On Fairness of Systemic Risk Measures

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Abstract

In our previous paper [7], we have introduced a general class of systemic risk measures that allow random allocations to individual banks before aggregation of their risks. In the present paper, we address the question of fairness of these allocations and we propose a fair allocation of the total risk to individual banks. We show that the dual problem of the minimization problem which identify the systemic risk measure, provides a valuation of the random allocations which is fair both from the point of view of the society/regulator and from the individual financial institutions. The case with exponential utilities which allows for explicit computation is treated in details.

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1 Introduction

A vector $\mathbf{X} = (X^1, \ldots, X^N) \in \mathcal{L}^0(\mathbb{R}^N)$ of N random variables denotes a configuration of risky factors at a future time T associated to a system of N entities/banks.

In the framework of Risk Measures, one of the first proposals, see [16], to measure the systemic risk of $\mathbf{X}$ was to consider the map

$$\rho(\mathbf{X}) := \inf\{m \in \mathbb{R} \mid \Lambda(\mathbf{X}) + m \in \mathbb{A}\}\,. \tag{1.1}$$

where

$$\Lambda : \mathbb{R}^N \to \mathbb{R}$$

is an aggregation rule that aggregates the N-dimensional risk factors into a univariate risk factor, and

$$\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$$

is one-dimensional acceptance set. Systemic risk can again be interpreted as the minimal cash amount that secures the system when it is added to the total aggregated system loss $\Lambda(\mathbf{X})$. The interpretation of (1.1) is that the systemic risk is the minimal capital needed to secure the system *after aggregating individual risks.*

It might be more relevant to measure systemic risk as the minimal capital that secures the aggregated system by injecting the capital into the single institutions *before aggregating the individual risks.* This way of measuring systemic risk can be expressed by

$$\rho(\mathbf{X}) := \inf\left\{\sum_{i=1}^{N} m_i \mid \mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{R}^N,\ \Lambda(\mathbf{X} + \mathbf{m}) \in \mathbb{A}\right\}. \tag{1.2}$$

Here, the amount m_i is added to the financial position X^i of institution $i \in \{1, \cdots, N\}$ before the corresponding total loss $\Lambda(\mathbf{X} + \mathbf{m})$ is computed (we refer to [3], [7] and [24]).

The main novelty of our paper [7] was the possibility of adding to $\mathbf{X}$ not merely a vector $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{R}^N$ of cash, but, more generally, a random vector $\mathbf{Y} \in \mathcal{C}$ in a class $\mathcal{C}$ such that

$$\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}} \cap \mathcal{L}, \text{ where } \mathcal{C}_{\mathbb{R}} := \left\{\mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \sum_{n=1}^{N} Y^n \in \mathbb{R}\right\}$$

and the subspace $\mathcal{L} \subseteq \mathcal{L}^0(\mathbb{R}^N)$ will be specified later. Here the notation $\sum_{n=1}^N Y^n \in \mathbb{R}$ means that $\sum_{n=1}^N Y^n$ is equal to some deterministic constant in $\mathbb{R}$, even though each single Y^n, $n = 1, \cdots, N$, is a random variable.

Then, the general systemic risk measure considered in [7] can be written as

$$\rho(\mathbf{X}) := \inf\left\{\sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C},\ \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}\right\}, \tag{1.3}$$

and can still be interpreted as the minimal total cash amount $\sum_{n=1}^{N} Y^n \in \mathbb{R}$ needed today to secure the system by distributing the cash at the future time T among the components of the risk vector $\mathbf{X}$. However, contrary to (1.2), in general the allocation $Y^i(\omega)$ to institution i does not need to be decided today but depends on the scenario ω that has been realized at time T. This corresponds to the situation of a lender of last resort who is equipped with a certain amount of cash today and who will allocate it according to where it serves the most depending on the scenario that has been realized at T. Of course, in general, the use of scenario dependent allocation $\mathbf{Y}$ as in (1.3) reduces, in comparison to the deterministic case in (1.2), the minimal amount of capital $\rho(\mathbf{X})$ needed to secure the system. Restrictions on the possible distributions of cash are given by the class $\mathcal{C}$, as shown in the Example 3.1.

Definition 1.1. (i) We say that the scenario dependent allocation $\mathbf{Y}_{\mathbf{X}} \in \mathcal{C}$ is an *optimal allocation* to $\rho(\mathbf{X})$, defined in (1.3), if it satisfies $\Lambda(\mathbf{X} + \mathbf{Y}_{\mathbf{X}}) \in \mathbb{A}$ and $\rho(\mathbf{X}) = \sum_{n=1}^{N} Y^n_{\mathbf{X}}$.
(ii) We say that a vector $(\rho^n(\mathbf{X}))_n \in \mathbb{R}^N$ is a *risk allocation* for $\rho(\mathbf{X})$ if $\sum_{n=1}^{N} \rho^n(\mathbf{X}) = \rho(\mathbf{X})$.

Even though, as mathematicians, we like well defined and sharp definitions, the analysis of a system of financial institutions suggests that the notion of *fairness* is a multi-faceted notion.

The aim of this paper is to analyze in detail the systemic risk measure in (1.3). In addition to several technical aspects regarding such systemic risk measures, we will answer the following main questions about fairness of risk allocations:

1. When is the *systemic valuation* $\rho(\mathbf{X})$ (*and its random allocation* $\sum_{n=1}^{N} Y^n_{\mathbf{X}}$) fair from the point of view of the *whole system*?

2. When is a *risk allocation* $(\rho^n(\mathbf{X}))_n \in \mathbb{R}^N$ of $\rho(\mathbf{X})$ fair from the point of view of the *whole system*?

3. When are the *systemic allocation* $\mathbf{Y} = (Y^n_{\mathbf{X}})_n \in \mathcal{C}_{\mathbb{R}}$ and the *risk allocation* $(\rho^n(\mathbf{X}))_n \in \mathbb{R}^N$ associated to $\rho(\mathbf{X})$, fair from the point of view of *each individual bank*?

We provide answers to these questions in the following introductory section without entering in the mathematical details of our analysis which will be provided in the subsequent sections. The optimal solution to the dual problem of the primal problem (1.3) will play a crucial role. It is a vector of probability measures $\mathbf{Q}_{\mathbf{X}} = (Q^1_{\mathbf{X}}, \cdots Q^N_{\mathbf{X}})$ which will provide the fair valuation of the optimal random allocations through the formula $\rho(\mathbf{X}) = \sum_{n=1}^{N} \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]$.

In the rest of the paper, we shall assume that the aggregation function Λ is of the form $\Lambda(\mathbf{x}) = \sum_{n=1}^{N} u_n(x_n)$ for utility functions u_n, $n = 1, \cdots, N$. The case with *exponential utilities* and *grouping of banks* will be treated in details in Section 6, where meaningful sensitivity properties will be established as well.

2 Fairness of systemic risk measures and allocations

The main objective of this paper is to discuss various aspects of fairness of the systemic risk measures $\rho(\mathbf{X})$, random allocations $\mathbf{Y} \in \mathcal{C}$, and risk allocations of the total systemic risk among individual banks. In this introductory section, we explain and motivate the various fairness properties, both from the point of view of the society/regulator and from the individual financial institutions. Precise definitions and statements, as well as detailed proofs, will be given in the course of the paper. For the remaining of this section, we assume that the infimum of the systemic risk measure

$$\rho(\mathbf{X}) := \inf_{\mathbf{Y} \in \mathcal{C} \subset \mathcal{C}_{\mathbb{R}}} \left\{ \sum_{n=1}^{N} Y^n \mid \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] \geq B \right\}, \tag{2.1}$$

is attained for an optimal (random) allocation $\mathbf{Y}_{\mathbf{X}} = (Y^1_{\mathbf{X}}, \cdots, Y^N_{\mathbf{X}}) \in \mathcal{C}$, which will turn out to be unique. Existence of such minimizer is discussed in Section 4.3. Note that (2.1) is a particular case of (1.3), where the function Λ is the sum of the utility functions u_n and $\mathbb{A}$ is a particular acceptance set. We first introduce the related optimization problem

$$\pi(\mathbf{X}) := \sup_{\mathbf{Y} \in \mathcal{C} \subset \mathcal{C}_{\mathbb{R}}} \left\{ \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] \mid \sum_{n=1}^{N} Y^n \leq A \right\}, \tag{2.2}$$

so that, if we interpret $\sum_{n=1}^{N} u_n(X^n + Y^n)$ as the aggregated utility of the system after allocating $\mathbf{Y}$, then $\pi(\mathbf{X})$ can be interpreted as the maximal expected utility of the system over all random allocations $\mathbf{Y} \in \mathcal{C}$ such that the aggregated budget constraint $\sum_{n=1}^{N} Y^n \leq A$ holds. In the following, we may write $\rho(\mathbf{X}) = \rho_B(\mathbf{X})$ and $\pi(\mathbf{X}) = \pi_A(\mathbf{X})$ in order to express the dependence on the minimal level of expected utility $B \in \mathbb{R}$ and on the maximal budget level $A \in \mathbb{R}$, respectively. We will see in Section 5 that

$$B = \pi_A(\mathbf{X}) \text{ if and only if } A = \rho_B(\mathbf{X}), \tag{2.3}$$

and, in these cases, the two problems $\pi_A(\mathbf{X})$ and $\rho_B(\mathbf{X})$ have the same unique optimal solution $\mathbf{Y}_{\mathbf{X}}$. From this, we infer that once a level $\rho(\mathbf{X})$ of total systemic risk has been determined, the optimal allocation $\mathbf{Y}_{\mathbf{X}}$ of ρ maximizes the expected system utility among all random allocations of cost less or equal to $\rho(\mathbf{X})$.

Once the total systemic risk has been identified as $\rho(\mathbf{X})$, the second essential question is how to allocate the total risk to the individual institutions. Recall that a vector $(\rho^1(\mathbf{X}), \cdots, \rho^N(\mathbf{X})) \in \mathbb{R}^N$ is called a systemic risk allocation (SRA) of $\rho(\mathbf{X})$ if $\sum_{n=1}^{N} \rho^n(\mathbf{X}) = \rho(\mathbf{X})$. For deterministic allocation, this property has been introduced in [13] as the "*Full Allocation*" property.

In the case of deterministic allocations $\mathbf{Y} \in \mathbb{R}^N$, i.e. $\mathcal{C} = \mathbb{R}^N$, the optimal deterministic

$\mathbf{Y_X}$ represents a canonical risk allocation $\rho^n(\mathbf{X}) := Y^n_{\mathbf{X}}$. For general (random) allocations $\mathbf{Y} \in \mathcal{C} \subset \mathcal{C}_{\mathbb{R}}$, we then follow the natural approach to consider risk allocations of the form

$$\rho^n(\mathbf{X}) := \mathbb{E}_{Q^n}[Y^n_{\mathbf{X}}] \quad \text{for } n = 1, \cdots, N, \tag{2.4}$$

where $\mathbf{Q} = (Q^1, \cdots, Q^N)$ is a given vector of probability measures such that $\sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n_{\mathbf{X}}] = \rho(\mathbf{X})$. In that way, $\rho^n(\mathbf{X}) = \mathbb{E}_{Q^n}[Y^n_{\mathbf{X}}]$ can be understood as a *systemic risk valuation* of $Y^n_{\mathbf{X}}$. Note that in our setting, besides providing a ranking in terms of systemic riskiness, a risk allocation $\rho^n(\mathbf{X})$ can be interpreted as a capital requirement for institution n in order to fund the total amount $\rho(\mathbf{X})$ of cash needed. In this sense, the vector $\mathbf{Q}$ allows for the monetary interpretation of a systemic pricing operator to determine the price (or cost) of (future) random allocations of the individual institutions. Obviously, it is of high interest to identify fairness criteria, acceptable both by the society and by the individual financial institutions, for such systemic valuation measures and their corresponding risk allocations.

Now, consider the situation where a valuation (or cost) operator $\mathbf{Q} = (Q^1, \cdots, Q^N)$ is given for the system. Then, a natural alternative formulation of the systemic risk measure and the related utility maximization problem in terms of the valuation provided by $\mathbf{Q}$ is

$$\rho^{\mathbf{Q}}(\mathbf{X}) = \rho^{\mathbf{Q}}_B(\mathbf{X}) \quad := \quad \inf_{\mathbf{Y} \in M^\Phi} \left\{ \sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] \mid \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \geq B \right\}, \tag{2.5}$$

$$\pi^{\mathbf{Q}}(\mathbf{X}) = \pi^{\mathbf{Q}}_A(\mathbf{X}) \quad := \quad \sup_{\mathbf{Y} \in M^\Phi} \left\{ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \mid \sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] \leq A \right\}. \tag{2.6}$$

Notice that in (2.5) and (2.6) the allocation $\mathbf{Y}$ is not required to belong to $\mathcal{C}_{\mathbb{R}}$ (that is adding up to a deterministic quantity) but to a vector space $\mathcal{L} = M^\Phi$ of random variables introduced later. Thus, for the systemic risk measure $\rho^{\mathbf{Q}}(\mathbf{X})$, we look for the minimal (systemic) cost $\sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n]$ among all $\mathbf{Y} \in M^\Phi$ satisfying the acceptability (utility) constraint $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \geq B$. Analogously, for $\pi^{\mathbf{Q}}(\mathbf{X})$ we maximize the expected systemic utility among all $\mathbf{Y} \in M^\Phi$ satisfying the budget constraint $\sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] \leq A$. Similarly as in (2.3), we will see in Section 5 that

$$B = \pi^{\mathbf{Q}}_A(\mathbf{X}) \text{ if and only if } A = \rho^{\mathbf{Q}}_B(\mathbf{X}), \tag{2.7}$$

and the two problems $\pi^{\mathbf{Q}}_A(\mathbf{X})$ and $\rho^{\mathbf{Q}}_B(\mathbf{X})$ have the same unique optimal solution.

The specific choice of a systemic valuation is the central question of this paper. It will turn out that the optimizer $\mathbf{Q_X} = (Q^1_{\mathbf{X}}, \cdots Q^N_{\mathbf{X}})$ of the dual problem of (2.1), presented in detail in Section 4.1 and in Corollary 4.16, provides a risk allocation $(\mathbb{E}_{Q^1_{\mathbf{X}}}[Y^1_{\mathbf{X}}], \cdots, (\mathbb{E}_{Q^N_{\mathbf{X}}}[Y^N_{\mathbf{X}}])$,

$$\sum_{n=1}^N \mathbb{E}_{Q_{\mathbf{X}^n}}[Y^n_{\mathbf{X}}] = \rho(\mathbf{X}),$$

satisfying

$$\rho_B(\mathbf{X}) = \rho_B^{Q_\mathbf{X}}(\mathbf{X}), \tag{2.8}$$

$$\pi_A(\mathbf{X}) = \pi_A^{Q_\mathbf{X}}(\mathbf{X}). \tag{2.9}$$

Furthermore, $\mathbf{Y_X}$ is the unique optimal solution for $\rho_B(\mathbf{X})$ and $\rho_B^{Q_\mathbf{X}}(\mathbf{X})$ in (2.8). Similarly, $\pi_A(\mathbf{X})$ and $\pi_A^{Q_\mathbf{X}}(\mathbf{X})$ in (2.9) have the same optimal solution $\mathbf{Y_X}$ (see Section 5).

We now discuss fairness properties of the systemic risk measure $\rho(\mathbf{X})$, the optimal allocation $\mathbf{Y_X}$, and the systemic probability measure $\mathbf{Q_X}$ with corresponding risk allocations $\mathbb{E}_{Q^1_\mathbf{X}}[Y^1]$,.. ..,$\mathbb{E}_{Q^N_\mathbf{X}}[Y^N]$, both from the perspective of the society/regulator and from the individual institutions.

Fairness from the perspective of the society/regulator. Consider the systemic risk measure $\rho(\mathbf{X})$ with $\mathcal{C} = \mathbb{R}^N$. In this case not only the total amount of cash $\rho(\mathbf{X})$ but also the individual cash amounts $\mathbf{Y} \in \mathbb{R}^N$ allocated to the institutions are already known today (i.e., they are deterministic). Such a risk measure only depends on the marginal distributions of $\mathbf{X}$ as can be seen from the constraint $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \geq B$ in (2.1) with Y^n deterministic. However, ignoring potential dependencies among the banks might be over-conservative and too costly. By considering scenario-dependent allocations $\mathbf{Y} \in \mathcal{C} \supsetneq \mathbb{R}^N$ (and by that considering the dependencies among the banks as was shown through examples in [7]), the consequential reduction of the overall cost of securing the system is beneficial to the society. Additionally, the requirement $\mathbf{Y} \in \mathcal{C}$ and $\mathcal{C} \subseteq \mathcal{C}_\mathbb{R}$ is important from the society's perspective as it guarantees that the cash amount $\rho(\mathbf{X})$ determined today is sufficient to cover the allocations $\mathbf{Y}$ at time T in any possible scenario. There might be cross-subsidization (in the sense of a risk exchange) among the banks at time T, but $\sum_{n=1}^N Y^n = \rho(\mathbf{X})$ means that the system clears and no additional external injections (or withdrawals) are necessary at time T. In that sense, the requirement $\mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_\mathbb{R}$ is fair from the society/regulator's perspective. Furthermore and most importantly, by (2.2) and (2.7), the optimal allocation $\mathbf{Y_X}$ maximizes the expected systemic utility among all allocations with total cost less or equal to $A = \rho(\mathbf{X})$.

Next, consider the systemic risk valuation using $\mathbf{Q_X}$. To explain one of the features of $\mathbf{Q_X}$, observe first that ρ in (1.3) keeps the classical cash additivity property

$$\rho(\mathbf{X} + \mathbf{m}) = \rho(\mathbf{X}) - \sum_{n=1}^N m^n \text{ for all } \mathbf{m} \in \mathbb{R}^N \text{ and all } \mathbf{X}, \tag{2.10}$$

which is a *global* property. The local version associated to (2.10) is

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}{+}\varepsilon\mathbf{m})|_{\varepsilon=0} = -\sum_{n=1}^N m^n \quad \text{ for } \mathbf{m} \in \mathbb{R}^N. \tag{2.11}$$

The expression $\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{m})|_{\varepsilon=0}$ represents the sensitivity of the risk of $\mathbf{X}$ with respect to the impact $\mathbf{m} \in\mathbb{R}^N$ and was named the *marginal risk contribution* by [3]. However, such property can not be immediately generalized to the case where $\mathbf{m} \in \mathbb{R}^N$ is replaced by random vectors $\mathbf{V}$, in particular when $\sum_{n=1}^N V^n$ is not a constant.

If the positions change from $\mathbf{X}$ to $\mathbf{X} + \varepsilon V^j \mathbf{e}_j$, where $\mathbf{e}_j$ is the jth unit vector and V^j is a random variable, then, we show in Section 4.4 that the riskiness of the entire system changes linearly by

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon V^j\mathbf{e}_j)|_{\varepsilon=0} = \mathbb{E}_{Q^j_{\mathbf{X}}}[-V^j], \tag{2.12}$$

which shows that $\mathbf{Q_X}$ can be naturally introduced as a systemic risk valuation operator.

Now, given a systemic risk valuation $\mathbf{Q}$, one is naturally led to the specification (2.5) for a systemic risk measure. Note, however, that in (2.5) the clearing condition $\sum_{n=1}^N Y^n = \rho(\mathbf{X})$ is not guaranteed since the optimization is performed over all $\mathbf{Y} \in M^\Phi$. Using the valuation with $\mathbf{Q_X}$ is then fair from the society/regulator's point of view since, by Proposition 4.15, the optimal allocation in (2.5) fulfills the clearing condition $\mathbf{Y} \in \mathcal{C}_\mathbb{R}$, and is in fact the same as the optimal allocation of the original systemic risk measure in (2.1). The equation

$$\sum_{n=1}^N Y^n_{\mathbf{X}} = \rho(\mathbf{X}) = \rho^{\mathbf{Q_X}}(\mathbf{X}) = \sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]$$

also shows that the selection of $\mathbb{E}_{Q_{\mathbf{X}}}[\cdot]$ as the valuation functional is as fair as computing $\rho(\mathbf{X})$ as the infimum of $\sum_{n=1}^N Y^n$, for admissible $\mathbf{Y}$, and supports the definition of $\mathbf{Q_X}$ as the systemic probability measure.

Fairness from the perspective of the individual institutions. The essential question for a financial institution is whether its allocated share of the total systemic risk determined by the risk allocation $(\mathbb{E}_{Q^1_{\mathbf{X}}}[Y^1_{\mathbf{X}}], \cdots, (\mathbb{E}_{Q^N_{\mathbf{X}}}[Y^N_{\mathbf{X}}])$ is fair.

For the banks, the clearing condition $\mathbf{Y} \in \mathcal{C}_\mathbb{R}$ is not relevant. Instead, given a vector $\mathbf{Q} = (Q^1, \cdots, Q^N)$ of valuation measures, the systemic risk measure $\rho^{\mathbf{Q}}_B(\mathbf{X})$ in (2.5) is more relevant. Thus, by choosing $\mathbf{Q} = \mathbf{Q_X}$, the requirements from both the society and the banks are reconciled as seen from (2.8). Furthermore, with the choice $\mathbf{Q} = \mathbf{Q_X}$, we have by (2.9)

$$\pi_A(\mathbf{X}) = \pi^{\mathbf{Q_X}}_A(\mathbf{X}) = \sup_{\sum_{n=1}^N a^n = A,} \sum_{n=1}^N \sup_{\mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n]=a_n} \mathbb{E}\left[u_n(X^n + Y^n)\right], \tag{2.13}$$

see Remark 5.1 for details. Choosing $A = \rho_B(\mathbf{X})$, we obtain by (2.7) and the fact that, then, $\mathbf{Y_X}$ is the optimal solution of $\pi^{\mathbf{Q_X}}_A(\mathbf{X})$, that $\mathbb{E}_{Q_{\mathbf{X}^n}}[Y^n_{\mathbf{X}}] = a_n$, $\sum_{n=1}^N \mathbb{E}_{Q_{\mathbf{X}^n}}[Y^n_{\mathbf{X}}] = A$ and (2.13) can be rewritten as

$$\pi_A(\mathbf{X}) = \pi^{\mathbf{Q_X}}_A(\mathbf{X}) = \sum_{n=1}^N \sup_{\mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n]=\mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]} \mathbb{E}\left[u_n(X^n + Y^n)\right].$$

This means that by using $\mathbf{Q_X}$ for valuation, the system utility maximization in (2.6) reduces to individual utility maximization problems for the banks without the "systemic" constraint $\mathbf{Y} \in \mathcal{C}$:

$$\forall n, \quad \sup_{Y^n} \left\{ \mathbb{E}\left[u_n(X^n + Y^n)\right] \mid \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n] = \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}] \right\}.$$

The optimal allocation $Y^n_{\mathbf{X}}$ and its value $\mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]$ can thus be considered fair by the n^{th} bank, as $Y^n_{\mathbf{X}}$ maximizes the individual expected utility of bank n among all random allocations (not constrained to be in $\mathcal{C}_{\mathbb{R}}$) with value $\mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]$. This finally argues for the fairness of the risk allocation $(\mathbb{E}_{Q^1_{\mathbf{X}}}[Y^1_{\mathbf{X}}], \cdots, \mathbb{E}_{Q^N_{\mathbf{X}}}[Y^N_{\mathbf{X}}])$ as fair valuation of the optimal allocation $(Y^1_{\mathbf{X}}, \cdots, Y^N_{\mathbf{X}})$.

