A fractional credit model with long range dependent default rate

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April 6, 2010

Abstract

Motivated by empirical evidence of long range dependence in macroeconomic variables like interest rates, domestic gross products or supply and demand rates, we propose a fractional Brownian motion (fBm) driven model to describe the dynamics of the short rate in a bond market as well as the default rate for possible default. We aim at results analogous to those achieved in recent years for affine models. We start with a bivariate fractional Vasicek model (with time dependent coefficient functions) for short and default rate, which allows for fairly explicit calculations. We calculate the prices of corresponding zero-coupon bonds by invoking Wick calculus. The mathematical challenges are the prediction of exponentials of two dependent fBm integrals, and to find closed formulas for the prices of defaultable zero-coupon bonds. Applying a pathwise Girsanov theorem we derive today's prices of European calls. More general options will be priced exploiting Fourier methods. We also compare our results to the classical Brownian motion driven Vasicek model.

2000 MSC Subject Classifications: primary: 60G12, 60G51, 60H10, 60H20, 91B70
secondary: 60G10, 60G99, 91B28

Keywords: credit risk, defaultable bond, default rate, derivatives pricing, fractional Brownian motion, fractional Vasicek model, hazard rate, interest rate, long range dependence, macroeconomic variables process, option pricing, prediction, short rate, Wick product

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1 Introduction

The ongoing financial crisis shows that the mostly used Gaussian or, more general, Markov models may not be sufficient to catch the market structure for credit derivatives. One reason for this may be the fact that short rates and/or default rates, which are driven by macroeconomic variables like domestic gross products, supply and demand rates or volatilities exhibit long range dependence, cannot be captured by Markov models. Empirical evidence has been reported over the years and we refer to Henry and Zaffaroni [16] for details and further references.

In this paper we start a thorough investigation considering bond and credit markets driven by fractional Brownian motions (fBms) with with Hurst index $H > \frac{1}{2}$. We aim at results analogous to those obtained in recent years for affine models; see e.g. Duffie [6] and Duffie, Filipovic and Schachermayer [7]. This idea has been present in Biagini, Fuschini and Klüppelberg [1], where the focus was, however, on credit contagion. In the present paper, we start again, this time with the focus on structural results in a fBm driven market. We are facing two problems. Firstly, the non-Markovianity implies that all past information will enter into prices. Secondly, our models are in general not semimartingales, so that we cannot use Itô calculus. However, we can apply pathwise or L^2 integration theory, also to obtain solutions to fBm driven stochastic differential equations (sde's); cf. Buchmann and Klüppelberg [5] based on previous work by Zähle [23].

In this paper we mainly focus on the pricing of defaultable derivatives depending on the short rate and the default rate. Moreover, we concentrate in this paper on the Vasicek model (with possibly time dependent coefficient functions). We are aware of the fact that, as a Gaussian process, the short rate as well as the default rate can also take negative values. However it is always possible to shift, and perhaps also scale, the model such that the probability of a path becoming negative is arbitrarily small. We leave more general models as, for instance, suggested in Buchmann and Klüppelberg [5] or the fractional Lévy driven versions of Fink and Klüppelberg [12] for future research.

Apart from the driving fBm the models we consider are well-known as two factor models in the literature and we refer to Filipovic [11], Schönbucher [21] and Bielecki and Rutkowski [3] for background reading on credit risk modeling. In Fink, Klüppelberg and Zähle [13] d factor models ($d \in \mathbb{N}$) are considered, however, the factors are assumed to be independent. We want to mention that in difference to our work, Ohashi [18] considered another way to build interest rate models with fBm using a Heath-Jarrow-Morton approach. In contrast to our approach he has to consider the possibility of arbitrage, while we can directly start to price bonds and derivates.

To calculate prices of (defaultable) zero-coupon bonds prediction theorems for exponential integrals of Vasicek models are essential. These can be obtained very elegantly by exploiting Wick calculus. Once the prediction problem has been solved, a Girsanov theorem and Fourier methods can be applied to find closed formulas for option prices in credit models.

Our paper is organized as follows. Section 2 will briefly recall the framework of the considered credit model. In section 3 we collect important facts about integration with respect to fBm and prediction of such integrals. We introduce a bivariate fractional time-dependent Vasicek model in Section 4 to describe the dynamics of the short and default rate. Zero coupon bond prices are calculated. In Section 5 we consider option pricing calculating today's price of a zero-coupon bond call directly and apply Fourier methods for more general options.

2 The credit model

The overall state of our stochastic system is described by the process $(r, H) = (r(t), H(t))_{t\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the filtration $(\mathcal{F}_t)_{t\geq 0}$, representing the complete market information and satisfying the usual conditions of completeness and right continuity. The stochastic process r models the *short rate*, and H is the *default indicator process* given by

$$H(t) = \mathbf{1}_{\{\tau \leq t\}}, \quad t \geq 0,$$

where τ is an $(\mathcal{F}_t)_{t\geq 0}$-stopping time, representing the default time of some firm or financial instrument. We denote by $(\mathcal{H}_t)_{t\geq 0}$ the filtration generated by H. We assume further that there exists a subfiltration $(\mathcal{G}_t)_{t\geq 0}$ of $(\mathcal{F}_t)_{t\geq 0}$ such that

$$\mathcal{F}_t := \mathcal{G}_t \vee \mathcal{H}_t\,, \quad t \geq 0.$$

Assumption 2.1. *Remaining in the framework of most reduced-form credit risk models in the literature we assume that there is a $(\mathcal{G}_t)_{t\geq 0}$-progressive stochastic process $\lambda = (\lambda_t)_{t\geq 0}$ modeling the intensity of H with the following properties (see also Corollary 5.1.5 of Bielecki and Rutkowski [3]): λ is positive, $\int_0^t \lambda(s)ds < \infty$ a.s. for all $t \in \mathbb{R}$ and it satisfies*

$$P(\tau > t \mid \mathcal{G}_t) = E[1 - H(t) \mid \mathcal{G}_t] = \exp\left\{-\int_0^t \lambda(s)ds\right\}. \tag{2.1}$$

Moreover, defining $\mathcal{G}_\infty := \bigvee_{t\geq 0} \mathcal{G}_t$, for all bounded $\mathcal{G}_\infty$-measurable random variables η, we have

$$E[\eta \mid \mathcal{G}_\infty] = E[\eta \mid \mathcal{G}_t]. \tag{2.2}$$

We call λ the default rate*. We further assume that also the* short rate *r is a $(\mathcal{G}_t)_{t\geq 0}$-progressive process.*

Assumption 2.2 (Market structure; cf. Frey and Backhaus [14], Ass. 3.1.)**.**

(1) The investor information at time t is given by the default history $(\mathcal{F}_t)_{t\geq 0}$. This means that the investor knows the short rate r, the default rate λ and the default indicator process H at time t.

(2) A risk neutral (martingale) pricing measure P exists and is known, such that the price of any $\mathcal{F}_T$-measurable claim $X \in L^1(\Omega)$ with maturity $T > 0$ at time t is given by $\mathcal{V}(t, T) = E[X \mid \mathcal{F}_t]$ for $0 \leq t \leq T$.

3 Integrals and prediction of fractional Brownian motion

There are many examples, which consider the short and default rate as functions of state vectors of Markov processes; see e.g. Duffie, Filipovic and Schachermayer [7] or Schönbucher [21], Chapter 7. Processes driven by Brownian motion are the most prominent ones. We will focus on the case where r and λ are given by Vasicek models, with possibly time-dependent coefficients, driven by fBms with Hurst indices strictly greater than $\frac{1}{2}$. This choice is motivated by the fact that macroeconomic variables like demand and supply, interest rates, or other economic activity measures often exhibit long range dependence.

We recall that fractional Brownian motion (fBm) is a zero mean Gaussian process starting in 0 with stationary increments satisfying $(B^H(ct))_{t\geq 0} \stackrel{d}{=} c^H(B^H(t))_{t\geq 0}$ for every $c > 0$. The parameter $H \in (0,1)$ is the Hurst index and $\stackrel{d}{=}$ means equality of finite dimensional distributions. We also assume that B^H is standard; i.e. that $E[B^H(1)^2] = 1$. For general background on fBm we refer to Biagini et al. [2] or Samorodnitsky and Taqqu [20]. For the present paper we shall heavily draw from Pipiras and Taqqu [19].

It is appropriate in our context to use fractional calculus, which suggests to replace H by the fractional parameter $\kappa = H - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2})$. In our long range dependence case, we shall work with $\kappa \in (0, \frac{1}{2})$, which implies that increments are positively correlated. We also recall that $\kappa = 0$ refers to standard Brownian motion and we shall write $B^0 = B$. In the sequel we shall work with two-sided processes, where a two-sided Brownian motion is defined as $B(t) = B_1(t) + B_2(-t)$ for $t \in \mathbb{R}$, where B_1 and B_2 are independent standard Bm's. We introduce then a bivariate fBm $(B^\kappa, \bar{B}^{\bar{\kappa}}) = (B^\kappa(t), \bar{B}^{\bar{\kappa}}(t))_{t\in\mathbb{R}}$ with $\kappa, \bar{\kappa} \in (0, \frac{1}{2})$. The dependence structure between the fBm's will be modeled as in Elliot and van der Hoek [10] by assuming that both processes arise through an integral representation driven by the same two-sided Bm $B = (B(t))_{t\in\mathbb{R}}$, which holds in $L^2(\Omega)$ and is stated in equation (3.7) of Pipiras and Taqqu [19]:

$$\begin{aligned} B^\kappa(t) &= c_\kappa \int_{-\infty}^{\infty} \mathcal{I}_-^\kappa \mathbf{1}_{(0,t)}(s) dB(s), \quad c_\kappa := \frac{\sqrt{\Gamma(2\kappa+2)\sin((\kappa+1/2)\pi)}}{\Gamma(\kappa+1)}, \\ \bar{B}^{\bar{\kappa}}(t) &= c_{\bar{\kappa}} \int_{-\infty}^{\infty} \mathcal{I}_-^{\bar{\kappa}} \mathbf{1}_{(0,t)}(s) dB(s), \quad c_{\bar{\kappa}} := \frac{\sqrt{\Gamma(2\bar{\kappa}+2)\sin((\bar{\kappa}+1/2)\pi)}}{\Gamma(\bar{\kappa}+1)}, \end{aligned} \tag{3.1}$$

for $t \in \mathbb{R}$, with gamma function Γ and the classical Riemann-Liouville fractional integral defined for $\alpha > 0$ by

$$(\mathcal{I}_-^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty f(t)(t-x)^{\alpha-1} dt$$

if the integrals exist for almost all $x \in \mathbb{R}$. We shall also need the fractional derivatives

$$(\mathcal{D}_-^\alpha f)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^\infty f(t)(t-x)^{-\alpha} dt$$

for $\alpha > -1$. The question of the existence of the fractional derivative $\mathcal{D}_-^\alpha f$ is more sophisticated and we refer to Zähle [23] for details. However, we will only take fractional derivatives of fractional integrals, where the orders fit together, therefore, existence will be always ensured.

