

EXISTENCE AND UNIQUENESS RESULTS FOR TIME-INHOMOGENEOUS TIME-CHANGE EQUATIONS AND FOKKER–PLANCK EQUATIONS

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ABSTRACT. We prove existence and uniqueness of solutions to Fokker–Planck equations associated to Markov operators multiplicatively perturbed by degenerate time-inhomogeneous coefficients. Precise conditions on the time-inhomogeneous coefficients are given. In particular, we do not necessarily require the coefficients to be neither globally bounded nor bounded away from zero. The approach is based on constructing random time-changes and studying related martingale problems for Markov processes with values in locally compact, complete and separable metric spaces.

Key words and phrases: Fokker–Planck equation, forward Kolmogorov equation, Markov process, martingale problem, random time-change, time-dependent operator.

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1. INTRODUCTION

The study of existence and uniqueness of solutions to Fokker–Planck equations, also known as forward Kolmogorov equations, is a classical topic of great current interest. One reason are the numerous applications which arose over the past decades such as in the theory of stochastic processes and of (partial) differential equations.

In this article, we consider the Fokker–Planck equation associated to a linear operator A which is assumed to be the (infinitesimal) generator of a Markov process with values in a metric space E (for instance $E = \mathbb{R}^d$ and A is an integro-differential operator), and to a degenerate *time-inhomogeneous* coefficient σ . More precisely, we establish sufficient conditions such that there exists a unique family of probability measures $(p(t, \cdot))_{t \in [0, t_0]}$ on E solving the Fokker–Planck equation

$$(1.1) \quad \int_E f(x) p(t, dx) - \int_E f(x) \mu_0(dx) = \int_0^t \int_E \sigma(s, x) \mathcal{A}f(x) p(s, dx) ds, \quad t \in [0, t_0],$$

for all “sufficiently nice” test functions f and given the initial condition μ_0 and $t_0 > 0$. We refer to Theorem 4.1 for the exact formulation of our existence and uniqueness result. Let us point out that the coefficient σ is not only time-dependent but also σ is neither (necessarily) globally bounded nor bounded away from zero.

For a *time-homogeneous* coefficient σ , the existence and uniqueness result provided in Theorem 4.1 is well-known, see, e.g., the book [EK86] or [Kur98]. Furthermore, classical results on multiplicative perturbations of Feller generators and time-changed Lévy processes allow for weak regularity assumptions on σ , see e.g. [BSW13, Thm. 4.1] and the original reference [Lum73], [ES85] and the references therein. However, note that these results do not deal with the question whether the Fokker–Planck equation uniquely determines the law of the time-changed process. For a *time-inhomogeneous* coefficient σ , existence and uniqueness has been studied in various situations, for instance, for σ bounded away from 0, we refer to [Str75], [Bas88], [BC09] and further references therein and, for globally bounded σ and \mathcal{A} being the generator of a diffusion,

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see [Fig08]. However, to the best of our knowledge, the previous literature does not cover our assumptions on the operator \mathcal{A} and σ .

An application, where such general conditions on the coefficient σ are essential, can be found in the accompanying paper [DGPR19] studying the solvability of the Skorokhod embedding problem (SEP) for Lévy processes L . Let us recall that the version of the SEP dealt with in [DGPR19] asks to find an integrable stopping time τ such that L_τ is distributed according to a given law μ_1 . The key idea of the approach developed in [DGPR19] is to find a construction of a coefficient σ such that there exists a unique solution $(p(t, \cdot))_{t \in [0, t_0]}$ to the Fokker–Planck equation (1.1) with $p(t_0, \cdot) = \mu_1$, where \mathcal{A} is assumed to be the generator of the given Lévy process. Needless to say, for this construction to work, weak regularity conditions on σ are crucially required. Moreover, the general setting considered in the present article paves the way to apply the approach of [DGPR19] to solve the Skorokhod embedding problem for other stochastic processes as well, such as Markov chains or multi-dimensional Brownian motion.

Apart from the indisputably justified interest of investigating Fokker–Planck equations on its own, further motivation to prove existence and uniqueness results for Fokker–Planck equations stems from its long list of recent applications. Just to name a few, let us mention the explicit construction of peacocks (see [HPRY11, Chapter 6]), the construction of (martingale) diffusions matching prescribed marginal distributions at given (random) times (see e.g. [CHO11] and [EHJT13]) or Dupire’s formula in mathematical finance (see [Dup94] and [CGMY04]). Further potential applications include probabilistic representations of the solution to irregular porous media type equations [BRR11] and measure-valued martingales [VBB⁺17]. The results obtained in the present article will be a useful tool when extending any of these applications to more general Markov processes such as, e.g., Lévy type processes and the associated generators.

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Organization of the paper: In Section 2 the notation, definitions and assumptions are introduced. Section 3 is devoted to the construction of a solution to the martingale problem and to the Fokker–Planck equation. The existence and uniqueness result (Theorem 4.1) for Fokker–Planck equations with degenerate time-inhomogeneous coefficients is proven in Section 4. Auxiliary results for the presented construction of random time-changes are provided in Section A.

2. NOTATION, DEFINITIONS AND ASSUMPTIONS

Our definitions and notation are precisely as in [EK86] with two exceptions pointed out in Remark 2.1.

Throughout the whole article, the underlying stochastic basis consists of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions of completeness and right-continuity. Moreover, we fix a locally compact, complete, separable metric space (E, d) with metric d and denote by $\mathcal{B}(E)$ its Borel σ -algebra. For $x \in E$ and $\varepsilon > 0$, set $B_\varepsilon(x) := \{y \in E : d(x, y) < \varepsilon\}$. $\mathcal{P}(E)$ denotes the set of probability measures on $(E, \mathcal{B}(E))$.

The space of continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfying $\lim_{|x| \rightarrow \infty} f(x) = 0$ is denoted by $C_0(\mathbb{R})$. For $n \in \mathbb{N}$ let $C_0^n(\mathbb{R})$ be the subset of functions $f \in C_0(\mathbb{R})$ such that f is n -times differentiable and all derivatives of order less or equal to n belong to $C_0(\mathbb{R})$ and we set $C_0^\infty(\mathbb{R}) := \bigcap_{n \in \mathbb{N}} C_0^n(\mathbb{R})$. The spaces of functions with compact support $C_c(\mathbb{R})$, $C_c^n(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$ are defined analogously. The space $D_E[0, \infty)$ stands for all maps $\omega: [0, \infty) \rightarrow E$ which are right-continuous and have a left-limit at each point $t \in [0, \infty)$ (short: RCLL paths). The space $D_E[0, \infty)$ is equipped with the Skorokhod topology ($J1$), see [EK86, Chap. 3, Sec. 5] for the precise definition. $B(E)$ denotes the space of real-valued, bounded, measurable functions on E and $\|\cdot\|$ is the sup-norm. $C_0(E)$ denotes the space of continuous functions that vanish at infinity and $C_b(E)$ the space of bounded continuous functions on E . A sequence $\{f_n\}_{n \in \mathbb{N}} \subset B(E)$ converges *boundedly and pointwise* to $f \in B(E)$ (denoted by $\text{bp-lim}_{n \rightarrow \infty} f_n = f$) if $\sup_n \|f_n\| < \infty$ and $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all

$x \in E$. $U \subset B(E)$ is called *bp-closed*, if $\{f_n\}_{n \in \mathbb{N}} \subset U$ and $\text{bp-lim}_{n \rightarrow \infty} f_n = f$ implies $f \in U$. For $V \subset B(E)$, we define $\text{bp-closure}(V)$ as the smallest subset of $B(E)$ which is bp-closed and contains V . A sequence $\{(f_n, g_n)\}_{n \in \mathbb{N}} \subset B(E) \times B(E)$ is said to converge to $(f, g) \in B(E) \times B(E)$ boundedly and pointwise (denoted by $\text{bp-lim}_{n \rightarrow \infty} (f_n, g_n) = (f, g)$) if $\text{bp-lim}_{n \rightarrow \infty} f_n = f$ and $\text{bp-lim}_{n \rightarrow \infty} g_n = g$. The definitions of bp-closed and bp-closure are then defined analogously for subsets of $B(E) \times B(E)$.

Following the martingale problem approach in [EK86], consider a given $D(\mathcal{L}) \subset C_b(E)$ and $\mathcal{L}: D(\mathcal{L}) \rightarrow C_b(E)$ linear. $(D(\mathcal{L}), \mathcal{L})$ is said to be *conservative* if

$$(2.1) \quad \text{there exists } \{h_n\}_{n \in \mathbb{N}} \subset D(\mathcal{L}) \text{ such that } (1, 0) = \text{bp-lim}_{n \rightarrow \infty} (h_n, \mathcal{L}h_n).$$

For a stochastic process $(X_t)_{t \geq 0}$ we set $\mathcal{F}_t^X := \sigma(X_s : s \leq t)$. A *solution to the martingale problem for (\mathcal{L}, μ)* is a progressively measurable E -valued stochastic process $(X_t)_{t \geq 0}$ defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that for each $h \in D(\mathcal{L})$, the process

$$(2.2) \quad h(X_t) - h(X_0) - \int_0^t \mathcal{L}h(X_s) ds, \quad t \geq 0,$$

is an $(\mathcal{F}_t^X)_{t \geq 0}$ -martingale and $\tilde{\mathbb{P}} \circ X_0^{-1} = \mu$. *Uniqueness* is said to hold for the martingale problem for (\mathcal{L}, μ) if any two solutions X, \tilde{X} have the same finite-dimensional distributions. The martingale problem for (\mathcal{L}, μ) is said to be *well-posed* if there exists a solution and uniqueness holds. A solution to the $D_E[0, \infty)$ -martingale problem (or RCLL-martingale problem) for (\mathcal{L}, μ) is an RCLL process that is a solution to the martingale problem for (\mathcal{L}, μ) . *Uniqueness* is said to hold for the RCLL-martingale problem for (\mathcal{L}, μ) if any two solutions to the RCLL-martingale problem for (\mathcal{L}, μ) have the same law on $D_E[0, \infty)$. The RCLL-martingale problem for (\mathcal{L}, μ) is said to be *well-posed* if there exists a solution and uniqueness holds.

Similarly, for a linear operator $(D(\mathcal{A}), \mathcal{A})$ and a measurable function $\sigma: [0, \infty) \times E \rightarrow [0, \infty)$, a solution to the (time-inhomogeneous) martingale problem for $(\sigma\mathcal{A}, \mu)$ is a progressively measurable E -valued process X defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that for each $f \in D(\mathcal{A})$ the process

$$(2.3) \quad f(X_t) - f(X_0) - \int_0^t \sigma(s, X_s) \mathcal{A}f(X_s) ds, \quad t \geq 0,$$

is an $(\mathcal{F}_t^X)_{t \geq 0}$ -martingale and $\tilde{\mathbb{P}} \circ X_0^{-1} = \mu$. Uniqueness, well-posedness and the corresponding concepts among $D_E[0, \infty)$ -processes are defined analogously to the time-homogeneous case. For $\mu \in \mathcal{P}([0, \infty))$ and $\nu \in \mathcal{P}(E)$ we write $\mu \otimes \nu$ for the product measure generated by μ and ν on $[0, \infty) \times E$. If F is a measurable space, δ_x denotes the Dirac measure at $x \in F$. For $V \subset B(E)$, $\text{span}(V)$ denotes the smallest linear subspace of $B(E)$ containing V , i.e. the set of all finite linear combinations of elements of V .

Remark 2.1. *The notation differs from the book [EK86] in two following respects: Firstly, in [EK86] $(D(\mathcal{L}), \mathcal{L})$ is said to be conservative if*

$$(2.4) \quad (1, 0) \in \text{bp-closure}(\{(h, \mathcal{L}h) : h \in D(\mathcal{L})\}).$$

While our requirement (2.1) implies (2.4), the converse is not true in general, cf. [EK86, Chap. 3, Sec. 4].

Secondly, (not necessarily RCLL) solutions to martingale problems are required to be progressively measurable in our context, as in [Kur98].

Remark 2.2. *The motivation for the definition of bp-closure is as follows: Suppose $(D(\mathcal{L}), \mathcal{L})$ is a linear operator on $C_b(E)$ and X is a solution to the martingale problem for (\mathcal{L}, μ) for some $\mu \in \mathcal{P}(E)$. Then by the dominated convergence theorem for conditional expectations, the set of*

functions $(h, g) \in B(E) \times B(E)$, for which

$$(2.5) \quad h(X_t) - h(X_0) - \int_0^t g(X_s) ds, \quad t \geq 0,$$

is an $(\mathcal{F}_t^X)_{t \geq 0}$ -martingale, is bp-closed and so (2.5) is a martingale for all $(h, g) \in \text{bp-closure}(\{(h, \mathcal{L}h) : h \in D(\mathcal{L})\})$.

Finally, we provide two definitions analogous to the conditions imposed in [Sat99, Thm. 35.4 (iii)] and [EK86, Chap. 6, Thm. 1.1]. The first one is a (strong) recurrence property, the second is essential for studying uniqueness of time-change equations (3.1). Denote by Z the coordinate process on $D_E[0, \infty)$.

Definition 2.3. A probability measure P on $(D_E[0, \infty), \mathcal{B}(D_E[0, \infty)))$ is called recurrent if

$$(2.6) \quad P \left(\int_0^\infty \mathbb{1}_{B_a(Z_0)}(Z_t) dt = \infty \right) = 1 \quad \text{for every } a > 0.$$

Definition 2.4. Let $H: E \rightarrow [0, \infty)$ be measurable and $P \in \mathcal{P}(D_E[0, \infty))$. H is called regular for P if P -a.s.

$$\inf \left\{ s \in [0, \infty) : \int_0^s H(Z_u)^{-1} du = \infty \right\} = \rho \quad \text{and} \quad H(Z_\rho) = 0 \quad \text{on } \{\rho < \infty\}$$

where

$$\rho := \inf \{s \in [0, \infty) : H(Z_s) = 0\}.$$

Example 2.5. Let $E = \mathbb{R}$ and Z be a P -Brownian motion. Suppose $H: \mathbb{R} \rightarrow [0, \infty)$ satisfies

$$(2.7) \quad \{x \in \mathbb{R} : H(x) = 0\} = I(H),$$

where $I(H)$ is the closed set

$$(2.8) \quad I(H) := \left\{ x \in \mathbb{R} : \forall \varepsilon > 0 : \int_{x-\varepsilon}^{x+\varepsilon} \frac{dy}{H(y)} = \infty \right\}.$$

Then H is regular for P . Indeed, since $I(H)$ in (2.8) is closed and (2.7) holds, $H^{-1}(\{0\})$ is closed. Hence, for P -a.e. ω with $\rho(\omega) < \infty$, $H(Z_{\rho(\omega)}(\omega)) = 0$ by (right-)continuity. Hence, the second part of the definition is established and the first part follows directly from (2.7) and [KS91, Chap. 5, Lem. 5.2].

2.1. Assumptions. The following assumptions are used at different places throughout the article. Our set of assumptions is split in such a way that we can distinguish as good as possible between assumptions on the stochastic process and on the time-inhomogeneous coefficient σ .

Assumption 2.6 (Regularity of σ). Let $t_0 > 0$ and $\sigma: [0, \infty) \times E \rightarrow [0, \infty)$ be given. Suppose σ is of the form $\sigma(t, x) := H(x)\tilde{\sigma}(t, x)$ for $(t, x) \in [0, \infty) \times E$ with $\tilde{\sigma}(t, x) \equiv 0$ for $t > t_0$ and such that

- (i) $H: E \rightarrow [0, \infty)$ is measurable,
- (ii) $\tilde{\sigma}: [0, t_0] \times E \rightarrow (0, \infty)$ is measurable and satisfies the following: for each compact set $K \subset E$ and $S \in (0, t_0)$ there exists $C_1, C_2, C_3 > 0$ such that

$$|\tilde{\sigma}(t, x) - \tilde{\sigma}(s, x)| \leq C_1|t - s| \quad \text{and} \quad C_2 \leq \tilde{\sigma}(t, x) \leq C_3,$$

for all $s, t \in [0, S]$ and for all $x \in K$, where C_3 does not depend on S (but it may depend on K).