Another desirable fairness property is *monotonocity*. It is clear that if $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_{\mathbb{R}}$, then $\rho_1(\mathbf{X}) \geq \rho_2(\mathbf{X})$ for the corresponding systemic risk measures

$$\rho_i(\mathbf{X}) := \inf \left\{ \sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C}_i, \, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A} \right\}, \quad i = 1, 2.$$

The two extreme cases occur for $\mathcal{C}_1 := \mathbb{R}^N$ (the deterministic case) and $\mathcal{C}_2 := \mathcal{C}_{\mathbb{R}}$ (the unconstraint scenario dependent case). Hence we know that when going from deterministic to scenario-dependent allocations the total systemic risk decreases. It is then desirable that each institution profits from this decrease in total systemic risk in the sense that also its individual risk allocation decreases:

$$\rho_1^n(\mathbf{X}) \geq \rho_2^n(\mathbf{X}) \text{ for each } n = 1, ..., N. \tag{2.14}$$

The opposite would clearly be perceived as unfair. This is discussed in the exponential setting of Section 6.2, where we show that (2.14) holds when $\rho_1^n(\mathbf{X}) := Y_1^n$ and $\rho_2^n(\mathbf{X}) := \mathbb{E}_{Q_2^n}[Y_2^n]$ (where Y_j^n is the optimal solution to the systemic risk measure $\rho_j(\mathbf{X})$ associated to $\mathcal{C}_j$, so that $Y_1^n \in \mathbb{R}$ is deterministic, and $\mathbf{Q}_2$ is the systemic probability measure associated to $\rho_2(\mathbf{X})$). By using a probability measure $\mathbf{R}$ different from $\mathbf{Q}_2$ to compute the risk allocation $\rho_2^n(\mathbf{X}) = \mathbb{E}_{R^n}[Y_{\mathbf{2}}^n]$, the property (2.14) is in general lost.

Additional fairness properties related to the systemic probability measure $\mathbf{Q_X}$ are addressed in Section 6.1, Proposition 6.5.

We conclude this Section with a literature overview on systemic risk. In [19], [12] and [18] one can find empirical studies on banking networks, while interbank lending has been studied via interacting diffusions and mean field approach in several papers like [28], [26], [15], [35], [6]. Among the many contributions on systemic risk modeling, we mention the classical contagion model proposed by [23], the default model of [31], the illiquidity cascade models of [30], [34] and [37], the asset fire sale cascade model by [17] and [14], as well as the model in [4] that additionally includes cross-holdings. Further works on network modeling are [1], [40], [2], [32], [5], [21] and [22]. See also the references therein. For an exhaustive overview on the literature on systemic risk we refer the reader to the recent volumes of [33] and of [27].

3 The setting

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider the space of random vectors

$$L^0(\mathbb{R}^N) := \{\mathbf{X} = (X^1, \dots, X^N) \mid X^n \in L^0(\Omega, \mathcal{F}, \mathbb{P}),\ n = 1, \cdots, N\}.$$

A vector $\mathbf{X} = (X^1, \dots, X^N) \in L^0(\mathbb{R}^N)$ denotes a configuration of risky factors at a future time T associated to a system of N entities. We assume that $L^0(\mathbb{R}^N)$ is a vector lattice equipped with the order relation

$$\mathbf{X}_1 \succeq \mathbf{X}_2 \quad if \quad X_1^i \geq X_2^i \ \mathbb{P} - a.s.\ \forall i = 1, \cdots, N. \tag{3.1}$$

Let $\mathcal{C}_{\mathbb{R}}$ be the linear space

$$\mathcal{C}_{\mathbb{R}} := \{\mathbf{Y} \in L^0(\mathbb{R}^N) \mid \sum_{n=1}^{N} Y^n \in \mathbb{R}\}. \tag{3.2}$$

Here we use the notation $\sum_{n=1}^N Y^n \in \mathbb{R}$ to denote that $\sum_{n=1}^N Y^n$ is equal to some deterministic constant in $\mathbb{R}$, even though each single Y^n, $n = 1, \cdots, N$, is a random variable.

By following [7], we consider systemic risk measures

$$\rho : \mathcal{L} \to \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$$

of the form

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C},\ \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A} \right\}, \tag{3.3}$$

where the map

$$\Lambda : \mathbb{R}^N \to \mathbb{R}$$

is an *aggregation rule* that aggregates the N-dimensional risk factor into a univariate risk factor, $\mathbb{A} \subseteq L^0(\mathbb{R})$ is the one dimensional *acceptance set* and the set $\mathcal{C}$ of *admissible* random elements satisfies $\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}} \cap \mathcal{L}$, where

$$\mathcal{L} \subseteq L^1(\mathbb{P}; \mathbb{R}^N)$$

is a vector subspace which will be specified in the sequel.

Example 3.1. We now introduce one relevant example for the set of admissible random elements, which we denote $\mathcal{C}^{(\mathbf{n})}$.

Definition 3.2. Set $n_0 = 0$. For $h \in \{1, \cdots, N\}$, let $\mathbf{n} := (n_1, \cdots, n_h) \in \mathbb{N}^h$, with $n_{m-1} < n_m$ for all $m = 1, \cdots, h$ and $n_h := N$, represent some partition of $\{1, \cdots, N\}$. We set $I_m := \{n_{m-1} + 1, \cdots, n_m\}$ for each $m = 1, \cdots, h$. The cardinality of each group is denoted with $N_m := n_m - n_{m-1}$. We introduce the following family of allocations $\mathcal{C}^{(\mathbf{n})} = \mathcal{C}_0^{(\mathbf{n})} \cap \mathcal{L}$, where

$$\mathcal{C}_0^{(\mathbf{n})} = \left\{ \mathbf{Y} \in L^0(\mathbb{R}^N) \mid \exists\, d = (d_1, \cdots, d_h) \in \mathbb{R}^h \ : \ \sum_{i \in I_m} Y^i = d_m \text{ for } m = 1, \cdots, h \right\} \subseteq \mathcal{C}_{\mathbb{R}}. \quad (3.4)$$

For a given $\mathbf{n} := (n_1, \cdots, n_h)$, the values $(d_1, \cdots, d_h)$ may change, but the number of elements in each of the h groups I_m is fixed by the partition $\mathbf{n}$. It is then easily seen that $\mathcal{C}^{(\mathbf{n})}$ is a linear space containing $\mathbb{R}^N$. Beside the obvious interpretation of the restrictions imposed to the elements $\mathbf{Y} \in \mathcal{C}^{(\mathbf{n})}$, we point out that the family $\mathcal{C}^{(\mathbf{n})}$ admits two extreme cases:

(i) the strongest restriction occurs when $h = N$, i.e. we consider exactly N groups, and in this case $\mathcal{C}^{(\mathbf{n})} = \mathbb{R}^N$ corresponds to the deterministic case;

(ii) on the opposite side, we have only one group $h = 1$ and $\mathcal{C}^{(\mathbf{n})} = \mathcal{C}_{\mathbb{R}} \cap \mathcal{L}$ is the largest possible class, corresponding to completely arbitrary random injection $\mathbf{Y} \in \mathcal{L}$ with the only requirement $\sum_{n=1}^N Y^n \in \mathbb{R}$.

3.1 Assumptions and properties of ρ

We now specify further properties of systemic risk measures of the form (3.3) under some additional, but still general hypotheses. In the sequel we will always work under the following

Assumption 3.3.

1. $\mathcal{C}_0 \subseteq \mathcal{C}_{\mathbb{R}}$ and $\mathcal{C} = \mathcal{C}_0 \cap \mathcal{L}$ is a convex cone satisfying $\mathbb{R}^N \subseteq \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$;
2. $\Lambda = \sum_{n=1}^N u_n$ where $u_n : \mathbb{R} \to \mathbb{R}$ is increasing, concave, $\lim_{x \to -\infty} \frac{u_n(x)}{x} = +\infty$.
3. $\mathbb{A} := \left\{ Z \in L^0(\mathbb{R}) \mid \mathbb{E}[Z] \geq B \right\}$.
4. There exists $\mathbf{M} \in \mathbb{R}^N$ such that $\Lambda(\mathbf{M}) \geq B$, and we write $\Lambda(+\infty) > B$.

As $\mathcal{C}$ is a convex cone containing $\mathbb{R}^N$, $\mathbf{Y} + \delta \in \mathcal{C}$ for every $\mathbf{Y} \in \mathcal{C}$ and any deterministic $\delta \in \mathbb{R}^N$.

Under Assumption 3.3, a systemic risk measure of the form (3.3) can be written as

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{n=1}^N Y^n \mid \mathbf{Y} \in \mathcal{C}, \, \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n) \right] \geq B \right\}. \quad (3.5)$$

Note that there is no loss in generality in assuming that $u_n(0) = 0$ (simply replace B with $B - \sum_{n=1}^{N} u_n(0)$) and that a natural selection for B is $B := \sum_{n=1}^{N} u_n(0)$. In this case $\rho(\mathbf{0}) \leq 0$. The proof of the following proposition, which exploits the behavior of u_n at $-\infty$, is postponed to the Appendix A.1.

Proposition 3.4. *For all* $\mathbf{X} \in \mathcal{L}$ *we have* $\rho(\mathbf{X}) > -\infty$.

The *domain* of ρ is defined by

$$dom(\rho) := \{\mathbf{X} \in \mathcal{L} \mid \rho(\mathbf{X}) < +\infty\}.$$

Proposition 3.5. *The map* $\rho : \mathcal{L} \to \mathbb{R} \cup \{+\infty\}$ *in* (3.5) *is convex, monotone decreasing, satisfies* $(L^{\infty}(\mathbb{R}^N) \cap \mathcal{L}) \subset dom(\rho)$ *and*

$$\rho(\mathbf{X}) = \widetilde{\rho}(\mathbf{X}) := \inf \left\{ \sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C},\, \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n) \right] = B \right\}, \quad \mathbf{X} \in dom(\rho).$$

If for each n, $u_n : \mathbb{R} \to \mathbb{R}$ *is strictly concave and there exists an optimal solution* $\mathbf{Y}_{\mathbf{X}} = \{Y^n_{\mathbf{X}}\}_n$, *i.e.,* $\rho(\mathbf{X}) = \sum_{n=1}^{N} Y^n_X$, *then it is unique.*

Proof. By Proposition 3.4, we know that $\rho : \mathcal{L} \to \mathbb{R} \cup \{+\infty\}$ and then convexity and monotonicity are straightforward. By assumption there exists $\mathbf{M} \in \mathbb{R}^N$ such that $\Lambda(\mathbf{M}) \geq B$. Let $\mathbf{X} \in L^{\infty}(\mathbb{R}^N)$ and set $Y^n := \| X^n \|_{\infty} + M^n$. Then

$$\mathbb{E}[\Lambda(\mathbf{X} + \mathbf{Y})] \geq \Lambda(\mathbf{0} + \mathbf{M}) \geq B.$$

Hence, for each $\mathbf{X} \in L^{\infty}(\mathbb{R}^N)$, $\{\mathbf{Y} \in \mathcal{C},\, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}\} \neq \emptyset$ and $L^{\infty}(\mathbb{R}^N) \subseteq dom(\rho)$. Now we prove that $\rho(\mathbf{X}) = \widetilde{\rho}(\mathbf{X})$. Clearly $\rho(\mathbf{X}) \leq \widetilde{\rho}(\mathbf{X})$. By contradiction assume that $\rho(\mathbf{X}) < \widetilde{\rho}(\mathbf{X})$. Then there exists $\varepsilon > 0$ and $\mathbf{Y} \in \mathcal{C}$ such that

$$\sum_{i=1}^{N} Y^i \leq \rho(\mathbf{X}) + \varepsilon < \widetilde{\rho}(\mathbf{X}) \text{ and } \mathbb{E}\left[\sum_{i=1}^{N} u_i(\mathbf{X} + \mathbf{Y}) \right] > B.$$

The continuity of u_i implies the existence of $\delta \in \mathbb{R}^N_+$, $\delta \neq \mathbf{0}$, such that

$$\mathbb{E}\left[\sum_{i=1}^{N} u_i(\mathbf{X} + \mathbf{Y} - \delta) \right] = B.$$

Then

$$\widetilde{\rho}(\mathbf{X}) \leq \sum_{i=1}^{N} (Y^i - \delta^i) < \sum_{i=1}^{N} Y^i \leq \rho(\mathbf{X}) + \varepsilon < \widetilde{\rho}(\mathbf{X}).$$

We now show uniqueness by contradiction. Suppose that $\rho(\mathbf{X})$ is attained by two distinct $\mathbf{Y}_1 \in \mathcal{C}$ and $\mathbf{Y}_2 \in \mathcal{C}$, so that $\mathbb{P}(\mathbf{Y}_1^j \neq \mathbf{Y}_2^j) > 0$ for some j. Then we have

$$\rho(\mathbf{X}) = \sum_{i=1}^{N} Y_1^i = \sum_{i=1}^{N} Y_2^i \quad \text{and} \quad \mathbb{E}\left[\sum_{i=1}^{N} u_i(\mathbf{X} + \mathbf{Y}_k) \right] = B \text{ for } k = 1, 2.$$

For $\lambda \in [0,1]$ set $\mathbf{Y}_\lambda := \lambda \mathbf{Y}_1 + (1-\lambda)\mathbf{Y}_2$. Then $\mathbf{Y}_\lambda \in \mathcal{C}$, as $\mathcal{C}$ is convex. This implies

$$\sum_{i=1}^N Y_\lambda^i = \lambda \sum_{i=1}^N Y_1^i + (1-\lambda)\sum_{i=1}^N Y_2^i = \rho(\mathbf{X}),\ \forall \lambda \in [0,1]$$

and for $\lambda \in (0,1)$

$$B = \lambda \mathbb{E}\left[\sum_{i=1}^N u_i(\mathbf{X}+\mathbf{Y}_1)\right] + (1-\lambda)\mathbb{E}\left[\sum_{i=1}^N u_i(\mathbf{X}+\mathbf{Y}_2)\right] <$$
$$< \mathbb{E}\left[\sum_{i=1}^N u_i(\lambda\mathbf{X}+\lambda\mathbf{Y}_1 + (1-\lambda)\mathbf{X}+(1-\lambda)\mathbf{Y}_2)\right] = \mathbb{E}\left[\sum_{i=1}^N u_i(\mathbf{X}+\mathbf{Y}_\lambda)\right]$$

where we used that u_j is strictly concave and $\mathbb{P}(\mathbf{Y}_1^j \neq \mathbf{Y}_2^j) > 0$. This is a contradiction with $\rho(\mathbf{X}) = \widetilde{\rho}(\mathbf{X})$ because

$$\rho(\mathbf{X}) = \sum_{i=1}^N Y_\lambda^i,\ \mathbf{Y}_\lambda \in \mathcal{C} \text{ and } \mathbb{E}\left[\sum_{i=1}^N u_i(\mathbf{X}+\mathbf{Y}_\lambda)\right] > B.$$

□

Lemma 3.6. *We have that*

$$\{\mathbf{X} \in \mathcal{L} \mid \mathbb{E}[\Lambda(\mathbf{X})] > -\infty\} \subset dom(\rho).$$

Proof. Let $\mathbf{1} = (1,\cdots,1)$ and $m \in \mathbb{R}$. Then $\mathbf{X} + m\mathbf{1} \uparrow +\infty$ $\mathbb{P}$-a.s. as $m \to +\infty$. As $\mathbb{E}[\Lambda(\mathbf{X})] > -\infty$, we have that $\mathbb{E}[\Lambda(\mathbf{X}+m\mathbf{1})] > -\infty$ for $m > 0$, and by monotone convergence it follows that $\mathbb{E}[\Lambda(\mathbf{X}+m\mathbf{1})]\uparrow\Lambda(+\infty) > B$. Since $\mathbb{R}_+^N \subseteq \mathcal{C}$, then $m\mathbf{1} \in \mathcal{C}$ and $\{\mathbf{Y} \in \mathcal{C},\ \Lambda(\mathbf{X}+\mathbf{Y}) \in \mathbb{A}\} \neq \emptyset$. □

3.2 Orlicz setting

We now study some important properties of systemic risk measures of the form (3.5) in a Orlicz space setting, see [38] for further details on Orlicz spaces. This presents several advantages. For a mathematical point of view, it is a more general setting than L^∞, but at the same time it simplifies the analysis, since the topology is order continuous and there are no singular elements in the dual space. Furthermore, it has been shown in [10] that the Orlicz setting is the natural one to embed utility maximization problems, as the natural integrability condition $\mathbb{E}[u(X)] > -\infty$ translates into $\mathbb{E}[\phi(X)] < +\infty$.

Let $u : \mathbb{R} \to \mathbb{R}$ be concave and increasing. Consider $\phi(x) := -u(-|x|) + u(0)$. Then ϕ is a Young function on $\mathbb{R}$, i.e., it is convex, symmetric, $\phi(0) = 0$ and $\lim_{x\to+\infty}\phi(x) = +\infty$. The Orlicz space L^ϕ and Orlicz Heart M^ϕ are respectively defined by

$$L^\phi := \left\{X \in L^0(\mathbb{R}) \mid \mathbb{E}[\phi(\alpha X)] < +\infty \text{ for some } \alpha > 0\right\}, \tag{3.6}$$
$$M^\phi := \left\{X \in L^0(\mathbb{R}) \mid \mathbb{E}[\phi(\alpha X)] < +\infty \text{ for all } \alpha > 0\right\} \tag{3.7}$$

and are Banach spaces when endowed with the Orlicz norm. The topological dual of M^ϕ is the Orlicz space L^{ϕ^*}, where the convex conjugate ϕ^* of ϕ, defined by

$$\phi^*(y) := \sup_{x\in\mathbb{R}} \{xy - \phi(x)\}, \ y \in \mathbb{R},$$

is also a Young function. Note that

$$\mathbb{E}[u(X)] > -\infty \text{ if } \mathbb{E}[\phi(X)] < +\infty. \tag{3.8}$$

Remark 3.7. It is well known that $M^\phi \subseteq L^\phi \subseteq L^1(\mathbb{P};\mathbb{R})$. In addition, from the Fenchel inequality $xy \le \phi(x) + \phi^*(y)$ we obtain

$$(\alpha|X|)\left(\lambda\frac{dQ}{d\mathbb{P}}\right) \le \phi(\alpha|X|) + \phi^*\left(\lambda\frac{dQ}{d\mathbb{P}}\right) \ \mathbb{P} \text{ a.s.}$$

and we immediately deduce that $\frac{dQ}{d\mathbb{P}} \in L^{\phi^*}$ implies $L^\phi \subseteq L^1(Q;\mathbb{R})$.

Given $u_1, \cdots, u_N : \mathbb{R} \to \mathbb{R}$ concave increasing functions with associated Young functions $\phi_1, \cdots, \phi_N$, we define

$$M^\Phi := M^{\phi_1} \times \cdots \times M^{\phi_N}, \quad L^\Phi := L^{\phi_1} \times \cdots \times L^{\phi_N} \tag{3.9}$$

and consider

$$\mathcal{L} = M^\Phi,$$

i.e. $\rho : M^\Phi \to \mathbb{R}\cup\{+\infty\}$. Under Assumption 3.3, M^Φ coincides with the domain of ρ and the systemic risk measures of the form (3.5) have good properties if restricted to M^Φ.

Proposition 3.8. *The map $\rho : M^\Phi \to \mathbb{R}\cup\{+\infty\}$ defined in* (3.5) *is finitely valued, monotone decreasing, convex, continuous and subdifferentiable on the Orlicz Heart $M^\Phi = dom(\rho)$.*

Proof. The equality $M^\Phi = dom(\rho)$, so that $\rho : M^\Phi \to \mathbb{R}$, follows from Lemma 3.6, the definition of M^Φ in (3.7), and (3.8). The remaining properties are a consequence of Proposition 3.5, Theorem A.3 in Appendix and the fact that M^Φ is a Banach space. □

4 The main results

4.1 Dual representation of ρ

We now investigate the dual representation of systemic risk measures of the form (3.5). When $\mathbf{Z} \in M^\Phi$ and $\xi \in L^{\Phi^*}$, we set $\mathbb{E}[\xi\mathbf{Z}] := \sum_{n=1}^N \mathbb{E}[\xi^n Z^n]$ and, for $\frac{d\mathbf{Q}}{d\mathbb{P}} \in L_+^{\Phi^*}$, $\mathbb{E}_{\mathbf{Q}}[\mathbf{Z}] = \sum_{n=1}^N \mathbb{E}_{\mathbf{Q}^n}[Z^n]$.

Proposition 4.1. *For any* $\mathbf{X} \in M^{\Phi}$,

$$\rho(\mathbf{X}) = \max_{\mathbf{Q}\in\mathcal{D}} \left\{ \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-X^n] - \alpha_{\Lambda,B}(\mathbf{Q}) \right\}, \tag{4.1}$$

where

$$\alpha_{\Lambda,B}(\mathbf{Q}) := \sup_{\mathbf{Z}\in\mathcal{A}} \left\{ \sum_{n=1}^{N} \mathbb{E}_{\mathbf{Q}^n}[-Z^n] \right\}, \ \mathbf{Q} \in \mathcal{D}, \tag{4.2}$$

with $\mathcal{A} := \left\{ \mathbf{Z} \in M^{\Phi} \mid \sum_{n=1}^{N} \mathbb{E}[u_n(Z^n)] \geq B \right\}$, *and*

$$\mathcal{D} := dom(\alpha_{\Lambda,B}) \cap \left\{ \frac{d\mathbf{Q}}{d\mathbb{P}} \in L_{+}^{\Phi^*} \mid Q^n(\Omega) = 1 \text{ and } \sum_{n=1}^{N} (\mathbb{E}_{Q^n}[Y^n] - Y^n) \leq 0 \text{ for all } \mathbf{Y} \in \mathcal{C} \right\}. \tag{4.3}$$

(i) *Suppose that for some* $i, j \in \{1, \cdots, N\}$, $i \neq j$, *we have* $\pm(e_i 1_A - e_j 1_A) \in \mathcal{C}$ *for all* $A \in \mathcal{F}$. *Then*

$$\mathcal{D} := dom(\alpha_{\Lambda,B}) \cap \left\{ \frac{d\mathbf{Q}}{d\mathbb{P}} \in L_{+}^{\Phi^*} \mid Q^n(\Omega) = 1,\ Q^i = Q^j \text{ and } \sum_{n=1}^{N} (\mathbb{E}_{Q^n}[Y^n] - Y^n) \leq 0 \text{ for all } \mathbf{Y} \in \mathcal{C} \right\}.$$

(ii) *Suppose that* $\pm(e_i 1_A - e_j 1_A) \in \mathcal{C}$ *for all* i, j *and all* $A \in \mathcal{F}$, *then*

$$\mathcal{D} := dom(\alpha_{\Lambda,B}) \cap \left\{ \frac{d\mathbf{Q}}{d\mathbb{P}} \in L_{+}^{\Phi^*} \mid Q^n(\Omega) = 1,\ Q^n = Q \right\}.$$

Proof. The dual representation (4.1) is a consequence of Proposition 3.8, Theorem A.3 and of Propositions 3.9 and 3.11 in [29], taking into consideration that $\mathcal{C}$ is a convex cone, the dual space of the Orlicz Heart M^{Φ} is the Orlicz space L^{Φ^*} and $M^{\Phi} = dom(\rho)$. Notice that from Theorem A.3 we know that the dual elements $\xi \in L_{+}^{\Phi^*}$ are positive but a priori not normalized. However, we obtain $\mathbb{E}[\xi^n] = 1$ by taking as $\mathbf{Y} = \pm e_j \in \mathbb{R}^N$, and using $\sum_{n=1}^{N} (\xi^n(Y^n) - Y^n) \leq 0$ for all $\mathbf{Y} \in \mathcal{C}$, so that $\xi^j(1) - 1 \leq 0$ and $\xi^j(-1) + 1 \leq 0$ imply $\xi^j(1) = 1$. This shows the form of the domain $\mathcal{D}$ in (4.3). Furthermore:

(i) Take $\mathbf{Y} := e_i 1_A - e_j 1_A \in \mathcal{C}$. From $\sum_{n=1}^{N} (Q^n(Y^n) - Y^n) \leq 0$ we obtain $Q^i(1_A) - 1_A + Q^j(-1_A) + 1_A \leq 0$, i.e., $Q^i(A) - Q^j(A) \leq 0$ and similarly taking $\mathbf{Y} := -e_i 1_A + e_j 1_A \in \mathcal{C}$ we get $Q^j(A) - Q^i(A) \leq 0$.

