Discrete approximation of stochastic integrals with respect to fractional Brownian motion of Hurst index $H > \frac{1}{2}$

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Abstract

In this paper we provide a discrete approximation for the stochastic integral with respect to the fractional Brownian motion of Hurst index $H > \frac{1}{2}$ defined in terms of the divergence operator. To determine the suitable class of integrands for which the approximation holds, we also investigate the relations among the spaces of Malliavin differentiable processes in the fractional and standard case.

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1 Introduction

Fractional Brownian motion (fBm) of Hurst index $H \in (0, 1)$ has been widely used in applications to model a number of phenomena, e.g. in biology, meteorology, physics and finance. A natural question is how to define a stochastic integral with respect to fBm and how to provide a natural discretization of it. The problem of defining a stochastic integral with respect to the fractional Brownian motion has been extensively studied in literature. Several different integrals has been introduced by exploiting different properties of the fractional Brownian motion: as Wiener integrals (exploiting the Gaussianity) in [12], [13], [24], as pathwise integrals (using fractional calculus and forward, backward and symmetric integrals) in [2], [3], [8], [9], [11], [20], [23], [26], [27], [28], as divergence (using fractional Malliavin calculus) in [3], [4] [10], [24], as stochastic integral with respect to the standard Brownian motion in [6], [16], as stochastic integral with respect to the fractional white noise in [14], [17]. For a complete survey on this subject, an extensive literature and relations among different definitions of integrals we refer to [7] and [24].
For applications, in particular for applications to finance, it is relevant to give an interpretation to a definition of stochastic integral. As it is the case of Itô and Skorohod integrals, this can be achieved by giving a procedure to obtain the integral as the limit of discrete approximations that are meaningful in the application field.

In this paper we address the problem of finding a discrete approximation of the stochastic integral defined as divergence. A discrete realization of the fractional Brownian motion for Hurst index $H > \frac{1}{2}$ is provided in different ways in [13] and [19]. This can be extended to find a discretization of stochastic integral of deterministic functions integrable with respect to the fractional Brownian motion. In [5] they study the convergence of Riemann-Stieltjes integrals with respect to the fractional Brownian motion for Hurst index $H > \frac{1}{2}$ when the integrands are almost surely Hölder-continuous of order $\lambda > 1 - H$. Other discrete approximations of stochastic integrals with respect to the fractional Brownian motion can be found in [3], [8], [14] and [22].
Here we focus on the case $H > \frac{1}{2}$ and consider the stochastic integral with respect to B^H defined in terms of the divergence operator δ^H relative to the fBm. We prove a discrete approximation for this kind of integral by means of the resolutions of the Fock space associated to B^H and by using its expression in terms of the divergence operator δ of the standard Brownian motion.
In order to find a suitable class of stochastic integrands for which our results hold, we also study the relations among the spaces of Malliavin differentiable processes with respect to B and to B^H with Hurst index $H > \frac{1}{2}$.

2 Basic definitions and notation

We recall in this section the basic definition and the main properties concerning fractional Brownian motion and the relative Malliavin calculus.

Definition 2.1. *The **fractional Brownian motion** (fBm) B^H of Hurst index $H \in (0,1)$ is a centered Gaussian process $\{B^H_t\}_{t\geq 0}$ with covariance given by*

$$R_H(s,t) := \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}). \tag{1}$$

As well-known, for $H = \frac{1}{2}$ we obtain the standard Brownian motion (Bm). By Kolmogorov's theorem we have that B^H admits a continuous modification for every $H \in (0,1)$. In the sequel we assume to work always with the continuous modification of B^H, that we denote again with B^H.
In particular we will focus on the case $H > \frac{1}{2}$. In this case the covariance (1) can be written as

$$R_H(s,t) = \alpha_H \int_0^t \int_0^s \phi(r,u)\,dr\,du \tag{2}$$

where ϕ is given by

$$\phi(r,u) = |r-u|^{2H-2} \tag{3}$$

and $\alpha_H = H(2H-1)$. By [24], we obtain that $R_H(s,t)$ can be expressed in terms of the deterministic kernel

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\,du \tag{4}$$

where $c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right)^{\frac{1}{2}}$, $t > s$, in the following way

$$R_H(t,s) = \int_0^{t\wedge s} K_H(t,u) K_H(s,u)\,du.$$

As in the standard Brownian motion case, we wish to associate to B^H a Gaussian Hilbert space in the following way. We denote $\mathcal{E}$ the space of step functions defined on $[0,T]$ and let $\mathcal{H}$ be the Hilbert space given by the completion of $\mathcal{E}$ with respect to the following inner product

$$\langle \chi_{[0,t]}, \chi_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s).$$

The function $\chi_{[0,t]} \mapsto B_t^H$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian Hilbert space generated by the fractional Brownian motion. We denote this isometry by B^H.
We recall that the following relation holds

$$L^2([0,T]) \subseteq \mathcal{H} \tag{5}$$

since

$$\|f\|_{\mathcal{H}} \leq k_H \|f\|_{L^2([0,T])} \tag{6}$$

for every $f \in L^2([0,T])$, where $k_H = \dfrac{T^{2H-1}}{H-\frac{1}{2}}$. For the proof we refer to [4] and [24].
We introduce now the operator K_H^*.

Definition 2.2. *Let K_H^* be the linear operator defined on $\mathcal{E}$*

$$K_H^* : \mathcal{E} \longrightarrow L^2([0,T])$$

such that

$$(K_H^* \varphi)(s) := \int_s^T \varphi(r) \frac{\partial K_H}{\partial r}(r,s) dr. \tag{7}$$

We note that

$$K_H^*(\chi_{[0,t]})(s) = K_H(t,s)\chi_{[0,t]}(s).$$

Proposition 2.3. *For every $\varphi, \psi \in \mathcal{E}$ we have*

$$\langle K_H^* \varphi, K_H^* \psi \rangle_{L^2([0,T])} = \langle \varphi, \psi \rangle_{\mathcal{H}}. \tag{8}$$

Proof. For the proof we refer to [4] and [23]. □

Hence we can extend the operator K_H^* to an isometry between the Hilbert space $\mathcal{H}$ and $L^2([0,T])$ (see [24] for further details). We define the process

$$B_t = B^H((K_H^*)^{-1}\chi_{[0,t]}) \quad t \in [0,T]. \tag{9}$$

We obtain that B_t is a Gaussian process, whose covariance is given by

$$\begin{aligned} \langle B_t, B_s \rangle_{L^2([0,T])} &= \langle B^H((K_H^*)^{-1}\chi_{[0,t]}), B^H((K_H^*)^{-1}\chi_{[0,s]}) \rangle_{L^2([0,T])} \\ &= \langle (K_H^*)^{-1}(\chi_{[0,t]}), (K_H^*)^{-1}(\chi_{[0,s]}) \rangle_{\mathcal{H}} \\ &= s \wedge t \end{aligned}$$

Hence B_t (or better its continuous modification) is a standard Brownian motion. Moreover for every $\varphi \in \mathcal{H}$ we have

$$B^H(\varphi) = \int_0^T (K_H^* \varphi)(t) dB_t$$

and in particular

$$B_t^H = B^H(\chi_{[0,t]}) = \int_0^T K_H(t,s) dB_s, \tag{10}$$

where the kernel K_H is defined in (4). As a consequence of (9) and (10), we immediately have that B^H and B generate the same filtration.

