \langle X_u, R(-y_u) \rangle \, du - \int_0^t H(X_u) \, du, \quad t \geq 0.$$

Since $\beta^j = 0$ for $j \in J$ and $x^i \geq 0$ for $i \in I$,

$$H(x) - F(-y_t) - \langle x, R(-y_t) \rangle = c(t) + x^\top \gamma(t) \geq 0$$

for all $(t, x) \in [0, T] \times D$. Thus $V_t \leq 0$ for $t \in [0, T]$ and $\exp(V_t)$ is bounded on $[0, T]$. Since E is a martingale by Proposition 4.2, the local martingale N satisfies

$$N_s = \exp(V_{t \wedge s}) E_{t \wedge s} f(s \wedge t, X_{s \wedge t}), \quad s \geq 0,$$

and is the product of a bounded process and a martingale. Thus N is a true martingale and combining this with $f(t, \cdot) = h$, the definition (3.8) and $f(s, \cdot) = P_{s,t} h$ (see Lemma 4.6) yields

$$(4.22) \quad \bar{T}_t^y h(x) = \mathbb{E}_x[U_t f(t, X_t)] = \mathbb{E}_x[N_t] = \mathbb{E}_x[N_0] = f(0, x) = P_{0,t} h(x).$$

\square

4.3. Proof of Theorem 3.1 and 3.5.

Proof of Theorem 3.1. We proceed in two steps: First (3.4) is verified under the assumption that a solution to (3.3) exists. In the second part, existence and uniqueness for (3.3) is established.

Expression for the Fourier coefficients: Since Ψ is continuously differentiable, each component is of finite variation and thus $[\Psi^j, X^j] = 0$ for all j . By the product rule and (3.3),

$$\begin{aligned}
(4.23) \quad (u + y_T)^\top X_T - \Psi(0, T, u)^\top X_0 &= \Psi(T, T, u)^\top X_T - \Psi(0, T, u)^\top X_0 \\
&= \int_0^T \Psi(s, T, u)^\top dX_s + \int_0^T X_s^\top \partial_s \Psi(s, T, u) ds \\
&= \int_0^T \Psi(s, T, u)^\top dX_s - \int_0^T X_s^\top (R(\Psi(s, T, u) - y_s) - \gamma_s) ds.
\end{aligned}$$

By Proposition 4.2 applied to the continuous function $g: \mathbb{R}_+ \rightarrow \mathbb{R}^d$ defined by

$$g(t) := \Psi(t \wedge T, T, u) - y_{t \wedge T},$$

the process

$$\tilde{E}_t := \exp \left(\int_0^t g_u^\top dX_u - \int_0^t F(g_u) + \langle X_u, R(g_u) \rangle du \right), \quad t \geq 0,$$

is a martingale.

Combining this with (4.23) and the definition of ρ we obtain

$$\begin{aligned}
(4.24) \quad \rho_T^x(f_u, y) &= \mathbb{E} \left[\exp \left((u + y_T)^\top X_T - \int_0^T y_s^\top dX_s - \int_0^T c_s + X_s^\top \gamma_s ds \right) \right] \\
&= \mathbb{E} \left[\exp \left(\Psi(0, T, u)^\top X_0 + \int_0^T (\Psi(s, T, u) - y_s)^\top dX_s \right. \right. \\
&\quad \left. \left. - \int_0^T c_s X_s^\top R(\Psi(s, T, u) - y_s) ds \right) \right] \\
&= \mathbb{E} \left[\tilde{E}_T \exp \left(\Psi(0, T, u)^\top X_0 + \int_0^T F(\Psi(s, T, u) - y_s) - c_s ds \right) \right] \\
&= \exp(\Phi(0, T, u) + \Psi(0, T, u)^\top x).
\end{aligned}$$

Existence and uniqueness of solutions to (3.3): Suppose first for some $T > 0$ there exists $\tilde{\Psi} \in C^1([0, T], \mathbb{R}^d)$ satisfying

$$(4.25) \quad \partial_t \tilde{\Psi}(t, u) = R(\tilde{\Psi}(t, u) + y_T - y_{T-t}) - \gamma_{T-t}, \quad \tilde{\Psi}(0, u) = u.$$

Then a solution to (3.3) is obtained by setting $\Psi(t, T, u) := \tilde{\Psi}(T - t, u) + y_T$ and

$$\Phi(t, T, u) = \int_t^T F(\Psi(s, T, u) - y_s) - c_s ds, \quad t \in [0, T].$$

Conversely, any solution to (3.3) gives rise to $\tilde{\Psi}$ satisfying (4.25) by setting $\tilde{\Psi}(t, u) := \Psi(T - t, T, u) - y_T$. Thus, to prove the theorem it suffices to construct $T > 0$ such that for all $u \in i\mathbb{R}^d$ there exists a unique $\tilde{\Psi}(\cdot, u) \in C^1([0, T], \mathbb{R}^d)$ satisfying (4.25). To do so, we will establish the following statements:

- (i) for any $T > 0$, $u \in \mathbb{C}^d$, there exists $t_+(u, T) \in (0, \infty]$ such that (4.25) has a unique solution on $[0, t_+(u, T))$. If $t_+(u, T) < \infty$, then $\lim_{t \uparrow t_+(u, T)} |\tilde{\Psi}(t, u)| = \infty$.
- (ii) there exists $T_0 > 0$ such that $t_+(0, T_0) > T_0$, i.e. the solution to (4.25) with $u = 0$, $T = T_0$ exists on $[0, T_0]$.
- (iii) for any $u \in i\mathbb{R}^d$, $t_+(u, T_0) > T_0$.

Then (iii) implies that for any $u \in i\mathbb{R}^d$ there exists a unique solution to (4.25) on $[0, T_0]$, which proves the theorem. We now show (i)-(iii). In what follows, we set $y_r := 0$ for $r < 0$ so that $y \in C(\mathbb{R}, \mathbb{R}^d)$.

(i) By [DFS03, Lemma 5.3] and (2.19), R is an analytic function. In particular it is locally Lipschitz continuous. Combining this with the fact that y is continuous, (i) follows from the global existence and uniqueness result for ordinary differential equations [Ama90, Theorem 7.6].