(ii) From (i), we obtain $Q^i = Q^j$. In addition, we get $\sum_{n=1}^{N} (\mathbb{E}_Q[Y^n] - Y^n) = \mathbb{E}_Q[\sum_{n=1}^{N} Y^n] - \sum_{n=1}^{N} Y^n) = 0$, as $\sum_{n=1}^{N} Y^n \in \mathbb{R}$.

□

Proposition 4.1 guarantees the existence of a maximizer $\mathbf{Q_X}$ to the dual problem (4.1) and that $\alpha_{\Lambda,B}(\mathbf{Q_X}) < +\infty$. Uniqueness will be proved in Corollary 4.16.

Definition 4.2. Let $\mathbf{X} \in M^{\Phi}$. An optimal solution of the dual problem (4.1) is a vector of probability measures $\mathbf{Q_X} = (Q^1_{\mathbf{X}}, \cdots, Q^N_{\mathbf{X}})$ verifying $\frac{d\mathbf{Q_X}}{d\mathbb{P}} \in L^{\Phi^*}_+$ and

$$\rho(\mathbf{X}) = \sum_{n=1}^{N} \mathbb{E}_{Q^n_{\mathbf{X}}}[-X^n] - \alpha_{\Lambda,B}(\mathbf{Q_X}). \tag{4.4}$$

In fact, there exists a simple relation among ρ_B, $\rho_B^{\mathbf{Q}}$ and $\alpha_{\Lambda,B}(\mathbf{Q})$ defined in (3.5), (2.5), and (4.2), respectively.

Proposition 4.3. *We have*

$$\rho_B^{\mathbf{Q}}(\mathbf{X}) = -\sum_{n=1}^{N} \mathbb{E}_{Q^n}[X^n] - \alpha_{\Lambda,B}(\mathbf{Q}) \tag{4.5}$$

and

$$\rho_B^{\mathbf{Q_X}}(\mathbf{X}) = \rho_B(\mathbf{X}), \tag{4.6}$$

where $\mathbf{Q_X}$ is an optimal solution of the dual problem (4.1).

Proof. We have

$$\begin{aligned} -\alpha_{\Lambda,B}(\mathbf{Q}) &= \inf\left\{\sum_{n=1}^{N} \mathbb{E}_{Q^n}[Z^n] \mid \mathbf{Z} \in M^{\Phi} \text{ and } \sum_{n=1}^{N} \mathbb{E}[u_n(Z^n)] \geq B\right\} \\ &= \inf\left\{\sum_{n=1}^{N} \mathbb{E}_{Q^n}[X^n + Y^n] \mid \mathbf{Y} \in M^{\Phi} \text{ and } \sum_{n=1}^{N} \mathbb{E}[u_n(X^n + Y^n)] \geq B\right\} \\ &= \sum_{n=1}^{N} \mathbb{E}_{Q^n}[X^n] + \rho_B^{\mathbf{Q}}(\mathbf{X}), \end{aligned}$$

which proves (4.5). Then from (4.5) and (4.4) we deduce

$$\rho_B^{\mathbf{Q_X}}(\mathbf{X}) = -\sum_{n=1}^{N} \mathbb{E}_{Q^n_{\mathbf{X}}}[X^n] - \alpha_{\Lambda,B}(\mathbf{Q_X}) = \rho_B(\mathbf{X}).$$

□

We now turn our attention to the uniqueness of the optimal solution to the problem (4.2). The proof employs the same arguments used in the proof of Proposition 3.5 and is postponed to Appendix A.2.1.

Lemma 4.4. *If each u_n is strictly concave then there exists at most one $\mathbf{Z} \in M^{\Phi}$ satisfying $\alpha_{\Lambda,B}(\mathbf{Q}) = \sum_{n=1}^{N} \mathbb{E}_{Q^n}[-Z^n]$ and*

$$\sum_{n=1}^{N} \mathbb{E}[u_n(Z^n)] \geq B. \tag{4.7}$$

Remark 4.5. (Uniqueness) Suppose that each u_n is strictly concave. The existence of the optimizer $\mathbf{Y_Q}$ for the problem $\rho_B^{\mathbf{Q}}(\mathbf{X})$ will be proved in Section 4.2. The uniqueness of $\mathbf{Z} \in M^\Phi$ for $\alpha_{\Lambda,B}(\mathbf{Q})$, shown in Lemma 4.4, also implies the uniqueness of the optimizer $\mathbf{Y_Q} \in M^\Phi$ for $\rho_B^{\mathbf{Q}}(\mathbf{X})$ thanks to Proposition 4.3.

Example 4.6. Consider the grouping Example 3.1. As $\mathcal{C}^{(\mathbf{n})}$ is a linear space containing $\mathbb{R}^N$, the dual representation (4.1) applies. In addition in each group we have $\pm(e_i 1_A - e_j 1_A) \in \mathcal{C}^{(\mathbf{n})}$ for all i, j *in the same group* and for all $A \in \mathcal{F}$. Therefore, in each group the components Q^i, $i \in I_m$, of the dual elements are all the same, i.e., $Q^i = Q^j \, \forall i, j \in I_m$, and representation (4.1) becomes

$$\rho(\mathbf{X}) = \max_{\mathbf{Q}\in\mathcal{D}} \left\{ \sum_{m=1}^{h} \sum_{k \in I_m} \left(\mathbb{E}_{Q^m}[-X^k] \right) - \alpha_{\Lambda,B}(\mathbf{Q}) \right\} = \max_{\mathbf{Q}\in\mathcal{D}} \left\{ \sum_{m=1}^{h} \mathbb{E}_{Q^m}[-\overline{X}_m] - \alpha_{\Lambda,B}(\mathbf{Q}) \right\}, \tag{4.8}$$

with

$$\mathcal{D} := dom(\alpha_{\Lambda,B}) \cap \left\{ \frac{d\mathbf{Q}}{d\mathbb{P}} \in L_+^{\Phi^*} \mid Q^i = Q^j \, \forall i, j \in I_m, Q^i(\Omega) = 1 \right\}$$

and $\overline{X}_m := \sum_{k \in I_m} X^k$. Indeed,

$$\sum_{n=1}^{N} (\mathbb{E}_{Q^n}[Y^n] - Y^n) = \sum_{m=1}^{h} \sum_{k \in I_m} (\mathbb{E}_{Q^m}[Y^k] - Y^k) = \sum_{m=1}^{h} \left(\mathbb{E}_{Q^m} \left[\sum_{k \in I_m} Y^k \right] - \sum_{k \in I_m} Y^k \right) = 0,$$

as $\sum_{k \in I_m} Y^k \in \mathbb{R}$. If we have only one single group, all components of a dual element $\mathbf{Q}$ are the same.

Remark 4.7. Consider the grouping Example 3.1. Let $\mathbf{Q} = (Q^1, \cdots, Q^n)_{n=1,\cdots,N}$ be a vector of probability measures with the property that in each group the components Q^i, $i \in I_m$, satisfy $Q^i = Q^m$ for all $i \in I_m$. Then $(\mathbb{E}_{\mathbf{Q}_1}[Y_{\mathbf{X}}^1], ..., \mathbb{E}_{\mathbf{Q}_N}[Y_{\mathbf{X}}^N])$ is a systemic risk allocation as in Definition (1.1), i.e.,

$$\rho(\mathbf{X}) = \sum_{n=1}^{N} \mathbb{E}_{Q^n}[Y_{\mathbf{X}}^n] = \sum_{m=1}^{h} \sum_{k \in I_m} \mathbb{E}_{Q^m}[Y_{\mathbf{X}}^k] = \sum_{m=1}^{h} d_m.$$

Indeed, for *such* a vector $(Q^1, \cdots, Q^m)$ of probability measures we have

$$\sum_{k \in I_m} \mathbb{E}_{Q^m}[Y_{\mathbf{X}}^k] = \mathbb{E}_{Q^m} \left[\sum_{k \in I_m} Y_{\mathbf{X}}^k \right] = \mathbb{E}_{Q^m}[d_m] = d_m.$$

Returning to our general setting, from now on, we work under the following two assumptions, with the understanding that Assumption 4.9 will hold with respect to the probability measures ($\mathbf{Q}$ or $\mathbf{Q_X}$) involved in the statements of the results.

Assumption 4.8. In addition to Assumptions 3.3, we assume that for any $n = 1, \cdots, N$, $u_n : \mathbb{R} \to \mathbb{R}$ is increasing, strictly concave, differentiable and satisfies the Inada conditions

$$u_n'(-\infty) := \lim_{x \to -\infty} u_n'(x) = +\infty, \; u_n'(+\infty) := \lim_{x \to +\infty} u_n'(x) = 0.$$

Some useful properties on the convex conjugate function $v_n(y) := \sup_{x \in \mathbb{R}} \{u_n(x) - xy\}$ are collected in Lemma A.6. The following additional Assumption 4.9 is related to the Reasonable Asymptotic Elasticity condition on utility functions, which was introduced in [41]. This assumption, even though quite weak (see [8] Section 2.2), is fundamental to guarantee the existence of the optimal solution to classical utility maximization problems (see [41] and [8]).

Assumption 4.9. For any $n = 1, \cdots, N$, v_n and $Q^n \ll \mathbb{P}$ satisfy

$$\mathbb{E}\left[v_n\left(\frac{dQ^n}{d\mathbb{P}}\right)\right] < \infty \quad \text{iff} \quad \mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{d\mathbb{P}}\right)\right] < \infty, \; \forall \lambda > 0.$$

From the Fenchel inequality

$$u_n(X^n) \le X^n \frac{dQ^n}{d\mathbb{P}} + v_n\left(\frac{dQ^n}{d\mathbb{P}}\right) \; \mathbb{P} \text{ a.s.}$$

we immediately deduce that if $X^n \in L^1(Q^n)$ and $\mathbb{E}\left[v_n\left(\frac{dQ^n}{d\mathbb{P}}\right)\right] < \infty$ then $\mathbb{E}\left[u_n\left(X^n\right)\right] < +\infty$.

Proposition 4.10. *When $\alpha_{\Lambda,B}(\mathbf{Q}) < +\infty$, the penalty function in* (4.2) *can be written as*

$$\alpha_{\Lambda,B}(\mathbf{Q}) := \sup_{\mathbf{Z} \in \mathcal{A}} \left\{\sum_{n=1}^N \mathbb{E}_{Q^n}[-Z^n]\right\} = \inf_{\lambda > 0}\left(-\frac{1}{\lambda}B + \frac{1}{\lambda}\sum_{n=1}^N \mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{d\mathbb{P}}\right)\right]\right). \tag{4.9}$$

Proof. In Appendix A. □

Proposition 4.11. *When $\alpha_{\Lambda,B}(\mathbf{Q}) < +\infty$, the infimum is attained in* (4.9)*, i.e.,*

$$\alpha_{\Lambda,B}\left(\mathbf{Q}\right) = \sum_{n=1}^N \mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}} v_n'\left(\widehat{\lambda}\frac{dQ^n}{d\mathbb{P}}\right)\right], \tag{4.10}$$

where $\widehat{\lambda} > 0$ is the unique solution of the equation[1]

$$-B + \sum_{n=1}^N \mathbb{E}\left[v_n\left(\lambda\frac{dQ^n}{d\mathbb{P}}\right)\right] - \lambda\sum_{n=1}^N \mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}} v_n'\left(\lambda\frac{dQ^n}{d\mathbb{P}}\right)\right] = 0. \tag{4.11}$$

[1] Note that $\widehat{\lambda}$ will depend on B, $(u_n)_{n=1,\cdots,N}$ and $\left(\frac{dQ_n}{d\mathbb{P}}\right)_{n=1,\cdots,N}$.

Proof. Set $\xi_n := \frac{dQ^n}{d\mathbb{P}} \geq 0$ a.s.. Recall from Lemma A.6 that v_n is strictly convex with $v_n(+\infty) = +\infty$, $v_n(0^+) = u_n(+\infty)$, $\lim_{z\to+\infty} \frac{v_n(z)}{z} = +\infty$ because of Assumption 3.3, and v_n is continuously differentiable. As $u_n'(+\infty) = 0$ and $u_n'(-\infty) = +\infty$, we get $v_n'(0) = -\infty$ and $v_n'(+\infty) = +\infty$.

Set $\eta = \frac{1}{\lambda} \in (0, +\infty)$ and consider the differentiable function $F : (0, +\infty) \to \mathbb{R}$ defined by

$$F(\eta) := -B\eta + \eta \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(\frac{1}{\eta}\xi_n\right)\right].$$

Then $\alpha_{\Lambda,B}(\xi) = \inf_{\eta>0} F(\eta)$ and (4.11) can be rewritten as

$$F'(\eta) = 0 \tag{4.12}$$

with

$$F'(\eta) = -B + \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(\frac{1}{\eta}\xi_n\right)\right] - \frac{1}{\eta}\sum_{n=1}^{N} \mathbb{E}\left[\xi_n v_n'\left(\frac{1}{\eta}\xi_n\right)\right].$$

Note that if $\widehat{\eta} > 0$ is the solution to (4.12), then by replacing such $\widehat{\eta}$ into $F(\eta)$ we immediately obtain (4.10).

Next, thanks to the integrability conditions provided by Lemma A.5, we show the existence of the solution $\widehat{\eta} > 0$ of (4.12). First we consider $\eta \to +\infty$. Since $\sum_{n=1}^{N} v_n(0^+) = \sum_{n=1}^{N} u_n(+\infty) > B$ by Assumption 3.3, we have that

$$\liminf_{\eta\to+\infty} -B + \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(\frac{1}{\eta}\xi_n\right)\right] > 0.$$

Moreover, $v_n'(0) = -\infty$ shows that

$$\liminf_{\eta\to+\infty} -\frac{1}{\eta}\sum_{n=1}^{N} \mathbb{E}\left[\xi_n v_n'\left(\frac{1}{\eta}\xi_n\right)\right] \geq 0.$$

Hence $\liminf_{\eta\to+\infty} F'(\eta) > 0$. We now look at $\eta \to 0$:

$$\begin{aligned}
\lim_{\eta\to 0} F'(\eta) &= -B + \lim_{\eta\to 0} \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(\frac{1}{\eta}\xi_n\right)\right] - \frac{1}{\eta}\sum_{n=1}^{N} \mathbb{E}\left[\xi_n v_n'\left(\frac{1}{\eta}\xi_n\right)\right] \\
&= -B + \lim_{t\to+\infty} \sum_{n=1}^{N} \mathbb{E}\left[v_n\left(t\xi_n\right)\right] - t\sum_{n=1}^{N} \mathbb{E}\left[\xi_n v_n'\left(t\xi_n\right)\right] \\
&= -B + \sum_{n=1}^{N} \lim_{t\to+\infty} \mathbb{E}\left[v_n\left(t\xi_n\right) - t\xi_n v_n'\left(t\xi_n\right)\right].
\end{aligned}$$

The convexity of v_n implies that for any fixed $z_0 > 0$ and $z > z_0$

$$v_n(z) - v_n(z_0) \leq v_n'(z)(z - z_0).$$

From $\lim_{z\to+\infty}\frac{v(z)}{z}=+\infty$, $v_n'(z)\to+\infty$ as $z\to+\infty$ and

$$v_n(z)-zv_n'(z)\le v_n(z_0)-z_0v_n'(z)\downarrow-\infty \text{ as } z\to+\infty,$$

we have by monotone convergence

$$\lim_{t\to+\infty}\mathbb{E}\left[v_n\left(t\xi_n\right)-t\xi_n v_n'\left(t\xi_n\right)\right]=-\infty,$$

so that $\liminf_{\eta\to 0}F'(\eta)=-\infty$. By the continuity of F' we obtain the existence of the solution $\widehat{\eta}>0$ for (4.12). Uniqueness follows from the strict convexity of F. □

Example 4.12. Let $\Lambda=\sum_{n=1}^N u_n$ with $u_n:\mathbb{R}\to\mathbb{R}$, $u_n(x)=-e^{-\alpha_n x}$, $\alpha_n>0$, for each n, and let $B<0$. Then, $v_n'(y)=\frac{1}{\alpha_n}\ln(\frac{y}{\alpha_n})$. From the first order condition (4.11) we obtain that the minimizer is $\widehat{\lambda}=-\frac{B}{\beta}$, with $\beta:=\sum_{n=1}^N\frac{1}{\alpha_n}$. Therefore, from (4.10) we have

$$\alpha_{\Lambda,B}(\mathbf{Q})=\sum_{n=1}^N\mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}}v_n'\left(\widehat{\lambda}\frac{dQ^n}{d\mathbb{P}}\right)\right]=\sum_{n=1}^N\frac{1}{\alpha_n}\left(H\left(Q^n,\mathbb{P}\right)+\ln\left(-\frac{B}{\beta\alpha_n}\right)\right), \tag{4.13}$$

where $H\left(Q^n,\mathbb{P}\right):=\mathbb{E}\left[\frac{dQ^n}{d\mathbb{P}}\ln\left(\frac{dQ^n}{d\mathbb{P}}\right)\right]$ is the relative entropy.

4.2 On the optimal solution of $\rho^{\mathbf{Q}}$ and comparison of optimal solutions

Theorem 4.13. *Suppose that $\alpha_{\Lambda,B}(\mathbf{Q})<+\infty$. Then the random vector $\mathbf{Y_Q}$ given by*

$$Y^n_{\mathbf{Q}}:=-X^n-v_n'\left(\widehat{\lambda}\frac{dQ^n}{d\mathbb{P}}\right),$$

where $\widehat{\lambda}$ is the unique solution to (4.11)*, satisfies $Y^n_{\mathbf{Q}}\in L^1(Q^n)$, $u_n(X^n+Y^n_{\mathbf{Q}})\in L^1(\mathbb{P})$, $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n_{\mathbf{Q}})\right]=B$ and*

$$\begin{aligned}\rho^{\mathbf{Q}}_B(\mathbf{X}) &= \inf_{\mathbf{Y}\in M^\Phi}\left\{\sum_{n=1}^N\mathbb{E}_{Q^n}\left[Y^n\right]\mid\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n)\right]\ge B\right\}=\sum_{n=1}^N\mathbb{E}_{Q^n}\left[Y^n_{\mathbf{Q}}\right]\\ &= \inf_{\mathbf{Y}\in L^1(\mathbf{Q};\mathbb{R}^N)}\left\{\sum_{n=1}^N\mathbb{E}_{Q^n}\left[Y^n\right]\mid\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n)\right]\ge B\right\}:=\widetilde{\rho}^{\mathbf{Q}}_B(\mathbf{X}),\end{aligned}$$

so that $\mathbf{Y_Q}$ is the optimal solution to the extended problem $\widetilde{\rho}^{\mathbf{Q}}_B(\mathbf{X})$.

Proof. The integrability conditions hold thanks to the results stated in Appendix A.3. From (4.5) and the expression (4.10) for the penalty, we compute:

$$\begin{aligned}\rho^{\mathbf{Q}}_B(\mathbf{X}) &= -\sum_{n=1}^N\mathbb{E}_{Q^n}[X^n]-\alpha_{\Lambda,B}(\mathbf{Q})=\\ &= \sum_{n=1}^N\mathbb{E}_{Q^n}\left[-X^n-v_n'\left(\widehat{\lambda}\frac{dQ^n}{d\mathbb{P}}\right)\right]=\sum_{n=1}^N\mathbb{E}_{Q^n}\left[Y^n_{\mathbf{Q}}\right].\end{aligned}$$

We show that $Y^n_{\mathbf{Q}}$ satisfies the budget constraint:

$$\begin{aligned}\sum_{n=1}^N \mathbb{E}\left[u_n\left(X^n+Y^n_{\mathbf{Q}}\right)\right] &= \sum_{n=1}^N \mathbb{E}\left[u_n\left(-v_n'\left(\widehat{\lambda}\frac{dQ^n}{d\mathbb{P}}\right)\right)\right]\\ &= -\widehat{\lambda}\sum_{n=1}^N \mathbb{E}_{Q^n}\left[v_n'\left(\widehat{\lambda}\frac{dQ^n}{d\mathbb{P}}\right)\right]+\sum_{n=1}^N \mathbb{E}\left[v_n\left(\widehat{\lambda}\frac{dQ^n}{d\mathbb{P}}\right)\right]=B\end{aligned}$$

due to $u(-v'(y))=-yv'(y)+v(y)$ (see Lemma A.6) and (4.11). The equality $\rho_B^{\mathbf{Q}}(\mathbf{X})=\widehat{\rho}_B^{\mathbf{Q}}(\mathbf{X})$, proved in (5.3), concludes the proof. □

Remark 4.14. (Uniqueness) As $\rho_B^{\mathbf{Q}}(\mathbf{X})=-\sum_{n=1}^N \mathbb{E}_{Q^n}[X^n]-\alpha_{\Lambda,B}(\mathbf{Q})$ by (4.5), by repeating step by step the proof of Lemma 4.4 we also conclude that the optimizers, when they exist, to both problems $\rho_B^{\mathbf{Q}}(\mathbf{X})$ and $\alpha_{\Lambda,B}(\mathbf{Q})$ are unique in $L^1(\mathbf{Q})$, not only in M^Φ (see Remark 4.5).

When both solutions to the problems $\rho_B(\mathbf{X})$ and $\rho_B^{Q_{\mathbf{X}}}(\mathbf{X})$ exist, then they coincide.

Proposition 4.15. *Let $\mathbf{Y_X}$ be the optimal solution of $\rho_B(\mathbf{X})$, $\mathbf{Q_X}$ be an optimal solution to the dual problem (4.1) and $\mathbf{Y_{Q_X}}$ be the optimal solution of $\rho_B^{Q_{\mathbf{X}}}(\mathbf{X})$. Then:*

$$\mathbf{Y_X}=\mathbf{Y_{Q_X}}=-X^n-v_n'\left(\widehat{\lambda}\frac{dQ^n_{\mathbf{X}}}{d\mathbb{P}}\right).$$

Proof. Notice that $\mathbf{Y_X}$ and $\mathbf{Y_{Q_X}}$ satisfies the same constraint, i.e.,

$$\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n_{\mathbf{X}})\right]\geq B \text{ and } \mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n_{\mathbf{Q_X}})\right]\geq B \tag{4.14}$$

and in addition

$$\sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]\leq \sum_{n=1}^N Y^n_{\mathbf{X}}, \tag{4.15}$$

as $\mathbf{Y_X}\in\mathcal{C}$ and $\mathbf{Q_X}\in\mathcal{D}$. From the definitions of $\mathbf{Y_X}$ and $\mathbf{Y_{Q_X}}$, from (4.5) and (4.4) we deduce that

$$\sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{Q_X}}]=\rho_B^{\mathbf{Q_X}}(\mathbf{X})=-\sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[X^n]-\alpha_{\Lambda,B}(\mathbf{Q_X})=\rho_B(\mathbf{X})=\sum_{n=1}^N Y^n_{\mathbf{X}}. \tag{4.16}$$

As $\mathbf{Y_X}$ satisfies the constraint (4.14), by definition of $\rho_B^{\mathbf{Q_X}}(\mathbf{X})$ we have

$$\sum_{n=1}^N Y^n_{\mathbf{X}}=\rho_B(\mathbf{X})=\rho_B^{\mathbf{Q_X}}(\mathbf{X})\leq\sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}],$$

which shows, together with (4.15), that

$$\sum_{n=1}^N Y^n_{\mathbf{X}}=\sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[Y^n_{\mathbf{X}}]. \tag{4.17}$$

From (4.16) we then deduce

$$\begin{aligned}\alpha_{\Lambda,B}(\mathbf{Q_X}) &= -\sum_{n=1}^{N}\mathbb{E}_{Q^n_{\mathbf{X}}}\left[X^n+Y^n_{\mathbf{Q_X}}\right]\\ \alpha_{\Lambda,B}(\mathbf{Q_X}) &= -\sum_{n=1}^{N}\left(\mathbb{E}_{Q^n_{\mathbf{X}}}\left[X^n\right]+Y^n_{\mathbf{X}}\right) = -\sum_{n=1}^{N}\mathbb{E}_{Q^n_{\mathbf{X}}}\left[X^n+Y^n_{\mathbf{X}}\right].\end{aligned}$$

As both $(\mathbf{X}+\mathbf{Y_X})$ and $(\mathbf{X}+\mathbf{Y_{Q_X}})$ satisfy the budget constraints associated to $\alpha_{\Lambda,B}(\mathbf{Q_X})$ in equation (4.7), this implies that $\alpha_{\Lambda,B}(\mathbf{Q_X})$ is attained by both $(\mathbf{X}+\mathbf{Y_X})$ and $(\mathbf{X}+\mathbf{Y_{Q_X}})$. The uniqueness shown in Lemma 4.4 allows us to conclude that $\mathbf{Y_X}=\mathbf{Y_{Q_X}}$. □

We now show that the maximizer of the dual representation is unique.