We denote by $\mathcal{S}_H$ the set of regular cylindric random variables S of the form

$$S = f(W_H(h_1), \ldots, W_H(h_n)), \tag{11}$$

where $h_i \in \mathcal{H}$, $n \geq 1$, and f belongs to the set $\mathcal{C}_b^\infty(\mathbb{R}^n, \mathbb{R})$ of bounded continuous functions with bounded continuous partial derivatives of every order.

Definition 2.4. *Let $H \in [\frac{1}{2}, 1)$ and $S \in \mathcal{S}_H$.*

(1) The Malliavin derivative of S with respect to B^H is defined as the $\mathcal{H}-$valued random variable

$$D^H S := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W^H(h_1), \ldots, W^H(h_n))h_i. \tag{12}$$

(2) The space $\mathcal{S}_H$ is endowed with the norm $\mathbb{D}_H^{1,2}$ such that

$$\|S\|^2_{\mathbb{D}_H^{1,2}} := \|S\|^2_{L^2(\Omega)} + \|D^H S\|^2_{L^2(\Omega,\mathcal{H})}.$$

We denote by $\mathbb{D}_H^{1,2}$ the closure of the space $\mathcal{S}_H$ with respect to the norm $\|\cdot\|_{\mathbb{D}_H^{1,2}}$ and extend by density the derivative operator to the space $\mathbb{D}_H^{1,2}$.
We introduce now the divergence operator. Let $H \in [\frac{1}{2}, 1)$.

Definition 2.5. *(1) The **domain** $\mathrm{dom}\delta^H$ of the divergence operator is defined as the set of the random variables $F \in L^2(\Omega, \mathcal{H})$ such that*

$$|E(\langle D^H S, F\rangle_{\mathcal{H}})| \leq c_F \|S\|_{L^2(\Omega)} \quad \forall S \in \mathcal{S}_H, \tag{13}$$

for some constant c_F depending on F.

*(2) The **divergence operator** $\delta^H : \mathrm{dom}\delta^H \longrightarrow L^2(\Omega)$ is defined by the duality relation*

$$\langle D^H G, F\rangle_{L^2(\Omega,\mathcal{H})} = \langle G, \delta^H F\rangle_{L^2(\Omega)} \tag{14}$$

for every $G \in \mathbb{D}_H^{1,2}$.

Hence the divergence operator is the adjoint of the derivative operator.
From now on, to simplify the notation in the case $H = \frac{1}{2}$, we omit to write the index H and denote

$$\mathbb{L}^{1,2} := \mathbb{D}^{1,2}_{1/2}(L^2([0, T])).$$

We conclude this section by recalling a result that will play a key role in the sequel.

Proposition 2.6. *We have that $F \in \mathrm{dom}\delta^H$ iff $\mathbb{K}_H^* F \in \mathrm{dom}\delta$ and*

$$\delta^H(F) = \delta(\mathbb{K}_H^* F). \tag{15}$$

Proof. For the proof we refer to [3] and [24]. □

According to the approach of [12] and [15], we define the stochastic integral with respect to B^H for $H > \frac{1}{2}$ as follows.

Definition 2.7. *Let B^H be a fractional Brownian motion of Hurst index $H > \frac{1}{2}$ and F a stochastic process belonging to domδ^H. We define **stochastic integral** of F with respect to B^H the divergence $\delta^H(F)$ of F introduced in Definition 2.5.*

3 Relations between the standard case and the case $H > \frac{1}{2}$

Before proceeding to prove our main results concerning the discretization of the stochastic integral as divergence, we need to clarify some relations between the standard and the fractional case.

3.1 Chaos expansions

By the Chaos Expansion Theorem, for every $F \in L^2(\Omega, L^2([0,T]))$ we have

$$\forall t \in [0,T], \quad F_t = \sum_{q=0}^{n} I_q(f_{q,t}) \quad \text{in} \quad L^2(\Omega) \tag{16}$$

where $f_{q,t}(t_1,\dots,t_q) := f_q(t_1,\dots,t_q,t) \in L^2([0,T]^{q+1})$ are symmetric with respect to the first q variables $(t_1,\dots,t_q)$ and I_q denotes the q-dimensional iterated Wiener integral with respect to $(t_1,\dots,t_q)$

$$I_q(f_{q,t}) := \int_0^T \left(\int_0^{t_q} \dots \int_0^{t_2} f_{q,t}(t_1,\dots,t_q) dB_{t_1} \dots dB_{t_{q-1}} \right) dB_{t_q}$$

(for further details see [18] and [23]).
We introduce the function

$$\mathbb{K}_H^* : L^2(\Omega, \mathcal{H}) \longrightarrow L^2(\Omega, L^2([0,T])) \tag{17}$$

that associates to $F \in L^2(\Omega, \mathcal{H})$ the random variable $\mathbb{K}_H^* F \in L^2(\Omega, L^2([0,T]))$ defined as $(\mathbb{K}_H^* F)(\omega) := K_H^*(F(\omega))$.

Proposition 3.1. *The following properties hold*

(i) $\mathbb{K}_H^* : L^2(\Omega, \mathcal{H}) \longrightarrow L^2(\Omega, L^2([0,T]))$ *is an isometry;*

(ii) $L^2(\Omega, L^2([0,T])) \subseteq L^2(\Omega, \mathcal{H})$ *and*

$$\|F\|_{L^2(\Omega,\mathcal{H})} \le k_H \|F\|_{L^2(\Omega,L^2([0,T]))}.$$

Proof. The proof follows by (5), (6) and (8). □

Proposition 3.2. *Let F belong to $L^2(\Omega, L^2([0,T]))$ with chaos expansion given by* (16)*. Then*

$$\mathbb{K}_H^* F = \sum_{q=0}^{\infty} I_q(K_{H,\cdot}^*(f_{q,\cdot})) \quad \textit{in} \quad L^2(\Omega, L^2([0,T])). \tag{18}$$

Moreover for almost every $t \in [0,T]$

$$(\mathbb{K}_H^* F)_t = \sum_{q=0}^{\infty} I_q((K_{H,\cdot}^* f_{q,\cdot})_t) \quad \textit{in} \quad L^2(\Omega), \tag{19}$$

where by (7) *we have*

$$(K_{H,\cdot}^* f_{q,\cdot})_t = \int_t^T f_q(t_1,\dots,t_q,r) \frac{\partial K_H}{\partial r}(r,t) dr.$$

Proof. We start with some preliminary remarks.