The two fBms arising from the same Bm have the economical interpretation that short rate and default rate are driven by the same market noise. However, the influence of this noise may be different and depends on the long range dependence parameters as well as on the coefficient functions of the Langevin equations (as will be seen in section 4).

From now on we will understand integration with respect to fBm in the pathwise or L^2-sense, which is the same for deterministic integrands if both integrals exists. The following proposition is a consequence of (3.13) and Theorem 3.2 of Pipiras and Taqqu [19].

Proposition 3.1. *Let $(B_t)_{t\in\mathbb{R}}$ be the two-sided Bm of (3.1) and $\kappa \in (0, \frac{1}{2})$. For every $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the following integrals are equal in the $L^2(\Omega)$-sense:*

$$\int_{\mathbb{R}} f(s)dB^\kappa(s) \quad = \quad c_\kappa \Gamma(\kappa+1) \int_{\mathbb{R}} \mathcal{I}^\kappa_-(f)(s)dB(s)$$

A similar result holds true for $\bar{B}^{\bar{\kappa}}$.

For $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\kappa \in (0, \frac{1}{2})$ the following inner product is finite:

$$\langle f, g\rangle_\kappa := \kappa(2\kappa+1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)g(v)|u-v|^{2\kappa-1}dudv.$$

We shall denote the induced norm by $\|\cdot\|_\kappa$.

For the prediction problem we want to solve, we have to work on the compact interval $[0, T]$ for some $T > 0$ and define the fractional integral with finite time horizon for $\alpha > 0$,

$$(I^\alpha_{T-}f)(s) = \frac{1}{\Gamma(\alpha)} \int_s^T f(r)(r-s)^{\alpha-1}dr, \quad 0 \le s \le T. \tag{3.2}$$

For $f \in L^2(\mathbb{R})$ this integral always exists. We shall also need the fractional derivative with finite time horizon for $0 < \alpha < 1$

$$(D^\alpha_{T-}g)(u) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{du} \int_u^T g(s)(s-u)^{-\alpha}ds, \quad 0 < u < T. \tag{3.3}$$

As usual, we shall often write $I^{-\alpha}_{T-} = D^\alpha_{T-}$. Recall that $\overline{\mathrm{sp}}_{[0,T]}(B^\kappa)$ is the closure in $L^2(\Omega)$ of all possible linear combinations of the increments of fBm on $[0, T]$.

Derivatives pricing essentially means prediction, given information of the past. In a Markov model, pricing formulas rely on the Markov property, which can certainly not be applied in models involving long range dependence processes.

Suppose we want to calculate the prediction

$$X^\kappa(t, T) := E[B^\kappa(T) \mid B^\kappa(s), s \in [0, t]], \quad 0 < t < T.$$

If $X^\kappa(t, T) \in \overline{\mathrm{sp}}_{[0,t]}(B^\kappa)$, then we hope that there exists some function $f \in L^2[0, T]$ such that $X^\kappa(t, T) = \int_0^t f(u)dB^\kappa(u)$. This is indeed true, and the formula has been derived by Gripenberg and Norros [15]. The following version is Theorem 7.1 of Pipiras and Taqqu.

Proposition 3.2. *Let $0 \le t < T$ and $\kappa \in (-\frac{1}{2}, \frac{1}{2})$. Then*

$$E[B^\kappa(T) \mid B^\kappa(s), s \in [0,t]] = B^\kappa(t) + \int_0^t \Psi^\kappa(t,T,u) dB^\kappa(u)\,, \tag{3.4}$$

where for $u \in (0,t)$,

$$\begin{aligned} \Psi^\kappa(t,T,u) &= u^{-\kappa}(I_{t-}^{-\kappa}(I_{T-}^{\kappa}(\cdot)^\kappa \mathbf{1}_{(t,T)}(\cdot)))(u) \\ &= \frac{\sin(\pi\kappa)}{\pi} u^{-\kappa}(t-u)^{-\kappa} \int_t^T \frac{z^\kappa (z-t)^\kappa}{z-u} dz\,. \end{aligned} \tag{3.5}$$

Moreover, $\Psi^\kappa(t,T,\cdot) \in L^2[0,t]$ for all $0 \le t \le T$.

Writing

$$E[B^\kappa(T) - B^\kappa(t) \mid B^\kappa(s), s \in [0,t]] = \int_0^t \Psi^\kappa(t,T,u) dB^\kappa(u)$$

it is immediately clear, how to extend this prediction formula to integrals of fBm as has been done in Lemma 1 of Duncan [8]:

Proposition 3.3. *For $0 \le t < T$ and $\kappa \in (0, \frac{1}{2})$ let $c \in L^2[t,T]$. Then*

$$E\left[\int_t^T c(r) dB^\kappa(r) \Big| B^\kappa(r), r \in [0,t]\right] = \int_0^t \Psi_c^\kappa(t,T,u) dB^\kappa(u), \tag{3.6}$$

where

$$\begin{aligned} \Psi_c^\kappa(t,T,u) &= u^{-\kappa}(I_{t-}^{-\kappa}(I_{T-}^{\kappa}(\cdot)^\kappa c(\cdot) \mathbf{1}_{(t,T)}(\cdot)))(u) \\ &= \frac{\sin(\pi\kappa)}{\pi} u^{-\kappa}(t-u)^{-\kappa} \int_t^T \frac{z^\kappa (z-t)^\kappa}{z-u} c(z) dz. \end{aligned} \tag{3.7}$$

Moreover, $\Psi_c^\kappa(t,T,\cdot) \in L^2[0,t]$ for all $0 \le t \le T$.
Finally, the integral in (3.6) is normally distributed with mean 0 and variance $\|\Psi_c^\kappa(t,T,\cdot)\mathbf{1}_{(0,t)}(\cdot)\|_\kappa^2$.

Predicting exponentials of fBm driven integrals is more challenging and has been considered in Duncan [8]. However, Proposition 2 of that paper is not correctly formulated. This can be seen immediately, because its result suggests that the prediction is deterministic. The correct version can be found in the Appendix. A more general version, also including the case $\kappa \in (-\frac{1}{2}, 0)$, has been shown in Fink et al. [13]. However, this approach does not apply to our situation here.

From now on we will always assume that $\kappa \in (0, \frac{1}{2})$. Before stating the appropriate prediction formula, we recall some basic properties of the Wick product for fBm. We refer to Biagini et al. [2], Section 3, Elliot and van der Hoek [10] or Duncan, Hu and Pasik-Duncan [9] for details and background.

There are various ways to introduce the Wick product and we will follow mainly Section 3.1 of Biagini et al. [2]. Let $\kappa \in (0, \frac{1}{2})$. First we consider for $g : \mathbb{R} \to \mathbb{R}$ with $\|g\|_\kappa < \infty$ exponentials of the form

$$\varepsilon(g) := \exp\left\{\int_{\mathbb{R}} g(s) dB^\kappa(s) - \frac{1}{2}\|g\|_\kappa\right\} \tag{3.8}$$

like in (3.7) of [2]. The set $\mathcal{E}$ of linear combinations of these exponentials is dense in $L^p(\mathbb{R})$ for all $p \geq 1$.

Definition 3.4. *For $g, h : \mathbb{R} \to \mathbb{R}$ with $\|g\|_\kappa, \|h\|_\kappa < \infty$ the* Wick product *of the exponentials of g, h is defined as*

$$\varepsilon(g) \diamond \varepsilon(h) := \varepsilon(g + h). \tag{3.9}$$

By bilinearity the Wick product is defined on the whole of $\mathcal{E}$. A classical density argument (see Theorem 3.1 of [2]) extends this definition now to L^p for all $p \geq 1$. The two main properties of the Wick product we need in this paper are summarized in the next proposition.

Proposition 3.5. *Let $c : \mathbb{R} \to \mathbb{R}$ with $\|c\|_\kappa < \infty$.*
(1) Define the Wick exponential by $\exp^\diamond(\cdot) := \sum_{i=0}^\infty ((\cdot)^{\diamond i}/i!)$. Then

$$\exp^\diamond \left\{ \int_\mathbb{R} c(s) dB^\kappa(s) \right\} = \exp\left\{ \int_\mathbb{R} c(s) dB^\kappa(s) - \frac{1}{2}\|c\|_\kappa \right\} = \varepsilon(c). \tag{3.10}$$

(2) Define $\mathcal{G}_t := \sigma\{B^\kappa(s), s \in (a, t]\}$ for $-\infty \leq a < t < \infty$. Then

$$E\left[\exp^\diamond \left\{ \int_\mathbb{R} c(s) dB^\kappa(s) \right\} \middle| \mathcal{G}_t \right] = \exp^\diamond \left\{ E\left[\int_\mathbb{R} c(s) dB^\kappa(s) \middle| \mathcal{G}_t \right] \right\}$$

Proof. Equation (1) is equation is given by (3.25) of Biagini et al. [2] and (2) is a consequence of (17) of Duncan [8] and the uniform convergence of the exponential Wick series. The argument is pretty much the same as in the corrected version of Proposition 3.2 of Duncan [8] in the Appendix below. □

Now we can predict exponentials.

Proposition 3.6. *For $0 \leq t < T$ define $\mathcal{G}_t := \sigma\{B^\kappa(s), s \in [0, t]\}$. Let $c \in L^2[t, T]$. Then*

$$E\left[\exp\left\{ \int_t^T c(s) dB^\kappa(s) \right\} \middle| \mathcal{G}_t \right] = e^{\frac{1}{2}\left(\|c(\cdot)\mathbf{1}_{(t,T)}(\cdot)\|_\kappa - \|\Psi_c^\kappa(t,T,\cdot)\mathbf{1}_{(0,t)}(\cdot)\|_\kappa \right)} \exp\left\{ \int_0^t \Psi_c^\kappa(t, T, u) dB^\kappa(u) \right\},$$

where $\Psi_c^\kappa(t, T, \cdot)$ is given in (3.7).