Assumption 2.7. Let $\mathcal{D} \subset C_0(E)$ and $\mathcal{A}: \mathcal{D} \rightarrow C_0(E)$ be linear so that

- (i) $(\mathcal{D}, \mathcal{A})$ is conservative, \mathcal{D} is dense in $C_0(E)$ and an algebra¹ in $C_0(E)$,
- (ii) for any $\mu_0 \in \mathcal{P}(E)$, the RCLL-martingale problem for (\mathcal{A}, μ_0) is well-posed.

¹That means $\mathcal{D} \subset C_0(E)$ is an algebra with respect to the addition and multiplication induced by $C_0(E)$.

Remark 2.8. *The most important special case of Assumption 2.7 (ii) is $\mu_0 = \delta_x$. In this case the corresponding law on $D_E[0, \infty)$ of the RCLL-solution to the martingale problem is denoted by P_x .*

Example 2.9. *Let \bar{A} be the generator of a Feller semigroup on $C_0(E)$ with domain $D(\bar{A})$ and \mathcal{D} be a core for \bar{A} (see [EK86]). Suppose \mathcal{D} is an algebra in $C_0(E)$ and denote by \mathcal{A} the restriction of \bar{A} to \mathcal{D} . Then \mathcal{D} is dense, $(\mathcal{D}, \mathcal{A})$ satisfies the assumptions of [EK86, Chap. 4, Thm. 2.2] and thus, by [EK86, Chap. 4, Cor. 2.8], $(\mathcal{D}, \mathcal{A})$ is conservative in the sense of (2.4) (from the proof of [EK86, Chap. 4, Cor. 2.8] also conservative in our sense (2.1)). Furthermore, by [EK86, Chap. 4, Thm. 2.7 and Thm. 4.1], for any $\mu_0 \in \mathcal{P}(E)$ the martingale problem for (\mathcal{A}, μ_0) is well-posed and the solution has sample paths in $D_E[0, \infty)$. Hence, Assumption 2.7 indeed holds.*

Assumption 2.10 (Regularity of H). *Let $(P_x)_{x \in E}$ as in Assumption 2.7 and $H: E \rightarrow [0, \infty)$ measurable. Assume that for any $x \in E$, H is regular for P_x (in the sense of Definition 2.4).*

Assumption 2.10 is needed to guarantee uniqueness of the time-change equations (3.1). Proposition A.2 provides a useful criterion to verify it.

Assumption 2.11 (Recurrence and boundedness). *Let $(\mathcal{D}, \mathcal{A})$ and $(P_x)_{x \in E}$ be as in Assumption 2.7 and Remark 2.8 and let $t_0 > 0$ and $\sigma = H\tilde{\sigma}$ be as in Assumption 2.6. Assume that*

- (i) *for any $x \in E$, P_x is recurrent (in the sense of Definition 2.3) and H is bounded on compacts,*
- (ii) *$\sigma \mathcal{A}f \in C_0([0, t_0] \times E)$ for all $f \in \mathcal{D}$.*

Assumption 2.11 (ii) can be seen as a weak locality assumption, which is always satisfied for Brownian motion:

Example 2.12. *Set $\mathcal{D} := C_c^\infty(\mathbb{R})$ and $\mathcal{A}f(x) := \frac{1}{2}f''(x)$ for $f \in \mathcal{D}$. By [EK86, Chap. 5, Prop. 1.1] and Example 2.9, Assumption 2.7 is satisfied and under P_x the canonical process Z is a Brownian motion started from x so (the first part of) Assumption 2.11 (i) holds. A sufficient condition for Assumption 2.11 (ii) to hold is that σ is continuous on $[0, t_0] \times \mathbb{R}$: If this is true, then $(t, x) \mapsto \sigma(t, x)\mathcal{A}f(x)$ is continuous and even compactly supported for all $f \in \mathcal{D}$.*

Assumption 2.13 (Boundedness of σ). *$\sigma: [0, \infty) \times E \rightarrow [0, \infty)$ is bounded.*

As we will see in Section 3, Assumption 2.11 (i) or 2.13 ensure that there exists a non-exploding solution to the time-change equation (3.1) below.

3. TIME-INHOMOGENEOUS TIME-CHANGES FOR MARKOV PROCESSES

Given a Markov process M with generator \mathcal{A} and a sufficiently regular time-inhomogeneous coefficient σ , our aim is to obtain a Markov process X with generator $\sigma \mathcal{A}$. The new Markov process X is identified as a time-change of M , where the time-change τ is characterized by the pathwise Carathéodory differential equation

$$(3.1) \quad \tau(t) = \int_0^t \sigma(s, M_{\tau(s)}) ds, \quad t \in [0, t_0].$$

3.1. Constructing the Time-Change. Due to the time-inhomogeneity of the coefficient σ in the differential equation (3.1), we need to include the time variable t in the state space of the time-changed process X . This time-inhomogeneity prevents us to directly rely on well-known results as for example [EK86, Chap. 6, Thm. 1.1]. Therefore, we verify as a first step that equation (3.1) indeed has a solution.

Lemma 3.1. *Let M be an E -valued process on $(\Omega, \mathcal{F}, \mathbb{P})$ with \mathbb{P} -almost surely RCLL sample paths. Denote by P the law on $D_E[0, \infty)$ of M . Assume that*

- $t_0 > 0$ and $\sigma = H\tilde{\sigma}$ are given as in Assumption 2.6,

- H is regular for P (in the sense of Definition 2.4),
- either H is bounded on compacts and P is recurrent (see Definition 2.3) or Assumption 2.13 holds.

Then there exists a family of random times $(\tau(t))_{t \in [0, t_0]}$ such that

- (i) $\tau: [0, t_0] \rightarrow [0, \infty)$ is non-decreasing and absolutely continuous, \mathbb{P} -a.s.,
- (ii) $\tau(t_0)$ is finite, \mathbb{P} -a.s.,
- (iii) τ solves the Carathéodory differential equation (3.1) for M .

Proof. Using the conventions $\inf \emptyset := \infty$ and $[0, 0) := \{0\}$, we define the random time

$$(3.2) \quad \rho := \inf \left\{ s \in [0, \infty) : \int_0^s H(M_u)^{-1} du = \infty \right\}$$

and the random times

$$(3.3) \quad \tau(t) := \begin{cases} \inf\{s \in [0, \rho) : \mathcal{T}(s) \geq t\} \wedge \rho & \text{if } t \in [0, t_0) \\ \sup_{s \in [0, t_0)} \inf\{r \in [0, \rho) : \mathcal{T}(r) \geq s\} \wedge \rho & \text{if } t = t_0 \end{cases}$$

where \mathcal{T} is given by the Carathéodory differential equation

$$(3.4) \quad \mathcal{T}(s) = \int_0^s \sigma(\mathcal{T}(r), M_r)^{-1} dr, \quad s \in [0, \tau(t_0)).$$

In other words, τ is the right inverse of \mathcal{T} until σ becomes 0, then one sets $\tau = \rho$. To prove that $(\tau(t))_{t \in [0, t_0]}$ is well-defined, it is sufficient to show that equation (3.4) has indeed a unique solution on the interval $[0, \tau(t) \wedge T]$ for every $t \in [0, t_0)$ and $T \in [0, \rho)$. Let us fix $t \in (0, t_0)$, an RCLL sample path of M denoted by $(M_s(\omega))_{s \in [0, T]}$ for $\omega \in \Omega$ and $T \in [0, \rho(\omega))$. By the RCLL property, the path $(M_s(\omega))_{s \in [0, T]}$ is contained in a compact set $K \subset E$, i.e. $\{M_s(\omega) : s \in [0, T]\} \subset K$, and $N(\omega) := \{s \in [0, T] : H(M_s(\omega)) = 0\}$ is a Lebesgue null set by definition of ρ . Therefore, $\sigma(r, M_s(\omega)) > 0$ and so $\gamma(r, s) := \sigma(r, M_s(\omega))^{-1}$ is well-defined for $r \in [0, t]$ and $s \in [0, T] \setminus N(\omega)$ and we set $\gamma(\cdot, s) := 1$ for $s \in N(\omega)$. Since $\{M_s(\omega) : s \in [0, T]\} \subset K$, by Assumption 2.6 on $\sigma = H\tilde{\sigma}$ there exist $C_1, C_2 > 0$ such that $C_2 H(M_t)^{-1} \leq \gamma(u, s)$ and

$$|\gamma(u, s) - \gamma(v, s)| = H(M_s(\omega))^{-1} \left| \frac{\tilde{\sigma}(u, M_s(\omega)) - \tilde{\sigma}(v, M_s(\omega))}{\tilde{\sigma}(u, M_s(\omega))\tilde{\sigma}(v, M_s(\omega))} \right| \leq C_2^2 C_1 H(M_s(\omega))^{-1} |u - v|$$

for all $u, v \in [0, t]$ and $s \in [0, T]$. Thus, γ satisfies the assumptions of Lemma A.1, which says that there exists a unique solution \mathcal{T} of the Carathéodory differential equation (3.4) on the interval $[0, \tau(t) \wedge T]$. Moreover, since now $\tau(s)$ is well-defined for all $s \in [0, t_0)$ and $\tau(t_0) = \sup_{s < t_0} \tau(s)$, $\tau(t_0)$ is also well-defined.

Note that if Assumption 2.6 (ii) holds also for $S = t_0$, we set

$$\tau(t) := \inf\{s \in [0, \rho) : \mathcal{T}(s) \geq t\} \wedge \rho, \quad t \in [0, t_0],$$

instead of (3.3) and the above argument works for $t = t_0$ as well.

(i) By definition of \mathcal{T} through equation (3.4), \mathcal{T} is absolutely continuous and strictly increasing on $[0, \tau(t))$ for every $t \in [0, t_0)$ and thus invertible with $\tau(s) = \mathcal{T}^{-1}(s)$ for $s \in [0, t)$. This implies that τ is also non-decreasing and absolutely continuous on $[0, t_0]$ (cf. [Leo09, Thm. 1.7 and Ex. 3.21]).

(ii) To verify that $\tau(t_0) < \infty$, \mathbb{P} -a.s., suppose first H is bounded on compacts and P is recurrent. Let $N := \{\omega \in \Omega : \tau(t_0)(\omega) = \infty\}$. Then for $\omega \in N$, equation (3.4) has a solution on $[0, \infty)$ and $\mathcal{T}(t)(\omega) < t_0$ for all $t \geq 0$. Fixing some $a > 0$, we notice that by Assumption 2.6 and since H is bounded on compacts, there exists a constant $C_1 > 0$ such that $\sigma(\mathcal{T}(s), M_s)^{-1} \geq C_1$ for

$s \in \{t \geq 0 : M_t(\omega) \in B_a(M_0(\omega))\}$ and $\omega \in N$. We therefore have

$$(3.5) \quad \begin{aligned} t_0 \geq \lim_{t \rightarrow \infty} \mathcal{T}(t) &= \int_0^\infty \sigma(\mathcal{T}(s), M_s)^{-1} ds \geq \int_0^\infty \mathbb{1}_{B_a(M_0)}(M_s) \sigma(\mathcal{T}(s), M_s)^{-1} ds \\ &\geq C_1 \int_0^\infty \mathbb{1}_{B_a(M_0)}(M_s) ds \end{aligned}$$

on N . However, the right-hand side of (3.5) is infinite, \mathbb{P} -a.s., by the recurrence assumption and hence (3.5) can only hold on a null set. Thus N is a \mathbb{P} -null set, i.e. $\tau(t_0) < \infty$, \mathbb{P} -a.s., as claimed. Supposing Assumption 2.13 holds, a similar argument works.

(iii) For every $t \in [0, t_0]$ such that $\tau(t) < \rho$, one observes that

$$(3.6) \quad 1 = \frac{d}{ds}(\mathcal{T}(\tau(s))) = \frac{d}{ds}\mathcal{T}(\tau(s)) \frac{d}{ds}\tau(s) = \sigma(\tau(s), M_{\tau(s)})^{-1} \frac{d}{ds}\tau(s)$$

for almost all $s \in [0, \mathcal{T}(\tau(t))]$, where the chain rule (c.f. [Leo09, Thm. 3.44]) and (3.4) was used. Therefore, the absolute continuity of τ and identity (3.6) show that τ indeed solves the desired integral equation (3.1) since

$$\tau(t) = \int_0^t \frac{d}{ds}\tau(s) ds = \int_0^t \sigma(s, M_{\tau(s)}) ds.$$

For every $t \in [0, t_0]$ such that $\tau(t) \geq \rho$, we denote $\chi := \inf\{t \in [0, t_0] : \tau(t) \geq \rho\}$. Notice that τ is constant (equal to ρ) on $[\chi, t_0]$ and, \mathbb{P} -a.s., $H(M_\rho) = 0$, by the assumption that H is regular for the law of M .² In particular, $\tau(t)$ satisfies equation (3.1) for every $t \in [\chi, t_0]$ as well. Recall that the assumption on H means that ρ defined in (3.2) satisfies

$$(3.7) \quad H(M_\rho) = 0 \text{ on } \{\rho < \infty\}, \quad \mathbb{P}\text{-a.s.},$$

$$(3.8) \quad \rho = \inf\{s \in [0, \infty) : H(M_s) = 0\}, \quad \mathbb{P}\text{-a.s.}$$

□

In order to create a better understanding of the time-change τ (defined by the Carathéodory differential equation (3.1)) and the assumptions of Lemma 3.1, two remarks are provided for the special case of Brownian motion.

Remark 3.2. *As we have seen in the proof of Lemma 3.1, (2.6) or Assumption 2.13 is required to ensure that the random time $\tau(t_0)$ is \mathbb{P} -almost surely finite. More precisely, we used*

$$(3.9) \quad \mathcal{T}(t) = \int_0^t \sigma(\mathcal{T}(s), M_s)^{-1} ds = t_0$$

for some finite $t \in [0, \infty)$, \mathbb{P} -a.s. For example, if M is a Brownian motion under \mathbb{P} , it is possible to verify condition (3.9) a posteriori by the uniqueness in law of the time-changed process $X_s := M_{\tau(s)}$ for $s \in [0, t_0]$ as done by Bass [Bas83].

Remark 3.3. *Let us consider the time-homogeneous case where M is a Brownian motion under \mathbb{P} . Suppose that $\sigma = H$ and we have a unique finite solution $(\tau(t))_{t \in [0, t_0]}$ to the differential equation (3.1). Then the time-changed process $X_t := M_{\tau(t)}$ is a weak solution to the Brownian stochastic differential equation (SDE)*

$$dX_t = \sqrt{H(X_t)} dM_t, \quad X_0 = x_0 \in \mathbb{R}, \quad t \in [0, t_0].$$

From Engelbert and Schmidt [ES85] (see also [KS91, Chap. 5, Thm. 5.7]) we know that a solution to this SDE exists and uniqueness in law holds for this SDE if and only if $\{x \in \mathbb{R} : H(x) = 0\} = I(H)$. Such a function H is regular for (the law of) M under \mathbb{P} in the sense of Definition 2.4, see Example 2.5. For example, if $H(x) = |x|^\alpha$ for $\alpha \in (0, 1)$, then there is no uniqueness in

² In fact, here only (3.7) is used and so the statement of the Lemma 3.1 could be modified accordingly. (3.8) is only used for uniqueness of the time-change in Lemma A.3 below.

law. However, the so-called fundamental solution in sense of [ES85] is unique in law: The weak solution of the SDE satisfying $H(X_s(\omega)) > 0$ Leb \otimes \mathbb{P} -almost surely is unique in law, i.e. a solution that does not spend time at the zeros of H . This is exactly how we construct the time-change: No time is spent at the zeros of H until the first time at which $1/H$ is not integrable anymore, then we stop.

3.2. Solution to the Associated Martingale Problem. In the next lemma we link the martingale problem for the given process M to the martingale problem for the time-changed process X .