(ii) For $(t, z, T) \in \mathbb{R} \times \mathbb{C}^d \times \mathbb{R}$, set

$$f(t, z, T) := R(z + y_T - y_{T-t}) - \gamma_{T-t}.$$

Then $f \in C(\mathbb{R} \times \mathbb{C}^d \times \mathbb{R}, \mathbb{C}^d)$ and, since R is locally Lipschitz-continuous, the prerequisites of [Ama90, Theorem 8.3] are satisfied. Thus, the set

$$D := \{(t, \tau, u, T) \in \mathbb{R} \times \mathbb{R} \times \mathbb{C}^d \times \mathbb{R} : t \in J(\tau, u, T)\}$$

is open, where $J(\tau, u, T)$ is the maximal interval of existence of the (unique) solution to

$$\dot{x}(t) = f(t, x(t), T), \quad x(\tau) = u.$$

Since $(0, 0, 0, 0) \in D$ and D is open, $(T_0, 0, 0, T_0) \in D$ for $T_0 > 0$ small enough. Thus $T_0 \in J(0, 0, T_0)$ and, since the right endpoint of the open interval $J(0, 0, T_0)$ is $t_+(0, T_0)$, the claim follows.

(iii) Fix $u \in i\mathbb{R}^d$. By (ii), $t_+(0, T_0) > T_0$ and so it suffices to show that $t_+(u, T_0) \geq t_+(0, T_0)$ or, by (i), that $|\tilde{\Psi}(t, u)|$ does not explode on $[0, T_0]$. Consider the J -components first. By (2.8), for $j \in J$ (4.25) is given as

$$(4.26) \quad \partial_t \tilde{\Psi}_j(t, u) = \langle \beta^j, \tilde{\Psi}_J(t, u) + y_J(T) - y_J(T-t) \rangle - (\gamma_{T-t})_j, \quad \tilde{\Psi}_j(0, u) = u_j$$

and, as this is a system of first order linear equations, $\tilde{\Psi}_J(t, u)$ exists for all $t \geq 0$. Thus it remains to analyze the I -components. We claim that there exists constants $c_0, c_1 > 0$ such that for all $t \in [0, T_0 \wedge t_+(u, T_0))$

$$(4.27) \quad \partial_t |\tilde{\Psi}_I(t, u)|^2 \leq c_0(c_1 + |\tilde{\Psi}_I(t, u)|^2).$$

Assuming that (4.27) has been established, Gronwall's inequality applied to $c_1 + |\tilde{\Psi}_I(t, u)|^2$ implies

$$(4.28) \quad |\tilde{\Psi}_I(t, u)|^2 \leq (c_1 + |u_I|^2) \exp(c_0 t) - c_1$$

for all $t \in [0, T_0 \wedge t_+(u, T_0))$. This allows to conclude (iii) by contradiction: If $T_0 \geq t_+(u, T_0)$, then (4.28) holds for all $t \in [0, t_+(u, T_0))$ and the left hand side of (4.28) explodes as $t \uparrow T_0$, whereas the right hand side is bounded by its value at T_0 . Hence, by contradiction $T_0 < t_+(u, T_0)$ as claimed.

Therefore it suffices to establish (4.27). To do so, we follow the proof of [DFS03, Proposition 6.1] and [KRM15, Proposition 5.1]. For $t \in \mathbb{R}$, set $\bar{y}(t) := y_{T_0} - y_{T_0-t}$. As argued above, (4.26) implies that $\tilde{\Psi}_J(t, u)$ exists for all $t \geq 0$. Furthermore, the real part of (4.26) does not depend on u and therefore

$$(4.29) \quad \operatorname{Re} \tilde{\Psi}_J(t, u) = \operatorname{Re} \tilde{\Psi}_J(t, 0) = \tilde{\Psi}_J(t, 0).$$

Set $T := t_+(0, T_0) \wedge t_+(u, T_0)$ and for $(t, x) \in [0, T] \times \mathbb{R}^m$,

$$f(t, x) := R_I((x, \tilde{\Psi}_J(t, 0)) + \bar{y}(t)) - (\gamma_{T_0-t})_I.$$

Then by [KRM15, Lemma 5.7], continuity of y and Lipschitz continuity of R_I , f satisfies the conditions of the comparison result [MMKS11, Proposition A.2]. Furthermore, (4.29) and the inequality $\operatorname{Re} R_i(z) \leq R_i(\operatorname{Re}(z))$ (valid for all $z \in \mathbb{C}^d$) yield

$$\partial_t \operatorname{Re} \tilde{\Psi}_i(t, u) - f_i(t, \operatorname{Re} \tilde{\Psi}_I(t, u)) \leq 0 = \partial_t \tilde{\Psi}_i(t, 0) - f_i(t, \tilde{\Psi}_I(t, 0)), \quad \operatorname{Re} \tilde{\Psi}_i(0, u) = \tilde{\Psi}_i(0, 0)$$

for $t \in [0, T)$, $i \in I$. Hence the comparison result [MMKS11, Proposition A.2] implies

$$(4.30) \quad \operatorname{Re} \tilde{\Psi}_i(t, u) \leq \tilde{\Psi}_i(t, 0), \quad \forall i \in I, t \in [0, t_+(0, T_0) \wedge t_+(u, T_0))$$

For $t \in [0, T_0 \wedge t_+(u, T_0))$ one uses (4.25) to write

$$\begin{aligned}
(4.31) \quad \frac{1}{2} \partial_t |\tilde{\Psi}_I(t, u)|^2 &= \operatorname{Re} \langle \overline{\tilde{\Psi}_I(t, u)}, \partial_t \tilde{\Psi}_I(t, u) \rangle \\
&= \operatorname{Re} \langle \overline{\tilde{\Psi}_I(t, u) + \bar{y}(t)}, R_I(\tilde{\Psi}(t, u) + \bar{y}(t)) \rangle - \langle \bar{y}(t), \operatorname{Re} R_I(\tilde{\Psi}(t, u) + \bar{y}(t)) \rangle \\
&\quad - \langle \operatorname{Re} \tilde{\Psi}_I(t, u), (\gamma_{T_0-t})_I \rangle \\
&= I_1 - I_2 - I_3,
\end{aligned}$$

where each I_i denotes an inner product. The three inner products in (4.31) can be estimated separately:

For the first one, denote by $g \in C(\mathbb{R}^d, \mathbb{R}_+)$ the function from Lemma 4.3, write $x^{+,I} := (x_I^+, x_J)$ for $x \in \mathbb{R}^d$ and recall $g(x) = g(x^{+,I})$. By (4.30) and the fact that $\tilde{\Psi}_J(t, u)$ exists for all $t \geq 0$, there exists $K \subset \mathbb{R}^d$ compact such that $(\operatorname{Re} \tilde{\Psi}(t, u) + \bar{y}(t))^{+,I} \in K$ for all $t \in [0, T_0 \wedge t_+(u, T_0))$. Hence Lemma 4.3 yields

$$\begin{aligned}
(4.32) \quad I_1 &\leq g(\operatorname{Re} \tilde{\Psi}(t, u) + \bar{y}(t))(1 + |\tilde{\Psi}_J(t, u) + \bar{y}_J(t)|^2)(1 + |\tilde{\Psi}_I(t, u) + \bar{y}_I(t)|^2) \\
&\leq 4g((\operatorname{Re} \tilde{\Psi}(t, u) + \bar{y}(t))^{+,I})(1 + |\tilde{\Psi}_J(t, u) + \bar{y}_J(t)|^2)(C_1 + |\tilde{\Psi}_I(t, u)|^2) \\
&\leq C_0(C_1 + |\tilde{\Psi}_I(t, u)|^2)
\end{aligned}$$

where $C_0 := (4 \sup_{x \in K} g(x) \wedge 1) \sup_{t \in [0, T_0]} (1 + |\tilde{\Psi}_J(t, u) + \bar{y}_J(t)|^2)$ and $C_1 := 1 + 2 \sup_{t \in [0, T_0]} |\bar{y}(t)|^2$.

For the second one, Lemma 4.4, the fact that $\tilde{\Psi}_J(t, u)$ exists for all $t \geq 0$ and (4.30) yield that there exists $C > 0$ such that

$$\begin{aligned}
(4.33) \quad -I_2 &\leq |\bar{y}(t)| |R_I(\tilde{\Psi}(t, u) + \bar{y}(t))| \\
&\leq CC_1(1 + |\tilde{\Psi}(t, u) + \bar{y}(t)|^2) \\
&\leq 4CC_1(C_0 + C_1 + |\tilde{\Psi}_I(t, u)|^2)
\end{aligned}$$

and for the last one

$$(4.34) \quad -I_3 \leq |\gamma_{T_0-t}| |\tilde{\Psi}_I(t, u)| \leq \sup_{s \in [0, T_0]} |\gamma_s| (1 + |\tilde{\Psi}_I(t, u)|^2).$$

Combining (4.31) with the estimates (4.32), (4.33) and (4.34) yields (4.27), as desired. \square

Proof of Theorem 3.5. Precisely as in the derivation of (4.24), one combines the definition (3.5), the product rule (4.23) for $u = 0$ and the definition of E_t in (4.8) to write

$$\begin{aligned}
\rho_t(G, y) &= \mathbb{E} \left[G(X_{[0,t]}) E_t \exp \left(\Psi(0, t, 0)^\top X_0 + \int_0^t F(g_s) - c_s \, ds \right) \right] \\
&= \exp(\Phi(0, t, 0)) \int_D \exp(\langle x, \Psi(0, t, 0) \rangle) \mathbb{E}_x [G(X_{[0,t]}) E_t] \pi_0(dx).
\end{aligned}$$

But for any $x \in D$,

$$\mathbb{E}_x [G(X_{[0,t]}) E_t] = \mathbb{E}_{\mathbb{Q}_x^{y,t}} [G(X_{[0,t]})]$$

with $\mathbb{Q}_x^{y,t} = \mathbb{Q}$ as in Lemma 4.2(ii). Thus the statement follows from the definition of $\bar{\pi}_t(G, y)$ and Lemma 4.2(ii). \square

5. ILLUSTRATION: FILTERING A COX-INGERSOLL-ROSS PROCESS

In this section the methodology developed in Section 3 is applied to the problem of filtering a Cox-Ingersoll-Ross process. We compare the approximation via our linearized filtering functional (LFF) (respectively the induced affine functional filter (AFF)) and other existing approximate filtering methods to the true solution.

5.1. Problem formulation. A Cox-Ingersoll-Ross (CIR) process is a weak solution to the stochastic differential equation

$$(5.1) \quad dX_t = (b + \beta X_t)dt + \sigma\sqrt{X_t}dB_t, \quad X_0 = x,$$

where $b \geq 0$, $\beta \in \mathbb{R}$, $\sigma > 0$ and B is a Brownian motion. Denoting by \mathbb{P}_x the law of X , this gives rise to a conservative affine process with state space $D = \mathbb{R}_+$. The parameters in (2.3) are given as $(0, \sigma^2, b, \beta, 0, 0, 0, 0)$. Let W a Brownian motion independent of X , $\Gamma > 0$ and set

$$(5.2) \quad Y_t = \int_0^t X_s ds + \Gamma W_t, \quad t \geq 0.$$

The goal is to calculate, for any $t \geq 0$, the distribution of X_t conditional on the σ -algebra generated by $(Y_s)_{s \in [0, t]}$ (see Section 2.2.1). In particular, we are interested in the conditional mean and variance

$$(5.3) \quad \begin{aligned} \hat{x}_t &= \mathbb{E}_x[X_t | \mathcal{F}_t^Y], \\ V_t &= \mathbb{E}_x[(X_t - \hat{x}_t)^2 | \mathcal{F}_t^Y], \quad t \geq 0. \end{aligned}$$