Proof. We use Proposition 3.5 and Proposition 3.3 to obtain

$$\begin{aligned} E\left[\exp\left\{ \int_t^T c(s) dB^\kappa(s) \right\} \middle| \mathcal{G}_t \right] &= e^{\frac{1}{2}\|c(\cdot)\mathbf{1}_{(t,T)}(\cdot)\|_\kappa} E\left[\exp^\diamond \left\{ \int_t^T c(s) dB^\kappa(s) \right\} \middle| \mathcal{G}_t \right] \\ &= e^{\frac{1}{2}\|c(\cdot)\mathbf{1}_{(t,T)}(\cdot)\|_\kappa} \exp^\diamond \left\{ E\left[\int_t^T c(s) dB^\kappa(s) \middle| \mathcal{G}_t \right] \right\} \\ &= e^{\frac{1}{2}\|c(\cdot)\mathbf{1}_{(t,T)(\cdot)}\|_\kappa} \exp^\diamond \left\{ \int_0^t \Psi_c^\kappa(t, T, u) dB^\kappa(u) \right\} \\ &= e^{\frac{1}{2}\left(\|c(\cdot)\mathbf{1}_{(t,T)}(\cdot)\|_\kappa - \|\Psi_c^\kappa(t,T,\cdot)\mathbf{1}_{(0,t)}(\cdot)\|_\kappa \right)} \exp\left\{ \int_0^t \Psi_c^\kappa(t, T, u) dB^\kappa(u) \right\}. \end{aligned}$$

□

4 Pricing a defaultable zero coupon bond

Turning now to the credit model from Section 2 we know that for any progressively measurable short rate process $(r(t))_{t\geq 0}$ and default rate process $(\lambda(t))_{t\geq 0}$ the price of a defaultable zero coupon bond with maturity T at time $t=0$ is given by

$$\overline{B}(0,T) = E\Big[e^{-\int_0^T r(s)ds}\mathbf{1}_{\{\tau>T\}}\Big] = E\Big[e^{-\int_0^T (r(s)+\lambda(s))ds}\Big].$$

Now we have to specify the dynamics of r and λ. Recall the bivariate fBm from Section 3. The dependence between B^κ and $\overline{B}^\kappa$ is then given by the covariance function (see (2.17) of Elliot and van der Hoek [10]) for $s,t\in\mathbb{R}$ as

$$\mathrm{Cov}(B^\kappa(t),\bar{B}^{\bar\kappa}(s)) = \frac{c_\kappa c_{\bar\kappa}\Gamma(\kappa+1)\Gamma(\bar\kappa+1)}{2\sin(\pi(\kappa+\bar\kappa)/2)\Gamma(\kappa+\bar\kappa+1)}[|t|^{\kappa+\bar\kappa+1}+|s|^{\kappa+\bar\kappa+1}+|t-s|^{\kappa+\bar\kappa+1}].$$

Using (3.1) we can show the following proposition by approximation with step functions.

Proposition 4.1. *Let $f,g\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Then*

$$\begin{aligned} & E\left[\int_{\mathbb{R}} f(s)dB^\kappa(s)\int_{\mathbb{R}} g(s)d\bar{B}^{\bar\kappa}(s)\right] \\ = & \frac{c_\kappa c_{\bar\kappa}\Gamma(\kappa+1)\Gamma(\bar\kappa+1)}{\sin(\pi(\kappa+\bar\kappa)/2)\Gamma(\kappa+\bar\kappa+1)}(\kappa+\bar\kappa)(\kappa+\bar\kappa+1)\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)|u-v|^{\kappa+\bar\kappa-1}dudv. \end{aligned}$$

Now we model the short rate r and the default rate λ as pathwise solutions to Langevin equations

$$\begin{aligned} dr(t) &= (k(t)-a(t)r(t))dt+\sigma(t)dB^\kappa(t), \quad r(0)=r_0\in\mathbb{R}, && (4.1)\\ d\lambda(t) &= (\bar k(t)-\bar a(t)\lambda(t))dt+\bar\sigma(t)d\bar{B}^{\bar\kappa}(t) \quad \lambda(0)=\lambda_0\in\mathbb{R}, && (4.2) \end{aligned}$$

where $k(\cdot),\bar k(\cdot),a(\cdot),\bar a(\cdot)$ are continuous and locally integrable. Further we assume that $\sigma(\cdot),\bar\sigma(\cdot)>0$ are continuous and that $\sigma(\cdot),1/\sigma(\cdot)$ are of bounded p-variation for some $0<p<1/(1-\kappa)$ and that $\bar\sigma(\cdot),1/\bar\sigma(\cdot)$ are of bounded $\bar p$-variation for some $0<\bar p<1/(1-\bar\kappa)$.

Although both fBms are driven by the same noise, its influence can vary through different coefficient functions of the Langevin equations.

Note that it is also possible to model different dynamics in r and λ by adding several independent factors driven by independent Brownian motions.

Lemma 4.2. *Under the above conditions the pathwise solutions of the SDEs* (4.1) *are given for* $0\leq t\leq T$ *by*

$$\begin{aligned} r(T) &= r(t)e^{-\int_t^T a(u)du}+\int_t^T e^{-\int_s^T a(u)du}k(s)ds+\int_t^T e^{-\int_s^T a(u)du}\sigma(s)dB^\kappa(s), && (4.3)\\ \lambda(T) &= \lambda(t)e^{-\int_t^T \bar a(u)du}+\int_t^T e^{-\int_s^T \bar a(u)du}\bar k(s)ds+\int_t^T e^{-\int_s^T \bar a(u)du}\bar\sigma(s)d\bar{B}^{\bar\kappa}(s), && (4.4) \end{aligned}$$

where the fBm integrals can be considered in the L^2- or pathwise sense, cf. Young [22].

Now the non-observable fBms can be replaced by observable processes given by solutions to (4.1).

Proposition 4.3. *Under the above conditions we have for* $0 \le t \le T$

$$dB^\kappa(t) = \left(-\frac{k(t)}{\sigma(t)} + \frac{a(t)}{\sigma(t)} r(t)\right) dt + \frac{1}{\sigma(t)} dr(t) \quad \text{and} \quad d\bar{B}^{\bar{\kappa}}(t) = \left(-\frac{\bar{k}(t)}{\bar{\sigma}(t)} + \frac{\bar{a}(t)}{\bar{\sigma}(t)} \lambda(t)\right) dt + \frac{1}{\bar{\sigma}(t)} d\lambda(t).$$

Proof. By (4.1) we have for $0 \le t \le T$

$$\int_t^T e^{-\int_s^T a} \sigma(s) dB^\kappa(s) = r(T) - r(t) e^{-\int_t^T a} - \int_t^T e^{-\int_s^T a} k(s) ds$$

and, applying a density formula (which can be applied by Theorem A.4 of Fink and Klüppelberg [12]) we get for $0 \le t \le T$

$$\begin{aligned} B^\kappa(T) - B^\kappa(t) &= \int_t^T \frac{e^{\int_u^T a}}{\sigma(u)} d\left(-\int_u^T e^{-\int_s^T a} \sigma(s) dB^\kappa(s)\right) \\ &= \int_t^T \frac{e^{\int_u^T a}}{\sigma(u)} d\left(\int_u^T e^{-\int_s^T a} k(s) ds + r(u) e^{-\int_u^T a} - r(T)\right) \\ &= -\int_t^T \frac{k(u)}{\sigma(u)} du + \int_t^T \frac{a(u)}{\sigma(u)} r(u) du + \int_t^T \frac{1}{\sigma(u)} dr(u). \end{aligned}$$

The second equation can be obtained similarly. □

Corollary 4.4. *Let* $0 \le t \le s$*. Then the sum* $r(s) + \lambda(s)$ *is normally distributed with mean zero and variance given by*

$$\begin{aligned} & \kappa(2\kappa+1) \int_t^s \int_t^s e^{-\int_u^s a - \int_v^s a} \sigma(u)\sigma(v) |u-v|^{2\kappa-1} du dv \\ + \quad & 2\rho(\kappa+\bar{\kappa})(\kappa+\bar{\kappa}+1) \int_t^s \int_t^s e^{-\int_u^s a - \int_v^s \bar{a}} \sigma(u)\bar{\sigma}(v) |u-v|^{\kappa+\bar{\kappa}-1} du dv \qquad (4.5) \\ + \quad & \bar{\kappa}(2\bar{\kappa}+1) \int_t^s \int_t^s e^{-\int_u^s \bar{a} - \int_v^s \bar{a}} \bar{\sigma}(u)\bar{\sigma}(v) |u-v|^{2\bar{\kappa}-1} du dv, \end{aligned}$$

where the covariance of the two integrals is given in line (4.5)*. Here*

$$\rho = \frac{c_\kappa c_{\bar{\kappa}} \Gamma(\kappa+1)\Gamma(\bar{\kappa}+1)}{\sin(\pi(\kappa+\bar{\kappa})/2)\Gamma(\kappa+\bar{\kappa}+1)} \ge 0.$$

Remark 4.5. Corollary 4.4 implies that short rate and default rate are positively correlated, which makes sense economically. A high default rate indicates a higher probability of default before maturity. An investor will, therefore, request a compensation by a higher interest rate before taking this risk.

The information filtration given by the short rate and the default rate processes is now

$$\begin{aligned} \mathcal{G}_t &= \sigma\{(r_s, \lambda_s), s \in [0,t]\} = \sigma\{(B_s^\kappa, \overline{B}_s^{\bar{\kappa}}), s \in [0,t]\} \\ &= \sigma\{B_s^\kappa, s \in [0,t]\} = \sigma\{\overline{B}_s^{\bar{\kappa}}, s \in [0,t]\}, \quad t \ge 0. \end{aligned}$$

The following prediction formula is a simple consequence of Propositions 3.2 and 3.3.

Corollary 4.6. *For $0 \le t < T$ let $c, \bar{c} \in L^2[t,T]$. Then*

$$\begin{aligned} & E\left[\int_t^T c(r)dB^\kappa(r) + \int_t^T \bar{c}(r)d\bar{B}^{\bar{\kappa}}(r) \Big| \Big(B^\kappa(r), \bar{B}^{\bar{\kappa}}(r)\Big), r \in [0,t]\right] \\ = \;& E\left[\int_t^T c(r)dB^\kappa(r)\Big| B^\kappa(r), r\in[0,t]\right] + E\left[\int_t^T \bar{c}(r)d\bar{B}^{\bar{\kappa}}(r)\Big|\bar{B}^{\bar{\kappa}}(r), r \in [0,t]\right] \\ = \;& \int_0^t \Psi_c^\kappa(t,T,u)dB^\kappa(u) + \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,u)d\bar{B}^{\bar{\kappa}}(u), \end{aligned}$$

where $\Psi_c^\kappa(t,T,\cdot)$ and $\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot)$ are as in (3.7) and belong to $L^2[0,t]$ for all $0 \le t \le T$.

Next we need an analog of Proposition 3.6, where now in the exponential there is the sum of two integrals as in Corollary 4.6, and the dependence between B^κ and $B^{\bar{\kappa}}$ matters. We shall proceed as follows. First we transform both integrals with respect to B^κ and $\bar{B}^\kappa$, respectively, into one integral with respect to B^κ and invoke afterwards Proposition 3.6.