(A) If we put

$$(S_Q)_t := \sum_{q=0}^{Q} I_q(f_{q,t}), \quad Q \in \mathbb{N},$$

then by the Chaos Expansion Theorem (see [23]) we obtain the following convergences

$$\lim_{Q\to\infty} \|(S_Q)_t - F_t\|_{L^2(\Omega)} = 0 \qquad \forall t \in [0,T] \tag{20}$$

$$\lim_{Q\to\infty} \|S_Q - F\|_{L^2(\Omega,L^2([0,T]))} = 0. \tag{21}$$

(B) Consider $F^* = \mathbb{K}_H^* F$. Then F^* admits chaos expansion given by

$$F^* = \sum_{q=0}^{\infty} I_q(f^*_{q,\cdot}) \qquad \text{in} \qquad L^2(\Omega, L^2([0,T])),$$

where $f^*_{q,\cdot} \in L^2([0,T]^{q+1})$ are symmetric function with respect to the variables $t_1, \dots, t_q$, for every $q \in \mathbb{N}$. Hence if we set

$$(S_Q^*)_t := \sum_{q=0}^{Q} I_q(f^*_{q,t}), \quad Q \in \mathbb{N},$$

the following convergences hold

$$\lim_{Q\to\infty} \|(S_Q^*)_t - F_t^*\|_{L^2(\Omega)} = 0 \tag{22}$$

$$\lim_{Q\to\infty} \|S_Q^* - F^*\|_{L^2(\Omega, L^2([0,T]))} = 0. \tag{23}$$

(C) By the stochastic version of the Fubini-Tonelli theorem ([25]), we can exchange the order of integration between the multiple integral I_q and the integral operator K_H^*

$$[K^*_{H,\cdot}(I_q((f_{q,\cdot}))]_t = I_q(K^*_{H,\cdot}(f_{q,\cdot}))(t), \tag{24}$$

for every $t \in [0,T]$.

By (8) and Proposition 3.1 we obtain that

$$\begin{aligned} \|\mathbb{K}_H^* S_Q - \mathbb{K}_H^* F\|^2_{L^2(\Omega, L^2([0,T]))} &= \|\mathbb{K}_H^*(S_Q - F)\|^2_{L^2(\Omega, L^2([0,T]))} \\ &\le k_H \|S_Q - F\|^2_{L^2(\Omega, L^2([0,T]))}. \end{aligned}$$

Hence by (21) we have

$$\lim_{Q\to\infty} \|\mathbb{K}_H^* S_Q - \mathbb{K}_H^* F\|_{L^2(\Omega, L^2([0,T]))} = 0. \tag{25}$$

Now it remains only to prove that

$$\lim_{Q\to\infty} \|(\mathbb{K}_H^* S_Q)_t - (\mathbb{K}_H^* F)_t\|_{L^2(\Omega)} = 0, \qquad \text{for a.e.} \quad t \in [0,T]. \tag{26}$$

By (23) and (25) we get immediately

$$\lim_{Q\to\infty} \|S_Q^* - \mathbb{K}_H^* S_Q\|^2_{L^2(\Omega, L^2([0,T]))} = 0. \tag{27}$$

Moreover we know

$$\|S_Q^* - \mathbb{K}_H^* S_Q\|^2_{L^2(\Omega, L^2([0,T]))} = \left\| \sum_{q=0}^{Q} I_q(f^*_{q,\cdot} - K_H^* f_{q,\cdot}) \right\|^2_{L^2(\Omega, L^2([0,T]))}. \tag{28}$$

and that the following relation holds between the norms:

$$\|I_q(f_{q,t})\|^2_{L^2(\Omega, L^2([0,T]))} = q! \|(f_{q,t})\|^2_{L^2([0,T]^{q+1})}, \tag{29}$$

if $(f_{q,t})$ is symmetric with respect to the first q variables $t_1, \ldots, t_q$, for every $q \in \mathbb{N}$. Since also $f^*_{q,\cdot} - K^*_{H,\cdot} f_{q,\cdot}$ are symmetric with respect to $(t_1, \ldots, t_q)$, for every $q \in \mathbb{N}$, we get

$$\left\| \sum_{q=0}^{Q} I_q(f^*_{q,\cdot} - K^*_{H,\cdot} f_{q,\cdot}) \right\|^2_{L^2(\Omega, L^2([0,T]))} = \sum_{q=0}^{Q} q! \| f^*_{q,\cdot} - K^*_{H,\cdot} f_{q,\cdot} \|^2_{L^2([0,T]^{q+1})}$$

and by (27)

$$\sum_{q=0}^{\infty} q! \| f^*_{q,\cdot} - K^*_{H,\cdot} f_{q,\cdot} \|^2_{L^2([0,T]^{q+1})} = 0.$$

Finally

$$\| f^*_{q,\cdot} - K^*_{H,\cdot} f_{q,\cdot} \|^2_{L^2([0,T]^{q+1})} = \int_0^T \left(\int_{[0,T]^q} (f^*_{q,\cdot} - K^*_{H,\cdot} f_{q,\cdot})^2 (t_1, \ldots, t_q, t) dt_1 \ldots dt_q \right) dt = 0$$

for every $q \in \mathbb{N}$. We conclude that for almost every $t \in [0,T]$

$$f^*_{q,t} - (K^*_{H,\cdot} f_{q,\cdot})_t = 0 \quad \text{in} \quad L^2([0,T]^q)$$

for every $q \in \mathbb{N}$. Finally for almost every $t \in [0,T]$ we have that $(S^*_Q)_t = (\mathbb{K}^*_H S_Q)_t$, for every $Q \in [0,T]$, and then

$$\lim_{Q \to \infty} \| (\mathbb{K}^*_H S_Q)_t - (\mathbb{K}^*_H F)_t \|_{L^2(\Omega)} = \lim_{Q \to \infty} \| (S^*_Q)_t - (\mathbb{K}^*_H F)_t \|_{L^2(\Omega)}.$$