Lemma 3.4. *Let σ , M and $(\mathcal{F}_t)_{t \geq 0}$ be given as in Lemma 3.1. For $(\tau(t))_{t \in [0, t_0]}$ as in Lemma 3.1 the process $(X_t)_{0 \leq t \leq t_0}$ is defined by $X_t := M_{\tau(t)}$. Suppose that for some $f, g \in C_0(E)$ the process*

$$M_t^{f,g} := f(M_t) - f(M_0) - \int_0^t g(M_s) ds, \quad t \in [0, t_0],$$

is an (\mathcal{F}_t) -martingale and σg is bounded. Then the process $(\tilde{M}_s^{f,g})_{0 \leq s \leq t_0}$, given by

$$\tilde{M}_t^{f,g} := f(X_t) - f(X_0) - \int_0^t \sigma(s, X_s) g(X_s) ds, \quad t \in [0, t_0],$$

is a martingale w.r.t. the right-continuous completion of the filtration generated by X .

Proof. First observe that $\tilde{M}_t^{f,g} = M_{\tau(t)}^{f,g}$ for $t \in [0, t_0]$. Indeed, since $\tau(\cdot)$ is monotone and absolutely continuous on $[0, t_0]$ and the function $s \mapsto g(M_s)$ is integrable on $[0, \tau(t_0)]$, \mathbb{P} -a.s., a change of variables (cf. [Leo09, Cor. 3.57]) leads to

$$\int_0^{\tau(t)} g(M_s) ds = \int_0^t g(M_{\tau(s)}) \sigma(s, M_{\tau(s)}) ds = \int_0^t \sigma(s, X_s) g(X_s) du, \quad t \in [0, t_0],$$

and thus $\tilde{M}_t^{f,g} = M_{\tau(t)}^{f,g}$.

Therefore, it is sufficient to verify that $(M_{\tau(t)}^{f,g})_{t \in [0, t_0]}$ is a martingale. For this purpose we rely on the optional sampling theorem (see e.g. [EK86, Chap. 2, Thm. 2.13]) and check its conditions: Because $f \in C_0(E)$ and σg is bounded, there exists a constant $C := C(f, g) > 0$ with $|\tilde{M}_t^{f,g}| \leq C$ for $t \in [0, t_0]$ and in particular $\sup_{t \in [0, t_0]} \mathbb{E}[|\tilde{M}_t^{f,g}|] < \infty$. Since $\tau(s)$ is finite for every $s \in [0, t_0]$, \mathbb{P} -a.s., and since for every $T < \tau(s)$ there exists $\tilde{s} \in [0, s]$ such that $\tau(\tilde{s}) = T$, we have $|M_T^{f,g}| = |M_{\tau(\tilde{s})}^{f,g}| = |\tilde{M}_{\tilde{s}}^{f,g}| \leq C$ and

$$\lim_{T \rightarrow \infty} \mathbb{E}[|M_T^f| \mathbb{1}_{\{\tau(s) > T\}}] \leq C \lim_{T \rightarrow \infty} \mathbb{P}(\tau(s) > T) = 0.$$

Hence, for $u, v \in [0, t_0]$ with $u \leq v$ the optional sampling theorem gives

$$\mathbb{E}[M_{\tau(v)}^{f,g} | \mathcal{F}_{\tau(u)}] = M_{\tau(u)}^{f,g},$$

which means that $(\tilde{M}_t^{f,g})_{t \in [0, t_0]}$ is an $(\mathcal{F}_{\tau(t)})$ -martingale. Because $\tilde{M}_t^{f,g}$ is measurable with respect to $\sigma(X_s : s \leq t)$ for $t \in [0, t_0]$, $\tilde{M}_t^{f,g}$ is also a martingale with respect to the usual augmentation of the filtration generated by X , see e.g. [RY99, Thm. II.2.8] or [RW00, Lem. II.67.10]. \square

Based on the previous Lemma 3.4, the time-changed process X is a solution to the “time-changed” martingale problem and the marginal distributions of X satisfy the corresponding Fokker–Planck equation:

Proposition 3.5. *Let $\mu_0 \in \mathcal{P}(E)$, $\mathcal{D} \subset C_0(E)$ and $\mathcal{A}: \mathcal{D} \rightarrow C_0(E)$ be linear. Let σ and M be given as in Lemma 3.1 and assume in addition that M is a solution on $(\Omega, \mathcal{F}, \mathbb{P})$ to the RCLL-martingale problem for (\mathcal{A}, μ_0) and either Assumption 2.11 (ii) or Assumption 2.13 holds. Let us denote by $p(t, \cdot)$ the law of $X_t = M_{\tau(t)} = M_{\int_0^t \sigma(s, X_s) ds}$ as constructed in Lemma 3.4 with $p(0, \cdot) = \mu_0$ and $\tau(t) := \tau(t_0)$ for $t > t_0$, where t_0 is as in Assumption 2.6. Then one has:*

- X is a solution to the (time-inhomogeneous) $D_E[0, \infty)$ -martingale problem for $(\sigma\mathcal{A}, \mu_0)$,
- for any $g \in B([0, \infty) \times E)$ the function $s \mapsto \int_E g(s, x) p(s, dx)$ is measurable,
- $(p(s, dx))_{s \in [0, t_0]}$ satisfies the Fokker–Planck equation, i.e., for any $f \in \mathcal{D}$,

$$(3.10) \quad \int_E f(x) p(t, dx) - \int_E f(x) \mu_0(dx) = \int_0^t \int_E \sigma(s, x) \mathcal{A}f(x) p(s, dx) ds, \quad t \in [0, t_0].$$

Proof. If $f \in \mathcal{D}$, then $\sigma\mathcal{A}f$ is bounded by Assumption 2.11 (ii) or 2.13. Combining this with our assumption that M is a solution to the RCLL-martingale problem for (\mathcal{A}, μ_0) and with Lemma 3.4, we obtain that

$$\tilde{M}_t^f := f(X_t) - f(X_0) - \int_0^t \sigma(s, X_s) \mathcal{A}f(X_s) ds, \quad t \geq 0,$$

is a martingale. In particular, one has $\mathbb{E}[\tilde{M}_t^f] = 0$ for all $t \in [0, t_0]$. Since $\sigma\mathcal{A}f$ is bounded, applying Fubini's theorem yields (3.10). Finally, $X: \Omega \times [0, \infty) \rightarrow E$ is measurable and thus so is $(\omega, s) \mapsto (s, X_s(\omega))$. Hence, for $g \in B([0, \infty) \times E)$ also $(\omega, s) \mapsto g(s, X_s(\omega))$ is measurable, and so, by the measurability statement in Fubini's theorem, also $s \mapsto \int_E g(s, x) p(s, dx)$ is measurable. \square

4. A UNIQUENESS RESULT FOR THE FOKKER–PLANCK EQUATION WITH DEGENERATE COEFFICIENTS

If X is a solution to the martingale problem for $(\sigma\mathcal{A}, \mu_0)$ and $p(t, \cdot)$ is the law of X_t , then according to the proof of Proposition 3.5,

$$(4.1) \quad \text{for any } g \in B([0, \infty) \times E), s \mapsto \int_E g(s, x) p(s, dx) \text{ is measurable}$$

and for all f nice enough it holds that

$$(4.2) \quad \int_E f(x) p(t, dx) - \int_E f(x) \mu_0(dx) = \int_0^t \int_E \sigma(s, x) \mathcal{A}f(x) p(s, dx) ds, \quad t \in [0, t_0].$$

Conversely, one may ask if solutions $(p(t, \cdot))_{t \in [0, t_0]}$ to (4.2) can arise differently. In this section, we provide sufficient conditions which guarantee that the Fokker–Planck equation (4.2) (also called Kolmogorov forward equation) uniquely characterizes the law of X , i.e. the one-dimensional marginal laws of X are the only family of probability measures that satisfy (4.2) for a large class of functions f . More precisely, we prove the following result on (existence and) uniqueness result to the Fokker–Planck equation for time-inhomogeneous operators:

Theorem 4.1. *Suppose σ and $(\mathcal{D}, \mathcal{A})$ satisfy Assumptions 2.6, 2.7, 2.10 and either 2.11 or 2.13. Let t_0 be as in Assumption 2.6 and $\mu_0 \in \mathcal{P}(E)$. Then existence and uniqueness hold for (4.2):*

- There exists a family of probability measures $(p(t, \cdot))_{0 \leq t \leq t_0}$ on E which satisfies (4.1) and (4.2) for all $f \in \mathcal{D}$ and $p(0, \cdot) = \mu_0$.*
- If $(q(t, \cdot))_{0 \leq t \leq t_0}$ and $(p(t, \cdot))_{0 \leq t \leq t_0}$ are two families of probability measures on E which both satisfy (4.1) and (4.2) for all $f \in \mathcal{D}$ and $q(0, \cdot) = \mu_0 = p(0, \cdot)$, then $q(s, \cdot) = p(s, \cdot)$ for all $s \in [0, t_0]$.*

While the existence part (Theorem 4.1 (i)) follows immediately from Proposition 3.5, the rest of the section is devoted to prove the uniqueness result in Theorem 4.1.

As we will see, existence and uniqueness of solutions to the time-inhomogeneous Fokker–Planck equation (4.2) is closely related to existence and uniqueness of solutions to the martingale problem for the time-homogeneous operator $\sigma\mathcal{A} + \partial_t$ on $C_0([0, \infty) \times E)$ defined in equation (4.6) below. We show that the martingale problem for this operator is well-posed and the associated time-homogeneous Fokker–Planck equation determines the marginal laws of the solution uniquely.

In the present context, mainly two difficulties arise: Firstly, σ is only locally bounded, time-inhomogeneous and $\{(t, x) \in [0, \infty) \times E : \sigma(t, x) = 0\} \neq \emptyset$. Secondly, even if well-posedness for the martingale problem associated to $\sigma\mathcal{A}$ can be established, it is not automatic that any solution to the Fokker–Planck equation corresponds to a solution to the martingale problem for $\sigma\mathcal{A}$.

The proof of Theorem 4.1 (ii) relies on the following theorem on uniqueness for the Fokker–Planck equation corresponding to time-homogeneous operators cited³ from [Kur98] (see also [BK93, Thm. 4.1]).

Theorem 4.2 ([Kur98, Thm. 2.6 (c)]). *Let (E_0, d_0) be a locally compact, complete, separable metric space, $D(\mathcal{L}) \subset C_b(E_0)$ and $\mathcal{L}: D(\mathcal{L}) \rightarrow C_b(E_0)$ be linear. Let $\nu \in \mathcal{P}(E)$ and suppose that*

- (i) $D(\mathcal{L})$ is an algebra and separates points,
- (ii) there exists a countable subset $\{h_k : k \geq 1\} \subset D(\mathcal{L})$ such that

$$(4.3) \quad \text{bp-closure}(\text{span}(\{(h_k, \mathcal{L}h_k) : k \geq 1\})) \supset \{(h, \mathcal{L}h) : h \in D(\mathcal{L})\},$$

- (iii) for each $y \in E_0$, there exists a RCLL-solution to the martingale problem for (\mathcal{L}, δ_y) ,
- (iv) uniqueness holds for the martingale problem for (\mathcal{L}, ν) .

Then uniqueness holds for the Fokker–Planck equation for (\mathcal{L}, ν) : Suppose $\{\nu_t\}_{t \geq 0} \subset \mathcal{P}(E_0)$ is such that

$$(4.4) \quad \text{for any } g \in B(E_0), s \mapsto \int_{E_0} g(y) \nu_s(dy) \text{ is measurable}$$

and

$$(4.5) \quad \int_{E_0} h d\nu_t = \int_{E_0} h d\nu + \int_0^t \int_{E_0} \mathcal{L}h d\nu_s ds, \quad t \geq 0,$$

for all $h \in D(\mathcal{L})$. If $\{\mu_t\}_{t \geq 0} \subset \mathcal{P}(E_0)$ also satisfies (4.4) and (4.5), then $\mu_t = \nu_t$ for all $t \geq 0$.

The rest of Section 4 is devoted to the proof of Theorem 4.1 (ii). The argument is split into three parts and we will only get to the actual proof in the third part. The procedure is as follows:

- In Section 4.1 the time-inhomogeneous problem is put into the time-homogeneous setup by including time as an additional state variable. The associated generator \mathcal{L} is defined in (4.6).
- In Section 4.2 well-posedness of the martingale problem for \mathcal{L} is proved.
- In Section 4.3 the results from Section 4.1 and 4.2 are used to show that Theorem 4.2 can indeed be applied to prove Theorem 4.1 (ii).

4.1. Reducing the Problem to the Time-Homogeneous Setup. Fix $(\mathcal{D}, \mathcal{A})$ as in Assumption 2.7 and a measurable function $\sigma: [0, \infty) \times E \rightarrow [0, \infty)$. For $f \in \mathcal{D}$ and $\gamma \in C_c^1[0, \infty)$, define the operator \mathcal{L} by

$$(4.6) \quad \mathcal{L}(f\gamma)(t, x) := \gamma(t)\sigma(t, x)\mathcal{A}f(x) + f(x)\gamma'(t), \quad t \in [0, \infty), x \in E,$$

and linearly extend \mathcal{L} to $D(\mathcal{L}) := \text{span}\{f\gamma : f \in \mathcal{D}, \gamma \in C_c^1[0, \infty)\} \subset C_0([0, \infty) \times E)$.

The following lemma relates the Fokker–Planck equation (4.2) and the martingale problem for the time-inhomogeneous operator $\sigma\mathcal{A}$ to the Fokker–Planck equation and the martingale problem for the time-homogeneous operator \mathcal{L} on $C_0([0, \infty) \times E)$. Furthermore, sufficient conditions for (4.3) are provided.

Lemma 4.3. *Suppose $\sigma: [0, \infty) \times E \rightarrow [0, \infty)$ is measurable, $(\mathcal{D}, \mathcal{A})$ is as in Assumption 2.7 and $(D(\mathcal{L}), \mathcal{L})$ as in (4.6). Further suppose $\sigma\mathcal{A}f$ is bounded for all $f \in \mathcal{D}$. Let $s_0 \geq 0$, $\mu_0 \in \mathcal{P}(E)$ and define $\sigma_{s_0}(s, x) := \sigma(s_0 + s, x)$. Then the following hold:*

³ More precisely, instead of our hypothesis (iii), in [Kur98] the weaker requirement that \mathcal{L} is a *pre-generator* is imposed. However, as explained in [Kur98, Sec. 2] or [KS01, Remark 1.1], (iii) implies that \mathcal{L} is a pre-generator.

- (i) If σ is bounded, then $(D(\mathcal{L}), \mathcal{L})$ is conservative, i.e. (2.1) holds.
- (ii) If X is a solution to the (time-inhomogeneous) RCLL-martingale problem for $(\sigma_{s_0}\mathcal{A}, \mu_0)$, then $(s_0 + t, X_t)_{t \geq 0}$ is a solution to the RCLL-martingale problem for $(\mathcal{L}, \delta_{s_0} \otimes \mu_0)$.
- (iii) If σ is bounded and (T, X) is a solution to the RCLL-martingale problem for $(\mathcal{L}, \delta_{s_0} \otimes \mu_0)$, then X is a solution to the (time-inhomogeneous) RCLL-martingale problem for $(\sigma_{s_0}\mathcal{A}, \mu_0)$ and T is indistinguishable from $(s_0 + t)_{t \geq 0}$.
- (iv) Suppose $(p(t, \cdot))_{t \in [0, t_0]}$ satisfies (4.1) and (4.2) for all $f \in \mathcal{D}$ and define $p(t, \cdot) := p(t_0, \cdot)$ and $\sigma(t, \cdot) := 0$ for $t > t_0$ and for all $t \in [0, \infty)$ the measures $\nu_t := \delta_t \otimes p(t, \cdot)$ on $E_0 := [0, \infty) \times E$. Then $\{\nu_t\}_{t \geq 0}$ satisfies (4.4) and (4.5) for all $h \in D(\mathcal{L})$.
- (v) Suppose either $\sigma\mathcal{A}f \in C_0([0, t_0] \times E)$ for all $f \in \mathcal{D}$ or σ is bounded. Then there exists a countable subset $\{h_k : k \in \mathbb{N}\} \subset D(\mathcal{L})$ such that (4.3) holds.