Lemma 4.7. *For $0 \le t \le T$ let $c \in L^2[t,T]$. Let B^κ and $\bar{B}^{\bar{\kappa}}$ be fBm's as in (3.1). Assume further that $\kappa \le \bar{\kappa}$. Then the equality of both integrals holds in the $L^2(\Omega)$-sense:*

$$\int_t^T c(v)d\bar{B}^{\bar{\kappa}}(v) = \frac{c_{\bar{\kappa}}\Gamma(\bar{\kappa}+1)}{c_\kappa\Gamma(\kappa+1)} \int_{\mathbb{R}} \mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(t,T)}(\cdot)c(\cdot))(v)dB^\kappa(v). \tag{4.6}$$

Proof. Set $a_\kappa := c_\kappa\Gamma(\kappa+1)$ and $a_{\bar{\kappa}} := c_{\bar{\kappa}}\Gamma(\bar{\kappa}+1)$. Then using repeatedly Proposition 3.1 we get

$$\begin{aligned} \int_t^T c(v)d\bar{B}^{\bar{\kappa}}(v) &= \int_{\mathbb{R}} \mathbf{1}_{(t,T)}(v)c(v)d\bar{B}^{\bar{\kappa}}(v) \\ &= a_{\bar{\kappa}} \int_{\mathbb{R}} \mathcal{I}_-^{\bar{\kappa}}(\mathbf{1}_{(t,T)}(\cdot)c(\cdot))(v)dB(v) \\ &= \frac{a_{\bar{\kappa}}}{a_\kappa}\int_{\mathbb{R}} \mathcal{D}_-^\kappa\mathcal{I}_-^{\bar{\kappa}}(\mathbf{1}_{(t,T)}(\cdot)c(\cdot))(v)dB^\kappa(v) \\ &= \frac{a_{\bar{\kappa}}}{a_\kappa}\int_{\mathbb{R}} \mathcal{D}_-^\kappa\mathcal{I}_-^\kappa\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(t,T)}(\cdot)c(\cdot))(v)dB^\kappa(v) \\ &= \frac{a_{\bar{\kappa}}}{a_\kappa}\int_{\mathbb{R}} \mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(t,T)}(\cdot)c(\cdot))(v)dB^\kappa(v). \end{aligned}$$

□

Proposition 4.8. *For $0 \le t < T$ let $c, \bar{c} \in L^2[t,T]$. Further let B^κ and $\bar{B}^{\bar{\kappa}}$ be fBm's as in (3.1). Then*

$$\begin{aligned} & E\left[\exp\left\{\int_t^T c(v)dB^\kappa(v) + \int_t^T \bar{c}(v)d\bar{B}^{\bar{\kappa}}(v)\right\}\Big|\,\mathcal{G}_t\right] \\ = \;& e^{W(t,T)-V(t,T)}\exp\left\{\int_0^t \Psi_c^\kappa(t,T,v)dB^\kappa(v) + \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)d\bar{B}^{\bar{\kappa}}(v)\,|\mathcal{G}_t\right\}. \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} W(t,T) &= \frac{1}{2}\left(\left\|\mathbf{1}_{(t,T)}(\cdot)c(\cdot)\right\|_\kappa^2 + 2\left\langle\mathbf{1}_{(t,T)}(\cdot)c(\cdot), \mathbf{1}_{(t,T)}(\cdot)\bar{c}(\cdot)\right\rangle_{\frac{\kappa+\bar{\kappa}}{2}} + \left\|\mathbf{1}_{(t,T)}(\cdot)\bar{c}(\cdot)\right\|_{\bar{\kappa}}^2\right), \\ V(t,T) &= \frac{1}{2}\left(\left\|\mathbf{1}_{(0,t)}(\cdot)\Psi_c^\kappa(t,T,\cdot)\right\|_\kappa^2 + 2\left\langle\mathbf{1}_{(0,t)}(\cdot)\Psi_c^\kappa(t,T,\cdot), \mathbf{1}_{(0,t)}(\cdot)\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot)\right\rangle_{\frac{\kappa+\bar{\kappa}}{2}}\right. \\ &\quad \left. + \left\|\mathbf{1}_{(0,t)}(\cdot)\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot)\right\|_{\bar{\kappa}}^2\right). \end{aligned}$$

and $\Psi_c^\kappa(t,T,\cdot)$, $\Psi_{\bar c}^{\bar\kappa}(t,T,\cdot)$ *are as in* (3.7) *and belong to* $L^2[0,t]$ *for all* $0 \le t \le T$.

Proof. To predict the exponential we transform it into a Wick exponential using Lemma 4.7 and then Proposition 3.5 as follows. W.l.o.g. assume that $\kappa \le \bar\kappa$. Define $a_\kappa = c_\kappa \Gamma(\kappa+1)$ and $a_{\bar\kappa} = c_{\bar\kappa}\Gamma(\bar\kappa+1)$. Then by Lemma 4.7 and Proposition 3.5

$$\begin{aligned} & \exp\left\{\int_t^T c(v)dB^\kappa(v) + \int_t^T \bar c(v) d\bar B^{\bar\kappa}(v)\right\} \\ = \; & \exp\left\{\int_{\mathbb{R}} \left(\mathbf{1}_{(t,T)}(v)c(v) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar c(\cdot))(v)\right) dB^\kappa(v)\right\} \\ = \; & e^{W(t,T)} \exp^\diamond\left\{\int_t^T \left(c(v) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar c(\cdot))(v)\right) dB^\kappa(v)\right\} \end{aligned}$$

and, as preliminary version,

$$\begin{aligned} W(t,T) &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\mathbf{1}_{(t,T)}(u)c(u) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar c(\cdot))(u)\right) \\ &\quad \times \left(\mathbf{1}_{(t,T)}(v)c(v) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar c(\cdot))(v)\right) |u-v|^{2\kappa-1} du dv \\ &= \frac{1}{2}\left(\left\|\mathbf{1}_{(t,T)}(\cdot)c(\cdot)\right\|_\kappa^2 + 2\frac{a_{\bar\kappa}}{a_\kappa}\left\langle \mathbf{1}_{(t,T)}(\cdot)c(\cdot), \mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar c(\cdot))(\cdot)\right\rangle_\kappa \right. \\ &\quad \left. + \left(\frac{a_{\bar\kappa}}{a_\kappa}\right)^2 \left\|\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar c(\cdot))(\cdot)\right\|_\kappa^2 \right) \end{aligned}$$

Next we take conditional expectation of the exponential integral which is nothing else than an L^2 projection. Therefore, Proposition 3.5 applies giving

$$\begin{aligned} & E\left[\exp\left\{\int_t^T c(v)dB^\kappa(v) + \int_t^T \bar c(v)d\bar B^{\bar\kappa}(v)\right\}\middle| \mathcal{G}_t\right] \\ = \; & e^{W(t,T)} E\left[\exp^\diamond\left\{\int_{\mathbb{R}}\left(\mathbf{1}_{(t,T)}(v)c(v) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar c(\cdot))(v)\right)dB^\kappa(v)\right\}\middle| \mathcal{G}_t\right] \\ = \; & e^{W(t,T)} \exp^\diamond\left\{E\left[\int_t^T \left(c(v) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar c(\cdot))(v)\right)dB^\kappa(v)\middle| \mathcal{G}_t\right]\right\}. \end{aligned}$$

Now transform the integral in the conditional expectation back and apply Corollary 4.6. Transforming the Wick exponential in a classical exponential yields the term

$$\begin{aligned} V(t,T) &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \left(\mathbf{1}_{(0,t)}(u)\Psi_c^\kappa(t,T,u) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(0,t)}(\cdot)\Psi_{\bar c}^{\bar\kappa}(t,T,\cdot))(u)\right) \\ &\quad \times \left(\mathbf{1}_{(0,t)}(v)\Psi_c^\kappa(t,T,v) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(0,t)}(\cdot)\Psi_{\bar c}^{\bar\kappa}(t,T,\cdot))(v)\right) |u-v|^{2\kappa-1} du dv \\ &= \frac{1}{2}\left(\left\|\mathbf{1}_{(0,t)}(\cdot)\Psi_c^\kappa(t,T,\cdot)\right\|_\kappa^2 + 2\frac{a_{\bar\kappa}}{a_\kappa}\left\langle \mathbf{1}_{(0,t)}(\cdot)\Psi_c^\kappa(t,T,\cdot), \mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(0,t)}(\cdot)\Psi_{\bar c}^{\bar\kappa}(t,T,\cdot))(\cdot)\right\rangle_\kappa \right. \\ &\quad \left. + \left(\frac{a_{\bar\kappa}}{a_\kappa}\right)^2 \left\|\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(0,t)}(\cdot)\Psi_{\bar c}^{\bar\kappa}(t,T,\cdot))(\cdot)\right\|_\kappa^2 \right). \end{aligned}$$

Finally, we transform the indefinite integral $\mathcal{I}_-^{\bar\kappa-\kappa}$ within the conditional expectation back using Lemma 4.7. Combining these two steps yields (4.7). The final versions of $V(t,T)$ and $W(t,T)$ can be calculated by Lemma A.1 of the Appendix. □

By Lemma 13.2 of Filipovic [11] the price of a defaultable zero-coupon bond is for $0 \le t \le T$ given by

$$\overline{B}(t,T) \;=\; E\Big[\mathbf{1}_{\{\tau>t\}} e^{-\int_t^T r(s)ds}\Big|\mathcal{F}_t\Big] = \mathbf{1}_{\{\tau>t\}} E\Big[e^{-\int_t^T (r(s)+\lambda(s))ds}\Big|\mathcal{G}_t\Big].$$

The following is our main result of this section, manifesting a similar structure for the price as in the affine Markovian case.

Theorem 4.9. *Let $0 \le t < T$. Set $D(t,T) := \int_t^T e^{-\int_t^s a} ds$, $\bar{D}(t,T) := \int_t^T e^{-\int_t^s \bar{a}} ds$ and assume that $D(\cdot,T)\sigma(\cdot), \bar{D}(\cdot,T)\bar{\sigma}(\cdot) \in L^2[t,T]$. Then*

$$\overline{B}(t,T) \;=\; \mathbf{1}_{\{\tau>t\}} e^{-A(t,T)-D(t,T)r(t)-\bar{D}(t,T)\lambda(t)}, \tag{4.8}$$

where

$$\begin{aligned} A(t,T) \;&=\; V(t,T) - W(t,T) + \int_t^T \big(D(v,T)k(v) + \bar{D}(v,T)\bar{k}(v)\big)dv \\ &\quad + \int_0^t \Psi_c^\kappa(t,T,v)dB^\kappa(v) + \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)d\bar{B}^{\bar{\kappa}}(v). \end{aligned}$$