By (22) we obtain the result. □

3.2 Relation between $\mathbb{D}^{1,2}_{\mathbf{H}}(\mathcal{H})$ and $\mathbb{L}^{1,2}$

We can generalize the Definition 2.5 to the case of $\mathcal{H}-$valued random variables. In this case the space $\mathcal{S}_H(\mathcal{H})$ is defined as the set of random variables U with values in $\mathcal{H}$ of the form

$$U = \sum_{j=1}^{n} F_j v_j, \tag{30}$$

where $F_j \in \mathbb{D}^{1,2}_H$, $v_j \in \mathcal{H}$, $\forall j \in \{1, \ldots, n\}$. If $U \in \mathcal{S}_H(\mathcal{H})$, the Malliavin derivative $D^H U$ of U is defined as the element in $L^2(\Omega, \mathcal{H} \otimes \mathcal{H})$ given by

$$D^H U := \sum_{j=1}^{n} D^H F_j \otimes v_j, \tag{31}$$

where $\otimes$ denotes the Hilbert-tensor product between the Hilbert spaces $\mathcal{H}$. We refer to [1] for the definition and the properties of the tensor product.
The derivative operator

$$D^H : \mathcal{S}_H(\mathcal{H}) \longrightarrow L^2(\Omega, \mathcal{H} \otimes \mathcal{H})$$

induces on $\mathcal{S}_H(\mathcal{H})$ the norm

$$\|U\|^2_{\mathbb{D}^{1,2}_H(\mathcal{H})} := \|U\|^2_{L^2(\Omega, \mathcal{H})} + \|D^H U\|^2_{L^2(\Omega, \mathcal{H} \otimes \mathcal{H})}.$$

We extend Definition 2.5 as follows.

Definition 3.3. *The space* $\mathbb{D}_H^{1,2}(\mathcal{H})$ *is defined as the closure of* $\mathcal{S}_H(\mathcal{H})$ *with respect to the norm* $\|\cdot\|_{\mathbb{D}_H^{1,2}(\mathcal{H})}$*. The derivative operator* D^H *can be extended to* $\mathbb{D}_H^{1,2}(\mathcal{H})$*.*

We note that the space $\mathbb{D}_H^{1,2}(\mathcal{H})$ is an Hilbert space with respect to the inner product

$$\langle F, G\rangle_{\mathbb{D}_H^{1,2}(\mathcal{H})} := \langle F, G\rangle_{L^2(\Omega,\mathcal{H})} + \langle D_{\mathcal{H}}^H F, D_{\mathcal{H}}^H G\rangle_{L^2(\Omega,\mathcal{H}\otimes\mathcal{H})}, \tag{32}$$

for every $F, G \in \mathbb{D}_H^{1,2}(\mathcal{H})$.
In particular we have

Proposition 3.4.

$$\mathbb{D}_H^{1,2}(\mathcal{H}) \subseteq \mathrm{dom}\delta^H \quad \textit{and} \quad \|\delta^H F\|_{L^2(\Omega)} \leq \|F\|_{\mathbb{D}_H^{1,2}(\mathcal{H})}.$$

Proof. For the proof and further details, see [18]. □

Consider now the operator $(\mathbb{K}_H^*)^{\otimes 2} : L^2(\Omega, \mathcal{H}\otimes\mathcal{H}) \to L^2(\Omega, L^2([0,T]^2))$, defined on $F \in L^2(\Omega, \mathcal{H}\otimes\mathcal{H})$ as follows

$$(\mathbb{K}_H^*)^{\otimes 2}(\omega) := (K_H^*)^{\otimes 2}(F(\omega)).$$

Proposition 3.5. *The operator* $\mathbb{K}_H^*$ *has the following properties:*

(i) Let $F \in L^2(\Omega, \mathcal{H})$*. Then*

$$F \in \mathbb{D}_H^{1,2}(\mathcal{H}) \Longleftrightarrow \mathbb{K}_H^* F \in \mathbb{L}^{1,2} \tag{33}$$

and

$$(\mathbb{K}_H^*)^{\otimes 2}(D^H F) = D(\mathbb{K}_H^* F). \tag{34}$$

(ii) $\mathbb{K}_H^* : \mathbb{D}_H^{1,2}(\mathcal{H}) \longrightarrow \mathbb{L}^{1,2}$ *is an isometry with respect to the inner product defined in* (32)*, i.e.*

$$\langle \mathbb{K}_H^* F, \mathbb{K}_H^* G\rangle_{\mathbb{L}^{1,2}} = \langle F, G\rangle_{\mathbb{D}_H^{1,2}(\mathcal{H})}$$

for every $F, G \in \mathbb{D}_H^{1,2}(\mathcal{H})$*.*

Proof. (i) Let $F \in \mathbb{D}_H^{1,2}(\mathcal{H}) \subset L^2(\Omega, \mathcal{H})$. Then exists a sequence (U_n) in $\mathcal{S}_H(\mathcal{H})$, such that

$$\begin{cases} U_n \longrightarrow F & \text{in} \quad L^2(\Omega, \mathcal{H}) \\ D^H U_n \longrightarrow D^H F & \text{in} \quad L^2(\Omega, \mathcal{H}\otimes\mathcal{H}) \end{cases} \tag{35}$$

for $n \to \infty$. We recall that by (30) the term U_n is given by

$$U_n = \sum_{j=1}^{k_n} F_j^n h_j^n \qquad n \in \mathbb{N}$$

where $F_j^n \in \mathbb{D}^{1,2}$ and $h_j \in \mathcal{H}$, $\forall j \in \{1, \ldots, n\}$.

We define $U_n^* := K_H^* U_n, n \in \mathbb{N}$. By Proposition 3.1 and (35) we have

$$\lim_{n\to\infty} \|U_n - F\|_{L^2(\Omega,\mathcal{H})} = 0.$$

That implies

$$U_n^* \longrightarrow \mathbb{K}_H^* F \quad \text{in} \quad L^2(\Omega, L^2([0,T])) \tag{36}$$

for $n \to \infty$.

In [24] is proved that $\mathbb{D}^{1,2}_H = \mathbb{D}^{1,2}$ and in particular

$$K^*_H(D^H F) = DF, \quad \forall F \in \mathbb{D}^{1,2}_H = \mathbb{D}^{1,2}. \tag{37}$$

By (37) we obtain the following relation

$$(\mathbb{K}^*_H)^{\otimes 2}(D^H U_n) = \sum_{j=1}^{k_n} DF^n_j \otimes K^*_H h^n_j = D(U^*_n).$$

Then

$$\begin{aligned} & \|D(U^*_n) - (K^*_H)^{\otimes 2}(D^H F)\|^2_{L^2(\Omega, L^2([0,T]^2))} = \\ = \; & \|(K^*_H)^{\otimes 2}(D^H U_n) - (K^*_H)^{\otimes 2}(D^H F)\|^2_{L^2(\Omega, L^2([0,T]^2))} \\ = \; & \|D^H U_n - D^H F\|^2_{L^2(\Omega, \mathcal{H}\otimes\mathcal{H})}. \end{aligned}$$

The last term goes to zero by (35) and we have

$$D(U^*_n) \longrightarrow (K^*_H)^{\otimes 2}(D^H F) \quad \text{in} \quad L^2(\Omega, L^2([0,T]^2)) \tag{38}$$

for $n \to \infty$.
By (36) and (38) we obtain that (34) holds. If we now assume $\mathbb{K}^*_H F \in \mathbb{L}^{1,2}$, we can proceed in the same way using $(K^*_H)^{-1}$ instead of K^*_H. Hence we have proved also (33).