There are various methods available to numerically approximate (5.3). For any of these methods one has to pass to a setup of discrete-time observations at some stage. To do this we fix $T > 0$, $N \in \mathbb{N}$ and a time-grid $0 = t_0 < t_1 < \dots < t_N = T$. Instead of observing the entire path (5.2), one observes at time t_i the random variable

$$(5.4) \quad y_i = X_{t_i}(t_i - t_{i-1}) + \Gamma\sqrt{t_i - t_{i-1}}\varepsilon_i,$$

for $i = 1, \dots, N$, where $\varepsilon_1, \dots, \varepsilon_N$ are i.i.d. standard normal random variables. This amounts to discretizing the integral in (5.2) using a Riemann sum and setting $y_i = Y_{t_i}^{disc} - Y_{t_{i-1}}^{disc}$. The filtering distribution is then approximated as $\mathbb{E}_x[f(X_{t_n}) | \mathcal{F}_{t_n}^Y] \approx \pi_{t_n}^N(f)$ with

$$(5.5) \quad \pi_{t_n}^N(f) := \mathbb{E}_x[f(X_{t_n}) | \mathcal{F}_{t_n}^{Y, N}],$$

for any measurable $f: \mathbb{R}_+ \rightarrow \mathbb{C}$ satisfying $\mathbb{E}_x[|f(X_t)|] < \infty$, where $\mathcal{F}_{t_n}^{Y, N} = \sigma(y_1, \dots, y_n)$ and $n = 1, \dots, N$. In particular, instead of (5.3) in what follows we will denote

$$(5.6) \quad \begin{aligned} \hat{x}_t &= \mathbb{E}_x[X_t | \mathcal{F}_t^{Y, N}], \\ V_t &= \mathbb{E}_x[(X_t - \hat{x}_t)^2 | \mathcal{F}_t^{Y, N}], \quad t \in \{t_0, \dots, t_N\}. \end{aligned}$$

5.2. Numerical solution: Approximate filtering methods. There are various methods at hand to numerically approximate (5.5) and (5.6). To illustrate the quality of these we first generate a sample path of the signal and observation process. More precisely, a sample of $(X_{t_0}, X_{t_1}, \dots, X_{t_N})$ is generated by (exact) sampling from the transition density (see [Gla04, Section 3.4]). Based on this sample, a sample of (y_1, \dots, y_N) is generated using (5.4).

For this sample observation we now compare different methods for approximating (5.5) and (5.6). As a benchmark we calculate (5.6) using a (bootstrap) particle filter with sufficiently many particles (10^6 in the examples below), see [BC09, Chapter 10]. In the plots these results will be denoted by \hat{x} and V by slight abuse of notation.

This benchmark is now compared to the approximation using the linearized filtering functional (LFF, developed in the present paper) and two standard approximations (explained in more detail below): A Gamma-approximation ([Bat06]) and a normal approximation ([GP99], see also [BH98]). The respective approximations to (5.6) are denoted as follows:

$$\begin{aligned} \text{Normal} & \hat{x}^{(EKF)}, V^{(EKF)} \\ \text{Gamma} & \hat{x}^{(G)}, V^{(G)} \\ \text{LFF} & \hat{x}^{(LFF)}, V^{(LFF)}. \end{aligned}$$

Firstly, let us explain the approximations from [Bat06] and [GP99] in more detail. In both cases basic idea is to *postulate* that (at each time-step t_n) the conditional distribution in (5.5) belongs to a certain two-parameter family of probability distributions (Normal in [GP99] and Gamma in [Bat06]). Then (at each time-step t_n) one only needs to approximate (5.6) and determine the two parameters from this. In [GP99] the updating procedure for (5.6) is based on the exact formulas for the mean and variance of a CIR process and the Kalman filter. This can be seen as a version of the extended Kalman filter. In [Bat06] numerical integration on the level of characteristic functions is used to update (5.6). We refer to these articles for more details. Both approximations [Bat06] and [GP99] can be viewed as special cases of the projection filter (first introduced in [BHL98]), see [BH98].

Finally, the unconditional mean and variance are denoted by $\bar{x}_t := \mathbb{E}_x[X_t]$ and $v_t := \mathbb{E}_x[(X_t - \bar{x}_t)^2]$. Since these correspond to a situation where no observations are available, a comparison of (\bar{x}, v) and (\hat{x}, V) shows how much information the (sample path of the) observation (y_1, \dots, y_N) contains about X . Therefore, these are also shown in the plots below.

5.3. Discussion. We now compare the methods introduced above for two sets of parameters. For both settings the following choices have been made:

- instead of a constant x , the signal process X is started from $X_0 = \max(0, Z)$, where $Z \sim \mathcal{N}(x_0, s_0^2)$ is independent of B and W ,
- the time horizon is $T = 1$ and the discretization uses an equidistant grid $t_i = iT/N$, $i = 0, \dots, N$,
- $N = 1000$, $\sigma = 0.04$, $\beta = -0.2$ and $s_0 = 2 \cdot 10^{-5}$.

The remaining parameter values differ for the two settings; they are indicated in the caption of the figures.

Case 1 We choose $b = 10^{-6}$, $\Gamma = x_0 = 0.005$. Figures 1 and 2 show the same sample path of a CIR process. The sample of observations is not shown in the plot, but one clearly sees that for t sufficiently large the conditional mean \hat{x} is neither very close to X nor very close to the mean \bar{x} . Thus, the filtering problem is indeed not trivial: the posterior distribution in (5.5) is neither close to the distribution of X_{t_n} nor concentrated at X_{t_n} .

In both figures the conditional mean \hat{x} is shown along with (dotted) “confidence bounds” given by $\hat{x} + \sqrt{V}$ and $\hat{x} - \sqrt{V}$. This allows to show both conditional mean and variance in the same plot. The analogous bounds are also shown for the unconditional mean and the different approximations.

The two figures illustrate that the linearized filtering functional provides a more accurate approximation for (5.6) than the standard methods.

Case 2 We choose $b = 2 \cdot 10^{-5}$, $\Gamma = x_0 = 0.0001$. In this case both the approximation using the linearized filtering functional (LFF) and the normal approximation are not very good. However, it appears that the LFF-approximation becomes better as t approaches 1. Although this behaviour is typical in the present parameter regime, a precise explanation (possibly based on ergodicity properties of the CIR process) is presently not available.

6. ILLUSTRATION: FILTERING A WISHART PROCESS

So far this article has been concerned with the filtering problem for $\mathbb{R}_+^m \times \mathbb{R}^{d-m}$ -valued affine processes. We now test the methodology on Wishart processes, an S_d^+ -valued generalization of the CIR process (as studied in Section 5). Here S_d^+ denotes the set of all symmetric, positive semidefinite $d \times d$ matrices. Wishart processes were introduced in [Bru91] and are commonly used for multivariate stochastic volatility modeling. They are a subclass of S_d^+ -valued affine processes as characterized in [CFMT11].