Here $V(t,T)$, $W(t,T)$ are given in Proposition 4.8 and $\Psi_c^\kappa(t,T,\cdot)$, $\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot)$ are as in (3.7) *with $c(\cdot) = D(\cdot,T)\sigma(\cdot)$, $\bar{c}(\cdot) = \bar{D}(\cdot,T)\bar{\sigma}(\cdot)$. Furthermore* $\log(\overline{B}(t,T))$ *is normally distributed with*

$$\begin{aligned} E[\log(\overline{B}(t,T))] \;&=\; -D(t,T)e^{-\int_0^t a} r(0) - \bar{D}(t,T)e^{-\int_0^t \bar{a}}\lambda(0) \\ &\quad -D(t,T)\int_0^t e^{-\int_v^t a}k(v)dv - \bar{D}(t,T)\int_0^t e^{-\int_v^t \bar{a}}\bar{k}(v)dv \\ &\quad -\int_t^T (D(v,T)k(v) + \bar{D}(v,T)\bar{k}(v))dv - V(t,T) + W(t,T)\,, \\ \mathrm{Var}(\log(\overline{B}(t,T))) \;&=\; \Big\|\Big(\Psi_c^\kappa(t,T,\cdot) + D(t,T)e^{-\int_\cdot^t a}\sigma(\cdot)\Big)\mathbf{1}_{[0,t]}(\cdot)\Big\|_\kappa^2 \\ &\quad +2\Big\langle\Big(\Psi_c^\kappa(t,T,\cdot) + D(t,T)e^{-\int_\cdot^t a}\sigma(\cdot)\Big)\mathbf{1}_{[0,t]}(\cdot), \\ &\quad \Big(\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot) + \bar{D}(t,T)e^{-\int_\cdot^t \bar{a}}\bar{\sigma}(\cdot)\Big)\mathbf{1}_{[0,t]}(\cdot)\Big\rangle_{\frac{\kappa+\bar{\kappa}}{2}} \\ &\quad +\Big\|\Big(\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot) + \bar{D}(t,T)e^{-\int_\cdot^t \bar{a}}\bar{\sigma}(\cdot)\Big)\mathbf{1}_{[0,t]}(\cdot)\Big\|_{\bar{\kappa}}^2. \end{aligned}$$

Proof. The case $t = 0$ is trivial by (4.5), so let $t > 0$. We obtain from Lemma 4.2 and Fubini's Theorem

$$\begin{aligned} \int_t^T (r(s)+\lambda(s))ds \;&=\; \int_t^T \Big[r(t)e^{-\int_t^s a} + \int_t^s e^{-\int_v^s a}k(v)dv + \int_t^s e^{-\int_v^s a}\sigma(v)dB^\kappa(v)\Big]ds \\ &\quad + \int_t^T \Big[\lambda(t)e^{-\int_t^s \bar{a}} + \int_t^s e^{-\int_v^s \bar{a}}\bar{k}(v)dv + \int_t^s e^{-\int_v^s \bar{a}}\bar{\sigma}(v)d\bar{B}^{\bar{\kappa}}(v)\Big]ds \\ &=\; D(t,T)r(t) + \int_t^T D(v,T)k(v)dv + \int_t^T D(v,T)\sigma(v)dB^\kappa(v) \\ &\quad +\bar{D}(t,T)\lambda(t) + \int_t^T \bar{D}(v,T)\bar{k}(v)dv + \int_t^T \bar{D}(v,T)\bar{\sigma}(v)d\bar{B}^{\bar{\kappa}}(v). \end{aligned} \tag{4.9}$$

By Proposition 4.8 we have

$$\begin{aligned}
& E\left[\exp\left\{\int_t^T D(v,T)\sigma(v)dB^\kappa(v) + \int_t^T \bar{D}(v,T)\bar{\sigma}(v)d\bar{B}^{\bar{\kappa}}(v)\right\}\Big|\,\mathcal{G}_t\right] \\
= \;& e^{W(t,T)-V(t,T)}\exp\left\{\int_0^t \Psi_c^\kappa(t,T,v)dB^\kappa(v) + \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)d\bar{B}^{\bar{\kappa}}(v)\right\}.
\end{aligned}$$

Now we get for the price of the defaultable zero-coupon bond by

$$\begin{aligned}
\overline{B}(t,T) &= \mathbf{1}_{\{\tau>t\}}E\left[e^{-\int_0^T(r(s)+\lambda(s))ds}\Big|\mathcal{G}_t\right] \\
&= \mathbf{1}_{\{\tau>t\}}e^{-D(t,T)r(t)-\int_t^T D(v,T)k(v)dv-\bar{D}(t,T)\lambda(t)-\int_t^T \bar{D}(v,T)\bar{k}(v)dv} \\
&\quad \times E\left[\exp\left\{-\int_t^T D(v,T)\sigma(v)dB^\kappa(v) - \int_t^T \bar{D}(v,T)\bar{\sigma}(v)d\bar{B}^{\bar{\kappa}}(v)\right\}\Big|\mathcal{G}_t\right] \\
&= \mathbf{1}_{\{\tau>t\}}e^{-D(t,T)r(t)-\int_t^T D(v,T)k(v)dv-\bar{D}(t,T)\lambda(t)-\int_t^T \bar{D}(v,T)\bar{k}(v)dv} \\
&\quad \times e^{W(t,T)-V(t,T)}\exp\left\{-\int_0^t \Psi_c^\kappa(t,T,v)dB^\kappa(v) - \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)d\bar{B}^{\bar{\kappa}}(v)\right\} \\
&= \mathbf{1}_{\{\tau>t\}}e^{-A(t,T)-D(t,T)r(t)-\bar{D}(t,T)\lambda(t)}
\end{aligned}$$

with $A(t,T)$, c and $\bar{c}$ as given in the assertion. The formulas for the expectation and variance of $\log(\overline{B}(t,T))$ can be obtained by simple calculations. □

Remark 4.10. If we compare (4.8) with Proposition 7.2 of Schönbucher [21], we realize that in the case $t = 0$ the zero coupon bond prices differ only by a deterministic factor. However, if we calculate the price at time $t > 0$, the whole paths of the fractional Brownian motions up to time t enter because of the dependent increments. Those integrals do not appear in a Markovian model.

By Proposition 4.3 we rewrite the bond price in terms of r and λ.

Corollary 4.11. *In the situation of Theorem 4.9 we have for* $0 \le t < T$

$$\begin{aligned}
\overline{B}(t,T) &= \mathbf{1}_{\{\tau>t\}}\exp\Big\{-\widetilde{A}(t,T) - D(t,T)r(t) - \bar{D}(t,T)\lambda(t)\Big\} \\
&\quad \times\exp\Big\{-\int_0^t (\Psi_c^\kappa(t,T,v)\frac{a(t)}{\sigma(t)}r(t) + \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)\frac{\bar{a}(t)}{\bar{\sigma}(v)}\lambda(t))dv\Big\} \\
&\quad \times\exp\Big\{-\int_0^t \Psi_c^\kappa(t,T,v)\frac{1}{\sigma(v)}dr(v) - \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)\frac{1}{\bar{\sigma}(v)}d\lambda(v)\Big\}
\end{aligned}$$

where $\Psi_c^\kappa(t,T,\cdot)$, $\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot)$ *are as in* (3.7) *with* $c(\cdot) = D(\cdot,T)\sigma(\cdot)$, $\bar{c}(\cdot) = \bar{D}(\cdot,T)\bar{\sigma}(\cdot)$ *and*

$$\widetilde{A}(t,T) = V(t,T) - W(t,T) + \int_t^T \big(D(v,T)k(v) + \bar{D}(v,T)\bar{k}(v)\big)dv$$

with $W(t,T)$ *and* $V(t,T)$ *as in Proposition 4.8.*

5 Option pricing

In this section we explain how derivatives prices can be calculated. First we aim for a European call price with a defaultable zero-coupon bond as underlying. Today's price can be found similar to the classical Brownian case and a closed formula is obtained. For more general options and times, Fourier techniques can be applied and we show, how to do this.

In Theorem 5.2 below we will price a European call option invoking a change of numéraire. Therefore, we need a Girsanov theorem. For the elementary case, where we the drift of a fBm is changed by a deterministic factor, the measure change has been derived in Norros, Valkeila and Virtamo [17], Theorem 4.1, using pathwise integration. In our case we need some result for the other direction. We need to know the distribution of a fBm after a given measure change. Theorem 3.3 of Duncan, Hu and Pasik-Duncan [9] considers a general situation, which we can use. Moreover, their result also covers the result of Norros, Valkeila and Virtamo [17].

Proposition 5.1. *Let $0 \le t < T \le S$. At time t the price $\mathcal{V}(t,T,S)$ of a European call at strike $K > 0$ and maturity T based on a defaultable zero-coupon bond maturing at time S as underlying is given by*

$$\begin{aligned} \mathcal{V}(t,T,S) &= \mathbf{1}_{\{\tau>t\}} E\left[e^{-\int_t^T (r(s)+\lambda(s))ds} (\overline{B}(T,S)-K)_+ \middle| \mathcal{G}_t \right] \\ &= \mathbf{1}_{\{\tau>t\}} \overline{B}(t,T) E^T \left[(\overline{B}(T,S)-K)_+ \middle| \mathcal{G}_t \right], \end{aligned} \tag{5.1}$$

where E^T is the expectation with respect to the T-forward measure defined by the Radon-Nikodym derivative

$$\frac{d\mathbb{P}^T}{d\mathbb{P}} = \exp\left\{ -\int_0^T (r(s)+\lambda(s))ds \right\} e^{-\overline{B}(0,T)}. \tag{5.2}$$

Proof. As in the classical Bm case we calculate the European call price by means of a T-forward measure (using the expressions defined in Theorem 4.9)

$$\begin{aligned} \frac{d\mathbb{P}^T}{d\mathbb{P}} &= \exp\left\{ -\int_0^T (r(s)+\lambda(s))ds \right\} e^{-\overline{B}(0,T)} \\ &= \exp\left\{ -\int_0^T D(v,T)\sigma(v)dB^{\kappa}(v) - \int_0^T \bar{D}(v,T)\bar{\sigma}(v)d\bar{B}^{\bar{\kappa}}(v) - W(0,T) \right\} \end{aligned}$$

Using Bayes' theorem for conditional expectations we obtain (5.1). □

Denote by N the standard normal distribution function.