Corollary 4.16. *Suppose that there exists an optimal solution $\mathbf{Y_X}$ to $\rho_B(\mathbf{X})$. Then the optimal solution $\mathbf{Q_X}=(Q^1_{\mathbf{X}},\cdots,Q^N_{\mathbf{X}})$ of the dual problem (4.1) is unique.*

Proof. Suppose that $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are two optimizers of the dual problem (4.1). Then $\alpha_{\Lambda,B}(\mathbf{Q}_1)<+\infty$, $\alpha_{\Lambda,B}(\mathbf{Q}_2)<+\infty$ and, by Proposition 4.15, we have, for each n :

$$-X^n - v_n'\left(\widehat{\lambda}_1\frac{dQ^n_1}{d\mathbb{P}}\right) = Y^n_{\mathbf{Q}_1} = Y^n_{\mathbf{X}} = Y^n_{\mathbf{Q}_2} = -X^n - v_n'\left(\widehat{\lambda}_2\frac{dQ^n_2}{d\mathbb{P}}\right),\quad \mathbb{P}\text{ a.s.}$$

As v_n' is invertible, we conclude that $\widehat{\lambda}_1\frac{dQ^n_1}{d\mathbb{P}}=\widehat{\lambda}_2\frac{dQ^n_2}{d\mathbb{P}}$, $\mathbb{P}$ a.s., which then implies $Q^n_1=Q^n_2$, as $\mathbb{E}\left[\frac{dQ^n_i}{d\mathbb{P}}\right]=1$. □

4.3 On the existence of the optimal solution to the primal problem $\rho(X)$

In order to prove the existence of the optimal solution to problem $\rho_B(\mathbf{X})$, we will proceed in two steps.

As for the problem $\rho^{\mathbf{Q}}_B(\mathbf{X})$, we can not expect to find an optimal solution that lies the space M^Φ, but only in the larger space of integrable random variables.

We first prove the existence of $\mathbf{Y}\in L^1(\mathbb{P};\mathbb{R}^N)$, which is the candidate solution, as specified in Theorem 4.17, to the problem $\rho_B(\mathbf{X})$. We already know that the optimal solution to $\rho_B(\mathbf{X})$, when it exists, coincides with the optimal solution $\mathbf{Y_{Q_X}}$ to problem $\rho^{\mathbf{Q_X}}_B(\mathbf{X})$, and that this solution $\mathbf{Y_{Q_X}}$ belongs to $L^1(\mathbf{Q_X};\mathbb{R}^N)$. So in the second step, we show, under proper conditions on $\mathcal{C}$, that $\mathbf{Y}\in L^1(\mathbf{Q_X};\mathbb{R}^N)$.

W.l.o.g. we may assume that $u_i(0)=0$, $1\le i\le N$ and observe that then

$$u_i(x_i)=u_i(x_i^+)+u_i(-x_i^-). \tag{4.18}$$

Theorem 4.17. *For $\mathcal{C}\subseteq\mathcal{C}_{\mathbb{R}}\cap M^\Phi$ and for any $\mathbf{X}\in M^\Phi$ there exists $\mathbf{Y}\in L^1(\mathbb{P};\mathbb{R}^N)$ such that*

$$\sum_{n=1}^{N}Y^n\in\mathbb{R},\quad \mathbb{E}\left[\sum_{n=1}^{N}u_n(X^n+Y^n)\right]\ge B,$$

$$\rho_B(\mathbf{X}) := \inf\left\{\sum_{n=1}^{N} Z^n \mid \mathbf{Z}\in\mathcal{C},\, \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n{+}Z^n)\right] \geq B\right\} = \sum_{n=1}^{N} Y^n,$$

and a sequence $\{\mathbf{Y}_k\}_{k\in\mathbb{N}}\subset\mathcal{C}$*,* $\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n{+}Y_k^n)\right]\geq B$ *and*

$$\mathbf{Y}_k \to \mathbf{Y}\ \mathbb{P}\text{-}a.s.$$

Remark 4.18. Recall that $\mathcal{C} := \mathcal{C}_0 \cap M^\Phi$ and that $\mathcal{C}_0 \subseteq \mathcal{C}_\mathbb{R}$ represents the effective constraint on the admissible injections, except for the integrability restriction expressed by M^Φ. Assume further that $\mathcal{C}_0$ is closed in $L^0(\mathbb{P})$, which is a reasonable assumption and holds true if $\mathcal{C}=\mathcal{C}^{(\mathbf{n})}$, in which case $\mathcal{C}_0^{(\mathbf{n})}$ is defined in (3.4). Then the random vector $\mathbf{Y}$ in Theorem 4.17 would also belong to $\mathcal{C}_0$, but in general not to $\mathcal{C}$ (as M^Φ is in general not closed for $\mathbb{P}$-a.s. convergence). The conclusion is that $\mathbf{Y}$ satisfies all the conditions in the definition of $\rho_B(\mathbf{X})$, with the only exception for the integrability condition $\mathbf{Y}\in M^\Phi$, which is replaced by $\mathbf{Y}\in L^1(\mathbb{P};\mathbb{R}^N)$. In the next subsection we will show when such $\mathbf{Y}$ also belongs to $\mathcal{C}_0\cap L^1(\mathbf{Q}_\mathbf{X};\mathbb{R}^N)$.

It is now evident that when the cardinality of Ω is finite and the set $\mathcal{C}$ is closed for $\mathbb{P}$-a.s. convergence, then the random vector $\mathbf{Y}$ in Theorem (4.17) belongs to $\mathcal{C}$ and $\mathbf{Y}=\mathbf{Y}_\mathbf{X}=\mathbf{Y}_{\mathbf{Q}_\mathbf{X}}$.

Proof. Take a sequence of vectors $(\mathbf{V}_k)_{k\in\mathbb{N}}\in\mathcal{C}\subseteq\mathcal{C}_\mathbb{R}\cap M^\Phi\subseteq L^1(\mathbb{P};\mathbb{R}^N)$ such that $\mathbb{R}\ni c_k := \sum_{n=1}^N V_k^n \downarrow \rho_B(\mathbf{X})$ as $k\to+\infty$ and $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n{+}V_k^n)\right]\geq B$. The sequence $(\mathbf{V}_k)_{k\in\mathbb{N}}$ is bounded for the $L^1(\mathbb{P};\mathbb{R}^N)$ norm if and only if so is the sequence $(\mathbf{X}+\mathbf{V}_k)_{k\in\mathbb{N}}$. Given the following decomposition in positive and negative part

$$\sum_{n=1}^{N}\mathbb{E}[|X^n{+}V_k^n|] = \sum_{n=1}^{N}\mathbb{E}[(X^n{+}V_k^n)^+] + \sum_{n=1}^{N}\mathbb{E}[(X^n{+}V_k^n)^-], \tag{4.19}$$

we define the index sets:

$$\begin{aligned} N_\infty^+ &= \left\{n\in\{1,...,N\}\mid \limsup_{k\to+\infty}\mathbb{E}[(X^n{+}V_k^n)^+] = +\infty\right\},\\ N_b^+ &= \left\{n\in\{1,...,N\}\mid \limsup_{k\to+\infty}\mathbb{E}[(X^n{+}V_k^n)^+] < +\infty\right\},\end{aligned}$$

and, similarly, N_∞^- and N_b^- for the negative part. We can split the expression (4.19) as

$$\sum_{n\in N_\infty^+} E_\mathbb{P}[(X^n{+}V_k^n)^+] + \sum_{n\in N_b^+} E_\mathbb{P}[(X^n{+}V_k^n)^+] + \sum_{n\in N_\infty^-} E_\mathbb{P}[(X^n{+}V_k^n)^-] + \sum_{n\in N_b^-} E_\mathbb{P}[(X^n{+}V_k^n)^-].$$

If the sequence $(\mathbf{X}+\mathbf{V}_k)_{k\in\mathbb{N}}$ is not $L^1(\mathbb{P};\mathbb{R}^N)$ -bounded, then one of the sets N_∞^+ or N_∞^- must be nonempty and therefore, because of the constraint $\sum_{n=1}^N V_k^n = c_k$, both N_∞^+ and

N_∞^- must be nonempty. From Lemma A.8 and from Lemma A.1 with $M := 2A$, by Jensen inequality and (4.18) we obtain

$$\begin{aligned}
B &\le \sum_{n=1}^N \mathbb{E}[u_n(X^n+V_k^n)] \le \sum_{n=1}^N u_n\left(\mathbb{E}[X^n+V_k^n]\right)\\
&= \sum_{n=1}^N u_n\left(\mathbb{E}[(X^n+V_k^n)^+]\right) + \sum_{n=1}^N u_n\left(-\mathbb{E}[(X^n+V_k^n)^-]\right)\\
&\le A\left(\sum_{n\in N_\infty^+}\mathbb{E}[(X^n+V_k^n)^+] + \sum_{n\in N_b^+}\mathbb{E}[(X^n+V_k^n)^+]\right)\\
&\quad -2A\left(\sum_{n\in N_\infty^-}\mathbb{E}[(X^n+V_k^n)^-] + \sum_{n\in N_b^-}\mathbb{E}[(X^n+V_k^n)^-]\right) + const\\
&= A\left(c_k + \sum_{n=1}^N \mathbb{E}[X^n]\right) + const - A\left(\sum_{n\in N_\infty^-}\mathbb{E}[(X^n+V_k^n)^-] + \sum_{n\in N_b^-}\mathbb{E}[(X^n+V_k^n)^-]\right)
\end{aligned}$$

which is a contradiction, as the second term that multiplies A in not bounded from above. Hence we exclude that our minimizing sequence $(\mathbf{V}_k)_{k\in\mathbb{N}}$ has unbounded $L^1(\mathbb{P};\mathbb{R}^N)$ norm and we may apply a Komlós compactness argument, as stated below in Theorem 4.19, with $E = \mathbb{R}^N$. Applying this result to the sequence $(\mathbf{V}_k)_{k\in\mathbb{N}} \in \mathcal{C}$, we can find a sequence $\mathbf{Y}_k \in conv(\mathbf{V}_i, i \ge k) \in \mathcal{C}$, as $\mathcal{C}$ is convex, such that

$$\mathbf{Y}_k \text{ converges } \mathbb{P}\text{-}a.s. \text{ to } \mathbf{Y} \in L^1(\mathbb{P};\mathbb{R}^N).$$

Observe that by construction $\sum_{n=1}^N Y_k^n$ is $\mathbb{P}$-a.s. a real number and, as a consequence, so is $\sum_{n=1}^N Y^n$. As $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+V_k^n)\right] \ge B$, also the $\mathbf{Y}_k$ satisfy such constraint and therefore $\rho_B(\mathbf{X}) \le \sum_{n=1}^N Y_k^n$.

Let $\mathbf{Y}_k = \sum_{i\in J_k} \lambda_i^k \mathbf{V}_i \in conv(\mathbf{V}_i, i \ge k)$, for some finite convex combination $(\lambda_i^k)_{i\in J_k}$ such that $\lambda_i^k > 0$ and $\sum_{i\in J_k} \lambda_i^k = 1$, where J_k is a finite subset of $\{k, k+1, ...\}$. For any fixed k we compute

$$\sum_{n=1}^N Y_k^n = \sum_{n=1}^N \left(\sum_{i\in J_k} \lambda_i^k V_i^n\right)_j = \sum_{i\in J_k} \lambda_i^k \left(\sum_{n=1}^N V_i^n\right) = \sum_{i\in J_k} \lambda_i^k c_i \le c_k \left(\sum_{i\in J_k} \lambda_i^k\right) = c_k \quad (4.20)$$

and from $\rho_B(\mathbf{X}) \le \sum_{n=1}^N Y_k^n \le c_k$, we then deduce that $\sum_{n=1}^N Y^n = \rho_B(\mathbf{X})$.

We now show that $\mathbf{Y}$ also satisfies the budget constraint. In case that all utility functions are bounded from above, this is an immediate consequence of Fatou Lemma, since

$$\sum_{n=1}^N \mathbb{E}[-u_n(X^n+Y^n)] = \sum_{n=1}^N \mathbb{E}[\underline{\lim}_{k\to\infty}\,(-u_n(X^n+Y_k^n))] \le \underline{\lim}_{k\to\infty} \sum_{n=1}^N \mathbb{E}[-u_n(X^n+Y_k^n)] \le B.$$

In the general case, recall first that the sequence $\mathbf{V}_k$ is bounded in $L^1(\mathbb{P};\mathbb{R}^N)$, and the argument used in (4.20) shows that

$$\| \mathbf{X}+\mathbf{Y}_k \|_1 \leq \| \mathbf{X} \|_1 + \sup_k \| \mathbf{V}_k \|_1,$$

hence $\sup_k \| \mathbf{X}+\mathbf{Y}_k \|_1 < \infty$.

Now we need to exploit the Inada condition at $+\infty$. Applying the Lemma A.9 to the utility functions u_n, assumed null in 0, we get

$$-u_n(x) + \varepsilon x^+ + b(\varepsilon) \geq 0 \; \forall x \in \mathbb{R}.$$

Replacing $\mathbf{X}+\mathbf{Y}$ in the expression above, applying Fatou Lemma we have

$$\begin{aligned}
&\mathbb{E}\left[\sum_{n=1}^{N} -u_n(X^n+Y^n) + \varepsilon(X^n+Y^n)^+ + b(\varepsilon)\right] \\
&= \mathbb{E}\left[\underline{\lim}_{k\to\infty}\left(\sum_{n=1}^{N} -u_n(X^n+Y_k^n) + \varepsilon(X^n+Y_k^n)^+ + b(\varepsilon)\right)\right] \\
&\leq \underline{\lim}_{k\to\infty}\sum_{n=1}^{N}\mathbb{E}\left[-u_n(X^n+Y_k^n) + \varepsilon(X^n+Y_k^n)^+ + b(\varepsilon)\right] \leq -B + \varepsilon\left(\sup_k \| \mathbf{X}+\mathbf{Y}_k \|_1\right) + b(\varepsilon).
\end{aligned}$$

As the term $b(\varepsilon)$ simplifies in the above inequality, we conclude that for all $\varepsilon > 0$

$$\mathbb{E}\left[\sum_{n=1}^{N} -u_n(X^n+Y^n)\right] \leq -B + \varepsilon\left(\sup_k \| \mathbf{X}+\mathbf{Y}_k \|_1 - \sum_{n=1}^{N}\mathbb{E}\left[(X^n+Y^n)^+\right]\right),$$

and since $\sup_k \| \mathbf{X}+\mathbf{Y}_k \|_1 < \infty$ we obtain

$$\mathbb{E}\left[\sum_{n=1}^{N} -u_n(X^n+Y^n)\right] \leq -B,$$

so that $\mathbf{Y}$ satisfies the constraint. □

Theorem 4.19 (Theorem 1.4 [20]). *Let E be a Banach reflexive space and $(f_k)_k \subseteq L^1((\Omega,\mathcal{F},\mathbb{P});E) := L^1$ be a sequence with bounded L^1 norms. Then there exists a sequence $(g_k)_k$ and g_0 in L^1 such that $g_k \in conv(f_i, i \geq k)$ and $\| g_k - g \|_E \to 0$ $\mathbb{P}-a.s.$, as $k \to \infty$.*

4.3.1 On the integrability in $L^1(\mathbf{Q}_\mathbf{X})$

We proved in Theorem 4.17 the existence of $\mathbf{Y}$ satisfying

$$\rho_B(\mathbf{X}) := \inf\left\{\sum_{n=1}^{N} Z_n \mid \mathbf{Z} \in \mathcal{C},\, \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n+Z^n)\right] \geq B\right\} = \sum_{n=1}^{N} Y^n$$

with $\mathbf{Y} \in L^1(\mathbb{P};\mathbb{R}^N)$, $\sum_{n=1}^N Y^n \in \mathbb{R}$, $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n+Y^n)\right] \geq B$ and $\mathbf{Y}$ is the $\mathbb{P}$-a.s. limit of a sequence in $\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}} \cap M^{\Phi}$.

We show in the Lemma below that this already implies that the negative part $\mathbf{Y}$ is $\mathbf{Q_X}$ integrable. Recall that $\frac{d\mathbf{Q_X}}{d\mathbb{P}} \in L^{\Phi^*}$ and so, from Remark 3.7, $\mathbf{X} \in M^{\Phi} \subseteq L^{\Phi} \subseteq L^1(\mathbf{Q_X};\mathbb{R}^N) \cap L^1(\mathbb{P};\mathbb{R}^N)$.

Lemma 4.20. *For all* $1 \leq j \leq N$

$$(Y^j)^- \in L^1(Q^j_{\mathbf{X}}),\ 1 \leq j \leq N.$$

Proof. Applying (4.18) and $\phi_j(x) := -u_j(-|x|)$, note that for each fixed $1 \leq j \leq N$

$$\begin{aligned} 0 \ &\leq\ \mathbb{E}\left[\phi_j((X^j+Y^j)^-)\right] \leq \sum_{n=1}^N \mathbb{E}\left[\phi_n((X^n+Y^n)^-)\right] = \sum_{n=1}^N \mathbb{E}\left[-u_n(-(X^n+Y^n)^-)\right] \\ &=\ \sum_{n=1}^N \mathbb{E}\left[u_n(X^n+Y^n)^+\right] - \sum_{n=1}^N \mathbb{E}\left[u_n(X^n+Y^n)\right] \leq \sum_{n=1}^N u_n\left(\mathbb{E}\left[(X^n+Y^n)^+\right]\right) - B < \infty, \end{aligned}$$

where we used Jensen inequality and $\mathbf{X}+\mathbf{Y} \in L^1(\mathbb{P};\mathbb{R}^N)$. This yields $(X^j+Y^j)^- \in L^{\phi_j}$. Since $L^{\phi_j} \subseteq L^1(Q^j_{\mathbf{X}})$, we deduce

$$(X^j+Y^j)^- \in L^1(Q^j_{\mathbf{X}}),\ 1 \leq j \leq N.$$

From $Y^j = (X^j+Y^j)^+ - (X^j+Y^j)^- - X^j \geq -(X^j+Y^j)^- - X^j$ we get

$$0 \leq (Y^j)^- \leq (-(X^j+Y^j)^- - X^j)^- = ((X^j+Y^j)^- + X)^+.$$

Since, by assumption, $X^j \in M^{\phi_j} \subseteq L^1(Q^j_{\mathbf{X}})$, then also $((X^j+Y^j)^- + X^j)^+ \in L^1(Q^j_{\mathbf{X}})$ and so

$$(Y^j)^- \in L^1(Q^j_{\mathbf{X}}),\ 1 \leq j \leq N.$$

□

Proposition 4.21. *Let* $\mathcal{C} = \mathcal{C}^{(\mathbf{n})}$*, as in Definition 3.2, and* $\mathbf{Y}$ *be the random vector in Theorem 4.17. Then* $(Y^j)^+ \in L^1(Q^j_{\mathbf{X}})$*,* $1 \leq j \leq N$*.*

Proof. Let $\mathcal{C}^{(\mathbf{n})} = \mathcal{C}_0^{(\mathbf{n})} \cap M^{\Phi}$. As $\mathcal{C}_0^{(\mathbf{n})}$ is closed for the convergence in probability, and $\mathbf{Y}$ is the $\mathbb{P}$-a.s. limit of a sequence in $\mathcal{C}$, then $\mathbf{Y} \in \mathcal{C}_0^{(\mathbf{n})}$, and so

$$\rho_B(\mathbf{X}) = \sum_{n=1}^N Y^n = \sum_{m=1}^h d_m, \text{ where } d_m = \sum_{i \in I_m} Y^i.$$

From

$$d_m = \sum_{i \in I_m} Y^i = \sum_{i \in I_m} (X^i+Y^i)^+ - \sum_{i \in I_m} (X^i+Y^i)^- - \sum_{i \in I_m} X^i$$

we deduce for $j \in I_m$ that

$$0 \leq (X^j + Y^j)^+ \leq \sum_{i \in I_m} (X^i + Y^i)^+ = d_m + \sum_{i \in I_m} (X^i + Y^i)^- + \sum_{i \in I_m} X^I,$$

with $(X^i + Y^i)^- \in L^{\phi_i} \subseteq L^1(Q^i_{\mathbf{X}})$ and $X^i \in M^{\phi_i} \subseteq L^1(Q^i_{\mathbf{X}})$. In order to conclude that $(Y^j)^+ \in L^1(Q^j_{\mathbf{X}})$ it is sufficient to show $(X^j + Y^j)^+ \in L^1(Q^j_{\mathbf{X}})$, but the inequality above shows that we need some information on

$$\bigoplus_{i \in I_m} L^1(Q^i_{\mathbf{X}}).$$

However, in case $\mathcal{C} = \mathcal{C}^{(\mathbf{n})}$, $Q^i_{\mathbf{X}} = Q^j_{\mathbf{X}}$ for all $i, j \in I_m$, hence $\bigoplus_{i \in I_m} L^1(Q^i_{\mathbf{X}}) = L^1(Q^m_{\mathbf{X}})$. Then we conclude that $(Y^j)^+ \in L^1(Q^m_{\mathbf{X}})$ for all $j \in I_m$ and $\mathbf{Y}^+ \in L^1(\mathbf{Q_X})$. $\square$

Recall that if ϕ_1, ϕ_2 are two Young functions such that $\phi_1 \preceq \phi_2$, (i.e. there exist $a > 0$, $b > 0$, $y_0 > 0$ such that $b\phi_2(ay) \geq \phi_1(y)$ for all $y \geq y_0$) then the conjugates satisfy $\phi_2^* \preceq \phi_1^*$, $L^{\phi_2} \subseteq L^{\phi_1}$ and $M^{\phi_2} \subseteq M^{\phi_1}$.

Proposition 4.22. *Let $\mathcal{C} = \mathcal{C}_0 \cap M^\Phi$ and suppose that $\mathcal{C}_0 \subseteq \mathcal{C}_\mathbb{R} \subseteq L^0(\mathbb{R}^N)$ is closed for the convergence in probability. Assume that*

$$\phi_1 \preceq \phi_2 \preceq \dots \preceq \phi_N \preceq \phi_1, \tag{4.21}$$

where ϕ_n is the Young function associated to u_n, $n = 1, \cdots, N$. Then the random vector $\mathbf{Y}$ in Theorem 4.17 satisfies

$$\mathbf{Y} \in L^\Phi \subseteq L^1(\mathbf{Q_X}).$$

Proof. As $\mathcal{C}_0$ is closed for the convergence in probability, the random vector $\mathbf{Y}$ in Theorem 4.17 satisfies $\mathbf{Y} \in \mathcal{C}_0$ and $(\mathbf{Y})^- \in L^1(\mathbf{Q_X})$. The same argument applied in Proposition 4.21 shows that

$$0 \leq (X^j + Y^j)^+ \leq \sum_{i=1}^n (X^i + Y^i)^+ = \rho_B(\mathbf{X}) + \sum_{i=1}^n (X^i + Y^i)^- + \sum_{i=1}^n X^i.$$

In order to obtain $Y^j \in L^1(Q^j_{\mathbf{X}})$, it is then sufficient to have conditions on u_n that guarantee

$$\bigoplus_{i=1,\dots,N} L^{\Phi_i} \subseteq L^1(Q^j_{\mathbf{X}}).$$

Under our assumptions $L^{\phi_i} = L^{\phi_j} := L^{\phi_0}$, for any i, j, and hence we obtain

$$\bigoplus_{i=1,\dots,N} L^{\phi_i} = L^{\phi_0}$$

and the conclusion follows. $\square$

Remark 4.23. The Young functions associated to the exponential utility functions $u_n(x) = -e^{-\alpha_n x}$, $\alpha_n > 0$, $n = 1, \cdots, N$, satisfies assumption (4.21). This case is treated in Section 6.