Although in theory sequential Monte Carlo methods can be applied for numerically filtering Wishart processes, in practice this is infeasible for $d \geq 3$ (see below). Hence, so far no numerical method has been available for this problem. We fill this gap by introducing a linearized filtering

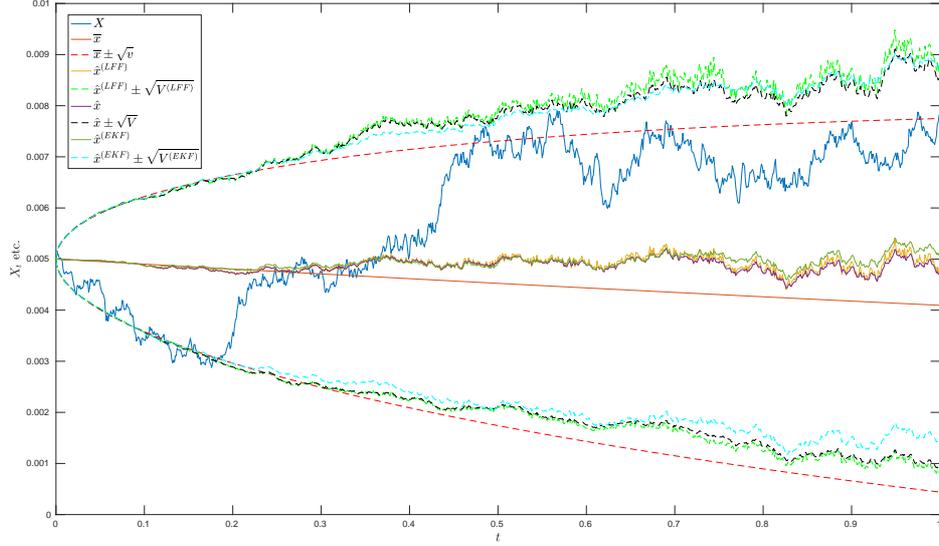


FIGURE 1. Case 1: Comparison with extended Kalman filter.

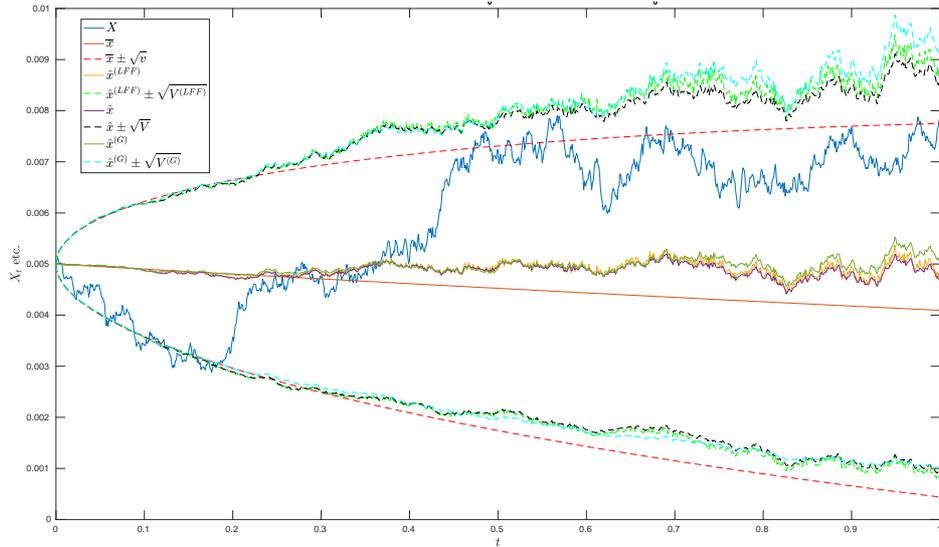


FIGURE 2. Case 1: Comparison with Gamma approximation.

functional analogous to (3.1) and perform numerical experiments for $d = 3$. This section contains simulation results. A generalization of the theory in Sections 3 and 4 to S_d^+ -valued affine processes will be subject of future work.

6.1. The signal process. Denote by S_d^+ the set of all symmetric, positive semidefinite $d \times d$ matrices and set $S_d^- = -S_d^+$. A Wishart process is (an S_d^+ -valued) weak solution to

$$(6.1) \quad dX_t = (b + HX_t + X_t H^\top)dt + \sqrt{X_t}dB_t\Sigma + \Sigma^\top dB_t^\top \sqrt{X_t}, \quad X_0 = x,$$

for B a $d \times d$ -matrix of independent standard Brownian motions and suitable $b \in S_d^+$, $x \in S_d^+$, $H \in \mathbb{R}^{d \times d}$, $\Sigma \in \mathbb{R}^{d \times d}$. For simplicity, we assume that $\Sigma \in S_d^+$, $H = 0$, $b = n\Sigma^2$ for some $n \in \mathbb{N}$ with $n \geq d + 1$ and that x has distinct eigenvalues. Then [Bru91, Proof of Theorem 2"] ensures that (6.1) has a unique strong solution for all $t \geq 0$. It also ensures that sample paths of X can be simulated easily: Given $z_0 \in \mathbb{R}^{n \times d}$ with $x = z_0^\top z_0$ and an $n \times d$ -Brownian motion W , set

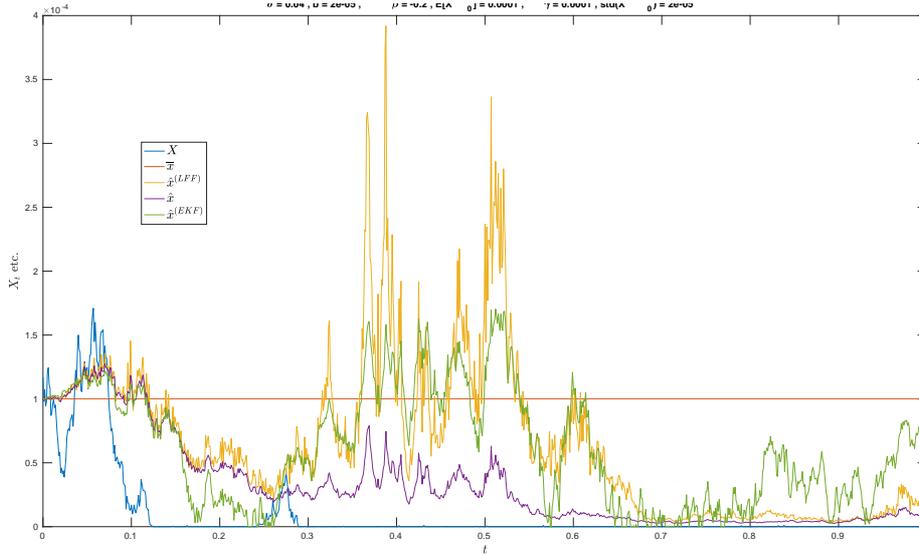


FIGURE 3. Case 2: Comparison with extended Kalman filter.