Theorem 5.2. *Let $0 < T \le S$. At time 0 the price $\mathcal{V}(0,T,S)$ of a European call at strike $K > 0$ and maturity T based on a defaultable zero-coupon bond maturing at time S as underlying is given by*

$$\begin{aligned} \mathcal{V}(0,T,S) &= \overline{B}(0,T) \\ &\times \left\{ e^{\frac{\Sigma(0,T,S)^2}{2} - A(0,T,S)} N\left(-\frac{A(0,T,S)+\log(K)}{\Sigma(0,T,S)} + \Sigma(0,T,S) \right) - KN\left(-\frac{A(0,T,S)+\log(K)}{\Sigma(0,T,S)} \right) \right\} \end{aligned}$$

with

$$
\begin{aligned}
A(0,T,S) &= V(T,S) - W(T,S) + \int_T^S \big(D(v,S)k(v) + \bar{D}(v,S)\bar{k}(v)\big)dv \\
&\quad + D(T,S)\left(r(0)e^{-\int_0^T a} + \int_0^T e^{-\int_v^T a}k(v)dv\right) \\
&\quad + \bar{D}(T,S)\left(\lambda(0)e^{-\int_0^T \bar{a}} + \int_0^T e^{-\int_v^T \bar{a}}\bar{k}(v)dv\right) \\
&\quad - \left\langle \Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_\kappa - \left\langle \Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), \bar{D}(\cdot,T)\bar{\sigma}(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\frac{\kappa+\bar{\kappa}}{2}} \\
&\quad - \left\langle \bar{\Phi}(\cdot)\mathbf{1}_{(0,T)}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\frac{\kappa+\bar{\kappa}}{2}} - \left\langle \bar{\Phi}(\cdot)\mathbf{1}_{(0,T)}(\cdot), \bar{D}(\cdot,T)\bar{\sigma}(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\bar{\kappa}}
\end{aligned}
$$

where $V(T,S), W(T,S)$ are as in Proposition 4.8. Furthermore,

$$
\begin{aligned}
\Sigma(0,T,S)^2 &= Var\left(-\int_0^T \Phi(v)dB^\kappa(v) - \int_0^T \bar{\Phi}(v)d\bar{B}^{\bar{\kappa}}(v)\right) \qquad (5.3)\\
&= \left\|\mathbf{1}_{(0,T)}(\cdot)\Phi(\cdot)\right\|_\kappa^2 + 2\left\langle \mathbf{1}_{(0,T)}(\cdot)\Phi(\cdot), \mathbf{1}_{(0,T)}(\cdot)\bar{\Phi}(\cdot)\right\rangle_{\frac{\kappa+\bar{\kappa}}{2}} + \left\|\mathbf{1}_{(0,T)}(\cdot)\bar{\Phi}(\cdot)\right\|_{\bar{\kappa}}^2.
\end{aligned}
$$

Here we have set

$$
\Phi(\cdot) := \Psi_c^\kappa(S,T,\cdot) + D(T,S)e^{-\int_\cdot^S a}\sigma(\cdot) \quad \text{and} \quad \bar{\Phi}(\cdot) := \Psi_{\bar{c}}^{\bar{\kappa}}(S,T,\cdot) + \bar{D}(T,S)e^{-\int_\cdot^S \bar{a}}\bar{\sigma}(\cdot), \qquad (5.4)
$$

where $\Psi_c^\kappa(S,T,\cdot)$, $\Psi_{\bar{c}}^{\bar{\kappa}}(S,T,\cdot)$ are as in (3.7) with $c(\cdot) = D(\cdot,S)\sigma(\cdot)$, $\bar{c}(\cdot) = \bar{D}(\cdot,S)\bar{\sigma}(\cdot)$.

Proof. W.l.o.g. assume $\bar{\kappa} \geq \kappa$. Recall $\overline{B}(S,T)$ from (4.8). We replace $r(S)$ and $\lambda(S)$ as in the proof of Theorem 4.9 by the solutions to the SDEs given in (4.4). Then we collect those terms, which are deterministic and those, which are not. This yields the following definition of a function F on the paths of the fBm B^κ as

$$
\begin{aligned}
&F(B^\kappa) := \\
&\left(\exp\left\{-\overline{A}(0,T,S) - \int_{\mathbb{R}}\left(\Phi(v)\mathbf{1}_{(0,T)}(v) + \frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(0,T)}(\cdot)\bar{\Phi}(\cdot))(v)\right)dB^\kappa(v)\right\} - K\right)_+.
\end{aligned}
$$

with Φ and $\bar{\Phi}$ as in (5.4) and

$$
\begin{aligned}
\overline{A}(0,T,S) &= V(T,S) - W(T,S) + \int_T^S \big(D(v,S)k(v) + \bar{D}(v,S)\bar{k}(v)\big)dv \\
&\quad + D(T,S)\left(r(0)e^{-\int_0^T a} + \int_0^T e^{-\int_v^T a}k(v)dv\right) \\
&\quad + \bar{D}(T,S)\left(\lambda(0)e^{-\int_0^T \bar{a}} + \int_0^T e^{-\int_v^T \bar{a}}\bar{k}(v)dv\right)
\end{aligned}
$$

where $V(T,S), W(T,S)$ are as in Proposition 4.8. Starting with (5.1) from Proposition 5.1 we obtain

$$
\begin{aligned}
\mathcal{V}(0,T,S) &= \overline{B}(0,T)E^T[(\overline{B}(T,S) - K)_+] \\
&= \overline{B}(0,T)E^T\left[F(B^\kappa)\right] \\
&= \overline{B}(0,T)E\left[F\left(B^\kappa + \kappa(2\kappa+1)\int_{-\infty}^{\cdot}\int_{-\infty}^{\infty}\Upsilon(v)|v-s|^{2\kappa-1}dv\right)ds\right].
\end{aligned}
$$

with

$$\Upsilon(v) := -\left(D(v,T)\sigma(v)\mathbf{1}_{(0,T)}(v) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(0,T)}(\cdot)\bar D(\cdot,T)\bar\sigma(\cdot))(v)\right).$$

For the last equality we applied Theorem 3.3 of Duncan, Hu and Pasik-Duncan [9] to calculate the expectation under the T-forward measure $\mathbb{P}^T$. (In fact, we have to extend their result to Wick exponentials defined on the whole of $\mathbb{R}$ as in (3.8).) We further calculate

$$\begin{aligned} & F\left(B^\kappa + \kappa(2\kappa+1)\int_{-\infty}^{\cdot}\int_{-\infty}^{\infty}\Upsilon(v)|v-s|^{2\kappa-1}dvds\right) \\ = & \left(\exp\left\{-\overline{A}(0,T,S) - \int_{\mathbb{R}}\left(\Phi(v)\mathbf{1}_{(0,T)}(\cdot) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(0,T)}(\cdot)\bar\Phi(\cdot))(v)\right)dB^\kappa(v)\right.\right. \\ & \left.\left. - \left\langle \Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(0,T)}(\cdot)\bar\Phi(\cdot))(\cdot), \Upsilon(\cdot)\right\rangle_\kappa\right\} - K\right)_+ . \end{aligned}$$

With Lemma A.1 of the Appendix one can show that

$$\begin{aligned} & -\left\langle \Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(0,T)}(\cdot)\bar\Phi(\cdot))(\cdot), \Upsilon(\cdot)\right\rangle_\kappa \\ = & \left\langle \Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_\kappa + \left\langle \Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), \bar D(\cdot,T)\bar\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\frac{\kappa+\bar\kappa}{2}} \\ & + \left\langle \bar\Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\frac{\kappa+\bar\kappa}{2}} + \left\langle \bar\Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), \bar D(\cdot,T)\bar\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\bar\kappa} . \end{aligned}$$

Collecting all terms and transforming the integral back we finally arrive at

$$\begin{aligned} & F\left(B^\kappa + \kappa(2\kappa+1)\int_{-\infty}^{\cdot}\left(\int_{-\infty}^{\infty}\Upsilon(v)|v-s|^{2\kappa-1}dv\right)ds\right) \\ = & \left(\exp\left\{-A(0,T,S) - \int_0^T\Phi(v)dB^\kappa(v) - \int_0^T\bar\Phi(v)d\bar B^{\bar\kappa}(v)\right\} - K\right)_+ , \end{aligned}$$

where

$$\begin{aligned} & A(0,T,S) \\ := & \overline{A}(0,T,S) - \left\langle \Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_\kappa - \left\langle \Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), \bar D(\cdot,T)\bar\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\frac{\kappa+\bar\kappa}{2}} \\ & - \left\langle \bar\Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\frac{\kappa+\bar\kappa}{2}} - \left\langle \bar\Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), \bar D(\cdot,T)\bar\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\bar\kappa} . \end{aligned}$$

Finally, we can calculate the expectation in the pricing formula. This works now exactly as in the case of the classical Black-Scholes setting, since the appearing integrals are Gaussian. This results in

$$\begin{aligned} \mathcal{V}(0,T,S) = & \overline{B}(0,T) \\ & \times E\left[\left(\exp\left\{-A(0,T,S) - \int_0^T\Phi(v)dB^\kappa(v) - \int_0^T\bar\Phi(v)d\bar B^{\bar\kappa}(v)\right\} - K\right)_+\right] \\ = & \overline{B}(0,T) \\ & \times e^{-A(0,T,S)}E\left[\left(\exp\left\{- \int_0^T\Phi(v)dB^\kappa(v) - \int_0^T\bar\Phi(v)d\bar B^{\bar\kappa}(v)\right\} - e^{A(0,T,S)}K\right)_+\right] \\ = & \overline{B}(0,T) \times \left\{e^{\frac{\Sigma(0,T,S)^2}{2} - A(0,T,S)}N\left(-\frac{A(0,T,S)+\log(K)}{\Sigma(0,T,S)} + \Sigma(0,T,S)\right)\right. \\ & \left. - KN\left(-\frac{A(0,T,S)+\log(K)}{\Sigma(0,T,S)}\right)\right\} \end{aligned}$$

where $\Sigma(0,T,S)^2$ is defined in (5.3). The expression for the variance can be deduced by calculating the characteristic functions analogously to the moment generating functions in Proposition 4.8, then apply Lemma A.1 of the Appendix to rewrite the appearing norms and scalar products. $\square$

Remark 5.3. We want to compare the price (5.3) to the European call price in a classical Brownian Vasicek model. For simplicity we choose a model with constant coefficient functions. Given two dependent standard Brownian motions $B, \overline{B}$ with correlation $\rho > 0$, we model the short and hazard rate by the SDEs

$$\begin{aligned} dr(t) &= (k - ar(t))dt + \sigma dB(t), \quad r(0) = r_0 \in \mathbb{R}, \\ d\lambda(t) &= (\bar{k} - \bar{a}\lambda(t))dt + \bar{\sigma} d\overline{B}(t), \quad \lambda(0) = \lambda_0 \in \mathbb{R}, \end{aligned}$$

where we will assume that $\sigma, \bar{\sigma} > 0$. We know by Proposition 5.3 of Schönbucher [21] that this model eventually boils down to a two-factor short rate model. Using for example Theorem 4.2.1 of Brigo and Mercurio [4], today's price of the defaultable zero coupon bond is given by

$$\overline{B}(0,T) = \exp\left\{ -A(0,T) - \frac{k}{a}\left[T - \frac{e^{-aT}-1}{a}\right] - \frac{\bar{k}}{\bar{a}}\left[T - \frac{e^{-\bar{a}T}-1}{\bar{a}}\right] - \frac{1-e^{-aT}}{a} r_0 - \frac{1-e^{-\bar{a}T}}{\bar{a}\lambda_0} \right\}$$

with

$$\begin{aligned} A(0,T) &= -\frac{1}{2}\left(\frac{\sigma^2}{a^2}\left[T + \frac{2}{a}e^{-aT} - \frac{1}{2a}e^{-2aT} - \frac{3}{2a}\right] + \frac{\bar{\sigma}^2}{\bar{a}^2}\left[T + \frac{2}{\bar{a}}e^{-\bar{a}T} - \frac{1}{2\bar{a}}e^{-2\bar{a}T} - \frac{3}{2\bar{a}}\right] \right. \\ &\quad \left. + 2\rho\frac{\sigma\bar{\sigma}}{a\bar{a}}\left[T + \frac{e^{-aT}-1}{a} + \frac{e^{-\bar{a}T}-1}{\bar{a}} - \frac{e^{-(a+\bar{a})T}-1}{a+\bar{a}}\right]\right). \end{aligned}$$