Proof. (i) Note that by Assumption 2.7 (i) and (2.1), there exists $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{D}$ such that $\text{bp-lim}_{n \rightarrow \infty} f_n = 1$ and $\text{bp-lim}_{n \rightarrow \infty} \mathcal{A}f_n = 0$. Furthermore, there exist $\gamma_n \in C_c^1[0, \infty)$, $n \in \mathbb{N}$, such that $\gamma_n(s) = 1$ for $s \in [0, n]$, $\gamma_n(s) = 0$ for $s \in [n + 1, \infty)$ and $\sup_{n \in \mathbb{N}} \sup_{s \in [0, \infty)} |\gamma_n'(s)| < \infty$.⁴ In particular, $\text{bp-lim}_{n \rightarrow \infty} \gamma_n' = 0$, $\text{bp-lim}_{n \rightarrow \infty} f_n \gamma_n = 1$ and, since σ is bounded, also $\text{bp-lim}_{n \rightarrow \infty} \mathcal{L}(f_n \gamma_n) = 0$. Thus, (2.1) holds with $h_n := f_n \gamma_n$.

(ii) By assumption $\sigma\mathcal{A}f$ is bounded for all $f \in \mathcal{D}$, thus $\mathcal{L}h \in B([0, \infty) \times E)$ for all $h \in D(\mathcal{L})$. Therefore, [EK86, Chap. 4, Thm. 7.1] implies that $(t, X_t)_{t \geq 0}$ is a solution to the martingale problem for $(D(\mathcal{L}), \mathcal{L}^{\sigma_{s_0}})$, where $\mathcal{L}^{\sigma_{s_0}}$ is given in (4.6) with σ replaced by σ_{s_0} . Inserting $h = f\tilde{\gamma}$ and $\mathcal{L}^{\sigma_{s_0}}$ in (2.3), this implies that

$$(4.7) \quad \tilde{\gamma}(t)f(X_t) - \tilde{\gamma}(0)f(X_0) - \int_0^t \tilde{\gamma}(s)\sigma(s_0 + s, X_s)\mathcal{A}f(X_s) ds - \int_0^t \tilde{\gamma}'(s)f(X_s) ds, \quad t \geq 0,$$

is an $(\mathcal{F}_t^X)_{t \geq 0}$ -martingale for all $f \in \mathcal{D}$ and $\tilde{\gamma} \in C_c^1[0, \infty)$. In particular, for given $\gamma \in C_c^1[0, \infty)$, we can use $\tilde{\gamma} := \gamma(\cdot + s_0)$ (which is again in $C_c^1[0, \infty)$) in (4.7) to see that

$$\gamma(s_0 + t)f(X_t) - \gamma(s_0)f(X_0) - \int_0^t \mathcal{L}(f\gamma)(s_0 + s, X_s) ds, \quad t \geq 0,$$

is an $(\mathcal{F}_t^X)_{t \geq 0}$ -martingale for all $f \in \mathcal{D}$ and $\gamma \in C_c^1[0, \infty)$. By linearity, this extends to all $h \in D(\mathcal{L})$ and therefore $(s_0 + t, X_t)_{t \geq 0}$ is a solution to the RCLL-martingale problem for $(\mathcal{L}, \delta_{s_0} \otimes \mu_0)$.

(iii) First note that both $A_1 := \{(\gamma, \gamma') : \gamma \in C_c^1[0, \infty)\}$ and $A_2 := \{(f, \sigma\mathcal{A}f) : f \in \mathcal{D}\}$ (viewed as subsets of $B([0, \infty) \times E)$) are contained in $\text{bp-closure}(\{(h, \mathcal{L}h) : h \in D(\mathcal{L})\})$. For A_1 , this follows by fixing γ and using $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{D}$ from the proof of (i) to obtain $\text{bp-lim}_{n \rightarrow \infty} (f_n \gamma) = \gamma$ and $\text{bp-lim}_{n \rightarrow \infty} \mathcal{L}(f_n \gamma) = \gamma'$ since σ is bounded. For A_2 , this follows analogously by using γ_n as defined in the proof of (i) and by noting that for any $f \in \mathcal{D}$, $\text{bp-lim}_{n \rightarrow \infty} (\gamma_n f, \mathcal{L}(\gamma_n f)) = (f, \sigma\mathcal{A}f)$. So, if (T, X) is a solution to the RCLL-martingale problem for $(\mathcal{L}, \delta_{s_0} \otimes \mu_0)$, then by Remark 2.2

$$(4.8) \quad h(T_t, X_t) - h(T_0, X_0) - \int_0^t g(T_s, X_s) ds, \quad t \geq 0,$$

is an $(\mathcal{F}_t^{(T, X)})_{t \geq 0}$ -martingale for all $(h, g) \in \text{bp-closure}(\{(h, \mathcal{L}h) : h \in D(\mathcal{L})\})$ and thus in particular for all $(h, g) \in A_1 \cup A_2$. Inserting $(\gamma, \gamma') \in A_1$ in (4.8) thus yields that

$$\gamma(T_t) - \gamma(T_0) - \int_0^t \gamma'(T_s) ds, \quad t \geq 0,$$

is an $(\mathcal{F}_t^{(T, X)})_{t \geq 0}$ -martingale and, since it is $(\mathcal{F}_t^T)_{t \geq 0}$ -adapted, also a martingale with respect to $(\mathcal{F}_t^T)_{t \geq 0}$. Thus, T is a solution to the RCLL-martingale problem for $(\partial_t, \delta_{s_0})$, where ∂_t has

⁴For instance the functions defined for $n \in \mathbb{N}$ by $\gamma_n(s) = 1$ for $s \in [0, n]$, $\gamma_n(s) = 0$ for $s \in [n + 1, \infty)$ and $\gamma_n(s) = 2(s - n)^3 - 3(s - n)^2 + 1$ for $s \in [n, n + 1]$ satisfy these properties.

domain $D(\partial_t) := C_c^1[0, \infty)$ and is defined as $\partial_t \gamma := \gamma'$ for $\gamma \in D(\partial_t)$. However, $(s_0 + t)_{t \geq 0}$ is also a solution to the RCLL-martingale problem for $(\partial_t, \delta_{s_0})$ since $\int_0^t \gamma'(s_0 + s) ds = \gamma(s_0 + t) - \gamma(s_0)$ for all $t \geq 0, \gamma \in D(\partial_t)$. By [EK86, Chap. 4, Thm. 4.1] uniqueness holds for the martingale problem for $(\partial_t, \delta_{s_0})$ and in particular T is indistinguishable from $(s_0 + t)_{t \geq 0}$.⁵

On the other hand, (4.8) is a martingale for all $(h, g) \in A_2$ as deduced above and so for each $f \in \mathcal{D}$,

$$f(X_t) - f(X_0) - \int_0^t \sigma(T_s, X_s) \mathcal{A}f(X_s) ds, \quad t \geq 0,$$

is a martingale. Since T is indistinguishable from $(s_0 + t)_{t \geq 0}$, the claim follows.

(iv) First notice that (4.2) actually holds for all $t \geq 0$ since $p(t, \cdot) = p(t_0, \cdot)$ and $\sigma(t, \cdot) = 0$ for $t > t_0$. Moreover, for any $f \in \mathcal{D}$, $\sigma \mathcal{A}f$ is bounded by assumption and so the function $s \mapsto \int_E \sigma(s, x) \mathcal{A}f(x) p(s, dx)$ is measurable by (4.1) and bounded. Thus, from (4.2) we see that $t \mapsto F(t) := \int_E f(x) p(t, dx)$ is absolutely continuous (c.f. [Leo09, Lem. 3.31]) with $F'(t) = \int_E \sigma(t, x) \mathcal{A}f(x) p(t, dx)$ for a.e. $t \geq 0$. Hence, for any $f \in \mathcal{D}$ and $\gamma \in C_c^1[0, \infty)$, we may integrate by parts (see [Leo09, Cor. 3.37]) to obtain

$$\begin{aligned} \gamma(t)F(t) - \gamma(0)F(0) &= \int_0^t \gamma(s)F'(s) ds + \int_0^t \gamma'(s)F(s) ds \\ &= \int_0^t \int_E \gamma(s) \sigma(s, x) \mathcal{A}f(x) p(s, dx) ds + \int_0^t \int_E \gamma'(s) f(x) p(s, dx) ds \\ &\stackrel{(4.6)}{=} \int_0^t \int_E \mathcal{L}(\gamma f)(s, x) p(s, dx) ds = \int_0^t \int_{E_0} \mathcal{L}(\gamma f) d\nu_s ds \end{aligned}$$

for any $t \geq 0$. But $\gamma(t)F(t) = \int_{E_0} (f\gamma) d\nu_t$ by definition, thus $\{\nu_t\}_{t \geq 0}$ satisfies (4.5) for all $h = f\gamma$ and by linearity also for all $h \in D(\mathcal{L})$. Finally, note that by definition of $\{\nu_t\}_{t \geq 0}$ the integral in (4.4) is the same as in (4.1) and so the result follows.

(v) Assume first $\sigma \mathcal{A}f \in C_0([0, t_0] \times E)$ for all $f \in \mathcal{D}$. Set $E_0 := [0, t_0] \times E$ and note that the spaces $C_0(E_0)$ and $C_0(E_0) \times C_0(E_0)$ are separable since $E, [0, t_0]$ and E_0 are separable and because products of separable spaces are separable. For $f \in \mathcal{D}$ and $\gamma \in C^1[0, t_0]$ our assumption and $\mathcal{D} \subset C_0(E)$ imply $f\gamma' \in C_0(E_0)$ and $\gamma \sigma \mathcal{A}f \in C_0(E_0)$. Hence, also $\mathcal{L}(f\gamma) \in C_0(E_0)$ and by linearity, $\mathcal{L}h \in C_0(E_0)$ for any $h \in D(\mathcal{L})$. Setting $G_0 := \{(h, \mathcal{L}h) : h \in D(\mathcal{L})\}$, this shows $G_0 \subset C_0(E_0) \times C_0(E_0)$. Since the latter space is separable (as argued above) and any subspace of a separable metric space is separable, we conclude that there exists $H_0 \subset G_0$, H_0 countable, such that each $(h, \mathcal{L}h) \in G_0$ is the limit in sup-norm of a sequence in H_0 . In particular, $G_0 \subset \text{bp-closure}(H_0)$, i.e. (4.3) holds.

Secondly, assume that σ is bounded. The same separability reasoning as above shows that there exist $\{\gamma_k\}_{k \in \mathbb{N}} \subset C_c^1[0, \infty)$ and $\{f_l\}_{l \in \mathbb{N}} \subset \mathcal{D}$ with the property that for any $\gamma \in C_c^1[0, \infty)$ and $f \in \mathcal{D}$, there exist $\{k_n\}_{n \in \mathbb{N}}, \{l_n\}_{n \in \mathbb{N}} \subset \mathbb{N}$ such that $\gamma = \lim_{n \rightarrow \infty} \gamma_{k_n}, \gamma' = \lim_{n \rightarrow \infty} \gamma'_{k_n}, f = \lim_{n \rightarrow \infty} f_{l_n}$ and $\mathcal{A}f = \lim_{n \rightarrow \infty} \mathcal{A}f_{l_n}$ in sup-norm. Since σ is bounded, this also implies $\text{bp-lim}_{n \rightarrow \infty} \sigma \mathcal{A}f_{l_n} = \sigma \mathcal{A}f$ and thus $\text{bp-lim}_{n \rightarrow \infty} \gamma_{k_n} f_{l_n} = \gamma f$ and $\text{bp-lim}_{n \rightarrow \infty} \mathcal{L}(\gamma_{k_n} f_{l_n}) = \mathcal{L}(\gamma f)$. Thus, we have shown

$$\{(f\gamma, \mathcal{L}(f\gamma)) : f \in \mathcal{D}, \gamma \in C_c^1[0, \infty)\} \subset \text{bp-closure}(\{(\gamma_k f_l, \mathcal{L}(\gamma_k f_l))\}_{k, l \in \mathbb{N}})$$

and by linearity this implies (4.3). \square

⁵To check the assumptions of [EK86, Chap. 4, Thm. 4.1] in more detail (see [EK86] for unexplained definitions), note that $[0, \infty)$ is locally compact, separable, $D(\partial_t)$ is dense in $C_0[0, \infty)$ and $C_0[0, \infty)$ is convergence determining (see [EK86, Chap. 3, Prop. 4.4]), hence separating. Furthermore, whenever $\gamma \in D(\partial_t)$, $t^* \geq 0$ satisfy $\gamma(t^*) = \sup_{t \geq 0} \gamma(t)$, then $\gamma'(t^*) = 0$. Thus ∂_t satisfies the positive maximum principle and is hence dissipative by [EK86, Chap. 4, Lem. 2.1]. Finally, fix $\lambda > 0$, then for any $g \in C_c^1[0, \infty)$, the function $\gamma(t) := \exp(\lambda t) \int_t^\infty g(s) \exp(-\lambda s) ds$ satisfies $\gamma \in D(\partial_t)$ and $\lambda \gamma - \partial_t \gamma = g$ so that the range of the operator $\lambda - \partial_t$ is $C_c^1[0, \infty)$ and in particular dense in $C_0[0, \infty)$.

4.2. Well-Posedness of the Martingale Problem. In this section, we show that the martingale problem for $(D(\mathcal{L}), \mathcal{L})$, see (4.6) above, is well-posed. The proof is split into three parts: Existence is established in Proposition 4.4, uniqueness is proved in Proposition 4.5 under the assumption that σ is bounded. Finally, in Proposition 4.7 the assumption of boundedness is removed.

To prove existence, we use the time-change construction from Lemma 3.1. Extra work is needed to incorporate the time-inhomogeneity.

Proposition 4.4. *Suppose σ and $(\mathcal{D}, \mathcal{A})$ are as in Theorem 4.1 and $(D(\mathcal{L}), \mathcal{L})$ is defined as in (4.6). Then for any $s_0 \geq 0$ and $x_0 \in E$ there exists a solution to the RCLL-martingale problem for $(\mathcal{L}, \delta_{s_0} \otimes \delta_{x_0})$.*

Proof. Define $\sigma_{s_0}(s, x) := \sigma(s_0 + s, x)$ for $s \geq 0$, $x \in E$. Assumption 2.11 (ii) or 2.13 implies that $\sigma_{s_0} \mathcal{A}f$ is bounded for any $f \in \mathcal{D}$ and so by Lemma 4.3 (ii) it suffices to show that there exists a solution to the (time-inhomogeneous) RCLL-martingale problem for $(\sigma_{s_0} \mathcal{A}, \delta_{x_0})$.

If $s_0 \geq t_0$, then $\sigma_{s_0}(s, x) = 0$ for all $(s, x) \in (0, \infty) \times E$, since $\sigma(t, \cdot) = 0$ for $t > t_0$. Setting $X_t := x_0$ for $t \geq 0$, by definition (c.f. (2.3)) it follows that X is a solution to the (time-inhomogeneous) RCLL-martingale problem for $(\sigma_{s_0} \mathcal{A}, \delta_{x_0})$.

If $s_0 < t_0$, set $\tilde{t}_0 := t_0 - s_0$ and note that σ_{s_0} satisfies Assumption 2.6 on $[0, \tilde{t}_0]$ (and $\sigma_{s_0}(t, \cdot) = 0$ for $t > \tilde{t}_0$), since σ satisfies Assumption 2.6. Furthermore, σ_{s_0} satisfies $\sigma_{s_0} \mathcal{A}f \in C_0([0, \tilde{t}_0] \times E)$ for all $f \in \mathcal{D}$ or σ_{s_0} is bounded, since σ satisfies Assumption 2.11 (ii) or 2.13. Let M denote the coordinate process on $D_E[0, \infty)$ and $\mathbb{P} = P_{x_0}$ (as defined in Assumption 2.7). Then σ_{s_0} , $(\mathcal{D}, \mathcal{A})$ and M satisfy the assumptions of Proposition 3.5, which implies that there exists a solution to the RCLL-martingale problem for $(\sigma_{s_0} \mathcal{A}, \delta_{x_0})$. \square

The next step is to prove uniqueness under the assumption that σ is bounded. Combined with Proposition 4.4, well-posedness of the RCLL-martingale problem for $(D(\mathcal{L}), \mathcal{L})$ follows.