$Z_t = W_t \Sigma + z_0$ for $t \geq 0$. Then $X := Z^\top Z$ is a weak solution to (6.1). Hence, to simulate a sample path of X one only needs to simulate a sample path of W and apply these two transformations. Finally, for $u, v \in S_d$ (the set of symmetric $d \times d$ -matrices) define $\langle u, v \rangle_{S_d} := \text{tr}(uv)$. Then for $t \geq 0$ the Laplace transform of X_t is given by

$$\mathbb{E}[e^{\langle u, X_t \rangle_{S_d}}] = \exp(\phi(t, u) + \langle \psi(t, u), x \rangle_{S_d}), \quad u \in S_d^-$$

for some $\phi : \mathbb{R}_{\geq 0} \times S_d^- \rightarrow \mathbb{R}_-$ and $\psi : \mathbb{R}_{\geq 0} \times S_d^- \rightarrow S_d^-$. In fact ϕ and ψ solve generalized Riccati equations (2.15) with $R(u) := 2u\Sigma^2u$, $F(u) := n\text{tr}(\Sigma^2u)$.

6.2. Numerical solution of the filtering problem. Fix $h : S_d^+ \rightarrow \mathbb{R}^m$ linear and $\Gamma \in \mathbb{R}^{m \times m}$ symmetric, invertible. The observation process Y is defined as

$$Y_t = \int_0^t h(X_s) ds + \Gamma W_t, \quad t \geq 0,$$

where W is an m -dimensional Brownian motion independent of X , and (the signal process) X is a solution to (6.1) with parameters as specified above (under \mathbb{P}). As before our goal is to numerically calculate the distribution of X_t conditional on the σ -algebra generated by $(Y_s)_{s \in [0, t]}$, for any $t \geq 0$. For this two methods are used: Firstly a bootstrap particle filter as in [BC09, Chapter 9] and secondly the approximate affine filter (AFF) induced by the linearized filtering functional (LFF). These are defined analogously to the case of a canonical state space. More precisely, fix $x_0 \in S_d^+$ and for $t \geq 0$, $y \in C(\mathbb{R}_+, \mathbb{R}^m)$ and $f \in B(S_d^+)$ define the LFF $\rho_t(\cdot, y)$ by

$$\rho_t(f, y) = \mathbb{E} \left[f(X_t) \exp \left(y_t^\top \Gamma^{-2} h(X_t) - \int_0^t y_s^\top \Gamma^{-2} dh(X_s) - \int_0^t h(x_0)^\top \Gamma^{-2} h(X_s) ds \right) \right]$$

and the AFF $\bar{\pi}_t(\cdot, y)$ by (3.2). As in Remark 3.10 the LFF is obtained by linearizing the pathwise filtering functional (associated to the observation process $\Gamma^{-1}Y$ and observation function $\Gamma^{-1}h$) at x_0 . Denoting by h^\top the adjoint⁶ of h and setting $\bar{y}_s = h^\top(\Gamma^{-2}y_s)$ and $\bar{x}_0 = h^\top(\Gamma^{-2}h(x_0))$ one rewrites $\rho_t(f, y)$ as

$$\rho_t(f, y) = \mathbb{E} \left[f(X_t) \exp \left(\langle \bar{y}_t, X_t \rangle_{S_d} - \int_0^t \langle \bar{y}_s, dX_s \rangle_{S_d} - \int_0^t \langle \bar{x}_0, X_s \rangle_{S_d} ds \right) \right]$$

⁶By definition, this is the unique linear map $h^\top : \mathbb{R}^m \rightarrow S_d$ such that $h(x)^\top y = \langle x, h^\top(y) \rangle_{S_d}$ for all $y \in \mathbb{R}^m$.

and based on Section 3 one expects

$$\bar{\pi}_t(f, y) = \mathbb{E}_{\mathbb{Q}_x^{y,t}}[f(X_t)],$$

where under $\mathbb{Q}_x^{y,t}$, X satisfies $X_0 = x$ and

$$(6.2) \quad dX_s = (n\Sigma^2 + H_s X_s + X_s H_s^\top) ds + \sqrt{X_s} dB_s \Sigma + \Sigma dB_s^\top \sqrt{X_s}, \quad s \in (0, t],$$

with $H_s = 2\Sigma^2(\Psi(s) - \bar{y}_s)$, B a $d \times d$ Brownian motion under $\mathbb{Q}_x^{y,t}$ and Ψ the solution to

$$(6.3) \quad \begin{aligned} -\partial_s \Psi(s) &= R(\Psi(s) - \bar{y}_s) - \bar{x}_0, \quad s \in [0, t] \\ \Psi(t) &= \bar{y}_t. \end{aligned}$$

In particular, (6.2) yields an ordinary differential equation for the approximate conditional mean $\hat{X}_t = \bar{\pi}_t(\text{id}, y)$ at time t : Formally taking expectations in (6.2) one obtains $\hat{X}_0 = x$ and

$$(6.4) \quad \frac{d\hat{X}_s}{ds} = (n\Sigma^2 + H_s \hat{X}_s + \hat{X}_s H_s^\top), \quad s \in (0, t].$$

6.3. Discussion. We now compare the two methods in an example. The following choices have been made: $h(x) := \text{vech}(x)$ is the half-vectorization operator (which takes the elements of x in the lower triangular part and writes them in an m -dimensional column vector) and $m = \frac{1}{2}d(d+1)$. Denote by I_d the $d \times d$ identity matrix. We choose $d = 3$, $\Gamma = \Gamma_0 I_3$, $\Sigma = \sigma I_3$ and the parameter values as shown in the following summary:

$$(6.5) \quad \begin{aligned} dX_t &= n\sigma^2 I_3 dt + \sigma X_t^{1/2} dB_t + \sigma dB_t^\top X_t^{1/2}, \quad X_0 = x_0 \\ dY_t &= \text{vech}(X_t) dt + \Gamma_0 dW_t, \quad Y_0 = 0 \\ (n, \sigma, x_0, \Gamma_0) &= (4, 0.04, \text{diag}(0.75^2, 0.5^2, 0.25^2), 0.06). \end{aligned}$$

The filtering problem is discretized analogously to the case of a CIR process discussed in detail in Section 5. We choose $T = 1$ and equidistant time-points $t_i = iT/N$, $i = 0, \dots, N$ with $N = 100$. (Exact) samples of $(X_{t_0}, X_{t_1}, \dots, X_{t_N})$ can be generated as explained in Section 6.1 and a spline interpolation is used to generate a continuous observation path y from discrete measurements.