(ii) This result follows immediately by (32), and from that the fact that K^*_H is an isometry.

□

We can prove now the following result, that is essential to characterize the space of integrands F, whose stochastic integral $\delta^H(F)$ admits the discretization provided in Section 4.

Proposition 3.6. *The following inclusion holds*

$$\mathbb{L}^{1,2} \subseteq \mathbb{D}^{1,2}_H(\mathcal{H}) \tag{39}$$

and

$$\|F\|_{\mathbb{D}^{1,2}_H(\mathcal{H})} \leq k_H \|F\|_{\mathbb{L}^{1,2}} \quad \forall F \in \mathbb{L}^{1,2}. \tag{40}$$

where $k_H = \frac{T^{2H-1}}{H-\frac{1}{2}}$ *as in* (6).

Proof. Let $F \in \mathbb{L}^{1,2}$. We recall that it is equivalent to the condition

$$\sum_{q=0}^{\infty} q(q!)\|f_{q,\cdot})\|^2_{L^2([0,T]^{q+1})} < \infty \tag{41}$$

(see [23]). By Proposition 3.2 and Proposition 3.5 we have that F belongs to $\mathbb{D}^{1,2}_H(\mathcal{H})$ iff

$$\sum_{q=0}^{\infty} q(q!)\|K^*_{H,\cdot} f_{q,\cdot})\|^2_{L^2([0,T]^{q+1})} < \infty.$$

By (6) we have

$$\begin{aligned} \|K^*_{H,\cdot} f_{q,\cdot}\|^2_{L^2([0,T]^{q+1})} &= \int_{[0,T]^q} \|f_q(t_1, ..., t_q)\|^2_{\mathcal{H}} dt_1 ... dt_q \\ &\leq k_H \|f_q\|^2_{L^2([0,T]^{q+1})} \end{aligned}$$

and then

$$\sum_{q=0}^{\infty} q(q!)\|K^*_{H,\cdot} f_{q,\cdot})\|^2_{L^2([0,T]^{q+1})} \leq k_H \sum_{q=0}^{\infty} q(q!)\|f_q\|^2_{L^2([0,T]^{q+1})}. \tag{42}$$

Since $F \in \mathbb{L}^{1,2}$, the series $\sum_{q=0}^{\infty} q(q!)\|f_q\|^2_{L^2([0,T]^{q+1})}$ is convergent in $L^2(\Omega, L^2([0,T]))$. Moreover (42) implies that

$$\sum_{q=0}^{\infty} q(q!)\|K^*_{H,\cdot} f_{q,\cdot})\|^2_{L^2([0,T]^{q+1})}$$

converges in $L^2(\Omega, L^2([0,T]))$.
We prove now (40). We note that

$$\|DF\|^2_{L^2(\Omega,L^2([0,T]^2))} = \sum_{q=0}^{\infty} q(q!)\|f_q\|^2_{L^2([0,T]^{q+1})}$$

and

$$\|D(\mathbb{K}^*_H F)\|^2_{L^2(\Omega,L^2([0,T]^2))} = \sum_{q=0}^{\infty} q(q!)\|K^*_{H,\cdot} f_{q,\cdot}\|^2_{L^2([0,T]^{q+1})}.$$

Hence by (42), it follows that

$$\|D(\mathbb{K}^*_H F)\|^2_{L^2(\Omega,L^2([0,T]^2))} \leq k_H \|DF\|^2_{L^2(\Omega,L^2([0,T]^2))}$$

Finally we get

$$\begin{aligned}
\|F\|^2_{\mathbb{D}^{1,2}_H(\mathcal{H})} &= \|F\|^2_{L^2(\Omega,\mathcal{H})} + \|D^H F\|^2_{L^2(\Omega,\mathcal{H}\otimes\mathcal{H})} \\
&= \|\mathbb{K}^*_H F\|^2_{L^2(\Omega,L^2([0,T]))} + \|(\mathbb{K}^*_H)^{\otimes 2}(D^H F)\|^2_{L^2(\Omega,L^2([0,T]^2))} \\
&= \|\mathbb{K}^*_H F\|^2_{L^2(\Omega,L^2([0,T]))} + \|D(\mathbb{K}^*_H F)\|^2_{L^2(\Omega,L^2([0,T]^2))} \\
&\leq k_H \|F\|^2_{L^2(\Omega,L^2([0,T]))} + k_H \|DF\|^2_{L^2(\Omega,L^2([0,T]^2))} \\
&= k_H \|F\|^2_{\mathbb{L}^{1,2}}.
\end{aligned}$$

□

Remark 3.7. *The space $\mathbb{L}^{1,2}$ is strictly included in $\mathbb{D}^{1,2}_H(\mathcal{H}) \cap L^2(\Omega, L^2([0,T]))$. We prove this by showing that there exists an element $F \in \mathbb{D}^{1,2}_H(\mathcal{H}) \cap L^2(\Omega, L^2([0,T]))$ that doesn't belong to $\mathbb{L}^{1,2}$. Let F have the following chaos expansion*

$$F = \sum_{q=0}^{\infty} I_q(f_{q,\cdot}) \quad in \quad L^2(\Omega, L^2([0,T]))$$

where

$$f_{q,t}(t_1,\ldots,t_q) = \frac{1}{\sqrt{T^q}} \frac{1}{\sqrt{q! q^\beta}} f(q^\alpha t) \qquad \alpha, \beta > 0. \tag{43}$$

Suppose that f satisfies the following hypotheses:

(i) $f \geq 0$*;*

(ii) $f \in L^2([0,\infty))$*;*

(iii) $\int_0^\infty \int_0^\infty f(u) f(v) \phi(u,v) du dv < \infty$*, where ϕ is defined in* (3)*.*

Since by (18) and (33) the following must hold

$$F \in \mathbb{L}^{1,2} \Longleftrightarrow \sum_{q=0}^{\infty} q(q!)\|f_{q,\cdot}\|^2_{L^2([0,T]^{q+1}} < \infty,$$

$$F \in \mathbb{D}^{1,2}_H(\mathcal{H}) \Longleftrightarrow \sum_{q=0}^{\infty} q(q!)\|K^*_{H,\cdot} f_{q,\cdot}\|^2_{L^2([0,T]^{q+1}} < \infty,$$

it is sufficient to prove that there exist $f_{q,\cdot}$ *of the form* (43) *such that*

$$\sum_{q=0}^{\infty} q(q!)\|f_{q,\cdot}\|^2_{L^2([0,T]^{q+1})} = \infty, \tag{44}$$

$$\sum_{q=0}^{\infty} q(q!)\|K^*_{H,\cdot} f_{q,\cdot}\|^2_{L^2([0,T]^{q+1})} < \infty. \tag{45}$$