Let $0 \le T \le S$. Applying Theorem 4.2.2 of Brigo and Mercurio [4] we get for the price $\mathcal{V}(0,T,S)$ of a call option with maturity T and strike K, written on a defaultable zero coupon bond maturing at time S:

$$\begin{aligned} &\mathcal{V}(0,T,S) \\ = \; &\overline{B}(0,S) N\left(\frac{\log\left(\frac{\overline{B}(0,S)}{K\overline{B}(0,T)}\right)}{\Sigma(0,T,S)} + \frac{1}{2}\Sigma(0,T,S) \right) - \overline{B}(0,T) K N\left(\frac{\log\left(\frac{\overline{B}(0,S)}{K\overline{B}(0,T)}\right)}{\Sigma(0,T,S)} - \frac{1}{2}\Sigma(0,T,S) \right), \end{aligned}$$

where

$$\begin{aligned} \Sigma^2(0,T,S) &= \frac{\sigma^2}{2a^3}\left(1-e^{-a(S-T)}\right)^2\left(1-e^{-2a(T-t)}\right) + \frac{\bar{\sigma}^2}{2\bar{a}^3}\left(1-e^{-\bar{a}(S-T)}\right)^2\left(1-e^{-2\bar{a}(T-t)}\right) \\ &= 2\rho\frac{\sigma\bar{\sigma}}{a\bar{a}(a+\bar{a})}\left(1-e^{-a(S-T)}\right)\left(1-e^{-\bar{a}(S-T)}\right)^2\left(1-e^{-(a+\bar{a})(T-t)}\right). \end{aligned}$$

Note now that the main structure of bond and call prices is the same in both models, especially today's bond prices differ only by a deterministic multiplicative factor; however, if we look further "into the future" the path of the fBm does matter, which results in a more complex option price.

We want to emphasize that we have in the situation of (5.3)

$$\overline{B}(0,T)e^{\frac{\Sigma(0,T,S)^2}{2}-A(0,T,S)} \neq \overline{B}(0,S)$$

and, therefore, cannot get exactly the same structure as in the Brownian case.

Numerical evaluations of the formulas in the fractional case are significantly more complicated than in the classical Brownian model. Especially calculating the norms $\|\cdot\|_\kappa$ is challenging due to the singularity of the weight function $(x,y)\mapsto |x-y|^{2\kappa-1}$ on the diagonal. For some graphs depicting bond prices for different fractional parameters we refer to Fink et al. [13].

The following pricing method allows for more general payoff functions, but it is less explicit. Note that it also includes the European call price calculated explicitly in Theorem 5.2.

Theorem 5.4. *Let $0\le t\le T$. Denote by X an $\mathcal{F}_T$-measurable payoff of the form*

$$X = f\left(\int_0^T \phi(s)dB^\kappa(s) + \int_0^T \bar{\phi}(s)d\overline{B}^{\bar{\kappa}}(s)\right)$$

for some $f:\mathbb{R}\to\mathbb{R}$ and $\phi,\bar{\phi}\in L^2[0,T]$. Assume further that there exist $b>0$ and $z\in\mathbb{R}$ such that $f_+^{b,z}(\cdot):=e^{-b\cdot}f(\cdot)\mathbf{1}_{[z,\infty)}(\cdot)$ and $f_-^{b,z}(\cdot):=e^{b\cdot}f(\cdot)\mathbf{1}_{(-\infty,z)}(\cdot)$ are in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Define for $\xi\in\mathbb{R}$ and $\star\in\{+,-\}$

$$\Phi^{\xi,\star}(\cdot) := D(\cdot,T)\sigma(\cdot) - (i\xi\star b)\phi(\cdot), \quad \bar{\Phi}^{\xi,\star}(\cdot) := \bar{D}(\cdot,T)\bar{\sigma}(\cdot) - (i\xi\star b)\bar{\phi}(\cdot). \tag{5.5}$$

Then the price of X at time t is given by

$$\begin{aligned}
&\mathcal{V}(t,T) = \mathbf{1}_{\{\tau>t\}}E\left[e^{-\int_t^T(r(s)+\lambda(s))ds}X\,\Big|\,\mathcal{G}_t\right] \qquad (5.6)\\
=\ & \mathbf{1}_{\{\tau>t\}}\exp\left\{-\int_t^T D(s,T)k(s)ds - \int_t^T \bar{D}(s,T)\bar{k}(s)ds - D(t,T)r(t) - \bar{D}(t,T)\lambda(t)\right\}\\
&\times\frac{1}{2\pi}\int_{\mathbb{R}}\left[\exp\left\{V^\xi(t,T) - W^\xi(t,T) - \int_0^t \Psi^\kappa_{c_\xi^+}(t,T,v)dB^\kappa(v) - \int_0^t \Psi^{\bar{\kappa}}_{\bar{c}_\xi^+}(t,T,v)d\overline{B}^{\bar{\kappa}}(v)\right\}\widehat{f_+^{b,z}}(\xi)\right.\\
&\left.+\exp\left\{V^\xi(t,T) - W^\xi(t,T) - \int_0^t \Psi^\kappa_{c_\xi^-}(t,T,v)dB^\kappa(v) - \int_0^t \Psi^{\bar{\kappa}}_{\bar{c}_\xi^-}(t,T,v)d\overline{B}^{\bar{\kappa}}(v)\right\}\widehat{f_-^{b,z}}(\xi)\right]d\xi
\end{aligned}$$

where $c_\xi^\star(\cdot)=\Phi^{\xi,\star}(\cdot)$ and $\bar{c}_\xi^\star(\cdot)=\bar{\Phi}^{\xi,\star}(\cdot)$,

$$\begin{aligned}
W^{\xi,\star}(t,T) &= \frac{1}{2}\left(\left\|\mathbf{1}_{(t,T)}(\cdot)\Phi^{\xi,\star}(\cdot)\right\|_\kappa^2 + 2\left\langle \mathbf{1}_{(t,T)}(\cdot)\Phi^{\xi,\star}(\cdot), \mathbf{1}_{(t,T)}(\cdot)\bar{\Phi}^{\xi,\star}(\cdot)\right\rangle_{\frac{\kappa+\bar{\kappa}}{2}}\right.\\
&\quad\left.+\left\|\mathbf{1}_{(t,T)}(\cdot)\bar{\Phi}^{\xi,\star}(\cdot)\right\|_{\bar{\kappa}}^2\right),\\
V^{\xi,\star}(t,T) &= \frac{1}{2}\left(\left\|\mathbf{1}_{(0,t)}(\cdot)\Psi^\kappa_{c_\xi^\star}(t,T,\cdot)\right\|_\kappa^2 + 2\left\langle \mathbf{1}_{(0,t)}(\cdot)\Psi^\kappa_{c_\xi^\star}(t,T,\cdot), \mathbf{1}_{(0,t)}(\cdot)\Psi^{\bar{\kappa}}_{\bar{c}_\xi^\star}(t,T,\cdot)\right\rangle_{\frac{\kappa+\bar{\kappa}}{2}}\right.\\
&\quad\left.+\left\|\mathbf{1}_{(0,t)}(\cdot)\Psi^{\bar{\kappa}}_{\bar{c}_\xi^\star}(t,T,\cdot)\right\|_{\bar{\kappa}}^2\right),
\end{aligned} \tag{5.7}$$

with $\widehat{f_+^{b,z}}$ and $\widehat{f_-^{b,z}}$ the Fourier transforms of $f_+^{b,z}$ and $f_-^{b,z}$ respectively.

Proof. Applying - as in the theorem before - Lemma 13.2 of Filipovic [11] we obtain (5.6). For some $a < 0$ and $z \in \mathbb{R}$ we have

$$f(x) = e^{bx}[e^{-bx}f(x)\mathbf{1}_{[z,\infty)}(x)] + e^{-bx}[e^{bx}f(x)\mathbf{1}_{(-\infty,z)}(x)] =: e^{bx}f_+^{b,z}(x) + e^{-bx}f_-^{b,z}(x). \quad (5.8)$$

Denote by $\widehat{f_+^{b,z}}$ and $\widehat{f_-^{b,z}}$ the Fourier transforms of $f_+^{b,z}$ and $f_-^{b,z}$ respectively. Using classical Fourier analysis we obtain for $\xi, x \in \mathbb{R}$ and $\star \in \{+,-\}$

$$\widehat{f_\star^{b,z}}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f_\star^{b,z}(x)dx, \quad f_\star^{b,z}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\widehat{f_\star^{b,z}}(\xi)d\xi,$$

where we used the fact that $f_+^{b,z}$ and $f_-^{b,z}$ are in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Set

$$J(t,T) := \int_t^T \phi(s)dB^\kappa(s) + \int_t^T \bar{\phi}(s)d\overline{B}^{\bar{\kappa}}(s).$$

We get by the definition and (5.8)

$$\begin{aligned} X &= f\left(J(0,T)\right) \\ &= e^{bJ(0,T)}f_+^{b,z}\left(J(0,T)\right) + e^{-bJ(0,T)}f_-^{b,z}\left(J(0,T)\right) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\left(e^{(i\xi+b)J(0,T)}\widehat{f_+^{b,z}}(\xi) + e^{(i\xi-b)J(0,T)}\widehat{f_-^{b,z}}(\xi)\right)d\xi \end{aligned}$$