In this setting the conditional mean \hat{x}_t (see (5.3) and (5.6)) is approximated by

- $\hat{x}_t^{(PF)}$ based on a bootstrap particle filter with N_p particles (as in [BC09, Chapter 10]),
- $\hat{x}_t^{(AFF)} = \hat{X}_t$ in (6.4).

The computation time required to calculate $\hat{x}_t^{(PF)}$ with a generic implementation on a standard laptop is enormous already for moderate N_p (e.g. for $N_p = 10^3, 10^4, 10^5$ it takes roughly 10 seconds, 1 minute, 10 minutes, respectively). On the other hand, in all these cases the approximation is very bad and so, in contrast to Section 5, here no benchmark is available. The two approximations are therefore compared based on their mean square error: We generate M sample paths of (6.5), calculate the approximate conditional mean with both methods and calculate the average at each time-point,

$$e_t^{(m)} = \frac{1}{M} \sum_{j=1}^M \|X_t^j - x_t^{(m),j}\|^2, \quad t = t_0, \dots, t_N$$

for $m \in \{PF, AFF\}$. Here X^j is the j -th sample path of X , $x_t^{(m),j}$ is the approximate conditional mean (calculated using method m) associated to it and $\|u\|^2 := \langle u, u \rangle_{S_d}$ for $u \in S_d$. By the law of large numbers and the definition of \hat{x} , a smaller value of $e_t^{(m)}$ indicates that (on average) \hat{x}_t and $x_t^{(m)}$ are closer.

Figure 4 shows a plot of $(t_i, e_{t_i}^{(m)})$, $i = 0, \dots, N$ for $m \in \{PF, AFF\}$, $N_p = 10^4$ and $M = 100$. For this number of particles the calculation of $\hat{x}_t^{(PF)}$ takes about 15 times longer than the calculation of $\hat{x}_t^{(AFF)}$ (on average). Nevertheless, the approximation quality of a bootstrap

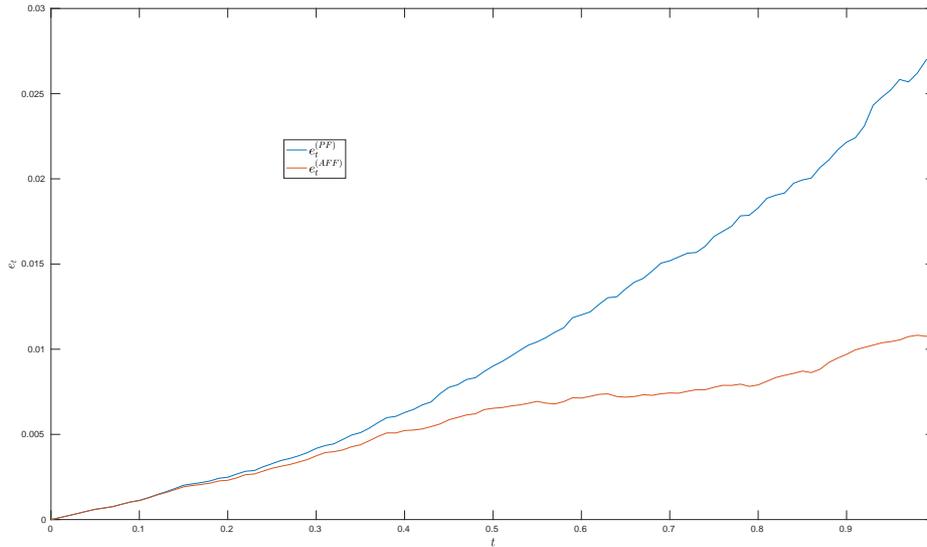


FIGURE 4. Comparison of mean square error for conditional mean of particle filter and AFF.

particle filter is considerably worse than that of the AFF, since the average mean-square error is significantly larger for longer time-periods, as shown in Figure 4.

REFERENCES

- [Ama90] H. Amann, *Ordinary differential equations: an introduction to nonlinear analysis*, vol. 13, Walter de Gruyter, 1990.
- [Bat06] D.S. Bates, *Maximum likelihood estimation of latent affine processes*, *Review of Financial Studies* **19** (2006), no. 3, 909–965.
- [BC09] A. Bain and D. Crisan, *Fundamentals of stochastic filtering*, Springer, 2009.
- [BH98] D. Brigo and B. Hanzon, *On some filtering problems arising in mathematical finance*, *Insurance: Mathematics and Economics* **22** (1998), no. 1, 53–64.
- [BHL98] D. Brigo, B. Hanzon, and F. LeGland, *A differential geometric approach to nonlinear filtering: the projection filter*, *IEEE Transactions on Automatic Control* **43** (1998), no. 2, 247–252.
- [Bru91] M.-F. Bru, *Wishart processes*, *Journal of Theoretical Probability* **4** (1991), no. 4, 725–751.
- [CC05] J.M.C. Clark and D. Crisan, *On a robust version of the integral representation formula of nonlinear filtering*, *Probability Theory and Related Fields* **133** (2005), no. 1, 43–56.
- [CDFO13] D. Crisan, J. Diehl, P.K. Friz, and H. Oberhauser, *Robust filtering: Correlated noise and multidimensional observation*, *Annals of Applied Probability* **23** (2013), 2139–2160.
- [CFMT11] C. Cuchiero, D. Filipović, E. Mayerhofer, and J. Teichmann, *Affine processes on positive semidefinite matrices*, *Annals of Applied Probability* **21** (2011), no. 2, 397–463.
- [CFY05] P. Cheridito, D. Filipović, and M. Yor, *Equivalent and absolutely continuous measure changes for jump-diffusion processes*, *Annals of Applied Probability* **15** (2005), no. 3, 1713–1732.
- [CIR85] J.C. Cox, J.E. Ingersoll, and S. Ross, *A theory of the term structure of interest rates*, *Econometrica* **53** (1985), no. 2, 385–407.
- [Cla78] J.M.C. Clark, *The design of robust approximations to the stochastic differential equations of nonlinear filtering*, *Communication systems and random process theory (Proc. 2nd NATO Advanced Study Inst., Darlington, 1977)*, NATO Advanced Study Inst. Ser., Ser. E: Appl. Sci., vol. 25, Sijthoff & Noordhoff, 1978, p. 721–734.
- [CM99] P. Carr and D.B. Madan, *Option valuation using the fast fourier transform*, *Journal of Computational Finance* **2** (1999), no. 4, 61–73.
- [CR11] D. Crisan and B. Rozovskii (eds.), *The oxford handbook of nonlinear filtering*, Oxford Handbooks in Mathematics, Oxford University Press, 2011.
- [Cri14] D. Crisan, *The stochastic filtering problem: a brief historical account*, *Journal of Applied Probability* **51A** (2014), 13–22.