The main idea of the proof is to show that any solution \tilde{X} to the RCLL-martingale problem for $(\mathcal{L}, \delta_{(s_0, x)})$ can be written as a time-change

$$\tilde{X}_t = M_{\int_0^t \sigma(s_0 + u, \tilde{X}_u) du}, \quad t \geq 0, \mathbb{P}\text{-a.s.},$$

for M which is a solution to the martingale problem for (\mathcal{A}, δ_x) . Corollary A.4 and Assumption 2.7 (ii) then allow us to conclude uniqueness.

Note that if $\sigma(t, x)$ did not depend on t , the proof could be simplified significantly by relying on [EK86, Chap. 6, Thm. 1.4].

Proposition 4.5. *Suppose σ and $(\mathcal{D}, \mathcal{A})$ satisfy Assumptions 2.6, 2.7, 2.13 and 2.10. Define $(D(\mathcal{L}), \mathcal{L})$ as in (4.6), then for each $\nu \in \mathcal{P}([0, \infty) \times E)$ the RCLL-martingale problem for (\mathcal{L}, ν) is well-posed.*

Proof. Firstly, note that it suffices to show that for each $(s_0, x_0) \in [0, \infty) \times E$ the RCLL-martingale problem for $(\mathcal{L}, \delta_{(s_0, x_0)})$ is well-posed: If this is established, we can combine [BK93, Thm. 2.1] and Lemma 4.3 (v) to conclude that also for any $\nu \in \mathcal{P}([0, \infty) \times E)$ the RCLL-martingale problem for (\mathcal{L}, ν) is well-posed.

From Proposition 4.4 it follows that for any $(s_0, x_0) \in [0, \infty) \times E$ there exists a solution to the RCLL-martingale problem for $(\mathcal{L}, \delta_{(s_0, x_0)})$. In order to prove the current proposition, by the above it is therefore sufficient to prove that for any $s_0 \in [0, \infty)$ and any $x_0 \in E$ uniqueness holds for the RCLL-martingale problem for $(\mathcal{L}, \delta_{s_0} \otimes \delta_{x_0})$. This will now be established.

Set $\mu_0 := \delta_{x_0}$ and suppose (T, X) is a solution RCLL-martingale problem for $(\mathcal{L}, \delta_{s_0} \otimes \mu_0)$ defined on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. By Lemma 4.3 (iii) it follows that, $\tilde{\mathbb{P}}$ -a.s., $T_t = t + s_0$ for all $t \geq 0$ and that X is a solution to the (time-inhomogeneous) RCLL-martingale problem for

$(\sigma(s_0 + \cdot)\mathcal{A}, \mu_0)$, i.e. for each $f \in \mathcal{D}$ the process

$$(4.9) \quad M_t^f := f(X_t) - f(X_0) - \int_0^t \sigma(s_0 + s, X_s) \mathcal{A}f(X_s) ds, \quad t \geq 0,$$

is a martingale. In the following, we show that this implies that X solves the time-change equation (A.5) for some process M that is a solution to the RCLL-problem for (\mathcal{A}, μ_0) . By the uniqueness result for the time-change equation in Corollary A.4, it then follows that the law of X is determined by σ and the law of M . Since uniqueness holds for the RCLL-martingale problem for (\mathcal{A}, μ_0) , it then follows that the law of X is uniquely determined by σ and (\mathcal{A}, μ_0) and thus the claim follows.

Before we start, let us consider the case $s_0 \geq t_0$. Since $\sigma(t, x) = 0$ for all $t > t_0, x \in E$, the integral term in (4.9) vanishes for any $f \in \mathcal{D}$. In particular, $f(X)$ is a martingale for any $f \in \mathcal{D}$. Combining this with our Assumption 2.7 (i) that \mathcal{D} is dense in $C_0(E)$, this implies (see [EK86, Chap. 3, Ex. 7]) that $X_t = X_0, \tilde{\mathbb{P}}$ -a.s., for any $t \geq 0$. However, X is RCLL and thus X is constant $\tilde{\mathbb{P}}$ -a.s. In particular, the law of (T, X) is uniquely determined.

Thus, we may assume $s_0 < t_0$ and in analogy to (3.1) define

$$\tau(t) := \int_0^t \sigma(s_0 + u, X_u) du,$$

for each $t \geq 0$, and set $\mathcal{T}(u) := \inf\{t \geq 0 : \tau(t) \geq u\}$ for each $u \geq 0$ and $Y_u := X_{\mathcal{T}(u)}$ for $u \leq \tau(t_0 - s_0)$. Note that σ is bounded and $\sigma(s, \cdot) = 0$ for $s > t_0$, thus $\tilde{\mathbb{P}}$ -a.s. $\tau(t_0 - s_0) < \infty$ and $\tau(s) = \tau(t_0 - s_0)$ for all $s > t_0 - s_0$. We now claim that

- (i) with probability one, X is constant on any interval $[t, u]$ with $\int_t^u \sigma(s_0 + s, X_s) ds = 0$,
- (ii) $\tilde{\mathbb{P}}$ -a.s., X satisfies the time-change equation $X_t = Y_{\tau(t)}$ for all $t \geq 0$ and Y is RCLL,
- (iii) on an extended probability space there exists a process \tilde{Y} that is a solution to the RCLL-martingale problem for (\mathcal{A}, μ_0) that satisfies $\tilde{Y}_u = Y_u$ for $u \leq \tau(t_0 - s_0)$ and such that $X = \tilde{Y}_{\tau(t)}$ still holds a.s. Furthermore, this implies the claim.

To prove (i), we define

$$\gamma_t := \inf\{u > t : \tau(u) > \tau(t)\} \quad \text{for each } t \geq 0$$

so that for any $u > t$, $\int_t^{\gamma_t \wedge u} \sigma(s_0 + s, X_s) ds = 0$ and thus (recall $\sigma \geq 0$) also

$$(4.10) \quad \sigma(s_0 + s, X_s) = 0 \quad \text{for a.e. } s \in [t, \gamma_t \wedge u], \tilde{\mathbb{P}}\text{-a.s.}$$

Furthermore, for $u < t$, $\{\gamma_t \leq u\} = \emptyset$ and for $u \geq t$,

$$(4.11) \quad \{\gamma_t < u\} = \{\tau(u) > \tau(t)\}$$

since τ is non-decreasing and continuous (which implies $\tau(\gamma_t) = \tau(t)$). Let us denote by $(\mathcal{F}_t)_{t \geq 0}$ the usual augmentation (in the sense of [RW00, Def. II.67.3]) of $(\mathcal{F}_t^X)_{t \geq 0}$. Since τ is adapted, (4.11) implies $\{\gamma_t < u\} \in \mathcal{F}_u$ for any $u \geq 0$ and so γ_t is an $(\mathcal{F}_t)_{t \geq 0}$ -stopping time. By right-continuity of X and [RW00, Lem. II.67.10], (4.9) is also a martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$. Hence, we can apply optional sampling to the martingales (4.9) to obtain for any $u > t$ and $f \in \mathcal{D}$ that

$$\begin{aligned} 0 &= \mathbb{E}[M_{\gamma_t \wedge u}^f - M_t^f | \mathcal{F}_t] = \mathbb{E} \left[f(X_{\gamma_t \wedge u}) - \int_t^{\gamma_t \wedge u} \sigma(s_0 + s, X_s) \mathcal{A}f(X_s) ds \Big| \mathcal{F}_t \right] - f(X_t) \\ &= \mathbb{E}[f(X_{\gamma_t \wedge u}) | \mathcal{F}_t] - f(X_t), \end{aligned}$$

where the last step follows from (4.10). Since $f \in \mathcal{D}$ was arbitrary and by Assumption 2.7 (i) \mathcal{D} is dense in $C_0(E)$, this implies (see [EK86, Chap. 3, Ex. 7]) that for fixed $u > t$, $X_{\gamma_t \wedge u} = X_t, \tilde{\mathbb{P}}$ -a.s. Thus we can find $\Omega_0 \in \tilde{\mathcal{F}}$ such that $\tilde{\mathbb{P}}(\Omega_0) = 1$ and on Ω_0 , we have $X_{\gamma_t \wedge u} = X_t$ for all $u > t \geq 0$ with $t, u \in \mathbb{Q}$. But then on Ω_0 this extends to all $u > t \geq 0$ by a standard argument: for $u > t \geq 0$, we find $\{u_n\} \subset \mathbb{Q}, \{t_n\} \subset \mathbb{Q}$ with $u_n \downarrow u, t_n \downarrow t$ as $n \rightarrow \infty$. Then $\gamma_{t_n} \wedge u_n \downarrow \gamma_t \wedge u$

as $n \rightarrow \infty$ and so we can use right-continuity of X for the first and last equality and our choice of Ω_0 for the second equality to obtain

$$(4.12) \quad X_t = \lim_{n \rightarrow \infty} X_{t_n} = \lim_{n \rightarrow \infty} X_{\gamma_{t_n} \wedge u_n} = X_{\gamma_t \wedge u}.$$

Thus, if $\omega \in \Omega_0$, $u > t \geq 0$ and $\int_t^u \sigma(s_0 + s, X_s(\omega)) ds = 0$, then $u \leq \gamma_t(\omega)$ and so by (4.12) indeed $X_t(\omega) = X_{\gamma_t(\omega) \wedge u} = X_u(\omega)$.

To prove (ii), set $\mathcal{T}_+(u) := \lim_{v \downarrow u} \mathcal{T}(v)$ and notice $\mathcal{T}_+(u) = \inf\{t \geq 0 : \tau(t) > u\}$. Then from (i) we get that, $\tilde{\mathbb{P}}$ -a.s., X is constant on the interval $[\mathcal{T}(u), \mathcal{T}_+(u)]$ for all $u \geq 0$. Hence, from $\mathcal{T}(\tau(t)) \leq t \leq \mathcal{T}_+(\tau(t))$ we obtain

$$X_t = X_{\mathcal{T}(\tau(t))} = Y_{\tau(t)}$$

for all $t \geq 0$. Furthermore, $u \downarrow u_0$ implies $\mathcal{T}(u) \downarrow \mathcal{T}_+(u_0)$ and so by right-continuity of X also

$$Y_u = X_{\mathcal{T}(u)} \rightarrow X_{\mathcal{T}_+(u_0)} = X_{\mathcal{T}(u_0)} = Y_{u_0}$$

and since \mathcal{T} is left-continuous and X has left-limits, the same reasoning shows that Y also has left-limits. Hence, Y is indeed RCLL.

For (iii), notice that $\sigma(s_0 + s, X_s) = 0$ for $s \in [\mathcal{T}(\tau(t_0 - s_0)), t_0 - s_0]$ and $s \in [\mathcal{T}(v), \mathcal{T}_+(v)]$ for any $v \geq 0$. Combining this with $\mathcal{T}(u) \leq t \Leftrightarrow \tau(t) \geq u$, one obtains

$$(4.13) \quad \begin{aligned} \int_0^{\mathcal{T}(u) \wedge (t_0 - s_0)} \sigma(s_0 + s, X_s) \mathcal{A}f(X_s) ds &= \int_0^{\mathcal{T}(u \wedge \tau(t_0 - s_0))} \sigma(s_0 + s, X_s) \mathcal{A}f(X_s) ds \\ &= \int_0^{\mathcal{T}_+(u \wedge \tau(t_0 - s_0))} \sigma(s_0 + s, X_s) \mathcal{A}f(X_s) ds, \end{aligned}$$

for $u \geq 0$, $f \in \mathcal{D}$. Since $\mathcal{T}_+(v) = \inf\{t \geq 0 : \tau(t) > v\}$, a change of variables as in [EK86, Chap. 6, Ex. 12] now allows to rewrite the last expression as

$$(4.14) \quad \int_0^{\mathcal{T}_+(u \wedge \tau(t_0 - s_0))} \sigma(s_0 + s, X_s) \mathcal{A}f(X_s) ds = \int_0^{u \wedge \tau(t_0 - s_0)} \mathcal{A}f(X_{\mathcal{T}_+(v)}) dv.$$

As argued in (ii), $X_{\mathcal{T}_+(v)} = Y_v$ for all $v \geq 0$ and by inserting this in the right-hand side of (4.14) and combining with (4.13), we get

$$(4.15) \quad \int_0^{\mathcal{T}(u) \wedge (t_0 - s_0)} \sigma(s_0 + s, X_s) \mathcal{A}f(X_s) ds = \int_0^{u \wedge \tau(t_0 - s_0)} \mathcal{A}f(Y_s) ds$$

for any $u \geq 0$, $f \in \mathcal{D}$. Furthermore, for any $t, u \geq 0$, $\{\mathcal{T}(u) \leq t\} = \{\tau(t) \geq u\} \in \mathcal{F}_t$ so that for each $u \geq 0$, $\mathcal{T}(u)$ is a stopping time. Using this, (4.15) and applying the optional sampling theorem to the martingales in (4.9), we therefore get that for any $f \in \mathcal{D}$ the process

$$(4.16) \quad M_{\mathcal{T}(u) \wedge (t_0 - s_0)}^f = f(Y_{u \wedge \tau(t_0 - s_0)}) - f(Y_0) - \int_0^{u \wedge \tau(t_0 - s_0)} \mathcal{A}f(Y_s) ds, \quad u \geq 0,$$

is a martingale with respect to the filtration $(\mathcal{F}_{\mathcal{T}(u) \wedge (t_0 - s_0)})_{u \geq 0}$ and thus also with respect to the filtration generated by $Y_{\cdot \wedge \tau(t_0 - s_0)}$. Let us denote by W the coordinate process on $D_E[0, \infty)$ and for $(\omega, \omega') \in \tilde{\Omega} \times D_E[0, \infty)$ define

$$\tilde{Y}_u(\omega, \omega') := \begin{cases} Y_u(\omega) & \text{for } u < \tau(t_0 - s_0)(\omega), \\ W_{u - \tau(t_0 - s_0)(\omega)}(\omega') & \text{for } u \geq \tau(t_0 - s_0)(\omega). \end{cases}$$

From (4.16) and Lemma 4.6 below (applied to the process $(Y_{u \wedge \tau(t_0 - s_0)})_{u \geq 0}$ and the random variable $\tau(t_0 - s_0)$) it follows that the process \tilde{Y} is a solution to the RCLL-martingale problem for (\mathcal{A}, μ_0) under a measure Q with $Q(A \times D_E[0, \infty)) = \tilde{\mathbb{P}}(A)$ for all $A \in \tilde{\mathcal{F}}$ and such that

$\tilde{Y}_{s \wedge \tau(t_0 - s_0)} = Y_{s \wedge \tau(t_0 - s_0)}$ for all $s \geq 0$, Q -a.s. Combining this with (ii) and $\tau(\cdot) \leq \tau(t_0 - s_0)$, it follows that

$$X_t = Y_{\tau(t)} = Y_{\tau(t) \wedge \tau(t_0 - s_0)} = \tilde{Y}_{\tau(t) \wedge \tau(t_0 - s_0)} = \tilde{Y}_{\tau(t)} = \tilde{Y}_{\int_0^t \sigma_{s_0}(s, X_s) ds}, \quad t \geq 0, \quad Q\text{-a.s.},$$

where $\sigma_{s_0}(s, x) := \sigma(s + s_0, x)$ for $s \geq 0, x \in E$. In particular, X satisfies a time-change equation (A.5) (with M replaced by \tilde{Y} and σ replaced by σ_{s_0}). By our assumptions on σ , σ_{s_0} satisfies Assumption 2.6 on $[0, t_0 - s_0]$ (and $\sigma_{s_0}(t, \cdot) = 0$ for $t > t_0 - s_0$) and 2.13. Since uniqueness holds for the RCLL-martingale problem for (\mathcal{A}, μ_0) , the law on $D_E[0, \infty)$ of \tilde{Y} under Q is given as P_{x_0} and thus, by Assumption 2.10, H is regular for Q . Altogether, Corollary A.4 can be applied to σ_{s_0} and \tilde{Y} , which implies that the law of X under Q is uniquely determined by σ_{s_0} and P_{x_0} . But the law of X under $\tilde{\mathbb{P}}$ is the same as under Q and so the claim follows. \square

For the well-posedness of the RCLL-martingale problem (see Proposition 4.5), we used the following auxiliary lemma. As the authors are not aware of a suitable reference, we also present its complete proof here.