Since by normality $E[e^{bJ(0,T)}] < \infty$ for all $b \in \mathbb{R}$ we can interchange expectation and integration as follows using (4.9)

$$\begin{aligned} \mathcal{V}(t,T) &= \mathbf{1}_{\{\tau>t\}}E\left[e^{-\int_t^T(r(s)+\lambda(s))ds}X\middle|\mathcal{G}_t\right] \\ &= \mathbf{1}_{\{\tau>t\}}E\left[e^{-\int_t^T(r(s)+\lambda(s))ds}\frac{1}{2\pi}\int_{\mathbb{R}}\left[e^{(i\xi+b)J(0,T)}\widehat{f_+^{b,z}}(\xi) + e^{(i\xi-b)J(0,T)}\widehat{f_-^{b,z}}(\xi)\right]d\xi\middle|\mathcal{G}_t\right] \\ &= \mathbf{1}_{\{\tau>t\}}e^{C(t,T)+D(t,T)r(t)+\bar{D}(t,T)\lambda(t)} \\ &\quad\times\frac{1}{2\pi}\int_{\mathbb{R}}\left[E\left[e^{G(t,T)+(i\xi+b)J(0,T)}\middle|\mathcal{G}_t\right]\widehat{f_+^{b,z}}(\xi) + E\left[e^{G(t,T)+(i\xi-b)J(0,T)}\middle|\mathcal{G}_t\right]\widehat{f_-^{b,z}}(\xi)\right]d\xi \end{aligned}$$

with $C(t,T) := -\int_t^T D(v,T)k(v)dv - \int_t^T \bar{D}(v,T)\bar{k}(v)dv$ and $G(t,T) := -\int_t^T D(s,T)\sigma(s)dB^\kappa(s) - \int_t^T \bar{D}(s,T)\bar{\sigma}(s)d\overline{B}^{\bar{\kappa}}(s)$. The case $t = 0$ is now again simple because we just need to calculate the expectations. Let further be $t > 0$. Prediction works now the same way as in Proposition 4.8 and we obtain for $\star \in \{+,-\}$ with $\Phi^{\xi,\star}$ and $\bar{\Phi}^{\xi,\star}$ as in (5.5):

$$\begin{aligned} &E\left[e^{G(t,T)+(i\xi\star b)J(0,T)}\middle|\mathcal{G}_t\right] = e^{(i\xi\star b)J(0,t)}E\left[e^{G(t,T)+(i\xi\star b)J(t,T)}\middle|\mathcal{G}_t\right] \\ = \;& e^{(i\xi\star b)J(0,t)}E\left[e^{-\int_t^T(D(s,T)\sigma(s)-(i\xi\star b)\phi(s))dB^\kappa(s)-\int_t^T(\bar{D}(s,T)\bar{\sigma}(s)-(i\xi\star b)\bar{\phi}(s))d\overline{B}^{\bar{\kappa}}(s)}\middle|\mathcal{G}_t\right] \\ = \;& e^{(i\xi\star b)J(0,t)}E\left[e^{-\int_t^T\Phi^{\xi,\star}(s)dB^\kappa(s)-\int_t^T\bar{\Phi}^{\xi,\star}(s)d\overline{B}^{\bar{\kappa}}(s)}\middle|\mathcal{G}_t\right] \\ = \;& e^{(i\xi\star b)J(0,t)+V^{\xi,\star}(t,T)-W^{\xi,\star}(t,T)}\exp\left\{-\int_0^t\Psi^\kappa_{c_\xi^\star}(t,T,v)dB^\kappa(v) - \int_0^t\Psi^{\bar{\kappa}}_{\bar{c}_\xi^\star}(t,T,v)d\overline{B}^{\bar{\kappa}}(v)\right\} \end{aligned}$$

where $c_\xi^\star(\cdot) = \Phi^{\xi,\star}(\cdot)$ and $\overline{c}_\xi^\star(\cdot) = \bar{\Phi}^{\xi,\star}(\cdot)$ and $W^{\xi,\star}(t,T)$, $V^{\xi,\star}(t,T)$ are as in (5.7). □

Acknowledgment

We thank Martina Zähle for interesting discussions which led to an improvement of this paper. We also thank Martina Zähle and Christian Bender for a discussion on Proposition A.2 below.

Appendix

Lemma A.1. *Given the situation of Section 4 we assume $\bar{\kappa} \geq \kappa$. Similar results hold true for $\bar{\kappa} < \kappa$. Let $f : \mathbb{R} \to \mathbb{R}$ with $||f(\cdot)||_\kappa < \infty$ and $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then we have*

$$\left(\frac{a_{\bar{\kappa}}}{a_\kappa}\right)^2 \left\|\mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(\cdot)\right\|_\kappa^2 = \|g(\cdot)\|_{\bar{\kappa}}^2 , \tag{A.1}$$

$$\frac{a_{\bar{\kappa}}}{a_\kappa} \left\langle f(\cdot), \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(\cdot)\right\rangle_\kappa = \langle f(\cdot), g(\cdot)\rangle_{\frac{\kappa+\bar{\kappa}}{2}} , \tag{A.2}$$

where $a_\kappa := c_\kappa \Gamma(\kappa+1)$ and $a_{\bar{\kappa}} := c_{\bar{\kappa}} \Gamma(\bar{\kappa}+1)$.

Proof. We know from Lemma 4.7 that in the L^2-sense

$$\int_{\mathbb{R}} g(v) d\overline{B}^{\bar{\kappa}}(v) = \frac{a_{\bar{\kappa}}}{a_\kappa} \int_{\mathbb{R}} \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(v) dB^\kappa(v)$$

and, therefore, variances are equal. Equation (A.1) follows. Furthermore we have with Lemma 4.7 again in the L^2-sense

$$\int_{\mathbb{R}} f(v) dB^\kappa(v) \int_{\mathbb{R}} g(v) d\overline{B}^{\bar{\kappa}}(v) = \int_{\mathbb{R}} f(v) dB^\kappa(v) \frac{a_{\bar{\kappa}}}{a_\kappa} \int_{\mathbb{R}} \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(v) dB^\kappa(v)$$

and, therefore, by Proposition 4.1

$$\begin{aligned} \langle f(\cdot), g(\cdot)\rangle_{\frac{\kappa+\bar{\kappa}}{2}} &= E\left[\int_{\mathbb{R}} f(v) dB^\kappa(v) \int_{\mathbb{R}} g(v) d\overline{B}^{\bar{\kappa}}(v)\right] \\ &= E\left[\int_{\mathbb{R}} f(v) dB^\kappa(v) \frac{a_{\bar{\kappa}}}{a_\kappa} \int_{\mathbb{R}} \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(v) dB^\kappa(v)\right] \\ &= \frac{a_{\bar{\kappa}}}{a_\kappa} \left\langle f(\cdot), \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(\cdot)\right\rangle_\kappa \end{aligned}$$

□

Now we consider and, in particular, correct Proposition 2 of Duncan [8]. We realized that in its orginal formulation it implies that expressions of the form $\exp\left\{\int_s^t b(v) dB^\kappa(v)\right\} \Big| \{B^\kappa(u), u \in [0, s]\}$ are deterministic (calculate for instance the characteristic function). In total we found several errors in the proof. We formulate the following corrected version.

Proposition A.2 (Corrected version of Proposition 2 of Duncan [8])**.** *Let B^κ be fBm with $\kappa \in (0, \frac{1}{2})$ and define*

$$X(t) = x_0 \exp\left\{\int_0^t a(v) dv + \int_0^t b(v) dB^\kappa(v) - \frac{1}{2}\|b(\cdot)\mathbf{1}_{(0,t)}(\cdot)\|_\kappa\right\}, \quad t \geq 0,$$

where $x_0 \in \mathbb{R}$, $a(\cdot)$ locally integrable and $b : \mathbb{R}_+ \to \mathbb{R}$ is continuous with $\|b(\cdot)\|_\kappa < \infty$ and $b(\cdot) > 0$. For $t > 0$ define $\mathcal{G}_t := \sigma\{X(s), s \in [0,t]\}$ and let $s \in (0,t)$ be fixed. Then

$$\begin{aligned}
& \mathbb{E}[X(t)|\mathcal{G}_s] \\
= \; & \mathbb{E}\left[X(s) \exp\left\{ \int_s^t a(v)dv + \int_s^t b(v)dB^\kappa(v) - \frac{1}{2}\|b(\cdot)\mathbf{1}_{(s,t)}(\cdot)\|_\kappa + \left\langle b(\cdot)\mathbf{1}_{(0,s)}(\cdot), b(\cdot)\mathbf{1}_{(s,t)}(\cdot)\right\rangle_\kappa \right\} \middle| \mathcal{G}_s \right] \\
= \; & X(s) \exp\left\{ \int_s^t a(v)dv - \left\langle b(\cdot)\mathbf{1}_{(0,s)}(\cdot), b(\cdot)\mathbf{1}_{(s,t)}(\cdot)\right\rangle_\kappa \right\} \exp^\diamond \left\{ \mathbb{E}\left[\int_s^t b(v)dB^\kappa(v) \middle| \mathcal{G}_s \right] \right\} \\
= \; & X(s) \exp\left\{ \int_s^t a(v)dv - \left\langle b(\cdot)\mathbf{1}_{(0,s)}(\cdot), b(\cdot)\mathbf{1}_{(s,t)}(\cdot)\right\rangle_\kappa - \frac{1}{2}\|\Psi_b^\kappa(t,s,\cdot)\mathbf{1}_{(0,s)}(\cdot)\|_\kappa^2 + \int_0^s \Psi_b^\kappa(t,s,v)dB^\kappa(v) \right\}
\end{aligned} \tag{A.3}$$

Proof. The proof works in principle like the proof of Proposition 3.6. The following points in Duncan's proof have to be taken care of.

First of all to apply Proposition 3.3 we need that $\mathcal{G}_t = \sigma\{B^\kappa(s), s \in [0,t]\}$, which cannot be deduced for all bounded measurable functions $b : \mathbb{R}_+ \to \mathbb{R}$ as suggested in Duncan (for $b(\cdot) \equiv 0$ the equality of filtrations does not hold). However, for b as in the assertion this works well.

Next, we have

$$X(t) = X(s) \exp\left\{ \int_s^t a(v)dv + \int_s^t b(v)dB^\kappa(v) - \frac{1}{2}\|b(\cdot)\mathbf{1}_{(s,t)}(\cdot)\|_\kappa + \left\langle b(\cdot)\mathbf{1}_{(0,s)}(\cdot), b(\cdot)\mathbf{1}_{(s,t)}(\cdot)\right\rangle_\kappa \right\},$$

where in Duncan [8] the term $\left\langle b(\cdot)\mathbf{1}_{(0,s)}(\cdot), b(\cdot)\mathbf{1}_{(s,t)}(\cdot)\right\rangle_\kappa$ is missing. Note that the identity

$$\|b(\cdot)\mathbf{1}_{(0,t)}(\cdot)\|_\kappa = \|b(\cdot)\mathbf{1}_{(0,s)}(\cdot)\|_\kappa + \|b(\cdot)\mathbf{1}_{(s,t)}(\cdot)\|_\kappa$$

does not hold, since $\left\langle b(\cdot)\mathbf{1}_{(0,s)}(\cdot), b(\cdot)\mathbf{1}_{(s,t)}(\cdot)\right\rangle_\kappa \neq 0$. This can be seen, for instance, by taking $b(\cdot) \equiv 1$:

$$\left\langle b(\cdot)\mathbf{1}_{(0,s)}(\cdot), b(\cdot)\mathbf{1}_{(s,t)}(\cdot)\right\rangle_\kappa = \mathbb{E}[B_s^\kappa(B_t^\kappa - B_s^\kappa)] \neq 0.$$

We use Proposition 3.5 (2) and Proposition 3.3 to get (A.3) exactly as in the proof of Proposition 3.6.

Finally, in Duncan [8] the term $\|\Psi_b^\kappa(t,s,\cdot)\mathbf{1}_{(0,s)}(\cdot)\|_\kappa^2$ is missing. □

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