- [CS03] R.-R. Chen and L. Scott, *Multi-factor cox-ingersoll-ross models of the term structure: Estimates and tests from a kalman filter model*, The Journal of Real Estate Finance and Economics **27** (2003), no. 2, 143–172.
- [CT13] C. Cuchiero and J. Teichmann, *Path properties and regularity of affine processes on general state spaces*, Séminaire de Probabilités XLV (Catherine Donati-Martin, Antoine Lejay, and Alain Rouault, eds.), Springer International Publishing, Heidelberg, 2013, pp. 201–244.
- [Dav80] M.H.A. Davis, *On a multiplicative functional transformation arising in nonlinear filtering theory*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete **54** (1980), no. 2, 125–139.
- [Dav11] ———, *Pathwise nonlinear filtering with correlated noise*, The Oxford Handbook of Nonlinear Filtering, Oxford University Press, 2011, pp. 403–42.
- [DFS03] D. Duffie, D. Filipović, and W. Schachermayer, *Affine processes and applications in finance*, Annals of Applied Probability **13** (2003), no. 3, 984–1053.
- [Duf01] D. Dufresne, *The integrated square-root process*, Centre for Actuarial Studies, University of Melbourne, 2001, Research Paper no. 90.
- [EK86] S.N. Ethier and T.G. Kurtz, *Markov processes. Characterization and convergence*, John Wiley & Sons, 1986.
- [Fil05] D. Filipovic, *Time-inhomogeneous affine processes*, Stochastic Processes and their Applications **115** (2005), no. 4, 639–659.
- [FKR10] P.K. Friz and M. Keller-Ressel, *Moment explosions*, John Wiley & Sons, Ltd, 2010.
- [FM09] D. Filipović and E. Mayerhofer, *Affine diffusion processes: theory and applications*, Advanced financial modelling, Radon Ser. Comput. Appl. Math., vol. 8, Walter de Gruyter, Berlin, 2009, pp. 125–164.
- [Gla04] P. Glasserman, *Monte carlo methods in financial engineering*, Applications of mathematics : stochastic modelling and applied probability, Springer, 2004.
- [GP99] A.L.J. Geyer and S. Pichler, *A state-space approach to estimate and test multifactor cox-ingersoll-ross models of the term structure*, Journal of Financial Research **22** (1999), no. 1, 107–130.
- [Hes93] S.L. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, The Review of Financial Studies **6** (1993), no. 2, 327–343.
- [JS03] J. Jacod and A.N. Shiryaev, *Limit theorems for stochastic processes*, 2nd ed., Springer, 2003.
- [KMK10] J. Kallsen and J. Muhle-Karbe, *Exponentially affine martingales, affine measure changes and exponential moments of affine processes*, Stochastic Processes and their Applications **120** (2010), no. 2, 163–181.
- [KO88] T.G. Kurtz and D.L. Ocone, *Unique characterization of conditional distributions in nonlinear filtering*, Annals of Probability **16** (1988), no. 1, 80–107.
- [KRM15] M. Keller-Ressel and E. Mayerhofer, *Exponential moments of affine processes*, Annals of Applied Probability **25** (2015), no. 2, 714–752.
- [KRST11] M. Keller-Ressel, W. Schachermayer, and J. Teichmann, *Affine processes are regular*, Probability Theory and Related Fields **151** (2011), no. 3, 591–611.
- [LS01] R.S. Liptser and A.N. Shiryaev, *Statistics of Random Processes, Volumes I and II.*, 2nd ed., Springer, 2001.
- [LSZ15] K. Law, A. Stuart, and K. Zygalkakis, *Data assimilation*, Springer-Verlag, New York, 2015.
- [MMKS11] E. Mayerhofer, J. Muhle-Karbe, and A.G. Smirnov, *A characterization of the martingale property of exponentially affine processes*, Stochastic Processes and their Applications **121** (2011), no. 3, 568–582.
- [Roc70] R.T. Rockafellar, *Convex analysis*, Princeton University Press, 1970.
- [RW00] L.C.G. Rogers and D. Williams, *Diffusions, Markov Processes, and Martingales*, second ed., vol. 1, Cambridge University Press, 2000.
- [RY99] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, 3rd ed., Springer, 1999.
- [SV10] P. Spreij and E. Veerman, *The affine transform formula for affine jump-diffusions with general closed convex state spaces.*, Preprint arXiv:1005.1099 (2010).
- [vH07] R. van Handel, *Filtering, stability, and robustness*, Ph.D. thesis, California Institute of Technology, 2007.

LUKAS GONON, EIDGENÖSSISCHE TECHNISCHE HOCHSCHULE ZÜRICH, SWITZERLAND
 E-mail address: lukas.gonon@math.ethz.ch

JOSEF TEICHMANN, EIDGENÖSSISCHE TECHNISCHE HOCHSCHULE ZÜRICH, SWITZERLAND
 E-mail address: josef.teichmann@math.ethz.ch