By standard integral calculations we get the following inequalities

$$q(q!)\|f_{q,\cdot}\|^2_{L^2([0,T]^{q+1})} \geq \|f\|^2_{L^2([0,T])} \frac{1}{q^{\alpha+\beta-1}} \tag{46}$$

and

$$q(q!)\|f_{q,\cdot}\|^2_{L^2([0,T]^{q+1})} \leq c_f \frac{1}{q^{2\alpha H+\beta-1}}. \tag{47}$$

where $c_f = \int_0^\infty \int_0^\infty f(u)f(v)\phi(u,v)dudv$.
Finally (44) *and* (45) *hold if we choose* α *e* β *such that*

$$\begin{cases} \alpha+\beta \leq 2, \\ 2\alpha H + \beta > 2. \end{cases} \tag{48}$$

Since $H > \frac{1}{2}$, *it is then sufficient to choose* $\alpha = \beta = 1$.

4 Discrete approximation for the stochastic integral

We can now prove a discrete approximation for the stochastic integral defined as divergence introduced in Definition 2.7.
By Proposition 2.6 it follows that the integral of F with respect to fBm coincides the Skorohod integral of $\mathbb{K}^*_H F$ with respect to the Bm. Hence one can provide a discretization of the integral $\delta^H(F)$ by using a discrete approximation of $\delta(\mathbb{K}^*_H F)$. However this is not satisfactory because the discretization depends on the Hurst index of the Brownian motion with respect to which the integral is defined. Therefore we define first a discretization of the process and then a procedure for obtaining the approximation of the integral.

4.1 Resolution of $[0, T]$

We introduce first the setting and some basic notions needed to prove the main result of this paper. For further details we refer to [21]. Consider the measure space $([0,T], \mathcal{B}, l)$, where $\mathcal{B}$ denotes the Borelian σ-algebra and l the Lebesgue measure.

Definition 4.1. *A **resolution** of* $L^2([0,T])$ *is an increasing family of* σ*-algebras generated by a finite number of sets*

$$\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset \mathcal{B}_n \subset \cdots \subset \mathcal{B}.$$

such that $\bigcup_{n\in\mathbb{N}} L^2([0,T],\mathcal{B}_n,l)$ *is dense in* $L^2([0,T])$*. We denote with* Γ_n *the finite set of generators of* $\mathcal{B}_n$*, that we assume to be a partition of the interval* $[0,T]$*.*

Given a resolution $\{\mathcal{B}_n\}$, consider the set $\Gamma_N := \bigcup_{n=0}^{N} \Gamma_n$ and the set $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$. We obtain on $\mathbb{R}^\Gamma$ a coherent system of marginal laws θ_N and by Kolmogorov theorem there exists a probability measure θ on $\mathbb{R}^\Gamma$ with this family of marginal laws. Hence we can associate a probability space $(\mathbb{R}^\Gamma, \theta)$ to a resolution $\{\mathcal{B}_n\}$.

Proposition 4.2. *The probability spaces associated to different resolutions are canonically isomorphic.*

Proof. For the proof see [21]. □

4.2 Fock space

We denote with $\mathcal{R}$ the set of all resolutions of $L^2([0,T])$.

Definition 4.3. *The **Fock space** is the set of all pairs* $(\Omega(r), \Psi_{r,r'})$ *where* $r, r' \in \mathcal{R}$*,* $\Omega(r)$ *denotes the probability space associated to the resolution* r *e* $\Psi_{r,r'}$ *is the canonical isomorphism between* $\Omega(r)$ *and* $\Omega(r')$*.*

All properties of a Fock space do not depend on the choice of the resolution. For further details, see [21]. Consider a subset $\gamma \in \mathcal{B}$. The space $L^2([0,T])$ can be decomposed as follows

$$L^2([0,T]) = L^2(\gamma) \oplus L^2(\gamma^c).$$

This decomposition induces a similar one on the relative Fock spaces

$$\mathrm{Foc}(L^2([0,T])) = \mathrm{Foc}(L^2(\gamma)) \times \mathrm{Foc}(L^2(\gamma^c)).$$

Definition 4.4. *We denote by*

$$E^{\gamma^c} : L^2(\mathrm{Foc}(L^2([0,T]))) \to L^2(\mathrm{Foc}(L^2(\gamma^c))) \tag{49}$$

the natural projection from $L^2(\mathrm{Foc}(L^2([0,T])))$ *onto* $L^2(\mathrm{Foc}(L^2(\gamma^c)))$*.*

We recall the construction of the Skorohod-Zakai-Nualart-Pardoux integral (in short, SZNP-integral) given in [21] for $F \in L^2(\Omega, L^2([0,T]))$. We define

$$I_n(F) := \sum_{\gamma\in\Gamma_n} E^{\gamma^c}(E^{\mathcal{B}_n}(F))(\gamma)B(\gamma),$$

where $\{\mathcal{B}_n\}$ is a resolution of $L^2([0,T])$ as introduced in Definition 4.1, E^{γ^c} denotes the projection (49) and $B(\gamma)$ is the usual Wiener integral of the indicator function χ_γ with respect to Bm, i.e.

$$B(\gamma) := \int_0^T \chi_\gamma(s)dB_s, \qquad \gamma \in \mathcal{B}.$$

The SZNP-integral is defined as the limit

$$\mathcal{I}(F) := \lim_{n\to\infty} I_n(F) \quad \text{in} \quad L^2(\Omega).$$

By [21] we have the following theorem

Theorem 4.5. *Let $F \in \mathbb{L}^{1,2}$. Then the SZNP-integral exists and*

$$\lim_{n\to\infty} I_n(F) = \delta(F) \quad in \quad L^2(\Omega). \tag{50}$$

Proof. For the proof we refer to [21]. □

Let $F \in \mathbb{L}^{1,2}$. Since by Proposition 3.6, $\mathbb{L}^{1,2} \subseteq \mathbb{D}_H^{1,2}(\mathcal{H})$ and by Proposition 3.4, $\mathbb{D}_H^{1,2}(\mathcal{H}) \subseteq \mathrm{dom}\delta^H$, the stochastic integral $\delta^H(F)$ is well defined.