Lemma 4.6. *Let (E, d) be a locally compact, complete, separable metric space, $\mathcal{D} \subset C_0(E)$ and $\mathcal{A}: \mathcal{D} \rightarrow C_0(E)$ linear. Suppose that the $D_E[0, \infty)$ -martingale problem for (\mathcal{A}, μ) is well-posed for any $\mu \in \mathcal{P}(E)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, τ be a $[0, \infty)$ -valued random variable, and Z be an E -valued RCLL process on $(\Omega, \mathcal{F}, \mathbb{P})$ with $Z_u = Z_{u \wedge \tau}$ for all $u \geq 0$, such that*

$$f(Z_u) - f(Z_0) - \int_0^{u \wedge \tau} \mathcal{A}f(Z_s) ds, \quad u \geq 0,$$

is a martingale for all $f \in \mathcal{D}$. Let us denote by X the coordinate process on $D_E[0, \infty)$. On $\Omega' := \Omega \times D_E[0, \infty)$ define the process Y via

$$Y_t(\omega, \omega') := \begin{cases} Z_t(\omega) & \text{for } t < \tau(\omega) \\ X_{t-\tau(\omega)}(\omega') & \text{for } t \geq \tau(\omega) \end{cases}$$

for $(\omega, \omega') \in \Omega'$ and $t \geq 0$. Furthermore, for each $x \in E$, denote by P_x the law of the solution of the RCLL-martingale problem for (\mathcal{A}, δ_x) and by \mathcal{S}_E the Borel σ -algebra in $D_E[0, \infty)$. Let us define the measure Q on $\mathcal{F} \times \mathcal{S}_E$ by

$$(4.17) \quad Q(A \times C) := \mathbb{E}[\mathbb{1}_A P_{Z_\tau}(C)]$$

for $A \in \mathcal{F}$, $C \in \mathcal{S}_E$ (and extend this to the product σ -algebra). Then under Q , Y is a solution to the RCLL-martingale problem for (\mathcal{A}, μ_0) , where μ_0 is the law of Z_0 . Furthermore, $Q(A \times D_E[0, \infty)) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$ and $Z_t = Y_{t \wedge \tau}$ for all $t \geq 0$, Q -a.s.

Proof. Essentially this is [EK86, Chap. 4, Lem. 5.16], the only difference is that we construct Y on $\Omega \times D_E[0, \infty)$ (instead of $D_E[0, \infty) \times [0, \infty) \times D_E[0, \infty)$).

To prove the claim, first notice that by [EK86, Chap. 4, Thm. 4.6], the map $x \mapsto P_x(C)$ is measurable for each $C \in \mathcal{S}_E$ and so Q is indeed well-defined. Furthermore, denoting by μ the law of Z_τ , also the measure \tilde{Q} defined on product sets as

$$\tilde{Q}(B \times C) := \int_E \mathbb{E}[\mathbb{1}_B(Z, \tau) | Z_\tau = x] P_x(C) \mu(dx),$$

for $B \in \mathcal{S}_E \times \mathcal{B}([0, \infty))$, $C \in \mathcal{S}_E$ is well-defined. Therefore, denoting by $(X^{(1)}, \eta, X^{(2)})$ the coordinate random variable on $D_E[0, \infty) \times [0, \infty) \times D_E[0, \infty)$, from [EK86, Chap. 4, Lem. 5.16, (5.52) and (5.53)] it follows that under \tilde{Q} the process $(\tilde{Y}_t)_{t \geq 0}$ defined as

$$\tilde{Y}_t := \begin{cases} X_t^{(1)} & \text{for } t < \eta \\ X_{t-\eta}^{(2)} & \text{for } t \geq \eta \end{cases}$$

is a solution to the RCLL-martingale problem for (\mathcal{A}, μ_0) . Thus, it remains to show that the law of Y under Q is the same as the law of \tilde{Y} under \tilde{Q} .

Firstly note that for $B \in \mathcal{S}_E \times \mathcal{B}([0, \infty))$ and $C \in \mathcal{S}_E$ one obtains

$$\begin{aligned} Q(\{(Z, \tau) \in B\} \times \{X \in C\}) &= \mathbb{E}[\mathbb{1}_{\{(Z, \tau) \in B\}} P_{Z_\tau}(C)] = \mathbb{E}[\mathbb{E}[\mathbb{1}_B(Z, \tau) | Z_\tau] \mathbb{E}[P_{Z_\tau}(C) | Z_\tau]] \\ &= \tilde{Q}(B \times C), \end{aligned}$$

where the last step uses [EK86, Chap. 4, Thm. 4.2 (c)]. Hence, \tilde{Q} coincides with the law of (Z, τ, X) on $D_E[0, \infty) \times [0, \infty) \times D_E[0, \infty)$ under Q . Recalling that $(X^{(1)}, \eta, X^{(2)})$ denotes the coordinate random variable on $D_E[0, \infty) \times [0, \infty) \times D_E[0, \infty)$, this implies in particular that for any $0 \leq t_1 < \dots < t_n$, $n \in \mathbb{N}$, $A_1, \dots, A_n \in \mathcal{B}(E)$,

$$\begin{aligned} Q(Y_{t_1} \in A_1, \dots, Y_{t_n} \in A_n) &= \sum_{I \subset \{1, \dots, n\}} Q(Z_{t_i} \in A_i, t_i < \tau \forall i \in I, X_{t_j - \tau} \in A_j, t_j \geq \tau \forall j \in I^c) \\ &= \sum_{I \subset \{1, \dots, n\}} \tilde{Q}(X_{t_i}^{(1)} \in A_i, t_i < \eta \forall i \in I, X_{t_j - \eta}^{(2)} \in A_j, t_j \geq \eta \forall j \in I^c) \\ &= \tilde{Q}(\tilde{Y}_{t_1} \in A_1, \dots, \tilde{Y}_{t_n} \in A_n). \end{aligned}$$

Thus, it follows that the law of \tilde{Y} under \tilde{Q} is the same as the law of Y under Q , hence the claim.

Finally, $Y_u = Z_u$ for $u < \tau$ and so we only need to show $Z_\tau = Y_\tau$, Q -a.s. This should be clear but we still give a formal argument: For any $A, B \in \mathcal{B}(E)$ we have, by (4.17) and $P_x(X_0 \in B) = \delta_x(B)$,

$$(4.18) \quad Q(\{Z_\tau \in A\} \times \{X_0 \in B\}) = \mathbb{E}[\mathbb{1}_A(Z_\tau) P_{Z_\tau}(X_0 \in B)] = \mathbb{P}(Z_\tau \in A \cap B).$$

Now denote by $\{x_m\}_{m \in \mathbb{N}}$ a countable dense subset of E . Then, for any $n \in \mathbb{N}$, $m \in \mathbb{N} \setminus \{n\}$ and $k \in \mathbb{N}$ with $d(x_n, x_m) > 2/k$, from (4.18) we have

$$Q(\{Z_\tau \in B_{1/k}(x_n)\} \cap \{X_0 \in B_{1/k}(x_m)\}) = \mathbb{P}(Z_\tau \in B_{1/k}(x_n) \cap B_{1/k}(x_m)) = \mathbb{P}(\emptyset) = 0.$$

Writing

$$\{Z_\tau \neq Y_\tau\} = \{Z_\tau \neq X_0\} = \bigcup_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N} \setminus \{n\}} \bigcup_{k \in \mathbb{N} : d(x_n, x_m) > 2/k} \{Z_\tau \in B_{1/k}(x_n)\} \cap \{X_0 \in B_{1/k}(x_m)\},$$

we see that $\{Z_\tau \neq Y_\tau\}$ is a countable union of Q -null sets and so the claim follows. \square

Finally, Proposition 4.5 is extended in two directions: Firstly, we localize to prove uniqueness for the RCLL-martingale problem when σ is unbounded, and secondly, we show that any progressively measurable (but not necessarily RCLL) solution to the martingale problem has an RCLL modification. Note that the last statement is not true in general (as discussed in [BK03]), but it has to be established to apply Theorem 4.2.

In the proof, the notion of a *stopped martingale problem* is used: Let (F, d) be a complete metric space, $(D(\mathcal{L}), \mathcal{L})$ an operator on $C_b(F)$ and U open in F . If X is an F -valued, RCLL process, then $\tau := \inf\{t \geq 0 : X_t \notin U \text{ or } X_{t-} \notin U\}$ is an $(\mathcal{F}_t^X)_{t \geq 0}$ -stopping time by [EK86, Chap. 2, Prop. 1.5 a)]. X is called a solution to the stopped martingale problem for (\mathcal{L}, ν_0, U) if $X_0 \sim \nu_0$, $X^\tau = X$ and

$$h(X_{\tau \wedge t}) - h(X_0) - \int_0^{\tau \wedge t} \mathcal{L}h(X_s) ds, \quad t \geq 0,$$

is an $(\mathcal{F}_t^X)_{t \geq 0}$ -martingale for any $h \in D(\mathcal{L})$.

Proposition 4.7. *Suppose σ and $(\mathcal{D}, \mathcal{A})$ are as in Theorem 4.1 and $(D(\mathcal{L}), \mathcal{L})$ is defined as in (4.6). Then for any $\mu_0 \in \mathcal{P}(E)$ the martingale problem for $(\mathcal{L}, \delta_0 \otimes \mu_0)$ is well-posed.*

Proof. Suppose first $\mu_0 = \delta_{x_0}$ for some $x_0 \in E$. By Proposition 4.4 and our assumptions, there exists a solution Z to the RCLL-martingale problem for $(\mathcal{L}, \delta_0 \otimes \mu_0)$. Therefore, it suffices to show that if \tilde{Z} is any (progressively measurable) solution to the martingale problem for $(\mathcal{L}, \delta_0 \otimes \mu_0)$,

then \tilde{Z} has the same finite-dimensional marginal distributions as Z . In order to prove this, we proceed as follows:

To start, by [EK86, Chap. 4, Cor. 3.7] and since $D(\mathcal{L})$ is dense in $C_0(\mathcal{L})$, \tilde{Z} has a modification (which we also denote by \tilde{Z}) with sample paths in $D_{E_0^\Delta}[0, \infty)$, where E_0^Δ is the one-point compactification of $E_0 := [0, \infty) \times E$. Furthermore, (2.2) remains valid for \tilde{Z} and all $h \in D(\mathcal{L})$, where we extend h to $C(E_0^\Delta)$ by $h(\Delta) := 0$. By assumption on E , there exists $\{V_n\}_{n \in \mathbb{N}} \subset E$ such that for any n , V_n is open, $\overline{V_n}$ is compact and $\cup_n V_n = E$. Define $U_n := [0, \infty) \times V_n$, $\tau_n := \inf\{t \geq 0 : Z_t \notin U_n \text{ or } Z_{t-} \notin U_n\}$ and $\tilde{\tau}_n := \inf\{t \geq 0 : \tilde{Z}_t \notin U_n \text{ or } \tilde{Z}_{t-} \notin U_n\}$. Since U_n is open in E_0 , it is also open in E_0^Δ and thus τ_n is an $(\mathcal{F}_t^Z)_{t \geq 0}$ -stopping time and $\tilde{\tau}_n$ is an $(\mathcal{F}_t^{\tilde{Z}})_{t \geq 0}$ -stopping time.

Suppose Assumption 2.11 holds. Then for any $n \in \mathbb{N}$ there exists $C_n > 0$ such that $|H(x)| \leq C_n$ and $|\tilde{\sigma}(t, x)| \leq C_n$ for all $(t, x) \in U_n$. Set $\tilde{\sigma}_n(t, x) := \min(\tilde{\sigma}(t, x), C_n)$, $H_n(x) := \min(H(x), C_n)$ and $\sigma_n := H_n \tilde{\sigma}_n$. In the other case, i.e. if Assumption 2.13 holds, set $\sigma_n := \sigma$. Then, in both cases, σ_n is bounded and coincides with σ in U_n . Define

$$\mathcal{L}^n(f\gamma)(t, x) := \gamma(t)\sigma_n(t, x)\mathcal{A}f(x) + f(x)\gamma'(t), \quad t \in [0, \infty), x \in E,$$

for $f \in \mathcal{D}$ and $\gamma \in C_c^1[0, \infty)$ and linearly extended to $D(\mathcal{L}^n) := \text{span}\{f\gamma : f \in \mathcal{D}, \gamma \in C_c^1[0, \infty)\}$ (and thus $D(\mathcal{L}^n) = D(\mathcal{L})$).

We now claim that:

- (i) Z^{τ_n} is a solution to the stopped martingale problem for $(\mathcal{L}^n, \delta_0 \otimes \mu_0, U_n)$ and this solution is unique in law,
- (ii) $\tilde{Z}^{\tilde{\tau}_n}$ takes values in $D_{E_0}[0, \infty)$ and is also a solution to the stopped martingale problem for $(\mathcal{L}^n, \delta_0 \otimes \mu_0, U_n)$ and thus, combining this with (i), we get that the finite-dimensional marginals of $\tilde{Z}^{\tilde{\tau}_n}$ and Z^{τ_n} agree,
- (iii) from (ii) it can be deduced that $\tilde{\tau}_n \rightarrow \infty$ as $n \rightarrow \infty$ and that \tilde{Z} and Z have the same distribution.

To show (i), notice that σ_n is bounded and satisfies Assumption 2.6 and Assumption 2.10 since they hold for H and $\tilde{\sigma}$. In particular, σ_n and $(\mathcal{D}, \mathcal{A})$ satisfy Assumptions 2.6, 2.7, 2.13 and 2.10 and thus by Proposition 4.5 the RCLL-martingale problem for $(\mathcal{L}^n, \delta_0 \otimes \mu_0)$ is well-posed. Therefore, by [EK86, Chap. 4, Thm. 6.1] for each $U \subset E_0$ open there exists a unique solution to the stopped martingale problem for $(\mathcal{L}^n, \delta_0 \otimes \mu_0, U)$. Applying optional sampling to τ_n and the martingales (2.2) and noticing

$$(4.19) \quad \int_0^{\tau_n \wedge t} \mathcal{L}h(Z_s) ds = \int_0^{\tau_n \wedge t} \mathcal{L}^n h(Z_s) ds,$$

we see that Z^{τ_n} is a (and hence the unique) solution to the stopped martingale problem for $(\mathcal{L}^n, \delta_0 \otimes \mu_0, U_n)$.