4.4 On local cash additivity and marginal risk contribution

We now show when the systemic risk measures of the form (3.3) are cash additive and local cash additive.

Lemma 4.24. *Define*

$$\mathcal{W}_{\mathcal{C}} := \{\mathbf{Z} \in \mathcal{C}_{\mathbb{R}} \mid \mathbf{Y} \in \mathcal{C} \Longleftrightarrow \mathbf{Y} - \mathbf{Z} \in \mathcal{C}\} \cap \mathcal{L}.$$

Then the risk measure ρ *defined in* (3.3) *is cash additive on* $\mathcal{W}_{\mathcal{C}}$*, i.e.,*

$$\rho(\mathbf{X} + \mathbf{Z}) = \rho(\mathbf{X}) - \sum_{n=1}^{N} Z^n \text{ for all } \mathbf{Z} \in \mathcal{W}_{\mathcal{C}} \text{ and all } \mathbf{X} \in \mathcal{L}.$$

Proof. Let $\mathbf{Z} \in \mathcal{W}_{\mathcal{C}}$. Then $\mathbf{W} := \mathbf{Z} + \mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$ for any $\mathbf{Y} \in \mathcal{C}$. For any $\mathbf{X} \in \mathcal{L}$ it holds

$$\begin{aligned}
\rho(\mathbf{X} + \mathbf{Z}) &= \inf\{\sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C},\ \Lambda(\mathbf{X} + \mathbf{Z} + \mathbf{Y}) \in \mathbb{A}_B\} \\
&= \inf\{\sum_{n=1}^{N} W^n - \sum_{n=1}^{N} Z^n \mid \mathbf{W} - \mathbf{Z} \in \mathcal{C},\ \Lambda(\mathbf{X} + \mathbf{W}) \in \mathbb{A}_B\} \\
&= \inf\{\sum_{n=1}^{N} W^n - \sum_{n=1}^{N} Z^n \mid \mathbf{W} \in \mathcal{C},\ \Lambda(\mathbf{X} + \mathbf{W}) \in \mathbb{A}_B\} \\
&= \rho(\mathbf{X}) - \sum_{n=1}^{N} Z^n.
\end{aligned}$$

□

Example 4.25. In case of the set $\mathcal{C}^{(\mathbf{n})}$ in Example 3.1, ρ is cash additive on

$$\mathcal{W}_{\mathcal{C}^{(\mathbf{n})}} = \mathcal{C}^{(\mathbf{n})}. \tag{4.22}$$

Note that equality (4.22) holds since we are assuming no restrictions on the vector $d = (d, \cdots, d_m) \in \mathbb{R}^m$, which determines the grouping. If for example, we restrict d to have non negative components, then it is no longer true that $\mathcal{W}_{\mathcal{C}^{(\mathbf{n})}} = \mathcal{C}^{(\mathbf{n})}$.

Corollary 4.26. *For the systemic risk measures of the form* (3.3) *we have:*

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})|_{\varepsilon=0} = -\sum_{n=1}^{N} V^n \tag{4.23}$$

for all $\mathbf{V}$ *such that* $\varepsilon\mathbf{V} \in \mathcal{W}_{\mathcal{C}}$ *for all* $\varepsilon \in (0, 1]$.

Proof. It follows from Lemma 4.24 which gives $\rho(\mathbf{X}+\varepsilon\mathbf{V}) = \rho(\mathbf{X}) - \varepsilon \sum_{n=1}^{N} V^n$. □

Remark 4.27. Note that Lemma 4.24 and Corollary 4.26 hold for systemic risk measures of the general form (3.3), without Assumption 3.3. Under Assumption 3.3 we have $\mathbb{R}^N \subseteq \mathcal{W}_\mathcal{C}$ and (4.23) holds for all $\mathbf{V} \in \mathbb{R}^N$.

The expression $\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})|_{\varepsilon=0}$ represents the sensitivity of the risk $\mathbf{X}$ with respect to the impact $\mathbf{V} \in L^0(\mathbb{R}^N)$. In the case of a deterministic $\mathbf{V} := \mathbf{m} \in \mathbb{R}^N$, it was called *marginal risk contribution* in [3]. Such property cannot be immediately generalized to the case of random vectors $\mathbf{V}$, also because in general $\sum_{n=1}^N V^n \notin \mathbb{R}$. In the following, we obtain the general local version of cash additivity, which extends the concept of marginal risk contribution to a random setting. In particular, (4.24) shows how the change in one component affects the change of the systemic risk measure.

Proposition 4.28. *Let $\mathbf{V} \in M^\Phi$ and $\mathbf{X} \in M^\Phi$. Let $\mathbf{Q_X}$ be the optimal solution to the dual problem (4.1) associated to $\rho(\mathbf{X})$ and assume that $\rho(\mathbf{X}+\varepsilon\mathbf{V})$ is differentiable with respect to ε at $\varepsilon = 0$, and $\frac{d\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}}}{d\mathbb{P}} \to \frac{d\mathbf{Q_X}}{d\mathbb{P}}$ in $\sigma^*(L^{\Phi^*}, M^\Phi)$, as $\varepsilon \to 0$. Then,*

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})|_{\varepsilon=0} = -\sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[V^n]. \tag{4.24}$$

Proof. As the penalty function $\alpha_{\Lambda,B}$ does not depend on $\mathbf{X}$, by (4.4) we deduce

$$\begin{aligned}
\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\left\{\sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}+\varepsilon\mathbf{V}}}[-X^n-\varepsilon V^n] - \alpha_{\Lambda,B}(\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}})\right\}|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\left\{\sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}+\varepsilon\mathbf{V}}}[-X^n] - \alpha_{\Lambda,B}(\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}})\right\}|_{\varepsilon=0} \\
&\quad + \sum_{n=1}^N \frac{d}{d\varepsilon}\left(\varepsilon\mathbb{E}_{Q^n_{\mathbf{X}+\varepsilon\mathbf{V}}}[-V^n]\right)|_{\varepsilon=0} \qquad (4.25) \\
&= 0 + \sum_{n=1}^N \lim_{\varepsilon\to 0}\mathbb{E}_{Q^n_{\mathbf{X}+\varepsilon\mathbf{V}}}[-V^n] = \sum_{n=1}^N \mathbb{E}_{Q^n_{\mathbf{X}}}[-V^n], \qquad (4.26)
\end{aligned}$$

where the equality between (4.25) and (4.26) is justified by the optimality of $\mathbf{Q_X}$ and the differentiability of $\rho(\mathbf{X}+\varepsilon\mathbf{V})$, while the last equality is guaranteed by the convergence of $\frac{d\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}}}{d\mathbb{P}}$. □

Remark 4.29. We emphasize that the generalization (4.24) of (4.23) holds because we are computing the expectation with respect to the systemic probability measure $\mathbf{Q_X}$. A relevant example where the assumptions of Proposition 4.28 hold is provided in Section 6.

5 Fairness in the details

We now turn to the details of the introductory Section 2.

Remark 5.1. Fix $\mathbf{Q} = (Q^1, ..., Q^N)$ such that $\frac{d\mathbf{Q}}{d\mathbb{P}} \in L^{\Phi^*}$. Then

$$M^{\Phi} = \left\{ \mathbf{Y} = \mathbf{a} + \mathbf{Z} \mid \mathbf{a} \in \mathbb{R}^N \text{ and } \mathbf{Z} \in M^{\Phi} \text{ such that } \mathbb{E}_{Q^n}[Z^n] = 0 \text{ for each } n \right\}.$$

Indeed, just take $\mathbf{Y} \in M^{\Phi}$ and let $a^n := \mathbb{E}_{Q^n}[Y^n] \in \mathbb{R}$ and $Z^n := Y^n - a^n \in M^{\phi_n}$. This justifies equation (2.13) in Section 2:

$$\begin{aligned}
\pi_A^Q(\mathbf{X}) &= \sup_{\mathbf{Y} \in M^{\Phi}} \left\{ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n) \right] \mid \sum_{n=1}^N \mathbb{E}_{Q^n}[Y^n] = A \right\} \\
&= \sup_{\sum_{n=1}^N a^n = A,\ Z^n \in M^{\phi_n},\ \mathbb{E}_{Q^n}[Z^n] = 0\ \forall n} \left\{ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + a^n + Z^n) \right] \right\} \\
&= \sup_{\sum_{n=1}^N a^n = A,} \sum_{n=1}^N \sup_{Z^n \in M^{\phi_n},\ \mathbb{E}_{Q^n}[Y^n] = a_n} \mathbb{E}\left[u_n(X^n + Y^n) \right]
\end{aligned}$$

We now establish important relations between primal problems (2.1) and (2.2), and problems (2.5) and (2.6).

5.1 Relations between primal problems

Note that in this section, we do not assume the existence of an optimizer for problems (2.1) or (2.2). We work under Assumptions 4.8 and 4.9.
Let $A \in \mathbb{R}$, $B \in \mathbb{R}$. As u_n is increasing, in both problems (2.1) and (2.2) we may replace the inequality in the constraints with an equality, and due to strict concavity the solution, if it exists, is unique (see Proposition 3.5). Recall that under Assumptions 3.3, $\mathcal{C}$ is a convex cone and therefore, if $\mathbf{Y} \in \mathcal{C}$, then $\mathbf{Y} + \delta \in \mathcal{C}$ for every deterministic $\delta \in \mathbb{R}^N$.

Proposition 5.2. *$B = \pi_A(\mathbf{X})$ if and only if $A = \rho_B(\mathbf{X})$, and in this case the unique optimal solution, if it exists, is the same for the two problems $\pi_A(\mathbf{X})$ and $\rho_B(\mathbf{X})$.*

Proof. $\Leftarrow$) Let $A = \rho_B(\mathbf{X})$ and suppose first that $\pi_A(\mathbf{X}) > B$. Then there must exist $\tilde{\mathbf{Y}} \in \mathcal{C}$ such that $\sum_{n=1}^N \tilde{Y}^n \leq A$ and $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] > B$. By continuity of u_n, then there exists $\epsilon > 0$ and $\widehat{\mathbf{Y}} := \tilde{\mathbf{Y}} - \epsilon \mathbf{1}$ such that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \tilde{Y}^n)\right] \geq B$ and $\sum_{n=1}^N \widehat{Y}^n < A$. This is in contradiction with $A = \rho_B(\mathbf{X})$.

Suppose now that $\pi_A(\mathbf{X}) < B$. Then there exists $\delta > 0$ such that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \leq B - \delta$ for all $\mathbf{Y} \in \mathcal{C}$ such that $\sum_{n=1}^N Y^n \leq A$. As $A = \rho_B(\mathbf{X})$, for all $\epsilon > 0$, there exists $\mathbf{Y}_\epsilon \in \mathcal{C}$ such that $\sum_{n=1}^N Y_\epsilon^n \leq A + \epsilon$ and $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y_\epsilon^n)\right] \geq B$. For any $\eta \geq \epsilon \geq \sum_{n=1}^N Y_\epsilon^n - A$ we get $\sum_{n=1}^N (Y_\epsilon^n - \frac{\eta}{N}) \leq A + \epsilon - \eta \leq A$. By continuity of u_n, we may select $\epsilon > 0$ and $\eta \geq \epsilon$ small enough so that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y_\epsilon^n - \frac{\eta}{N})\right] > B - \delta$. As $\widehat{\mathbf{Y}} := (Y_\epsilon^n - \frac{\eta}{N})_n \in \mathcal{C}$, we obtain a contradiction.

Suppose that there exists $\mathbf{Y} \in \mathcal{C}$ that is the optimal solution of problem (2.1). As $A := \rho_B(\mathbf{X})$, then $\sum_{n=1}^N Y^n = A$ and the constraint in problem (2.2) is fulfilled for $\mathbf{Y}$.

Hence, $B = \pi_A(\mathbf{X}) \geq \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \geq B$ and we deduce that $\mathbf{Y}$ is an optimal solution of problem (2.2).

$\Rightarrow$) Let $B = \pi_A(\mathbf{X})$ and suppose first that $\rho_B(\mathbf{X}) < A$. Then, there must exist $\tilde{\mathbf{Y}} \in \mathcal{C}$ such that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \widetilde{Y}^n)\right] \geq B$ and $\sum_{n=1}^N \widetilde{Y}^n < A$. Then, there exists $\epsilon > 0$ and $\widehat{\mathbf{Y}} := \tilde{\mathbf{Y}} + \epsilon \mathbf{1} \in \mathcal{C}$ such that $\sum_{n=1}^N \widehat{Y}^n \leq A$ and $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \widehat{Y}^n)\right] > B$. This is in contradiction with $B = \pi_A(\mathbf{X})$.

Suppose now that $\rho_B(\mathbf{X}) > A$. Then, there exists $\delta > 0$ such that $\sum_{n=1}^N Y^n \geq A + \delta$ for all $\mathbf{Y} \in \mathcal{C}$ such that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] \geq B$. As $B = \pi_A(\mathbf{X})$, for all $\epsilon > 0$ there exists $\mathbf{Y}_\epsilon \in \mathcal{C}$ such that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y_\epsilon^n)\right] > B - \epsilon$, and $\sum_{n=1}^N Y_\epsilon^n \leq A$. Define $\eta_\epsilon := \inf\left\{a > 0 : \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y_\epsilon^n + \frac{a}{N})\right] \geq B\right\}$ and note that $\eta_\epsilon \downarrow 0$ if $\epsilon \downarrow 0$. Select $\epsilon > 0$ such that $\eta_\epsilon < \delta$. Then, for any $0 < \beta < \delta - \eta_\epsilon$ we have: $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y_\epsilon^n + \frac{\eta_\epsilon+\beta}{N})\right] \geq B$ and $\sum_{n=1}^N (Y_\epsilon^n + \frac{\eta_\epsilon+\beta}{N}) \leq A + \eta_\epsilon + \beta < A + \delta$. As $(Y_\epsilon^n + \frac{\eta_\epsilon+\beta}{N}) \in \mathcal{C}$, we obtain a contradiction.

Suppose that there exists $\mathbf{Y} \in \mathcal{C}$ that is the optimal solution of problem (2.2) and set $B := \pi_A(\mathbf{X})$. Then $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] = B$ and the constraint in problem (2.1) is fulfilled for $\mathbf{Y}$. Hence, $A = \rho_B(\mathbf{X}) \leq \sum_{n=1}^N Y^n \leq A$ and we deduce that $\mathbf{Y}$ is an optimal solution of problem (2.1). As $\rho_B(\mathbf{X})$ admits at most one solution by Proposition 3.5, the same must be true for $\pi_A(\mathbf{X})$. □

Now consider the situation where a valuation operator $\mathbf{Q} = (Q^1, \cdots, Q^N)$ such that $\frac{d\mathbf{Q}}{d\mathbb{P}} \in L^{\Phi^*}$ is given for the system. First, we note that problem (2.5) admits a unique optimal solution denoted by $\mathbf{Y}_\mathbf{Q}$. Existence is given in Section 4.2 and uniqueness was derived in remarks 4.5 and 4.14.

Then, similarly as in Proposition 5.2, we obtain

Proposition 5.3. $B = \pi_A^{\mathbf{Q}}(\mathbf{X})$ *if and only if* $A = \rho_B^{\mathbf{Q}}(\mathbf{X})$ *and the two problems have the same optimal solution* $\mathbf{Y}_\mathbf{Q}$.

Proof. $\Leftarrow$) Let $\mathbf{Z}$ be an optimal solution of problem (2.5) and set $A := \rho_B^Q(\mathbf{X})$. Then $\sum_{n=1}^N \mathbb{E}_{Q^n}[Z^n] = A$ and the constraint in problem (2.6) is fulfilled for $\mathbf{Z}$. Hence, $\pi_A^Q(\mathbf{X}) \geq \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Z^n)\right] \geq B$. If $\pi_A^Q(\mathbf{X}) > B$, then there exists $\widetilde{\mathbf{Z}}$ such that $\sum_{n=1}^N \mathbb{E}_{Q^n}\left[\widetilde{Z}^n\right] \leq A$ and $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \widetilde{Z}^n)\right] > B$. Then, there exists $\epsilon > 0$ and $\widehat{\mathbf{Z}} := \widetilde{\mathbf{Z}} - \epsilon \mathbf{1}$ such that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \widehat{Z}^n)\right] \geq B$ and $\sum_{n=1}^N \mathbb{E}_{Q^n}\left[\widehat{Z}^n\right] < A$. This is in contradiction with $A = \rho_B^Q(\mathbf{X})$. Hence, $\pi_A^Q(\mathbf{X}) = B$ and $\pi_A^Q(\mathbf{X}) = \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Z^n)\right]$, and therefore, $\mathbf{Z}$ is an optimal solution of problem (2.6).

$\Rightarrow$) Let $\mathbf{Z}$ be an optimal solution of problem (2.6) and set $B := \pi_A^Q(\mathbf{X})$. Then we have $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Z^n)\right] = B$ and the constraint in problem (2.5) is fulfilled for $\mathbf{Z}$. Hence, $\rho_B^Q(\mathbf{X}) \leq \sum_{n=1}^N \mathbb{E}_{Q^n}[Z^n] \leq A$. If $\rho_B^Q(\mathbf{X}) < A$, then, $\exists \widetilde{\mathbf{Z}}$ such that $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \widetilde{Z}^n)\right] \geq B$ and $\sum_{n=1}^N \mathbb{E}_{Q^n}\left[\widetilde{Z}^n\right] < A$. Then, $\exists \epsilon > 0$ and $\widehat{\mathbf{Z}} := \widetilde{\mathbf{Z}} + \epsilon \mathbf{1}$ such that $\sum_{n=1}^N \mathbb{E}_{Q^n}\left[\widehat{Z}^n\right] \leq A$ and $\mathbb{E}\left[\sum_{n=1}^N u_n(X^n + \widehat{Z}^n)\right] > B$. This is in contradiction with $B = \pi_A^Q(\mathbf{X})$. Hence, $\rho_B^Q(\mathbf{X}) = A$ so that $\rho_B^Q(\mathbf{X}) = \sum_{n=1}^N \mathbb{E}_{Q^n}[Z^n]$ and $\mathbf{Z}$ is an optimal solution of problem (2.5). □

Recall that we denote by $\mathbf{Q_X} = (Q^1_{\mathbf{X}}, \cdots Q^N_{\mathbf{X}})$ an optimizer of the dual problem of (2.1), presented in detail in Section 4.1 and in Corollary 4.16.

The key relation (2.8) was shown in Proposition 4.3. We now prove the other key relation (2.9).

Proposition 5.4. *$\pi_A(\mathbf{X}) = \pi_A^{Q_{\mathbf{X}}}(\mathbf{X})$.*

Proof. Set $A := \rho_B(\mathbf{X})$. Then

$$\begin{aligned} A := \rho_B(\mathbf{X}) = \rho_B^{\mathbf{Q_X}}(\mathbf{X}), &\qquad \text{(by Proposition 4.3)},\\ B = \pi_A(\mathbf{X}), &\qquad \text{(by Proposition 5.2)},\\ B = \pi_A^{\mathbf{Q_X}}(\mathbf{X}), &\qquad \text{(by Proposition 5.3)}, \end{aligned}$$

and therefore, $\pi_A(\mathbf{X}) = \pi_A^{\mathbf{Q_X}}(\mathbf{X})$. □

5.1.1 Extension to integrable random variables

Define

$$\begin{aligned} \widetilde{\rho}_B(\mathbf{X}) \ &:= \inf_{\mathbf{Z} \in L^1(\mathbf{Q_X};\mathbb{R}^N)} \left\{ \sum_{n=1}^N Z^n \mid \mathbf{Z} \in \mathcal{C}_0,\, \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Z^n)\right] \geq B \right\},\\ \widetilde{\pi}_A(\mathbf{X}) \ &:= \sup_{Z \in L^1(\mathbf{Q_X};\mathbb{R}^N)} \left\{ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Z^n)\right] \mid \mathbf{Z} \in \mathcal{C}_0,\, \sum_{n=1}^N Z^n \leq A \right\} \end{aligned}$$

and

$$\begin{aligned} \widetilde{\rho}_B^{\mathbf{Q}}(\mathbf{X}) \ &:= \inf_{\mathbf{Z} \in L^1(\mathbf{Q};\mathbb{R}^N)} \left\{ \sum_{n=1}^N \mathbb{E}_{Q^n}[Z^n] \mid \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Z^n)\right] \geq B \right\},\\ \widetilde{\pi}_A^{\mathbf{Q}}(\mathbf{X}) \ &:= \sup_{\mathbf{Z} \in L^1(\mathbf{Q};\mathbb{R}^N)} \left\{ \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Z^n)\right] \mid \sum_{n=1}^N \mathbb{E}_{Q^n}[Z^n] \leq A \right\}. \end{aligned}$$

With the same arguments applied in the proof of Proposition 5.3 we can show that

$$B = \widetilde{\pi}_A^{\mathbf{Q}}(\mathbf{X}) \Leftrightarrow A = \widetilde{\rho}_B^{\mathbf{Q}}(\mathbf{X}). \tag{5.1}$$

Furthermore, the extension to $L^1(\mathbf{Q};\mathbb{R}^N)$ does not increment the optimal value of $\pi_A^{\mathbf{Q}}(\mathbf{X})$.

Proposition 5.5.

$$\widetilde{\pi}_A^Q(\mathbf{X}) = \pi_A^{\mathbf{Q}}(\mathbf{X}). \tag{5.2}$$

Proof. From Remark 5.1 and Proposition A.10 we deduce:

$$\begin{aligned} \pi_A^Q(\mathbf{X}) &= \sup_{\sum_{n=1}^N a^n = A,} \sum_{n=1}^N \sup_{Y^n \in M^{\phi_n},\ E_{Q^n}[Y^n]=a_n} \mathbb{E}\left[u_n(X^n+Y^n)\right] = \sup_{\sum_{n=1}^N a^n = A} \sum_{n=1}^N P_n(a_n) \\ &= \sup_{\sum_{n=1}^N a^n = A} \sum_{n=1}^N U_n(a_n) = \widetilde{\pi}_A^Q(\mathbf{X}), \end{aligned}$$

where we use the definitions in (A.10) and in (A.11) for $P_n(a_n)$ and $U_n(a_n)$. □

From (5.1), (5.2) and Proposition 5.3 we obtain

$$\widetilde{\rho}_B^{\mathbf{Q}}(\mathbf{X}) = \rho_B^{\mathbf{Q}}(\mathbf{X}). \tag{5.3}$$

The conclusion is that

$$\begin{aligned} \rho_B(\mathbf{X}) &= \rho_B^{\mathbf{Q_X}}(\mathbf{X}) = \widetilde{\rho}_B^{\mathbf{Q_X}}(\mathbf{X}), \\ \pi_A(\mathbf{X}) &= \pi_A^{Q_{\mathbf{X}}}(\mathbf{X}) = \widetilde{\pi}_A^{\mathbf{Q_X}}(\mathbf{X}). \end{aligned}$$

5.2 Fairness revisited

We assume that the primal problem (2.1) admits a solution, in other words an optimal allocation $\mathbf{Y_X} = (Y_{\mathbf{X}}^1, \cdots Y_{\mathbf{X}}^N) \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}} \cap M^{\Phi}$. By Proposition 3.5, this optimizer is unique. The question of its existence has been discussed in Section 4.3. The existence and uniquesness of the optimal solution $\mathbf{Q_X}$ to the dual problem was established in Proposition 4.1 and Corollary 4.16. In Section 4.2 we proved that $\mathbf{Y_X} = \mathbf{Y_{Q_X}}$ is optimal for $\rho^{\mathbf{Q_X}}(\mathbf{X})$. We proved in (4.17) that $(\mathbb{E}_{Q_{\mathbf{X}}^1}[Y_{\mathbf{X}}^1], \cdots, (\mathbb{E}_{Q_{\mathbf{X}}^N}[Y_{\mathbf{X}}^N])$ is a systemic risk allocation:

$$\sum_{n=1}^N \mathbb{E}_{Q_{\mathbf{X}}^n}[Y_{\mathbf{X}}^n] = \rho(\mathbf{X}).$$

Additionally, since $\sum_{n=1}^N Y_{\mathbf{X}}^n = \rho(\mathbf{X})$, the system clears at time T without any intervention of the central bank/regulator, which is fair from the point of view of the society. We refer to Section 2 for a detailed discussion, where we also argued that the risk allocation $(\mathbb{E}_{Q_{\mathbf{X}}^1}[Y_{\mathbf{X}}^1], \cdots, \mathbb{E}_{Q_{\mathbf{X}}^N}[Y_{\mathbf{X}}^N])$ is the fair valuation of the optimal allocation $(Y_{\mathbf{X}}^1, \cdots, Y_{\mathbf{X}}^N)$ from the point of view of the individual banks which perform their own utility maximization problems:

$$\forall n, \quad \sup_{Y^n} \left\{ \mathbb{E}\left[u_n(X^n+Y^n)\right] \mid \mathbb{E}_{Q_{\mathbf{X}}^n}[Y^n] = \mathbb{E}_{Q_{\mathbf{X}}^n}[Y_{\mathbf{X}}^n] \right\}.$$

6 The exponential case

In this section, we focus on a relevant case under Assumption 3.3, i.e., we set $\mathcal{C} = \mathcal{C}^{(\mathbf{n})}$, see Examples 3.1 and 4.6, and we choose $u_n(x) = -e^{-\alpha_n x}$, $\alpha_n > 0$, $n = 1, \cdots, N$, as in Example 4.12 . We select $B < \sum_{n=1}^N u_n(+\infty) = 0$. Under these assumptions, $\phi_n(x) := -u_n(-|x|) + u_n(0) = e^{\alpha_n |x|} - 1$,

$$M^{\phi_n} = M^{\phi_0} := \left\{ X \in L^0(\mathbb{R}) \mid \mathbb{E}[e^{c|X|}] < +\infty \text{ for all } c > 0 \right\},$$

the Orlicz Hearts M^{ϕ_n}, $n = 1, \cdots, N$, coincide with the single Orlicz Heart M^{ϕ_0} associated to the exponential function $\phi_0(x) := e^{|x|} - 1$ and the random variable $\overline{X} := \sum_n X^n \in M^{\phi_0}$ is well defined.