Definition 4.6. *Let $F \in \mathbb{L}^{1,2}$. We define*

- $I_n^H(F) := \sum_{\gamma\in\Gamma_n} E^{\gamma^c}(E^{\mathcal{B}_n}(\mathbb{K}_H^* F)(\gamma))B(\gamma)$,
- $J_n^H(F) := \sum_{\gamma\in\Gamma_n} E^{\mathcal{B}_n}(E^{\gamma^c}(\mathbb{K}_H^*(E^{\mathcal{B}_n}(F)))(\gamma)B(\gamma)$,

where $\{\mathcal{B}_n\}$ is a resolution of $L^2([0,T])$ as introduced in Definition 4.1 and E^{γ^c} denotes the projection (49).

By Theorem 4.5 we have that

$$\lim_{n\to\infty} I_n^H(F) = \delta(\mathbb{K}_H^* F) \quad \text{in} \quad L^2(\Omega), \tag{51}$$

and by Proposition 2.6 we get

$$\delta(\mathbb{K}_H^* F) = \delta^H(F).$$

The purpose is now to prove that $J_n^H(F)$ provides the desired discretization of $\delta^H(F)$.

Theorem 4.7. *Let $F \in \mathbb{L}^{1,2}$. Then*

$$\lim_{n\to\infty} J_n^H(F) = \delta^H(F) \qquad in \qquad L^2(\Omega).$$

Proof. Let $F \in \mathbb{L}^{1,2}$. We need to prove

$$\lim_{n\to\infty} \big(I_n^H(F) - J_n^H(F)\big) = 0 \quad \text{in} \quad L^2(\Omega).$$

By (16) the chaos expansion of F is given by

$$F_t = \sum_0^\infty I_q(f_{q,t}),$$

where $f_{q,t} \in L^2([0,T]^q + 1)$ are symmetric functions for every $q \in \mathbb{N}$.
By Proposition 3.2 we get

$$(\mathbb{K}_H^* F)_t = \sum_{q=0}^\infty I_q(K_{H,\cdot}^* f_{q,\cdot})(t).$$

We have that

$$E^{\mathcal{B}_n}(\mathbb{K}_H^* F)_t = \sum_{q=0}^\infty I_q(E^{\mathcal{B}_n}(K_{H,\cdot}^* f_{q,\cdot})(t), \tag{52}$$

$$E^{\mathcal{B}_n}(\mathbb{K}_H^* F)(\gamma) = \sum_{q=0}^\infty I_q(E^{\mathcal{B}_n}(f_{q,\cdot})(\gamma)), \tag{53}$$

$$E^{\gamma^c} E^{\mathcal{B}_n}(\mathbb{K}_H^* F)(\gamma) = \sum_0^\infty I_q\big(E^{\mathcal{B}_n}(K_{H,\cdot}^* f_{q,\cdot})(\gamma)\chi_{\gamma^c}^{\otimes q}\big). \tag{54}$$

Here we write $K^*_{H,\cdot}f_{q,\cdot}$ to indicate on which variable the operator K^*_H is operating. By [23] we obtain that the product of two Wiener integrals can be written as

$$I_q\big(E^{\mathcal{B}_n}(K^*_{H,\cdot}f_{q,\cdot}))(\gamma)\chi_{\gamma^c}^{\otimes q}\big)B(\gamma) = I_{q+1}\big(E^{\mathcal{B}_n}(K^*_{H,\cdot}f_{q,\cdot}))(\gamma)\chi_\gamma\chi_{\gamma^c}^{\otimes q}\big).$$

Hence $I^H_n(F)$ can be rewritten as

$$I^H_n(F) = \sum_{q=0}^{\infty} I_{q+1}\Big(\sum_{\gamma\in\Gamma_n} E^{\mathcal{B}_n}(K^*_{H,\cdot}f_{q,\cdot}))\chi_\gamma\chi_{\gamma^c}^{\otimes q}\Big)$$

and analogously

$$J^H_n(F) = \sum_{q=0}^{\infty} I_{q+1}\Big(\sum_{\gamma\in\Gamma_n} E^{\mathcal{B}_n}\big(K^*_{H,\cdot}(E^{\mathcal{B}_n}(f_{q,\cdot}))\big)\chi_\gamma\chi_{\gamma^c}^{\otimes q}\Big).$$

It follows that the difference between $I^H_n(F)$ and $J^H_n(F)$ is of the form

$$I^H_n(F) - J^H_n(F) = \sum_{q=0}^{\infty} I_{q+1}\Big(\sum_{\gamma\in\Gamma_n}\big[E^{\mathcal{B}_n}(K^*_{H,\cdot}f_{q,\cdot}) - E^{\mathcal{B}_n}\big(K^*_{H,\cdot}(E^{\mathcal{B}_n}(f_{q,\cdot}))\big)\big]\chi_\gamma\chi_{\gamma^c}^{\otimes q}\Big).$$

Taking the norms, we get

$$\|I^H_n(F) - J^H_n(F)\|^2_{L^2(\Omega)} = \sum_{q=0}^{\infty}(q+1)!\left\|\sum_{\gamma\in\Gamma_n}\left[E^{\mathcal{B}_n}(K^*_{H,\cdot}f_{q,\cdot})) - E^{\mathcal{B}_n}\left(K^*_{H,\cdot}(E^{\mathcal{B}_n}(f_{q,\cdot}))\right)\right]\chi_\gamma\chi_{\gamma^c}^{\otimes q}\right\|^2_{L^2([0,T]^{q+1})}.$$

Let $A_n : [0,T]^{q+1} \longrightarrow \mathbb{R}$ be the function

$$A_n := E^{\mathcal{B}_n}(K^*_{H,\cdot}f_{q,\cdot} - E^{\mathcal{B}_n}\big(K^*_{H,\cdot}(E^{\mathcal{B}_n}(f_{q,\cdot})\big).$$

Since for every $n\in\mathbb{N}$, the sets $\{\gamma\}_{\gamma\in\Gamma_n}$ are a partition of the interval $[0,T]$, we obtain

$$\begin{aligned}
\Big\|\sum_{\gamma\in\Gamma_n} A_n\chi_\gamma\chi_{\gamma^c}^{\otimes q}\Big\|^2_{L^2([0,T]^{q+1})} &= \int_{[0,T]^{q+1}}\Big(\sum_{\gamma\in\Gamma_n} A_n\chi_\gamma\chi_{\gamma^c}^{\otimes q}\Big)^2 dt_1\dots dt_q dt \\
&= \int_{[0,T]^{q+1}}\Big(\sum_{\gamma\in\Gamma_n} A_n^2\chi_\gamma\chi_{\gamma^c}^{\otimes q}\Big) dt_1\dots dt_q dt \\
&\le \int_{[0,T]^{q+1}}\Big(\sum_{\gamma\in\Gamma_n} A_n^2\chi_\gamma\Big) dt_1\dots dt_q dt \\
&= \int_{[0,T]^{q+1}} A_n(t_1,\dots t_q,t)\Big(\sum_{\gamma\in\Gamma_n}\chi_\gamma(t)\Big) dt_1\dots dt_q dt \\
&= \|A_n\|^2_{L^2([0,T]^{q+1})}.
\end{aligned}$$