To show (ii), note that, by definition of $\tilde{\tau}_n$, \tilde{Z} is RCLL and U_n -valued on $[0, \tilde{\tau}_n)$. Let us first show that actually $\tilde{\mathbb{P}}$ -a.s., for each n , $\tilde{Z}_{\tilde{\tau}_n} \in E_0$ (a priori, we could have $\tilde{Z}_{\tilde{\tau}_n} = \Delta$). To do so, note that by applying optional sampling and taking expectations in (2.2), we obtain

$$(4.20) \quad \mathbb{E}[h(\tilde{Z}_{\tilde{\tau}_n \wedge t})] = \mathbb{E}[h(\tilde{Z}_0)] + \mathbb{E}\left[\int_0^{\tilde{\tau}_n \wedge t} \mathcal{L}h(\tilde{Z}_s) ds\right] = \mathbb{E}[h(\tilde{Z}_0)] + \mathbb{E}\left[\int_0^{\tilde{\tau}_n \wedge t} \mathcal{L}^n h(\tilde{Z}_s) ds\right]$$

for all $h \in D(\mathcal{L})$ (with $C_0(E_0)$ extended to E_0^Δ as above). The second step in (4.20) follows as in (4.19). Since σ_n is bounded, $(D(\mathcal{L}^n), \mathcal{L}^n)$ is conservative by Lemma 4.3 (i) and so (for any $n \in \mathbb{N}$) there exists $\{h_k\}_{k \in \mathbb{N}} \subset D(\mathcal{L})$ such that $\text{bp-lim}_{k \rightarrow \infty} h_k = 1$ and $\text{bp-lim}_{k \rightarrow \infty} \mathcal{L}^n h_k = 0$. In particular, $\text{bp-lim}_{k \rightarrow \infty} h_k = \mathbb{1}_{E_0}$ in $C(E_0^\Delta)$. Inserting h_k in (4.20) and letting $k \rightarrow \infty$, dominated convergence gives

$$\mathbb{P}(\tilde{Z}_{\tilde{\tau}_n \wedge t} \in E_0) = \lim_{k \rightarrow \infty} \mathbb{E}[h_k(\tilde{Z}_{\tilde{\tau}_n \wedge t})] = \lim_{k \rightarrow \infty} \mathbb{E}[h_k(\tilde{Z}_0)] = \mathbb{P}(\tilde{Z}_0 \in E_0) = 1.$$

Therefore, for any $n \in \mathbb{N}$, \tilde{Z} is RCLL and U_n -valued on $[0, \tilde{\tau}_n]$. Thus, for any $n \in \mathbb{N}$, we may view $\tilde{Z}^{\tilde{\tau}_n}$ as a $D_{E_0}[0, \infty)$ -valued process and optional sampling applied to (2.2) (and the analogon of (4.19) for \tilde{Z}) shows that $\tilde{Z}^{\tilde{\tau}_n}$ is a solution to the stopped martingale problem for $(\mathcal{L}^n, \delta_0 \otimes \mu_0, U_n)$. Thus, by (i), the laws of $\tilde{Z}^{\tilde{\tau}_n}$ and Z^{τ_n} coincide.

To show (iii), first note that $\tilde{\tau}_n \leq \tilde{\tau}_{n+1}$ for all $n \in \mathbb{N}$ and thus $\tau := \lim_{n \rightarrow \infty} \tilde{\tau}_n$ is well-defined. Since \tilde{Z} has left-limits in E_0^Δ , also $Y_t := \lim_{n \rightarrow \infty} \tilde{Z}_{\tilde{\tau}_n \wedge t}$ is well-defined in E_0^Δ . Furthermore, $Y_t = \Delta$ if and only if $\tau \leq t$. Since Z has sample paths in $D_{E_0}[0, \infty)$, it holds that $\tau_n \rightarrow \infty$, \mathbb{P} -a.s., and so (ii) implies

$$(4.21) \quad \mathbb{E}[h(Y_t)] = \lim_{n \rightarrow \infty} \mathbb{E}[h(\tilde{Z}_t^{\tilde{\tau}_n})] = \lim_{n \rightarrow \infty} \mathbb{E}[h(Z_t^{\tau_n})] = \mathbb{E}[h(Z_t)]$$

for any $t \geq 0$ and $h \in C_0(E_0)$. Taking $\{h_k\}_{k \in \mathbb{N}} \subset C_0(E_0)$ with $\text{bp-lim}_{k \rightarrow \infty} h_k = 1$ (and thus $\text{bp-lim}_{k \rightarrow \infty} h_k = \mathbb{1}_{E_0}$ in $C(E_0^\Delta)$), inserting h_k in (4.21) and letting $k \rightarrow \infty$, one obtains

$$\mathbb{P}(\tau > t) = \mathbb{P}(Y_t \in E_0) = \lim_{k \rightarrow \infty} \mathbb{E}[h_k(Y_t)] = \lim_{k \rightarrow \infty} \mathbb{E}[h_k(Z_t)] = \mathbb{P}(Z_t \in E_0) = 1$$

for all $t \geq 0$. Hence, $\mathbb{P}(\tau = \infty) = 1$, $\tilde{\tau}_n \rightarrow \infty$ a.s. and \tilde{Z} does not explode, i.e. it has sample paths in $D_{E_0}[0, \infty)$. In particular, for any choice of $0 \leq t_0 < \dots < t_m$, $m \in \mathbb{N}$, $h_0, \dots, h_m \in C_0(E_0)$,

$$\mathbb{E} \left[\prod_{k=0}^m h_k(\tilde{Z}_{t_k}) \right] = \lim_{n \rightarrow \infty} \mathbb{E} \left[\prod_{k=0}^m h_k(\tilde{Z}_{t_k}^{\tilde{\tau}_n}) \right] \stackrel{(ii)}{=} \lim_{n \rightarrow \infty} \mathbb{E} \left[\prod_{k=0}^m h_k(Z_{t_k}^{\tau_n}) \right] = \mathbb{E} \left[\prod_{k=0}^m h_k(Z_{t_k}^{\tau_n}) \right].$$

Therefore, also the finite-dimensional marginals of \tilde{Z} and Z coincide.

Finally, since now well-posedness of the RCLL-martingale problem for $(\mathcal{L}, \delta_0 \otimes \mu_0)$ in the case $\mu_0 = \delta_{x_0}$ and $x_0 \in E$ is established, Lemma 4.3 (v) and [BK93, Thm. 2.1] imply that the RCLL-martingale problem for $(\mathcal{L}, \delta_0 \otimes \mu_0)$ is well-posed also for arbitrary $\mu_0 \in \mathcal{P}(E)$. The exact same argument as above now shows that uniqueness holds even in the class of progressively measurable solutions. \square

4.3. From Uniqueness of the Martingale Problem to Uniqueness for the Fokker–Planck Equation. Finally, we put together all results obtained in the previous sections. When σ is continuous, Theorem 4.2 can be applied. When σ is not continuous, the following extension of Theorem 4.2 will be required:⁶

Theorem 4.8 ([Kur98, Thm. 2.7]). *Let E_0 and F be locally compact, separable metric spaces, $D(\mathcal{L}^0) \subset C_b(E_0)$ and $\mathcal{L}^0: D(\mathcal{L}^0) \rightarrow C_b(E_0 \times F)$ linear. Let $\eta: E_0 \times \mathcal{B}(F) \rightarrow [0, 1]$ be a transition kernel and define*

$$(4.22) \quad \mathcal{L}_\eta f(x) := \int_F \mathcal{L}^0 f(x, y) \eta(x, dy), \quad f \in D(\mathcal{L}^0).$$

For any $y \in F$, define the linear operator \mathcal{L}_y^0 with domain $D(\mathcal{L}^0)$ on $C_b(E_0)$ by $f \mapsto \mathcal{L}^0 f(\cdot, y)$. Let $\nu \in \mathcal{P}(E_0 \times F)$ and suppose that

- (i) $D(\mathcal{L}^0)$ is an algebra and separates points,
- (ii) for any $x \in E_0$ and $y \in F$ there exists a solution to the RCLL-martingale problem for $(\mathcal{L}_y^0, \delta_x)$,
- (iii) \mathcal{L}_η and (\mathcal{L}_η, ν) satisfy the conditions (ii) and (iv) of Theorem 4.2.

Then the conclusion of Theorem 4.2 is valid, i.e. uniqueness holds for the forward equation for (\mathcal{L}_η, ν) .

In the next lemma we show how to obtain uniqueness for the Fokker–Planck equation for $(\mathcal{D}, \mathcal{A})$ from uniqueness for $(D(\mathcal{L}), \mathcal{L})$.

⁶See footnote 3 for a discussion why (ii) implies that \mathcal{L}_y^0 is a pre-generator (in the terminology of [Kur98]).

Lemma 4.9. *Suppose σ and $(\mathcal{D}, \mathcal{A})$ are as in Theorem 4.1 and $(D(\mathcal{L}), \mathcal{L})$ is defined as in (4.6). Let $\mu_0 \in \mathcal{P}(E)$, t_0 as in Assumption 2.6 and define $\nu := \delta_0 \otimes \mu_0$. Then the following statements hold:*

- (i) *If $\sigma \mathcal{A}f \in C_0([0, t_0] \times E)$ for all $f \in \mathcal{D}$, then the assumptions of Theorem 4.2 are satisfied.*
- (ii) *Suppose there exists $R > 0$ such that $|\sigma(t, x)| \leq R$ for all $(t, x) \in [0, \infty) \times E$. Define $E_0 := [0, \infty) \times E$, $F := [0, R]$, $D(\mathcal{L}^0) := D(\mathcal{L})$ and for $f \in \mathcal{D}$, $\gamma \in C_c^1[0, \infty)$, set*

$$\mathcal{L}^0(f\gamma)((t, x), v) := \gamma(t)v\mathcal{A}f(x) + f(x)\gamma'(t), \quad (t, x) \in E_0, v \in F,$$

and linearly extend this definition of \mathcal{L}^0 to $D(\mathcal{L}^0)$. Finally, define $\eta((t, x), \cdot) := \delta_{\sigma(t, x)}(\cdot)$ and \mathcal{L}_η as in (4.22). Then the assumptions of Theorem 4.8 are satisfied and $(D(\mathcal{L}), \mathcal{L})$ coincides with $(D(\mathcal{L}^0), \mathcal{L}_\eta)$.

In particular, in both cases the conclusion of Theorem 4.2 is valid, i.e. uniqueness holds for the forward equation (4.5) for (\mathcal{L}, ν) .

Proof. To prove (i), firstly note that by Assumption 2.7 (i) \mathcal{D} is an algebra and dense. Hence, $D(\mathcal{L})$ is an algebra and separates points. Secondly, by Lemma 4.3 (v) the condition (4.3) is indeed satisfied. Thirdly, by Proposition 4.4 existence holds and fourthly, by Proposition 4.7 uniqueness holds. Therefore, assumptions (i)-(iv) of Theorem 4.2 are indeed satisfied.

To prove (ii), notice that σ is a measurable function, and thus η is indeed a transition kernel. Furthermore, by definition we have

$$\mathcal{L}h(t, x) = \int_0^R \mathcal{L}^0h((t, x), v) \eta((t, x), dv) = \mathcal{L}_\eta h(t, x).$$

Hence, $(D(\mathcal{L}), \mathcal{L})$ and $(D(\mathcal{L}^0), \mathcal{L}_\eta)$ indeed coincide and it only remains to show that the assumptions of Theorem 4.8 are satisfied. Firstly, $D(\mathcal{L}^0) = D(\mathcal{L})$ is an algebra and separates points as argued in (i). Secondly, for any $v \in [0, R]$ and $(s_0, x) \in E_0$ a solution to the RCLL-martingale problem for $(\mathcal{L}_v^0, \delta_{(s_0, x)})$ can be constructed as follows: Let M be a solution to the RCLL-martingale problem for (\mathcal{A}, δ_x) and set $X_t := M_{vt}$. Then by elementary change of variable,

$$f(X_t) - \int_0^t v\mathcal{A}f(X_s) ds = f(M_{tv}) - \int_0^{tv} \mathcal{A}f(M_s) ds$$

for all $f \in \mathcal{D}$, $t \geq 0$, and thus X is a solution to the (time-inhomogeneous) RCLL-martingale problem for $(v\mathcal{A}, \delta_x)$. Therefore, by Lemma 4.3, $(t + s_0, X_t)_{t \geq 0}$ is a solution to the RCLL-martingale problem for $(\mathcal{L}_v^0, \delta_{(s_0, x)})$. Finally, since σ is bounded, $(D(\mathcal{L}), \mathcal{L})$ satisfies (4.3) by Lemma 4.3 (v) and the martingale problem for (\mathcal{L}, ν) is well-posed by Proposition 4.7. \square

After these preparations, we are ready to prove the main result in this section, Theorem 4.1 (ii).

Proof of Theorem 4.1 (ii). Suppose $(p(t, \cdot))_{t \in [0, t_0]}$ and $(\tilde{p}(t, \cdot))_{t \in [0, t_0]}$ both satisfy (4.1) and (4.2). Defining for any $t \geq 0$ the measures $\nu_t := \delta_t \otimes p(t, \cdot)$ and $\tilde{\nu}_t := \delta_t \otimes \tilde{p}(t, \cdot)$, by Lemma 4.3 (iv), $(\nu_t)_{t \geq 0}$ and $(\tilde{\nu}_t)_{t \geq 0}$ both satisfy (4.4) and (4.5). Under our assumptions, Lemma 4.9 implies that uniqueness holds for (4.5), i.e. $\tilde{\nu}_t = \nu_t$ for all $t \geq 0$ or $\delta_t \otimes p(t, \cdot) = \delta_t \otimes q(t, \cdot)$ for all $t \geq 0$. In particular, $\int_E f(x) p(s, dx) = \int_E f(x) \tilde{p}(s, dx)$ for all $f \in C_0(E)$ and all $s \in [0, t_0]$ and thus the assertion follows. \square

APPENDIX A. AUXILIARY RESULTS FOR THE CONSTRUCTION OF TIME-CHANGES

In order to ensure the existence of the time-change τ as defined by the random differential equation (3.1), we used the following lemma concerning so-called Carathéodory differential equations.

Lemma A.1. *Let $t_0 > 0$ and consider the Carathéodory differential equation*

$$(A.1) \quad \mathcal{T}(t) = \int_0^t \gamma(\mathcal{T}(s), s) ds,$$

where $\gamma: [0, t_0] \times [0, \infty) \rightarrow [0, \infty)$ and the integral is understood in the Lebesgue sense. For some $S \in (0, t_0]$ and $T > 0$ suppose that $\gamma(r, \cdot)$ is measurable for each $r \in [0, S]$, $\gamma(0, \cdot)$ is integrable on $[0, T]$ and there exists an integrable function $f: [0, T] \rightarrow [0, \infty)$ such that

$$|\gamma(r, t) - \gamma(s, t)| \leq f(t)|r - s| \quad \text{for all } r, s \in [0, S], t \in [0, T].$$

Then there exists a unique absolutely continuous function $\mathcal{T}: I \rightarrow [0, S]$ satisfying (A.1) for some interval $I \subset [0, \infty)$, where either there exists $T_0 \in (0, T]$ such that we may take $I = [0, T_0]$ and we have $\mathcal{T}(T_0) = S$ or we may take $I = [0, T]$ and have $\mathcal{T}(t) < S$ for all $t \leq T$.

Proof. Since

$$|\gamma(r, t)| \leq |\gamma(0, t)| + |f(t)|S \quad \text{for all } r \in [0, S]$$

and the right-hand side is integrable on $[0, T]$, γ satisfies the Carathéodory Conditions in [Fil88, Chap. 1] and thus [Fil88, Chap. 1, Thm. 1] guarantees the existence of a solution \mathcal{T} on an interval $[0, T_0]$ for some $T_0 > 0$. The solution \mathcal{T} can be extended either to the whole interval $[0, T]$ provided $\mathcal{T}(t) \leq S$ for all $t \in [0, T]$ or to the interval $[0, T_0]$ for some $T_0 \in (0, T]$ with $\mathcal{T}(T_0) = S$ (see e.g. [Fil88, Chap. 1, Thm. 4]). Uniqueness of the solution follows by [Fil88, Chap. 1, Thm. 2]. \square

Next, we provide a condition that is useful in verifying regularity of H (see Definition 2.4) needed for the existence of the time-change in Lemma 3.1 and the uniqueness in Lemma A.3 below.