The systemic risk measure (3.5) becomes

$$\begin{aligned} \rho(\mathbf{X}) &= \inf \left\{ \sum_{n=1}^N Y^n \mid \mathbf{Y} \in \mathcal{C}^{(\mathbf{n})},\, \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n) \right] = B \right\}, \\ &= \inf \left\{ \sum_{n=1}^N Y^n \mid \mathbf{Y} \in \mathcal{C}^{(\mathbf{n})},\, \mathbb{E}\left[-\sum_{n=1}^N \exp\left[-\alpha_n (X^n + Y^n)\right] \right] = B \right\}. \end{aligned} \tag{6.1}$$

For a given partition $\mathbf{n}$ and allocations $\mathcal{C}^{(\mathbf{n})}$, we can explicitly compute the unique optimal allocation $\mathbf{Y}$ of (6.1) and the corresponding systemic risk

$$\rho(\mathbf{X}) = \sum_{i=1}^N Y^i = \sum_{m=1}^h d_m.$$

Theorem 6.1. *We have that*

$$d_m = \beta_m \log\left(-\frac{\beta}{B} \mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m} \right) \right] \right) - A_m \tag{6.2}$$

for $m = 1, \cdots, h$, *and*

$$Y^k_m = -X^k + \frac{1}{\beta_m \alpha_k} \overline{X}_m + \frac{1}{\beta_m \alpha_k} d_m + \left(\frac{1}{\beta_m \alpha_k} A_m - A^k_m \right) \in M^{\phi_0} \tag{6.3}$$

for $k \in I_m$, *where* $\overline{X}_m = \sum_{k \in I_m} X^k$ *and*

$$\beta_m = \sum_{k \in I_m} \frac{1}{\alpha_k}, \qquad \beta = \sum_{i=1}^N \frac{1}{\alpha_i},$$

$$A^k_m = \frac{1}{\alpha_k} \log\left(\frac{1}{\alpha_k} \right), \qquad A_m = \sum_{k \in I_m} A^k_m.$$

Proof. In Appendix A. □

Remark 6.2. (i) Notice that if we arbitrarily change the components of the vector $\mathbf{X}$, but keep fixed the components in one given subgroup, say I_{m_0}, then the risk measure $\rho(\mathbf{X})$ will of course change, but d_{m_0} and $Y^k_{m_0}$ for $k \in I_{m_0}$ remain the same.

(ii) If $B := \sum_{n=1}^{N} u_n(0) = -N$, then $\rho(\mathbf{0}) = 0$.

From Propositions 3.8, 3.5 and Theorem 6.1 we deduce

Proposition 6.3. *The map ρ in* (6.1) *is finitely valued, monotone decreasing, convex, continuous and subdifferentiable on the Orlicz Heart $M^{\Phi} = (M^{\phi_0})^N$, and it has a unique optimal solution.*

Define:

$$\frac{dQ_{\mathbf{X}}^m}{d\mathbb{P}} := \frac{e^{-\frac{1}{\beta_m}\overline{X}_m}}{\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]} \quad m = 1, \cdots, h. \tag{6.4}$$

Proposition 6.4. *The vector $\mathbf{Q_X}$ of probability measures with densities given by (6.4) is the optimal solution of the dual problem (4.8), i.e.,*

$$\rho(\mathbf{X}) = \sum_{m=1}^{h} \mathbb{E}_{Q_{\mathbf{X}}^m}[-\overline{X}_m] - \alpha_{\Lambda,B}(\mathbf{Q_X}),$$

and $\mathbb{E}_{Q_{\mathbf{X}}^m}[Y_{\mathbf{X}}^n]$, $m = 1, \cdots, h$, $n \in I_m$, is a systemic risk allocation, as in Definition 1.1.

Proof. First note that

$$\sum_{i \in I_m} \frac{1}{\alpha_i} \ln\left(-\frac{B}{\beta\alpha_i}\right) = -\beta_m \ln\left(-\frac{\beta}{B}\right) + A_m,$$

and

$$H\left(Q_{\mathbf{X}}^m, \mathbb{P}\right) = \mathbb{E}_{Q_{\mathbf{X}}^m}\left[\ln\left(\frac{dQ_{\mathbf{X}}^m}{d\mathbb{P}}\right)\right] = \frac{1}{\beta_m}\mathbb{E}_{Q_{\mathbf{X}}^m}\left[-\overline{X}_m\right] - \ln\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]. \tag{6.5}$$

By (4.13), $\alpha_{\Lambda,B}(\mathbf{Q_X})$ can be rewritten as

$$\begin{aligned}
\alpha_{\Lambda,B}(\mathbf{Q_X}) &= \sum_{m=1}^{h}\sum_{i\in I_m}\left\{\frac{1}{\alpha_i}H\left(Q_{\mathbf{X}}^m, \mathbb{P}\right) + \frac{1}{\alpha_i}\ln\left(-\frac{B}{\beta\alpha_i}\right)\right\} \\
&= \sum_{m=1}^{h}\left(\beta_m H\left(Q_{\mathbf{X}}^m, \mathbb{P}\right) + \sum_{i\in I_m}\frac{1}{\alpha_i}\ln\left(-\frac{B}{\beta\alpha_i}\right)\right) \\
&= \sum_{m=1}^{h}\left(\mathbb{E}_{Q_{\mathbf{X}}^m}\left[-\overline{X}_m\right] - \beta_m\ln\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right] - \beta_m\ln\left(-\frac{\beta}{B}\right) + A_m\right) \\
&= \sum_{m=1}^{h}\left(\mathbb{E}_{Q_{\mathbf{X}}^m}\left[-\overline{X}_m\right] - \beta_m\log\left(-\frac{\beta}{B}\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]\right) + A_m\right) \\
&= \sum_{m=1}^{h}\left(\mathbb{E}_{Q_{\mathbf{X}}^m}\left[-\overline{X}_m\right] - d_m\right) = \sum_{m=1}^{h}\mathbb{E}_{Q_{\mathbf{X}}^m}\left[-\overline{X}_m\right] - \rho(\mathbf{X}).
\end{aligned}$$

Remark 4.7 concludes the proof. □

6.1 Sensitivity analysis

Let $\mathbf{X} \in M^{\Phi}$, $\mathbf{V} \in M^{\Phi}$ and set $\overline{V}_m := \sum_{k \in I_m} V_k$, for $m = 1, \cdots, h$. We consider a perturbation $\varepsilon \mathbf{V}$, $\varepsilon \in \mathbb{R}$, and perform a sensitivity analysis in the exponential case. Consider the optimal allocations $Y^i_{\mathbf{X}+\varepsilon\mathbf{V}}$ and the optimal solution $\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}}$ of the dual problem associated to $\rho(\mathbf{X} + \varepsilon\mathbf{V})$, see (6.4). By (6.3) and (6.2) we have

$$Y^n_{\mathbf{X}+\varepsilon\mathbf{V}} = -X^n - \varepsilon V^n + \frac{1}{\beta_m \alpha_n}\left(\overline{X}_m + \varepsilon \overline{V}_m\right) + \frac{1}{\beta_m \alpha_n} d_m(\mathbf{X} + \varepsilon \mathbf{V}) + \left(\frac{1}{\beta_m \alpha_n} A_m - A^n_m\right), \tag{6.6}$$

where

$$d_m(\mathbf{X} + \varepsilon\mathbf{V}) = \beta_m \log\left(-\frac{\beta}{B}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m + \varepsilon \overline{V}_m}{\beta_m}\right)\right]\right) - A_m. \tag{6.7}$$

Proposition 6.5. *Let ρ be the systemic risk measure defined in (6.1). Then*

1. *Marginal risk contribution of group m:*

$$\frac{d}{d\epsilon} d_m(\mathbf{X} + \varepsilon \mathbf{V})\Big|_{\varepsilon=0} = \mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m],$$

2. *Local causal responsibility:*

$$\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}}}[Y^n_{\mathbf{X}+\varepsilon\mathbf{V}}]\Big|_{\varepsilon=0} = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n],$$

3. $\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Z]\Big|_{\varepsilon=0} = -\frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, Z]$, *for any $Z \in M^{\Phi}(\mathbb{R})$,*

4. *Marginal risk allocation of institution n:*

$$\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^n_{\mathbf{X}+\varepsilon\mathbf{V}}]\Big|_{\varepsilon=0} = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n] - \frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, Y^n_{\mathbf{X}}] \tag{6.8}$$

$$= \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n] + \frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, X^n] - \frac{1}{\alpha_n}\frac{1}{\beta_m}\frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, \overline{X}_m], \tag{6.9}$$

5. *Sensitivity of the penalty function:*

$$\frac{d}{d\varepsilon}\alpha_{\Lambda,B}(\mathbf{Q}_{\mathbf{X}+\varepsilon\mathbf{V}})\Big|_{\varepsilon=0} = \sum_{m=1}^{h} \frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[\overline{V}_m, \overline{X}_m],$$

6. *Systemic marginal risk contribution:*

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon\mathbf{V})\Big|_{\varepsilon=0} = \sum_{m=1}^{h}\sum_{i \in I_m} \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^i] = \sum_{m=1}^{h} \mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m].$$

The proof is postponed to the Appendix. The interpretation of these formulas is not simple because we are *dealing with the systemic probability measure* $Q^m_{\mathbf{X}}$ *and not with the "physical" measure* $\mathbb{P}$. Indeed, $Q^m_{\mathbf{X}}$ is the "artificial" measure that emerges from the dual optimization (think of the difference between the physical measure $\mathbb{P}$ and a martingale measure). To fix the idea, let us take $\mathbf{V}$ with only one component different from 0, so that we write $\mathbf{V} = V^j \mathbf{e}_j$. From Item 1 (or Item 6), we see that

$$\frac{d}{d\varepsilon}\rho(\mathbf{X}+\varepsilon V^j \mathbf{e}_j)\Big|_{\varepsilon=0} = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^j] \text{ (with } j \text{ belonging to the group } m).$$

From this, we can interpret $Q^m_{\mathbf{X}}$ as systemic risk evaluation (systemic probability measure): i.e. if the position changes from $\mathbf{X}$ to $\mathbf{X}+\varepsilon V^j \mathbf{e}_j$ then the riskiness of the entire system changes linearly by $\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^j]$. In the following discussion, we have to keep in mind that $Q^m_{\mathbf{X}}$ already represents the systemic view of the system. If we replacing $\mathbf{Q_X}$ with $\mathbb{P}$, none of the results of Proposition 6.5 will hold in general.

Remark 6.6. We now comment on the results of Proposition 6.5.
The first term $\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n]$ in (6.8) or (6.9) is easy to interpret: it is not a systemic contribution, as it only involves the increment V^n in the (same) bank n. If we sum over all n in the same group, we obtain from (6.8) or (6.9)

$$\sum_{n\in I_m} \frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^n_{\mathbf{X}+\varepsilon\mathbf{V}}]\Big|_{\varepsilon=0} = \mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m] = \frac{d}{d\varepsilon}d_m(\mathbf{X}+\varepsilon V)\Big|_{\varepsilon=0}, \tag{6.10}$$

as it should be. So, this first term $\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n]$ is the contribution to the marginal risk allocation of bank n regardless of any systemic influence. When summing up we get the marginal risk allocation of the whole group. Equation (6.10) is the Local Casual Responsibility for the whole group, but not for the single bank inside each group. Note that the sign of the increment V^n in the first term of (6.8) is here relevant: an increment (positive) corresponds to a risk reduction, regardless of the dependence structure. If $\mathbf{V}$ is deterministic, the marginal risk allocation to bank n is exactly $\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n] = -V^n$ and no other correction terms are present.

To understand the other terms in (6.8) or (6.9), take $\mathbf{V} = V^j \mathbf{e}_j$ with $j \neq n$. In this way, the first term in (6.8) disappears ($V^n = 0$) and we obtain

$$\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon V^j \mathbf{e}_j}}[Y^n_{\mathbf{X}+\varepsilon V^j \mathbf{e}_j}]\Big|_{\varepsilon=0} = \frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[V^j, X^n] - \frac{1}{\alpha_n}\frac{1}{\beta_m}\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[V^j, \overline{X}_m].$$

To fix the ideas, suppose that $COV_{Q^m_{\mathbf{X}}}[V^j, X^n] < 0$, and examine for the moment only the contribution of $\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[V^j, X^n]$. This component does not depend on the "systemic relevance" of bank n (i.e. it does not depend on the specific α_n) but it depends on the dependence structure between (V^j, X^n). If the systemic risk evaluation $Q^m_{\mathbf{X}}$ attributes negative correlation to (V^j, X^n), then, from the systemic perspective this is good (independently of the sign of V^j): a decrement in bank j is balanced by bank n, and viceversa. If bank n is negatively correlated (as seen by $Q^m_{\mathbf{X}}$) with the increment of bank j, then the risk allocation of bank n should decrease. Therefore, bank n takes advantage of this, as its risk allocation is reduced ($\frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[V^j, X^n] < 0$). Since the overall marginal risk allocation of the group m is fixed (equal to $\mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m] = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^j]$, from (6.10)), someone else has to pay for such advantage to bank n. This is the last term in (6.9), discussed next.

For the third component $-\frac{1}{\alpha_n}\frac{1}{\beta_m}\frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[V^j, \overline{X}_m]$ in (6.9), we distinguish between the systemic component $-\frac{1}{\beta_m}\frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[V^j, \overline{X}_m]$, which only depends on the aggregate group $\overline{X}_m$, and the *systemic relevance* $\frac{1}{\alpha_n}$ of bank n. The systemic quantity $-\frac{1}{\beta_m}\frac{1}{\beta_m} COV_{Q^m_{\mathbf{X}}}[V^j, \overline{X}_m]$ is therefore distributed among the various banks according to $\frac{1}{\alpha_n}$. In addition, this term must compensate for the possible risk reduction term (the second term in (6.9)), as the overall risk allocation to group m is determined by $\mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m] = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^j]$.

Note that if $\alpha_n = \alpha$ then we may rewrite Item 4 as

$$\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^n_{\mathbf{X}+\varepsilon\mathbf{V}}]\Big|_{\varepsilon=0} = \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^n] + \alpha COV_{Q^m_{\mathbf{X}}}\left[\frac{\overline{V}_m}{N_m}, X^n\right] - \alpha COV_{Q^m_{\mathbf{X}}}\left[\frac{\overline{V}_m}{N_m}, \frac{\overline{X}_m}{N_m}\right]$$

where N_m is the number of banks in group I_m.

Finally Items 1 and 6 express the same property (which holds in general, as shown in Proposition 4.28) respectively for one group or for the entire system.

6.2 Monotonicity

Consider a fixed $\mathbf{X} \in M^\Phi$. For a given partition $\mathbf{n}$ and $\mathcal{C} = \mathcal{C}^{(\mathbf{n})}$, let Y^k_r, $k \in I_r$, $r = 1, \cdots, h$, be the corresponding optimal solution of the primal problem (6.1) and $Q^r_{\mathbf{X}}$, $r = 1, \cdots, h$, be the optimal solutions of the corresponding dual problem (4.8) (in this section we suppress the label $\mathbf{X}$ from the optimal solutions to the primal problem).

Consider for some $m \in \{1, \cdots, h\}$ a non empty subgroup I'_m of the group I_m. Set $I''_m := I_m \backslash I'_m$. Then the $(h+1)$ groups $I_1, I_2, \cdots, I'_m, I''_m, I_{m+1}, \cdots, I_h$ corresponds to a new partition $\mathbf{n}'$. The optimal solutions of the primal problem (6.1) with $\mathcal{C} = \mathcal{C}^{(\mathbf{n}')}$ coincide with Y^k_r, $k \in I_r$, for $r \neq m$.

The interpretation of the monotonicity condition (6.13) was already formulated at the end of Section 2. Its generalization in the context of h groups is formulated below in (6.11).

For $r = m$, $i \in I'_m$, we have the following.

Proposition 6.7. *Define with $Y^i_{m'}$, $i \in I'_m$, the optimal solution of the primal problem with $\mathcal{C} = \mathcal{C}^{(\mathbf{n}')}$. Then*

$$\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\sum_{i \in I'_m} Y^i_m\right] \leq \sum_{i \in I'_m} Y^i_{m'} := d'_m. \tag{6.11}$$

In particular, if the group I'_m consists of only one single element $\{i\}$, then $Y^i_{m'}$ is deterministic and

$$\mathbb{E}_{Q^m_{\mathbf{X}}}[Y^i_m] \le Y^i_{m'} \quad \text{for each } i \in I_m. \tag{6.12}$$

If we compare the deterministic optimal solution $\mathbf{Y}^*$ (corresponding to $\mathcal{C} = \mathbb{R}^N$) with the (random) optimal solutions $\mathbf{Y}$ associated to one single group ($\mathcal{C} = \mathcal{C}_{\mathbb{R}} \cap M^\Phi$), we conclude

$$\mathbb{E}_{Q_{\mathbf{X}}}[Y^n] \le (Y^*)^n \text{ for each } n = 1, \cdots, d, \tag{6.13}$$

where $Q_{\mathbf{X}}$ is the unique optimal solution of the dual problem associated to $\mathcal{C} = \mathcal{C}_{\mathbb{R}} \cap M^\Phi$.

Proof. Given the subgroup I'_m, define

$$\beta'_m := \sum_{k \in I'_m} \frac{1}{\alpha_k}; \quad A'_m := \sum_{k \in I'_m} A^k_m$$
$$A' := \sum_{k \in I'_m} \left(\frac{1}{\beta_m \alpha_k} A_m - A^k_m \right) = \frac{\beta'_m}{\beta_m} A_m - A'_m.$$

with $\beta_m = \sum_{k \in I_m} \frac{1}{\alpha_k}$. Then the optimal value with respect to $\mathcal{C}^{(\mathbf{n}')}$ is given by

$$d'_m = \beta'_m \ln \left\{ -\frac{\beta}{B} \mathbb{E} \left[\exp \left(-\frac{1}{\beta'_m} \sum_{k \in I'_m} X^k \right) \right] \right\} - A'_m.$$

Summing the components of the solutions relative to $\mathcal{C}^{(\mathbf{n})}$ over $k \in I'_m$, we get

$$\begin{aligned} \sum_{k \in I'_m} \mathbf{Y}^k_m &= \sum_{k \in I'_m} \left(\frac{1}{\beta_m \alpha_k} \overline{X}_m - X^k \right) + \sum_{k \in I'_m} \frac{1}{\beta_m \alpha_k} d_m + \sum_{k \in I'_m} \left(\frac{1}{\beta_m \alpha_k} A_m - A^k_m \right) \\ &= \left(\frac{\beta'_m}{\beta_m} \overline{X}_m - \sum_{k \in I'_m} X^k \right) + \frac{\beta'_m}{\beta_m} d_m + A'. \end{aligned}$$

Using Jensen inequality we obtain

$$
\begin{aligned}
&\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\sum_{k\in I'_m}\mathbf{Y}^k_m\right]\\
&=\beta'_m\ln\left\{\exp\left(\frac{1}{\beta'_m}\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\left(\frac{\beta'_m}{\beta_m}\overline{X}_m-\sum_{k\in I'_m}X^k\right)\right]\right)\right\}+\frac{\beta'_m}{\beta_m}\beta_m\log\left(-\frac{\beta}{B}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]\right)\\
&\quad-\frac{\beta'_m}{\beta_m}A_m+A'\\
&\le\beta'_m\ln\left\{\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\exp\left(\frac{1}{\beta_m}\overline{X}_m-\frac{1}{\beta'_m}\sum_{k\in I'_m}X^k\right)\right]\right\}+\beta'_m\log\left(-\frac{\beta}{B}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]\right)-A'_m\\
&=\beta'_m\ln\left\{\mathbb{E}\left[\frac{\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\exp\left(\frac{1}{\beta_m}\overline{X}_m\right)\exp\left(-\frac{1}{\beta'_m}\sum_{k\in I'_m}X^k\right)}{\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]}\right]\right\}\\
&\quad+\beta'_m\log\left(-\frac{\beta}{B}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]\right)-A'_m\\
&=\beta'_m\ln\left\{\mathbb{E}\left[\frac{\exp\left(-\frac{1}{\beta'_m}\sum_{k\in I'_m}X^k\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}\right]\frac{\beta}{\gamma}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]\right\}-A'_m\\
&=\beta'_m\ln\left\{-\frac{\beta}{B}\mathbb{E}\left[\exp\left(-\frac{1}{\beta'_m}\sum_{k\in I'_m}X^k\right)\right]\right\}-A'_m=d'_m.
\end{aligned}
$$

We have that (6.12) and (6.13) directly follow by (6.11). □

A Appendix

A.1 Properties

Lemma A.1. *Assumption 3.3 implies:*

(a) there exists $c\in\mathbb{R}$ and $b\in\mathbb{R}_+$ such that $u_n(x)\le c+bx$ for all $x\ge 0$ and all n.

(b) for all n there exists $A_n\in\mathbb{R}$ and $a_n\in\mathbb{R}_+$ such that $u_n(x)\le A_n+a_nx$ for all $x\in\mathbb{R}$.

(c) the constants b and a_n can be selected so that $a:=\min_n a_n>b$.

Proof. Notice that $dom(u_n)=\mathbb{R}$ for each n. Hereafter the left derivatives of the concave increasing functions u_n are denoted by u'_n and satisfy $u'_n(x)\ge 0$ for all $x\in\mathbb{R}$.

(a) The concavity of each u_n implies that $u_n(x)\le c_n+u'_n(0)x$ for all $x\in\mathbb{R}$ (for some c_n) and therefore, setting $b:=\max_n u'_n(0)\ge 0$ and $c:=\max_n c_n$, $u_n(x)\le c+bx$ for all $x\ge 0$.