Hence it follows

$$\|I^H_n(F) - J^H_n(F)\|^2_{L^2(\Omega)} \le \sum_{q=0}^{\infty}(q+1)!\|A_n\|^2_{L^2([0,T]^{q+1})}.$$

Put $A_q(n) := (q+1)!\|A_n\|^2_{L^2([0,T]^{q+1})}$. To prove that $\sum_{q=0}^{\infty} A_q(n)$ goes to zero in $L^2(\Omega)$ for $n\to\infty$, it is sufficient to verify the following conditions:

$$A_q(n) \longrightarrow 0 \qquad \text{as} \quad n\to\infty \qquad \forall q\ge 0, \tag{55}$$

$$\sum_{q=0}^{\infty} A_q(n) \quad \text{converges uniformly in} \quad n. \tag{56}$$

Condition (55) is equivalent to

$$\lim_{n\to\infty} \|A_n\|^2_{L^2([0,T]^{q+1})} = 0. \tag{57}$$

We compute the norm

$$\begin{aligned}
\|A_n\|^2_{L^2([0,T]^{q+1})} &= \int_{[0,T]^{q+1}} A_n^2(t_1,\ldots,t_q,t)dt_1\ldots dt_q dt \\
&= \int_{[0,T]^q} \Big(\int_0^T A_n^2(t_1,\ldots,t_q,t)dt\Big)dt_1\ldots dt_q.
\end{aligned}$$

To prove (57), we define the sequence

$$G_n(t_1,\ldots,t_q) := \int_0^T A_n^2(t_1,\ldots,t_q,t)dt \tag{58}$$

and show that it verifies the hypothesis of Lebesgue Theorem on bounded convergence. Indeed

$$\begin{aligned}
& G_n(t_1,\ldots,t_q) = \|A_n\|^2_{L^2([0,T])} = \\
= \; & \left\|E^{\mathcal{B}_n}(K^*_{H,\cdot}f_{q,\cdot})) - E^{\mathcal{B}_n}\left(K^*_{H,\cdot}(E^{\mathcal{B}_n}(f_{q,\cdot}))\right)\right\|^2_{L^2([0,T])} \\
= \; & \left\|E^{\mathcal{B}_n}\left[(K^*_{H,\cdot}f_{q,\cdot})) - (K^*_{H,\cdot}(E^{\mathcal{B}_n}(f_{q,\cdot})))\right]\right\|^2_{L^2([0,T])} \\
\le \; & \left\|K^*_{H,\cdot}f_{q,\cdot} - K^*_{H,\cdot}(E^{\mathcal{B}_n}(f_{q,\cdot}))\right\|^2_{L^2([0,T])} \\
= \; & \left\|K^*_{H,\cdot}\left(f_{q,\cdot} - E^{\mathcal{B}_n}(f_{q,\cdot})\right)\right\|^2_{L^2([0,T])} \\
= \; & \|f_{q,\cdot} - E^{\mathcal{B}_n}(f_{q,\cdot}))\|^2_{\mathcal{H}} \\
\le \; & k_H\|f_{q,\cdot} - E^{\mathcal{B}_n}(f_{q,\cdot}))\|^2_{L^2([0,T])}.
\end{aligned}$$

where $k_H = \frac{T^{2H-1}}{H-\frac{1}{2}}$ as in (6). The last term tends to zero when $n\to\infty$ and then

$$\lim_{n\to\infty} G_n(t_1,\ldots,t_q) = 0, \qquad \text{for a.e.} - (t_1,\ldots,t_q).$$

We need now to show that there exists a function $G \in L^1([0,T]^q)$ such that

$$|G_n| \le G, \qquad \text{for a.e.} - (t_1,\ldots,t_q).$$

The following relation holds

$$\begin{aligned}
|G_n(t_1,\ldots,t_q)| &= G_n(t_1,\ldots,t_q) \\
&\le c_H\|f_{q,\cdot} - E^{\mathcal{B}_n}(f_{q,\cdot}))\|^2_{L^2([0,T])} \\
&\le c_H\big(\|f_{q,\cdot}\|_{L^2([0,T])} + \|E^{\mathcal{B}_n}(f_{q,\cdot}))\|_{L^2([0,T])}\big)^2 \\
&\le 4c_H\|f_{q,\cdot}\|^2_{L^2([0,T])}.
\end{aligned}$$

Set $G(t_1,\ldots,t_q) := 4c_H\|f_q\|^2_{L^2([0,T])}$. We have

$$\begin{aligned}\int_{[0,T]^q} G(t_1,\ldots,t_q)dt_1\ldots dt_q &= 4c_H\int_{[0,T]^q}\|f_q\|^2_{L^2([0,T])}dt_1\ldots dt_q\\ &= 4c_H\int_{[0,T]^q}(\int_0^T f_q^2(t_1,\ldots,t_q,t)dt)dt_1\ldots dt_q\\ &= 4c_H\|f_q\|^2_{L^2([0,T]^{q+1})}<\infty, \qquad (59)\end{aligned}$$

since $F\in\mathbb{L}^{1,2}$. Hence the norm $\|G\|_{L^1([0,T]^q)}$ is finite and by the Lebesgue theorem applied to $G_n(t_1,\ldots,t_q)$ we can conclude that $\|A_n\|^2_{L^2([0,T]^{q+1})}$ converges to zero when $n\to\infty$. This ends the proof of (57) and hence of (55).
To prove (56) we use the following criterion of uniform convergence if

$$\sum_{q=0}^{\infty}\sup_n|A_q(n)|<\infty,$$

then $\sum_{q=0}^{\infty}A_q(n)$ converges uniformly in n. We have by (58) and (59) that

$$\begin{aligned}|A_q(n)| &= (q+1)!\|A_n\|^2_{L^2([0,T]^{q+1})}\\ &= (q+1)!G_n(t_1,\ldots,t_q)\\ &\le (q+1)!4c_H\|f_q\|^2_{L^2([0,T]^{q+1})}.\end{aligned}$$

Hence

$$\sup_n|A_q(n)|\le 4c_H(q+1)!\|f_q\|^2_{L^2([0,T]^{q+1})}.$$

Since $F\in\mathbb{L}^{1,2}$, the series $\sum_{q=0}\|f_q\|^2_{L^2([0,T]^{q+1})}$ is convergent and then also $\sum_{q=0}^{\infty}\sup_n|A_q(n)|<\infty$. This ends the proof. $\square$

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