Proposition A.2. *Let $\mathcal{D} \subset C_0(E)$ dense in $C_0(E)$ and $\mathcal{A}: \mathcal{D} \rightarrow C_0(E)$ be linear. Suppose M is a solution on $(\Omega, \mathcal{F}, \mathbb{P})$ to the RCLL-martingale problem for (\mathcal{A}, μ_0) , for some $\mu_0 \in \mathcal{P}(E)$. Denote by P the law on $D_E[0, \infty)$ of M . Then any $H \in \mathcal{D}$ with $H \geq 0$ is regular for P .*

Proof. Define ρ as in (3.2) and recall that, by Definition 2.4, (3.7) and (3.8) have to be verified. Set

$$\rho_0 := \inf \{s \in [0, \infty) : H(M_s) = 0\}.$$

Since H is continuous and M is RCLL, $H(M_\rho) = 0$ on $\{\rho < \infty\}$ and $\rho_0 \leq \rho$, \mathbb{P} -a.s. In particular, $\rho_0 = \rho$ on $\{\rho_0 = \infty\}$, and if $\{\rho_0 < \infty\}$ is a \mathbb{P} -null set, this already establishes the claim. Otherwise the probability measure $\tilde{\mathbb{P}}(\cdot) := \mathbb{P}(\cdot | \{\rho_0 < \infty\})$ is well-defined, $\rho_0 < \infty$, $\tilde{\mathbb{P}}$ -a.s. and to prove the proposition we only need to show $\tilde{\mathbb{P}}(\rho_0 \geq \rho) = 1$. To do so, on $\{\rho_0 < \infty\}$ define for any $t \geq 0$ the random time

$$\tau(t) := \inf \left\{ s \in [0, \infty) : \int_0^s H(M_{u+\rho_0})^{-1} du \geq t \right\}.$$

Since $H(M_{\rho_0}) = 0$ and $\rho_0 \leq \rho$, $\tilde{\mathbb{P}}$ -a.s., it suffices to establish that $\tilde{\mathbb{P}}$ -a.s. for any $t \geq 0$, $\tau(t) = 0$. For the proof of the last statement one proceeds as follows: Since H is bounded, $H(M_\rho) = 0$ on $\{\rho < \infty\}$, $\tilde{\mathbb{P}}$ -a.s., and by footnote 2, Lemma 3.1 can be applied to the RCLL process $(M_{u+\rho_0})_{u \geq 0}$ on $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ with $\tilde{\sigma} = 1$ and $\sigma = H$. This yields $\tilde{\mathbb{P}}$ -a.s.,

$$(A.2) \quad \tau(t) = \int_0^t H(M_{\tau(u)+\rho_0}) du, \quad t \geq 0,$$

and $\tau(t) < \infty$ for any $t \geq 0$. Denote by $(\mathcal{F}_t)_{t \geq 0}$ the \mathbb{P} -usual augmentation of the filtration generated by M . Then ρ and ρ_0 (possibly modified on a \mathbb{P} -null set, see [EK86, Chap. 4, Cor. 3.13]) are $(\mathcal{F}_t)_{t \geq 0}$ -stopping times and thus

$$\{\tau(t) + \rho_0 \leq s\} = \{\rho_0 \leq s\} \cap \left(\left\{ \int_{\rho_0}^s H(M_u)^{-1} du \geq t \right\} \cup \{\rho \leq s - \rho_0\} \right) \in \mathcal{F}_s$$

shows that also $\tau(t) + \rho_0$ is a stopping time. By assumption on H , M and \mathcal{A} the process

$$N_t := H(M_t) - H(M_0) - \int_0^t \mathcal{A}H(M_s) ds, \quad t \geq 0,$$

is an $(\mathcal{F}_t^M)_{t \geq 0}$ -martingale and thus, by [RW00, Lem. II.67.10], also an $(\mathcal{F}_t)_{t \geq 0}$ -martingale. By the optional sampling theorem, for any $r \geq 0$, \mathbb{P} -a.s.,

$$\mathbb{E}[N_{(\tau(t)+\rho_0) \wedge r} | \mathcal{F}_{\rho_0 \wedge r}] = N_{\rho_0 \wedge r}$$

or equivalently

$$\mathbb{E}[H(M_{(\tau(t)+\rho_0) \wedge r}) | \mathcal{F}_{\rho_0 \wedge r}] = H(M_{\rho_0 \wedge r}) + \mathbb{E} \left[\int_{\rho_0 \wedge r}^{(\tau(t)+\rho_0) \wedge r} \mathcal{A}H(M_u) du \middle| \mathcal{F}_{\rho_0 \wedge r} \right].$$

Multiplying by $\mathbb{1}_{\{\rho_0 \leq r\}}$, using $\{\rho_0 \leq r\} \in \mathcal{F}_{\rho_0 \wedge r}$ and taking expectations gives

$$\mathbb{E}[H(M_{(\tau(t)+\rho_0) \wedge r}) \mathbb{1}_{\{\rho_0 \leq r\}}] = \mathbb{E}[H(M_{\rho_0 \wedge r}) \mathbb{1}_{\{\rho_0 \leq r\}}] + \mathbb{E} \left[\int_{\rho_0 \wedge r}^{(\tau(t)+\rho_0) \wedge r} \mathcal{A}H(M_u) du \mathbb{1}_{\{\rho_0 \leq r\}} \right].$$

By assumption $\mathcal{D} \subset C_0(E)$ is dense and thus separating (see [EK86, Chap. 3, Ex. 11]) in the terminology of [EK86, Chap. 3, Sec. 4]. Therefore, by quasi-left continuity, [EK86, Chap. 4, Thm. 3.12],

$$\lim_{r \rightarrow \infty} M_{(\tau(t)+\rho_0) \wedge r} \mathbb{1}_{\{\rho_0 \leq r\}} = M_{\tau(t)+\rho_0} \mathbb{1}_{\{\rho_0 < \infty\}}, \quad \mathbb{P}\text{-a.s.},$$

and so using dominated convergence, boundedness and non-negativity of H , $H(M_{\rho_0}) = 0$ on $\{\rho < \infty\}$ and setting $C := \|\mathcal{A}H\|$, one estimates

$$\begin{aligned} \mathbb{E}[H(M_{\tau(t)+\rho_0}) \mathbb{1}_{\{\rho_0 < \infty\}}] &= \lim_{r \rightarrow \infty} \mathbb{E}[H(M_{(\tau(t)+\rho_0) \wedge r}) \mathbb{1}_{\{\rho_0 \leq r\}}] \\ \text{(A.3)} \quad &= \lim_{r \rightarrow \infty} \left| \mathbb{E} \left[\int_{\rho_0}^{(\tau(t)+\rho_0) \wedge r} \mathcal{A}H(M_u) du \mathbb{1}_{\{\rho_0 \leq r\}} \right] \right| \\ &\leq C \mathbb{E}[\tau(t) \mathbb{1}_{\{\rho_0 < \infty\}}]. \end{aligned}$$

Using (A.2) and Tonelli's theorem for the first and (A.3) for the second equality yields

$$\text{(A.4)} \quad \mathbb{E}[\tau(t) \mathbb{1}_{\{\rho_0 < \infty\}}] = \int_0^t \mathbb{E}[H(M_{\tau(u)+\rho_0}) \mathbb{1}_{\{\rho_0 < \infty\}}] du \leq C \int_0^t \mathbb{E}[\tau(u) \mathbb{1}_{\{\rho_0 < \infty\}}] du$$

and so Gronwall's lemma implies that the left-hand side of (A.4) is 0 for any $t \geq 0$. But this implies that $\tilde{\mathbb{P}}$ -a.s., $\tau(t) = 0$ for all $t \geq 0$ as desired. \square

A.1. Pathwise Uniqueness. To verify that the random times $(\tau(t))_{t \in [0, t_0]}$ solving the differential equation (3.1) are indeed stopping times with respect to the filtration generated by the process M , we show pathwise uniqueness of the time-changed Markov process $X_t := M_{\tau(t)}$ for $t \in [0, t_0]$.

Lemma A.3. *Let σ and M be given as in Lemma 3.1. $(\tau(t))_{t \in [0, t_0]}$ is the family of random times from Lemma 3.1 with $\tau(t) := \tau(t_0)$ for $t > t_0$ and the time-changed process X is given by $X_t := M_{\tau(t)}$ for $t \geq 0$. Suppose M is (\mathcal{F}_t) -adapted. Then the following holds:*

- (i) *The time-changed process X has RCLL sample paths, \mathbb{P} -a.s.*
- (ii) *Any RCLL process \tilde{X} satisfying*

$$\text{(A.5)} \quad \tilde{X}_t = M_{\int_0^t \sigma(u, \tilde{X}_u) du}, \quad t \in [0, \infty), \quad \mathbb{P}\text{-a.s.},$$

is indistinguishable from X .

- (iii) *The random times $(\tau(t))_{t \in [0, t_0]}$ are (\mathcal{F}_t) -stopping times.*

Proof. (i) Since M has RCLL sample paths and τ is non-decreasing and absolutely continuous by Lemma 3.1, the time-changed process X has RCLL sample paths.

(ii) Let \tilde{X} be an RCLL process satisfying equation (A.5). Define the random time

$$\tilde{\rho} := t_0 \wedge \inf \{t \geq 0 : H(\tilde{X}_t) = 0\}$$

and set

$$\tilde{\tau}(s) := \int_0^s \sigma(u, \tilde{X}_u) du, \quad s \in [0, \infty).$$

Notice that the integral is well-defined since σ is bounded on compacts and \tilde{X} is RCLL. Since $X_t = M_{\tau(t)}$ and $\tilde{X}_t = M_{\tilde{\tau}(t)}$, to verify that X and \tilde{X} are indistinguishable, it is sufficient to show that $\tau(t) = \tilde{\tau}(t)$ for every $t \in [0, \infty)$, \mathbb{P} -a.s.

By [Leo09, Lem. 3.31] $\tilde{\tau}$ is absolutely continuous with weak derivative $\tilde{\tau}'(u) = \sigma(u, \tilde{X}_u)$ for $u \in [0, \infty)$ and invertible on $[0, \tilde{\rho} \wedge t_0]$ by the definition of $\tilde{\rho}$. The inverse of $\tilde{\tau}$ is denoted by $\tilde{\mathcal{T}}$ with domain $[0, \tilde{\tau}(\tilde{\rho} \wedge t_0)]$. Because $\tilde{\mathcal{T}}$ is also strictly increasing and absolutely continuous, the chain rule (see [Leo09, Thm. 3.44]) gives

$$1 = \frac{d}{dt} \tilde{\tau}(\tilde{\mathcal{T}}(t)) = \sigma(\tilde{\mathcal{T}}(t), \tilde{X}_{\tilde{\mathcal{T}}(t)}) \frac{d}{dt} \tilde{\mathcal{T}}(t) \quad \text{for almost all } t \in [0, \tilde{\tau}(\tilde{\rho} \wedge t_0)].$$

Combining this with fundamental theorem of calculus (see [Leo09, Thm. 3.30]), one has that $\tilde{\mathcal{T}}$ satisfies the integral equation

$$\tilde{\mathcal{T}}(t) = \int_0^t \sigma(\tilde{\mathcal{T}}(s), \tilde{X}_{\tilde{\mathcal{T}}(s)})^{-1} ds, \quad t \in [0, \tilde{\tau}(\tilde{\rho} \wedge t_0)].$$

Moreover, notice that $M_t = M_{\tilde{\tau}(\tilde{\mathcal{T}}(t))} = \tilde{X}_{\tilde{\mathcal{T}}(t)}$ for $t \in [0, \tilde{\tau}(\tilde{\rho} \wedge t_0)]$. Therefore, $\mathcal{T}(t) = \tilde{\mathcal{T}}(t)$ for $t \in [0, \tilde{\tau}(\tilde{\rho} \wedge t_0) \wedge \tau(t_0)]$ since the solution to this equation is unique on $[0, \tau(t_0)]$, see (3.4). Furthermore, we have $\tilde{\tau}(\tilde{\rho} \wedge t_0) \leq \rho \wedge \tilde{\tau}(t_0)$ since $t < \tilde{\tau}(\tilde{\rho} \wedge t_0)$ implies $\tilde{\mathcal{T}}(t) < \tilde{\rho} \wedge t_0$ and thus $t < \rho \wedge \tilde{\tau}(t_0)$, where we recall ρ from (3.2) and that (3.7), (3.8) holds. In conclusion, $\mathcal{T}(t) = \tilde{\mathcal{T}}(t)$ for $t \in [0, \tilde{\tau}(\tilde{\rho} \wedge t_0)]$, which leads to $\tilde{\tau}(s) = \tau(s)$ for $s \in [0, \tilde{\rho} \wedge t_0]$.

To see $\tilde{\tau}(s) = \tau(s)$ for $s > \tilde{\rho} \wedge t_0$, we first observe that $1/(H(M_s) \vee \varepsilon)$ is bounded for every $\varepsilon > 0$ and $\tilde{\sigma}$ is bounded on compacts by Assumption 2.6. Applying a change of variables ([Leo09, Cor. 3.57]) and using monotone convergence gives

$$Ct \geq \lim_{\varepsilon \rightarrow 0} \int_0^t \frac{\sigma(s, \tilde{X}_s)}{H(\tilde{X}_s) \vee \varepsilon} ds = \lim_{\varepsilon \rightarrow 0} \int_0^{\tilde{\tau}(t)} \frac{1}{H(M_s) \vee \varepsilon} ds = \int_0^{\tilde{\tau}(t)} \frac{1}{H(M_s)} ds,$$

which ensures $\tilde{\tau}(t) \leq \rho$ for all $t \geq 0$. Assuming $\tilde{\rho} < t_0$, there exist $\{t_n\}_{n \in \mathbb{N}} \subset [\tilde{\rho}, t_0]$ with $t_n \downarrow \tilde{\rho}$ and $H(\tilde{X}_{t_n}) = 0$ and so $\rho \leq \tilde{\tau}(\tilde{\rho})$ by (A.5) and (3.7), (3.8). In this case $\tilde{\tau}(t) = \tilde{\tau}(\tilde{\rho}) = \rho = \tau(t)$ for all $t \geq \tilde{\rho}$. Assuming $\tilde{\rho} \geq t_0$, we have $\tilde{\tau}(t) = \tilde{\tau}(t_0)$ for all $t \geq t_0$ due to $\sigma(t, \cdot) = 0$ for $t > t_0$ and in particular $\tilde{\tau}(t) = \tilde{\tau}(t_0) = \tau(t)$ for $t \geq t_0$.

(iii) In order to apply a result from [EK86], we consider the two-dimensional process $Y_t := (t, M_t)$ and the time-changed process (t, X_t) for $t \in [0, T]$. Hence, [EK86, Chap. 6, Thm. 2.2 (b)] implies that $\tau(t)$ is a stopping time with respect to the usual augmentation of the filtration generated by M , and thus also an (\mathcal{F}_t) -stopping time, where we keep in mind that the first component of Y generates a trivial filtration. \square

Corollary A.4. *Let σ , M and \tilde{X} be given as in Lemma A.3 and denote by P the law (on $D_E[0, \infty)$) of M under \mathbb{P} . Then the law of \tilde{X} under \mathbb{P} is uniquely determined by P and σ .*

Proof. By Lemma A.3 (ii), the law of \tilde{X} is identical to the law of X under \mathbb{P} . To show explicitly that the latter is uniquely determined by P and σ , one proceeds as follows: Let $n \in \mathbb{N}$, $t_1, \dots, t_n \in [0, \infty)$, $B_1, \dots, B_n \in \mathcal{B}(E)$ and let $\pi_1: D_E[0, \infty) \times D_E[0, \infty) \rightarrow D_E[0, \infty)$ be the projection map on the first component. We have seen in the proof of Lemma A.3 that there exist a unique solution to the time-change equation for \mathbb{P} -a.e. sample path $M(\omega)$. Hence, as in the proof of [EK86, Chap. 6, Lem. 2.1], the map

$$\gamma: D_E[0, \infty) \times D_E[0, \infty) \rightarrow D_E[0, \infty), \quad \gamma(M, X) := M_{\int_0^\cdot \sigma(u, X_u) du},$$

is Borel measurable and the set

$$C := \{(m, x) \in D_E[0, \infty) \times D_E[0, \infty) : \gamma(m, x) = x, x_{t_1} \in B_1, \dots, x_{t_n} \in B_n\}$$

is in $\mathcal{B}(D_E[0, \infty)^2)$. Then [EK86, Appendix 11, Thm. 11.3] implies that $\pi_1 C$ is in the P -completion of $\mathcal{B}(D_E[0, \infty))$ and thus

$$\mathbb{P}(X_{t_1} \in B_1, \dots, X_{t_n} \in B_n) = P(\pi_1 C)$$

is indeed uniquely determined by P and σ . \square

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