(b) From $\lim_{x\to-\infty}\frac{u_n(x)}{x}=+\infty$ we obtain $u'_n(x)\uparrow+\infty$ as $x\downarrow-\infty$. Therefore, for each n there exists $x_n\in\mathbb{R}$ such that $u'_n(x)>b$ for all $x\le x_n$. Then, for $x_0:=\min\{x_1,\cdots,x_N\}$,

$u_n'(x) > b$ for all $x \leq x_0$. Set $a_n := u_n'(x_0)$. Then the concavity of u_n implies: $u_n(x) \leq A_n + a_n x$ for all $x \in \mathbb{R}$ (for some A_n).

(c) Finally the construction above guarantees that $\min_n a_n = \min_n u_n'(x_0) > b$. $\square$

Proof of Proposition 3.4. By contradiction, we suppose that $\rho(\mathbf{X}) = -\infty$, for some $\mathbf{X} \in \mathcal{L} \subseteq L^1(\mathbb{P}, \mathbb{R}^N)$. Let $\mathbf{Y}_m \in \mathcal{C}$ satisfy $\sum_{n=1}^N Y_m^n \downarrow -\infty$, as $m \to +\infty$ and $\Lambda(\mathbf{X} + \mathbf{Y}_m) \in \mathbb{A}$ for each m. The condition $\sum_{n=1}^N Y_m^n \downarrow -\infty$, as $m \to +\infty$ implies $\sum_{n=1}^N \mathbb{E}[Y_m^n] \downarrow -\infty$, as $m \to +\infty$. Notice also that, by Jensen inequality,

$$B \leq \mathbb{E}[\Lambda(\mathbf{X} + \mathbf{Y}_m)] \leq \Lambda(\mathbb{E}[\mathbf{X} + \mathbf{Y}_m]) = \sum_{n=1}^N u_n(\mathbb{E}[X^n] + \mathbb{E}[Y_m^n]). \tag{A.1}$$

We now prove that $\sum_{n=1}^N u_n(\mathbb{E}[X^n] + \mathbb{E}[Y_m^n]) \downarrow -\infty$, as $m \to +\infty$, which is in contradiction with (A.1). Set $\mathbf{x}_m := (x_m^n)_{n=1}^N$ where $x_m^n := \mathbb{E}\left[Y_m^n\right]$. Since $\sum_{n=1}^N x_m^n \downarrow -\infty$, there must exist $n_0 \in \{1, \cdots, N\}$ and a subsequence $\mathbf{x}_{h_m}$ such that $x_{h_m}^{n_0} \downarrow -\infty$ as $m \to +\infty$. With an abuse of notation, denote again such subsequence $\mathbf{x}_{h_m}$ with $\mathbf{x}_m$. Then we have $x_m^{n_0} \downarrow -\infty$. If there exists another coordinate $n_1 \in \{1, \cdots, N\} \setminus n_0$ such that $\liminf_{m \to \infty} x_m^{n_1} = -\infty$, take the subsequence $\mathbf{x}_{k_m}$ such that $x_{k_m}^{n_1} \downarrow -\infty$. By diagonal procedure, we obtain one single sequence denoted again by $\mathbf{x}_m$ such that $x_m^{n_0} \downarrow -\infty$ and $x_m^{n_1} \downarrow -\infty$, as $m \to +\infty$. We may adopt this procedure (at most N times) also in the case $\limsup_{m \to \infty} x_m^{n_2} = +\infty$ for some coordinate n_2. At the end, we will obtain one single sequence $\mathbf{x}_m$ and three disjoint sets of coordinate indices N_-, N_+, N^* such that

$$\begin{aligned}
&x_m^n \downarrow -\infty && \text{if } n \in N_- \subseteq \{1, \cdots, N\}, \\
&x_m^n \uparrow +\infty && \text{if } n \in N_+ \subseteq \{1, \cdots, N\}, \\
&\mid x_m^n \mid \leq K && \text{for all } m \text{ and all } n \in N^* = \{1, \cdots, N\} \setminus (N_- \cup N_+),
\end{aligned}$$

where K is a constant independent of m. We know that $N_- \neq \emptyset$, since $n_0 \in N_-$ (but the other two sets N_+ and N^* may be empty). Since $\sum_{n=1}^N x_m^n \downarrow -\infty$, we deduce that, for large m, $\sum_{n=1}^N x_m^n \leq 0$ so that

$$\sum_{n \in N_+} x_m^n \leq -\sum_{n \in N_-} x_m^n - \sum_{n \in N^*} x_m^n \leq -\sum_{n \in N_-} x_m^n + NK, \text{ for each fixed (large) } m. \tag{A.2}$$

Now we use the inequalities of Lemma A.1. From $a_n \geq a$, we get (for large m) $a_n x_m^n \leq a x_m^n$ when $n \in N_-$ (as $x_m^n \leq 0$); for $n \in N_+$ (and large m) we have $(\mathbb{E}[X^n] + x_m^n) \geq 0$ and we can use inequality (a) in Lemma A.1. Letting $d^n := \mathbb{E}[X^n]$, we obtain, for each fixed large

m, that

$$\begin{aligned}
\sum_{n=1}^{N} u_n(\mathbb{E}[X^n] + \mathbb{E}[Y_m^n]) &= \sum_{n\in N_+} u_n(d^n + x_m^n) + \sum_{n\in N_-} u_n(d^n + x_m^n) + \sum_{n\in N^*} u_n(d^n + x_m^n) \\
&\leq \sum_{n\in N_+} (c + b(d^n + x_m^n)) + \sum_{n\in N_-} (A_n + a_n(d^n + x_m^n)) + \sum_{n\in N^*} u_n(d^n + K) \\
&\leq C + \sum_{n\in N_+} b x_m^n + \sum_{n\in N_-} a_n x_m^n \\
&\leq C + bNK + (a-b) \sum_{n\in N_-} x_m^n, \qquad \text{(A.3)}
\end{aligned}$$

where we use (A.2) in inequality (A.3) and $C := \sum_{n\in N_+}(c + bd^n) + \sum_{n\in N_-}(A_n + a_n d^n) + \sum_{n\in N^*} u_n(d^n + K)$ is independent of m. Then $(a-b)\sum_{n\in N_-} x_m^n \downarrow -\infty$, as $m \to +\infty$, since $a > b$ by Lemma A.1 and $x_m^n \downarrow -\infty$ for each $n \in N_-$. This concludes the proof. □

Remark A.2. The condition $\rho(\mathbf{X}) > -\infty$ is essentially a condition on the behavior of $\Lambda : \mathbb{R}^N \to \mathbb{R}$ at $-\infty$. Note that if the condition $\lim_{x\to-\infty} \frac{u_n(x)}{x} = +\infty$ is not satisfied, there might be a problem. Take $N = 2$ and the increasing concave functions

$$u_1(x) = 3x,\ u_2(x) = x.$$

Take $x_m^1 = m$, $x_m^2 = -2m$. As

$$x_m^1 + x_m^2 = -m \to -\infty, \text{ but } \Lambda(x_m) = u_1(x_m^1) + u_2(x_m^2) = 3m - 2m = m \to +\infty,$$

we cannot control $\Lambda(x)$ as in (A.3).

A.2 Orlicz Setting

We now recall an important result for the characterization of systemic risk measures of the form (3.5) on the Orlicz Heart.

Theorem A.3. ([9]) *Suppose that $\mathcal{L}$ is a Fréchet lattice and $\rho : \mathcal{L} \to \mathbb{R} \cup \{+\infty\}$ is convex and monotone decreasing. Then*

1. *ρ is continuous in the interior of dom(ρ), with respect to the topology of $\mathcal{L}$,*
2. *ρ is subdifferentiable in the interior of dom(ρ),*
3. *for all $\mathbf{X} \in int(dom(\rho))$*

$$\rho(\mathbf{X}) = \max_{Q\in\mathcal{L}_+^*} \{Q(-X) - \alpha(Q)\},$$

where $\mathcal{L}^*$ is the dual of $\mathcal{L}$ (for the topology for which $\mathcal{L}$ is a Frechet lattice), $\mathcal{L}_+^* = \{Q \in \mathcal{L}^* \mid Q \text{ is positive}\}$ and $\alpha : \mathcal{L}^* \to \mathbb{R} \cup \{+\infty\}$, defined by

$$\alpha(Q) = \sup_{\mathbf{X}\in\mathcal{L}} \{Q(-\mathbf{X}) - \rho(\mathbf{X})\},$$

is $\sigma(\mathcal{L}^*, \mathcal{L})$-lsc and convex.

A.2.1 Dual representation in the Orlicz setting

Proof. (of Lemma 4.4). First we show that

$$c(\mathbf{Q}) := -\alpha_{\Lambda,B}(\mathbf{Q}) = \inf\left\{\sum_{n=1}^{N}\mathbb{E}_{Q^n}[Z^n] \mid \mathbf{Z} \in M^{\Phi},\ \sum_{n=1}^{N}\mathbb{E}[u_n(Z^n)] \geq B\right\} = \widetilde{c}(\mathbf{Q})$$

where

$$\widetilde{c}(\mathbf{Q}) := \inf\left\{\sum_{n=1}^{N}\mathbb{E}_{Q^n}[Z^n] \mid \mathbf{Z} \in M^{\Phi},\ \sum_{n=1}^{N}\mathbb{E}[u_n(Z^n)] = B\right\}.$$

Clearly $c(\mathbf{Q}) \leq \widetilde{c}(\mathbf{Q})$. Assume by contradiction that $c(\mathbf{Q}) < \widetilde{c}(\mathbf{Q})$. Then, $\exists \varepsilon > 0$ and $\mathbf{Z} \in M^{\Phi}$ such that

$$\sum_{n=1}^{N}\mathbb{E}_{Q^n}[Z^n] \leq c(\mathbf{Q}) + \varepsilon < \widetilde{c}(\mathbf{Q}) \text{ and } \sum_{n=1}^{N}\mathbb{E}[u_n(Z^n)] > B.$$

The continuity of u_n implies the existence of $\delta \in \mathbb{R}^N_+$ such that $\sum_{n=1}^{N}\mathbb{E}[u_n(Z^n - \delta^n)] = B$. Then,

$$\widetilde{c}(\mathbf{Q}) \leq \sum_{n=1}^{N}\mathbb{E}_{Q^n}[Z^n - \delta^n] \leq \sum_{n=1}^{N}\mathbb{E}_{Q^n}[Z^n] \leq c(\mathbf{Q}) + \varepsilon < \widetilde{c}(\mathbf{Q}).$$

For the uniqueness, let us suppose that $c(\mathbf{Q})$ is attained by two distinct $\mathbf{Z}_1 \in M^{\Phi}$ and $\mathbf{Z}_2 \in M^{\Phi}$, so that $\mathbb{P}(Z_1^j \neq Z_2^j) > 0$ for some j. Then we have

$$c(\mathbf{Q}) = \sum_{n=1}^{N}\mathbb{E}_{Q^n}[Z_1^n] = \sum_{n=1}^{N}\mathbb{E}_{Q^n}[Z_2^n] \quad \text{and} \quad \sum_{n=1}^{N}\mathbb{E}\left[u_n(Z_k^n)\right] \geq B \text{ for } k = 1, 2.$$

For $\lambda \in [0, 1]$ set $\mathbf{Z}_\lambda := \lambda\mathbf{Z}_1 + (1-\lambda)\mathbf{Z}_2 \in M^{\Phi}$. Then

$$\sum_{n=1}^{N}\mathbb{E}_{Q^n}[Z_\lambda^n] = \lambda\sum_{n=1}^{N}\mathbb{E}_{Q^n}[Z_1^n] + (1-\lambda)\sum_{n=1}^{N}\mathbb{E}_{Q^n}[Z_2^n] = c(\mathbf{Q}),\ \forall\lambda \in [0, 1],$$

and for $\lambda \in (0, 1)$

$$B \leq \lambda\mathbb{E}\left[\sum_{n=1}^{N}u_n(\mathbf{Z}_1)\right] + (1-\lambda)\mathbb{E}\left[\sum_{n=1}^{N}u_n(\mathbf{Z}_2)\right] < \mathbb{E}\left[\sum_{n=1}^{N}u_n(Z_\lambda)\right],$$

where we used the strict convexity of u_j and $\mathbb{P}(Z_1^j \neq Z_2^j) > 0$. This is a contradiction with $c(\mathbf{Q}) = \widetilde{c}(\mathbf{Q})$ because

$$c(\mathbf{Q}) = \sum_{n=1}^{N}\mathbb{E}_{Q^n}[Z_\lambda^n],\ \mathbf{Z}_\lambda \in M^{\Phi} \text{ and } \mathbb{E}\left[\sum_{i=1}^{N}u_n(\mathbf{Z}_\lambda^n)\right] > B.$$

□

Proof. (of Proposition 4.10)

Consider the convex functional $\Theta_n : M^\Phi(\mathbb{R}) \to \mathbb{R}\cup\{+\infty\}$ defined by $\Theta_n(Z) := \mathbb{E}[-u_n(Z)]$ and let Θ_n^* be its convex conjugate. Then we have $\Theta_n^*(\xi) = \mathbb{E}[v_n(-\xi)]$, for $\xi \in L^{\Phi^*}(\mathbb{R})$ by [9], Section 5.2. Set $G(\mathbf{Z}) := \sum_{n=1}^N \mathbb{E}[-u_n(Z^n)] + B = \sum_{n=1}^N \Theta_n(Z^n) + B$, and observe that

$$\mathcal{A} := \left\{ \mathbf{Z} \in M^\Phi \mid \sum_{n=1}^N \mathbb{E}[u_n(Z^n)] \geq B \right\} = \left\{ \mathbf{Z} \in M^\Phi \mid G(\mathbf{Z}) \leq 0 \right\}.$$

We have that G is convex and decreasing with respect to the order relation (3.1). Let $G^*(\xi)$ be its convex conjugate, for $\xi \in L^{\Phi^*}$. We assume that $\xi \neq \mathbf{0}$. By the Fenchel inequality

$$\mathbb{E}[\mathbf{Z}\xi] \leq G(\mathbf{Z}) + G^*(\xi),$$

we obtain for all $\mathbf{Z} \in \mathcal{A}$ and $\lambda > 0$,

$$\mathbb{E}[-\mathbf{Z}\xi] = \lambda \mathbb{E}[\mathbf{Z}(-\frac{1}{\lambda}\xi)] \leq \lambda [G(\mathbf{Z}) + G^*(-\frac{1}{\lambda}\xi)] \leq \lambda G^*(-\frac{1}{\lambda}\xi), \ \mathbb{P}\text{-a.s.}$$

Hence

$$\alpha_{\Lambda,B}(\xi) := \sup_{\mathbf{Z}\in\mathcal{A}} \{\mathbb{E}[-\mathbf{Z}\xi]\} \leq \inf_{\lambda>0} \lambda G^*(-\frac{1}{\lambda}\xi). \tag{A.4}$$

By definition of the convex Fenchel conjugate and the fact that M^Φ is a product space, we have

$$\begin{aligned} G^*(\xi) \quad : &= \sup_{\mathbf{Z}\in M^\Phi} \{\mathbb{E}[\xi\mathbf{Z}] - G(\mathbf{Z})\} \\ &= -B + \sup_{\mathbf{Z}\in M^\Phi} \left\{ \sum_{n=1}^N \mathbb{E}[\xi_n Z^n] - \sum_{n=1}^N \Theta_n(Z^n) \right\} \\ &= -B + \sum_{n=1}^N \left(\sup_{Z\in M^\Phi(\mathbb{R})} \{\mathbb{E}[\xi_n Z] - \Theta_n(Z)\} \right) \\ &= -B + \sum_{n=1}^N \Theta_n^*(\xi_n), \end{aligned}$$

where we have used (3.9), and therefore

$$\inf_{\lambda>0} \lambda G^*(-\frac{1}{\lambda}\xi) = \inf_{\lambda>0} \left(-B\lambda + \lambda \sum_{n=1}^N \Theta_n^*(-\frac{1}{\lambda}\xi_n) \right) = \inf_{\lambda>0} \left(-B\lambda + \lambda \sum_{n=1}^N \mathbb{E}\left[v_n\left(\frac{1}{\lambda}\xi_n\right)\right]\right).$$

We need only to prove that there is no duality gap in (A.4), i.e., if $\alpha_{\Lambda,B}(\xi) < +\infty$ then

$$\alpha_{\Lambda,B}(\xi) = \inf_{\lambda>0} \lambda G^*(-\frac{1}{\lambda}\xi). \tag{A.5}$$

Observe that, by the definition of G^*, we have for each $\lambda > 0$

$$\lambda G^*(-\frac{1}{\lambda}\xi) := \sup_{\mathbf{Z}\in M^\Phi} \{\mathbb{E}[-\xi\mathbf{Z}] - \lambda G(\mathbf{Z})\}.$$

As ξ is not identically equal to $\mathbf{0}$ and M^Φ is a linear space, we have $\sup_{\mathbf{Z}\in M^\Phi}\{\mathbb{E}[-\xi\mathbf{Z}]\} = +\infty$ and therefore

$$\inf_{\lambda>0}\lambda G^*(-\frac{1}{\lambda}\xi) = \inf_{\lambda>0}\sup_{\mathbf{Z}\in M^\Phi}\{\mathbb{E}[-\xi\mathbf{Z}] - \lambda G(\mathbf{Z})\} = \inf_{\lambda\geq 0}\sup_{\mathbf{Z}\in M^\Phi}\{\mathbb{E}[-\xi\mathbf{Z}] - \lambda G(\mathbf{Z})\}.$$

We claim that

$$\inf_{\lambda\geq 0}\sup_{\mathbf{Z}\in M^\Phi}\{\mathbb{E}[-\xi\mathbf{Z}] - \lambda G(\mathbf{Z})\} = \sup_{\mathbf{Z}\in M^\Phi}\inf_{\lambda\geq 0}\{\mathbb{E}[-\xi\mathbf{Z}] - \lambda G(\mathbf{Z})\}. \tag{A.6}$$

Assuming (A.6), we may immediately conclude that

$$\begin{aligned}\inf_{\lambda>0}\lambda G^*(-\frac{1}{\lambda}\xi) &= \sup_{\mathbf{Z}\in M^\Phi}\inf_{\lambda\geq 0}\{\mathbb{E}[-\xi\mathbf{Z}] - \lambda G(\mathbf{Z})\} = \sup_{\mathbf{Z}\in M^\Phi}\left\{\mathbb{E}[-\xi\mathbf{Z}] - \sup_{\lambda\geq 0}\lambda G(\mathbf{Z})\right\} \\ &= \sup_{\mathbf{Z}\in\mathcal{A}}\{\mathbb{E}[-\xi\mathbf{Z}]\} := \alpha_{\Lambda,B}(\xi).\end{aligned}$$

We now prove (A.6) by showing the equivalent condition (simply multiply each side of (A.6) by -1):

$$\sup_{\lambda\geq 0}\inf_{\mathbf{Z}\in M^\Phi}\{\mathbb{E}[\xi\mathbf{Z}] + \lambda G(\mathbf{Z})\} = \inf_{\mathbf{Z}\in M^\Phi}\sup_{\lambda\geq 0}\{\mathbb{E}[\xi\mathbf{Z}] + \lambda G(\mathbf{Z})\}. \tag{A.7}$$

Let $f_0(\mathbf{Z}) = \mathbb{E}[\xi\mathbf{Z}]$ and, in order to make an easy comparison with the results in [39], rename G as $f_1 = G$. Consider the function $F : M^\Phi \times \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$, defined by

$$F(\mathbf{Z},u) = \begin{cases} f_0(\mathbf{Z}) & \text{if } \mathbf{Z}\in M^\Phi \text{ and } f_1(\mathbf{Z})\leq u, \\ +\infty & \text{otherwise,}\end{cases}$$

see (2.8) in [39], and the associated Lagrangian, see (4.4) in [39],

$$K(\mathbf{Z},\lambda) = \begin{cases} f_0(\mathbf{Z}) + \lambda f_1(\mathbf{Z}) & \text{if } \mathbf{Z}\in M^\Phi,\ \lambda\geq 0, \\ -\infty & \text{if } \mathbf{Z}\in M^\Phi,\ \lambda < 0, \\ +\infty & \text{if } \mathbf{Z}\notin M^\Phi.\end{cases}$$

Then (A.7) can be rewritten as

$$\sup_{\lambda\geq 0}\inf_{\mathbf{Z}\in M^\Phi}K(\mathbf{Z},\lambda) = \inf_{\mathbf{Z}\in M^\Phi}\sup_{\lambda\geq 0}K(\mathbf{Z},\lambda). \tag{A.8}$$

It is easily seen (e.g. see Lemma 8 in [9]) that the function $f_1 : M^\Phi \to \mathbb{R}$ is convex decreasing and finite valued on M^Φ. Therefore, Theorem A.3 guarantees that it is continuous on M^Φ (for the M^Φ-norm). Therefore, by [39] the function F is closed convex in $(\mathbf{Z},u)$.

Then the absence of duality gap, expressed by (A.8) follows from Theorems 17 and 18 of [39], provided that the (convex) optimal value function

$$\varphi(u) := \inf_{\mathbf{Z}\in M^\Phi}F(\mathbf{Z},u),\ u\in\mathbb{R},$$

is bounded from above in a neighborhood of 0. Clearly, it is sufficient to show the existence of an element $\mathbf{Z}_0 \in M^\Phi$ such that $u \to F(\mathbf{Z}_0, u)$ is bounded from above in a neighborhood of 0. The assumption $\Lambda(+\infty) > B$ guarantees the existence of $\mathbf{Z}_0 \in M^\Phi$ such that $\sum_{n=1}^N \mathbb{E}[u_n(Z_0^n)] > B$ (take Z_0^n equal to some large enough constant), i.e., $f_1(\mathbf{Z}_0) := \sum_{n=1}^N \mathbb{E}[-u_n(Z_0^n)] + B < 0$. Set $0 < \delta < |f_1(\mathbf{Z}_0)|$. Hence for all $u \in \mathbb{R}$ such that $|u| < \delta$ we have $f_1(\mathbf{Z}_0) < u$ and $F(\mathbf{Z}_0, u) = \mathbb{E}[\xi \mathbf{Z}_0] < +\infty$, as $\mathbf{Z}_0 \in M^\Phi$ and $\xi \in L^{\Phi^*}$. □

Remark A.4. In [25], (A.5) is deduced, by different means, in a $L^\infty(\mathbb{R})$ setting and in the one-dimensional case. In [3], (A.5) is obtained, by different means, in the multi-dimensional deterministic case, i.e. in $\mathbb{R}^N$.

A.3 Auxiliary results for existence

The following auxiliary results are standard and can be found in many articles on utility maximization. Recall that we are working under Assumptions 4.8 and 4.9.

Lemma A.5. *Let $\upsilon : \mathbb{R}_+ \to \mathbb{R}$ be a strictly convex differentiable function with $\upsilon'(0^+) = -\infty$, $\upsilon'(+\infty) = +\infty$ and let $Q \ll \mathbb{P}$. Then*
(a) $\upsilon'(\lambda \frac{dQ}{d\mathbb{P}}) \in L^1(Q)$ $\forall \lambda > 0$;
(b) $F(\lambda) \triangleq \mathbb{E}[\frac{dQ}{d\mathbb{P}} \upsilon'(\lambda \frac{dQ}{d\mathbb{P}})]$ defines a bijection between $(0, +\infty)$ and $(-\infty, +\infty)$.

Lemma A.6. *The convex conjugate function $v : \mathbb{R} \to (-\infty, +\infty]$ of u, given by $v(y) = \sup_{x \in \mathbb{R}} \{u(x) - xy\}$, is a proper lsc convex function, equal to $+\infty$ on $(-\infty, 0)$, bounded from below on $\mathbb{R}$, finite valued strictly convex, continuously differentiable on $(0, +\infty)$ and satisfying*

$$v(+\infty) = +\infty,\ v(0^+) = u(+\infty),\ v'(0^+) = -\infty,\ v'(+\infty) = +\infty,$$
$$u'(x) = (v')^{-1}(-x), u(-v'(y)) = -yv'(y) + v(y), \quad \forall y \geq 0,$$

where the usual rule $0 \cdot \infty = 0$ is applied.

Proposition A.7 (Prop. 3.6, [11])**.** *Let $Q \ll \mathbb{P}$. For all $c \in \mathbb{R}$ the optimizer $\lambda(c; Q)$ of*

$$\min_{\lambda > 0} \left\{ \mathbb{E}\left[v\left(\lambda \frac{dQ}{d\mathbb{P}} \right) \right] + \lambda c \right\}$$

is the unique positive solution of the first order condition

$$\mathbb{E}_Q\left[v'\left(\lambda \frac{dQ}{d\mathbb{P}} \right) \right] + c = 0. \tag{A.9}$$

If $\sup \left\{ \mathbb{E}[u(g)] \mid g \in L^1(Q) \text{ and } \mathbb{E}_Q[g] \leq c \right\} < u(+\infty)$, *the random variable* $\widehat{g} := -v'(\lambda(c; Q) \frac{dQ}{d\mathbb{P}})$ *belongs to the set* $\left\{ g \in L^1(Q) \mid \mathbb{E}_Q[g] = c \right\}$, *satisfies* $u(\widehat{g}) \in L^1(\mathbb{P})$, *and*

$$\min_{\lambda > 0} \left\{ \mathbb{E}\left[v\left(\lambda \frac{dQ}{d\mathbb{P}} \right) \right] + \lambda c \right\} = \sup \left\{ \mathbb{E}[u(g)] \mid g \in L^1(Q) \text{ and } \mathbb{E}_Q[g] \leq c \right\} = \mathbb{E}[u(\widehat{g})] < u(+\infty).$$

Lemma A.8. *If* $\lim_{x\to-\infty}\left(\frac{u_n(x)}{x}\right)=+\infty$*, then for every* $M>0$ *there exists a constant* $d>0$ *with* $u_n(x)\le Mx+d$ *for all* n *and* $x\le 0$.

Proof. The assumption implies that there exists $K>0$ (which depends on M) such that for all n $u_n(x)\le Mx$ for $x\le -K$. Hence $Mx-u_n(x)\ge 0$ for $x\in(-\infty,-K)$. It is clear now that since the function $Mx-u_n(x)$ is continuous on $[-K,0]$ we may add a properly chosen $d>0$ so that $Mx+d-u_n(x)\ge 0$ for all $x\in(-\infty,0]$ and all n. □

Lemma A.9. *Suppose that for every* $n\in\{1,...,N\}$ *the function* $f_n:\mathbb{R}\to\mathbb{R}$ *is continuous increasing and satisfies*

$$\lim_{x\to+\infty}\frac{f_n(x)}{x}=0.$$

Then for every $\varepsilon>0$ *there exists* $b=b(\varepsilon)>0$ *such that* $f_n(x)\le\varepsilon x+b$ *for* $x\ge 0$ *and all* n.

Proof. The assumption guarantees the existence of a constant $K>0$, which depends on ε, such that $f_n(x)\le\varepsilon x$ for $x\ge K$ and all n. Hence

$$f_n(x)\le\varepsilon x+K\varepsilon+\sup_n\left(\sup_{[0,K]}f_n(s)\right)\quad\forall\,x\ge 0.$$

□

A.3.1 On the extension of the domain

Assume that $\mathbf{X}\in M^{\Phi}$; $\mathbf{Q}=(Q^n)_n$, where $Q^n\ll P$ and $\frac{d\mathbf{Q}}{dP}\in L^{\Phi^*}$.

For $a_n\in\mathbb{R}$ consider the problem:

$$P_n(a_n):=\sup\left\{\mathbb{E}\left[u_n(X^n+W)\right]\mid W\in M^{\phi_n},\,\mathbb{E}_{Q^n}[W]\le a_n\right\}.\tag{A.10}$$

Notice that $\mathbb{E}[u_n(X^n+W)]\le u_n(\mathbb{E}[X^n+W])<+\infty$ for all $X^n,W\in M^{\phi_n}\subseteq L^1(\mathbb{P};\mathbb{R})$. In particular, the condition $X^n\in M^{\phi_n}$ implies that $\mathbb{E}[u(X^n+a_n)]>-\infty$, so that X^n does not lead to prohibitive punishments when $W=a_n$, which in turn implies that $P(a_n)>-\infty$. From Remark 3.7 we know that $\frac{d\mathbf{Q}}{dP}\in L^{\Phi^*}$ and $W\in M^{\phi_n}$ implies $W\in L^1(Q^n)$. This shows that the problem (A.10) is well posed. However, in general the optimal solution to (A.10) will only exist on a larger domain, as suggested by the well known result reported in Proposition A.7. This leads to introduce the auxiliary problem:

$$U_n(a_n):=\sup\left\{\mathbb{E}\left[u_n(X^n+W)\right]\mid W\in L^1(Q^n),\,\mathbb{E}_{Q^n}[W]\le a_n\right\}\le u_n(+\infty).\tag{A.11}$$

From $M^{\phi_n}\subseteq L^1(Q^n)$ we clearly have: $P_n(a_n)\le U_n(a_n)$, so that if $P_n(a_n)=u(+\infty)$ then $P_n(a_n)=U_n(a_n)=u(+\infty)$.

Proposition A.10.

$$P_n(a_n) = U_n(a_n). \tag{A.12}$$

Proof. By the Fenchel inequality we get

$$\mathbb{E}\left[u_n(X^n + W)\right] \leq \lambda\left(\mathbb{E}_{Q^n}[X^n] + \mathbb{E}_{Q^n}[W]\right) + \mathbb{E}\left[v_n\left(\lambda \frac{dQ^n}{dP}\right)\right],$$

and hence

$$P_n(a_n) \leq U_n(a_n) \leq \inf_{\lambda>0}\left\{\lambda\left(\mathbb{E}_{Q^n}[X^n] + a_n\right) + \mathbb{E}\left[v_n\left(\lambda \frac{dQ^n}{dP}\right)\right]\right\}.$$

The following claim concludes the proof: if $P_n(a_n) < u(+\infty)$ then:

$$P_n(a_n) = \inf_{\lambda>0}\left\{\lambda\left(\mathbb{E}_{Q^n}[X^n] + a_n\right) + \mathbb{E}\left[v_n\left(\lambda \frac{dQ^n}{dP}\right)\right]\right\}.$$

To show this, consider the integral functional $I : M^{\phi_n} \to \mathbb{R}$ defined by $I(X^n) = \mathbb{E}\left[u_n(X^n)\right]$ is finite valued, monotone increasing and concave on M^{ϕ_n} (as $\mathbb{E}[u_n(X^n)] \leq u_n(\mathbb{E}[X^n]) < +\infty$), and therefore, by the Theorem A.3, it is norm-continuous on M^{ϕ_n}. We can then follow the well known duality approach (see for example [11]). Consider the convex cone $D^0 := \left\{W \in M^{\phi_n} \mid \mathbb{E}_{Q^n}[W] \leq 0\right\}$ which is the polar cone of the one dimensional cone $D := \left\{\lambda \frac{dQ^n}{dP} \mid \lambda \geq 0\right\}$, so that the bipolar D^{00} coincide with D. Let $\delta_{D^0} : M^{\phi_n} \to \mathbb{R}\cup\{+\infty\}$ be the support functional of D^0. By [36], or directly by hand, the concave conjugate $I^* : L^{\phi_n^*} \to \mathbb{R}\cup\{-\infty\}$ is given by $I^*(\xi^n) = \mathbb{E}\left[-v_n(\xi^n)\right]$ and so, by Fenchel duality Theorem,

$$\begin{aligned}
P_n(a_n) &= \sup_{W\in D^0} \mathbb{E}\left[u_n(X^n + a_n + W)\right] = \sup_{Z\in D^0+X^n+a_n} \mathbb{E}\left[u_n(Z)\right] \\
&= \sup_{Z\in M^{\phi_n}} \left\{\mathbb{E}\left[u_n(Z)\right] - \delta_{D^0+X^n+a_n}(Z)\right\} = \min_{\xi^n\in L^{\phi_n^*}} \left\{\delta^*_{D^0+X^n+a_n}(\xi^n) - \mathbb{E}\left[-v_n(\xi^n)\right]\right\} \\
&= \min_{\xi^n\in L^{\phi_n^*}} \left\{\mathbb{E}[\xi^n(X^n + a_n)] + \delta_{D^{00}}(\xi^n) + \mathbb{E}\left[v_n(\xi^n)\right]\right\} \\
&= \min_{\xi^n\in D^{00}} \left\{\mathbb{E}[\xi^n(X^n + a_n)] + \mathbb{E}\left[v_n(\xi^n)\right]\right\} = \min_{\lambda>0}\left\{\lambda\left(\mathbb{E}_{Q^n}[X^n] + a_n\right) + \mathbb{E}\left[v_n\left(\lambda \frac{dQ^n}{dP}\right)\right]\right\},
\end{aligned}$$

where we used $\delta^*_{D^0} = \delta_{D^{00}}$, $D^{00} = D$ and the fact that the minimizer is obtained at $\lambda > 0$, otherwise if $\lambda = 0$ then $P_n(a_n) = \mathbb{E}\left[v_n(0)\right] = u_n(+\infty)$. □

A.4 The Exponential Case

Proof of Theorem 6.1. For the sake of simplicity we start by choosing $h = 1$. We note that

$$\begin{aligned}
&\rho_B(\mathbf{X}) \\
&= \inf\left\{\sum_{n=1}^N Y^n \mid \mathbf{Y} \in M^\Phi : \exists d \in \mathbb{R} \text{ s.t. } \sum_{n=1}^N Y^n = d \text{ and } \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] = B\right\} \\
&= \inf\left\{d \mid (d, \mathbf{Y}) \in \mathbb{R}\times M^\Phi \text{ s.t.} \sum_{n=1}^N Y^n = d \text{ and } \mathbb{E}\left[\sum_{n=1}^N u_n(X^n + Y^n)\right] = B\right\}.
\end{aligned}$$

Let $F : \mathbb{R}\times M^{\Phi} \to \mathbb{R}$ be given by

$$F(d, \mathbf{Y}) = d$$

and $f_1 : \mathbb{R}\times M^{\Phi} \to M^{\Phi}$ and $f_2 : \mathbb{R}\times M^{\Phi} \to \mathbb{R}$ be defined by

$$f_1(d, \mathbf{Y}) = \sum_{n=1}^{N} Y^n - d \quad \text{and} \quad f_2(d, \mathbf{Y}) = \mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] - B,$$

respectively. Then we can rewrite

$$\rho_B(\mathbf{X}) = \inf_{(d,\mathbf{Y})\in\mathbb{R}\times M^{\Phi}} \{F(d, \mathbf{Y}) \mid f_1(d, \mathbf{Y}) = 0,\ f_2(d, \mathbf{Y}) = 0\}$$

with associated Lagrangian $L(d, \mathbf{Y}, \mathbf{Z}, \mu) : \mathbb{R}\times M^{\Phi} \times (M^{\Phi})^* \times \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned} L(d, \mathbf{Y}, \mathbf{Z},\ \mu) &= F(d, \mathbf{Y}) + \mathbb{E}[Z f_1(d, \mathbf{Y})] + \mu f_2(d, \mathbf{Y}) \\ &= d + \mathbb{E}\left[Z\left(\sum_{n=1}^{N} Y^n - d\right)\right] + \mu\left(\mathbb{E}\left[\sum_{n=1}^{N} u_n(X^n + Y^n)\right] - B\right). \end{aligned}$$

The problem boils down to solve the system $\nabla L = 0$, taking derivatives with respect to each $(d, \mathbf{Y}, \mathbf{Z},\ \mu)$.

Consider now the general case $h > 1$. We have

$$\begin{aligned} L(\{d_m\}_{m=1}^{h}, \mathbf{Y}, \{Z^m\}_{m=1}^{h}, \mu) = &\sum_{m=1}^{h} d_m + \mathbb{E}\left[\sum_{m=1}^{h} Z^m(\overline{Y}_m - d_m)\right] \\ &+ \mu\left\{\mathbb{E}\left[\sum_{m=1}^{h}\sum_{k\in I_m} \exp(-\alpha_k(X_k + Y_k))\right] + B\right\}, \end{aligned}$$

with $\overline{Y}_m = \sum_{k\in I_m} Y_k$.

We compute the Gateaux derivative in the direction $\mathbf{V} \in M^{\Phi}$:

$$\begin{aligned} &\lim_{\epsilon\to 0} \frac{\mathcal{L}(\mathbf{Y} + \epsilon\mathbf{V}) - \mathcal{L}(\mathbf{Y})}{\epsilon} \\ &= \lim_{\epsilon\to 0} \mu\mathbb{E}\left[\sum_{m=1}^{h}\sum_{k\in I_m} \exp(-\alpha_k(X_k + Y_k))\frac{\exp(-\epsilon V_k\alpha_k) - 1}{\epsilon}\right] + \lim_{\epsilon\to 0} \frac{1}{\epsilon}\mathbb{E}\left[\epsilon\sum_{m=1}^{h}\sum_{k\in I_m} V_k Z^m\right] \qquad \text{(A.13)} \\ &= -\mu\mathbb{E}\left[\sum_{m=1}^{h}\sum_{k\in I_m} \exp(-\alpha_k(X_k + Y_k))V_k\alpha_k\right] + \mathbb{E}\left[\sum_{m=1}^{h} V_k Z^m\right] =: \phi_Y(\mathbf{V}), \end{aligned}$$

where in (A.13) we can apply the Dominated Convergence Theorem by using estimations similar to the ones in Remark A.11. We now show that $\phi_Y(\mathbf{V})$ is also the Fréchet derivative of L, i.e., that

$$\lim_{\|\mathbf{V}\|_{L^{\Phi}}\to 0} \frac{L(\mathbf{Y} + \mathbf{V}) - L(\mathbf{Y}) - \phi_Y(\mathbf{V})}{\|\mathbf{V}\|_{L^{\Phi}}} = 0.$$

We have

$$L(\mathbf{Y}+\mathbf{V})-L(\mathbf{Y})-\phi_Y(\mathbf{V}) = \mu\mathbb{E}\left[\sum_{m=1}^{h}\sum_{k\in I_m} \exp(-\alpha_k(X_k+Y_k))(\exp(-v^k\alpha_k)-1+v^k\alpha_k)\right],$$

and we obtain

$$\begin{aligned}
&\mathbb{E}[\exp(-\alpha_k(X_k+Y_k))(\exp(-\alpha_k V_k)-1+\alpha_k V_k)] \\
&\leq \mathbb{E}[|\exp(-\alpha_k(X_k+Y_k))(\exp(-\alpha_k V_k)-1+\alpha_k V_k)|] \\
&\leq K_1\mathbb{E}[\exp(-\alpha_k(X_k+Y_k-|V_k|))V_k^2] \\
&\leq K_2\mathbb{E}[\exp(-2\alpha_k(X_k+Y_k-|V_k|))]^{\frac{1}{2}}\mathbb{E}[V_k^4]^{\frac{1}{2}} \\
&\leq K_2\mathbb{E}[\exp(-4\alpha_k(X_k+Y_k))]^{\frac{1}{4}}\mathbb{E}[\exp(4|V_k|)]^{\frac{1}{4}}\mathbb{E}[V_k^4]^{\frac{1}{2}} \\
&= K_3\left\|V_k\right\|^2_{L^4(\mathbb{R})},
\end{aligned}$$

where we use twice the Hölder inequality. Since

$$K_3\left\|V_k\right\|^2_{\mathcal{L}^4(\mathbb{R})} \leq K_4\left\|V_k\right\|^2_{L^{\phi_k}},$$

we have

$$|L(\mathbf{Y}+\mathbf{V})-L(\mathbf{Y})-\phi_Y(\mathbf{V})| \leq K_4\left\|\mathbf{V})\right\|^2_{L^\Phi}.$$

To conclude the proof, it is then sufficient to substitute $\mathbf{Y}$ of the form (6.3) in $\phi_Y(\mathbf{V})$ to verify that $\phi_Y(\mathbf{V})=0$ for all $\mathbf{V}\in M^\Phi$. □

Proof of Proposition 6.5. The following results hold because $\mathbf{X},\mathbf{V}\in M^\Phi$ and Remark A.11.

1. By (6.7) we get

$$\left.\frac{d}{d\epsilon}d_m(\mathbf{X}+\epsilon\mathbf{V})\right|_{\epsilon=0} = \beta_m\frac{\frac{d}{d\epsilon}\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]} = -\frac{\mathbb{E}\left[-\overline{V}_m\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]} = \mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m]. \tag{A.14}$$

2. By (6.7) and (A.14) we deduce

$$\begin{aligned}
\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}}}[Y^i_{\mathbf{X}+\varepsilon\mathbf{V}}]|_{\varepsilon=0} &= \lim_{\epsilon\to 0}\frac{1}{\varepsilon}\left\{\mathbb{E}_{Q^m_{\mathbf{X}}}\left[Y^i_{\mathbf{X}+\varepsilon\mathbf{V}}-Y^i_{\mathbf{X}}\right]\right\} \\
&= \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^i]+\frac{1}{\beta_m\alpha_i}\mathbb{E}_{Q^m_{\mathbf{X}}}[\overline{V}_m]+\frac{1}{\beta_m\alpha_i}\left.\frac{d}{d\epsilon}d_m(\mathbf{X}+\epsilon\mathbf{V})\right|_{\epsilon=0} \\
&= \mathbb{E}_{Q^m_{\mathbf{X}}}[-V^i].
\end{aligned} \tag{A.15}$$

3. Note that

$$
\begin{aligned}
&\frac{d}{d\epsilon}\left(\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}\right)\\
&=\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}\left(-\frac{\overline{V}_m}{\beta_m}\right)-\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]^2}\frac{d}{d\epsilon}\left(\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]\right)\\
&=\frac{1}{\beta_m}\left\{-\overline{V}_m\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}+\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}\mathbb{E}\left[\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}\overline{V}_m\right]\right\}\\
&=\frac{1}{\beta_m}\left\{-\overline{V}_m\frac{dQ^m_{\mathbf{X}+\epsilon\mathbf{V}}}{d\mathbb{P}}+\frac{dQ^m_{\mathbf{X}+\epsilon\mathbf{V}}}{d\mathbb{P}}\mathbb{E}_{Q^m_{\mathbf{X}+\epsilon\mathbf{V}}}[\overline{V}_m]\right\}.
\end{aligned}
$$

Hence we have by Remark A.11 that

$$
\begin{aligned}
\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\mathbb{E}_{Q^m_{\mathbf{X}+\epsilon\mathbf{V}}}[Z]-\mathbb{E}_{Q^m_{\mathbf{X}}}[Z]\right)&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\mathbb{E}_{\mathbb{P}}\left[\left(\frac{dQ^m_{\mathbf{X}+\epsilon\mathbf{V}}}{d\mathbb{P}}-\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}\right)Z\right]\right)\\
&=\mathbb{E}_{\mathbb{P}}\left[\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\frac{dQ^m_{\mathbf{X}+\epsilon\mathbf{V}}}{d\mathbb{P}}-\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}\right)Z\right] \qquad \text{(A.16)}\\
&=\mathbb{E}_{\mathbb{P}}\left[\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}-\frac{\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}\right)Z\right]\\
&=\mathbb{E}_{\mathbb{P}}\left[\frac{d}{d\epsilon}\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}\Bigg|_{\epsilon=0}Z\right]\\
&=\frac{1}{\beta_m}\mathbb{E}_{\mathbb{P}}\left[\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\overline{V}_m\right]Z\right]-\mathbb{E}_{\mathbb{P}}\left[\overline{V}_m\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}Z\right]\\
&=\frac{1}{\beta_m}\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\overline{V}_m\right]\mathbb{E}_{Q^m_{\mathbf{X}}}[Z]-\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\overline{V}_mZ\right]=-\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m,Z]. \qquad \text{(A.17)}
\end{aligned}
$$

4. By (A.14) and (A.15) we obtain

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\varepsilon}\left(\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^i_{\mathbf{X}+\varepsilon\mathbf{V}}]-\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^i_{\mathbf{X}}]\right)\\
&=\lim_{\epsilon\to 0}\left\{\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[-V^i]+\frac{1}{\beta_m\alpha_i}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[\overline{V}_m]\right\}+\frac{1}{\beta_m\alpha_i}\frac{d}{d\epsilon}d_m(\mathbf{X}+\epsilon\mathbf{V})\Bigg|_{\epsilon=0}\\
&=\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^i]+\frac{1}{\beta_m\alpha_i}\mathbb{E}_{Q^m_{\mathbf{X}}}[\overline{V}_m]+\frac{1}{\beta_m\alpha_i}\mathbb{E}_{Q^m_{\mathbf{X}}}[-\overline{V}_m]=\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^i]. \qquad \text{(A.18)}
\end{aligned}
$$

By (A.17) and (A.18) we have

$$\begin{aligned}
&\frac{d}{d\varepsilon}\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^i_{\mathbf{X}+\varepsilon\mathbf{V}}]|_{\varepsilon=0}\\
&=\lim_{\epsilon\to 0}\frac{1}{\varepsilon}\left(\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^i_{\mathbf{X}+\varepsilon\mathbf{V}}]-\mathbb{E}_{Q^m_{\mathbf{X}}}[Y^i_{\mathbf{X}}]\right)\\
&=\lim_{\epsilon\to 0}\frac{1}{\varepsilon}\left(\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^i_{\mathbf{X}+\varepsilon\mathbf{V}}]-\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^i_{\mathbf{X}}]\right)+\lim_{\epsilon\to 0}\frac{1}{\varepsilon}\left(\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}[Y^i_{\mathbf{X}}]-\mathbb{E}_{Q^m_{\mathbf{X}}}[Y^i_{\mathbf{X}}]\right)\\
&=\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^i]-\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m,Y^i_{\mathbf{X}}]\\
&=\mathbb{E}_{Q^m_{\mathbf{X}}}[-V^i]+\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m,X^i]-\frac{1}{\beta_m}\frac{1}{\beta_m}\frac{1}{\alpha_i}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m,\overline{X}_m],
\end{aligned}$$

where the last equation follows from (6.3).

5. Set

$$\theta_m(\varphi_m):=\left(\beta_m H(Q^m,\mathbb{P})+\sum_{i\in I_m}\frac{1}{\alpha_i}\ln\left(-\frac{B}{\beta\alpha_i}\right)\right).$$

By (6.5) we then have

$$\begin{aligned}
&\theta_m\left(\frac{dQ^m_{\mathbf{X}+\varepsilon\mathbf{V}}}{d\mathbb{P}}\right)-\theta_m\left(\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}\right)\\
&=-\beta_m\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}\left[\left(\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]+\beta_m\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\frac{\overline{X}_m}{\beta_m}\right]\\
&\quad-\beta_m\ln\left(\mathbb{E}\left[e^{-\frac{1}{\beta_m}\left(\overline{X}_m+\epsilon\overline{V}_m\right)}\right]+\beta_m\ln\left(\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]\right)\right)\\
&=-\left(\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}\left[\overline{X}_m\right]-\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\overline{X}_m\right]\right)-\epsilon\mathbb{E}_{Q^m_{\mathbf{X}+\varepsilon\mathbf{V}}}\left[\overline{V}_m\right]-\beta_m\ln\left(\frac{\mathbb{E}\left[e^{-\frac{1}{\beta_m}\left(\overline{X}_m+\epsilon\overline{V}_m\right)}\right]}{\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]}\right).
\end{aligned}\tag{A.19}$$

By De L'Hôpital it follows

$$\lim_{\varepsilon\to 0}\ln\left(\frac{\mathbb{E}\left[e^{-\frac{1}{\beta_m}\left(\overline{X}_m+\epsilon\overline{V}_m\right)}\right]}{\mathbb{E}\left[e^{-\frac{1}{\beta_m}\overline{X}_m}\right]}\right)=-\frac{1}{\beta_m}\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\overline{V}_m\right],\tag{A.20}$$

Hence by (A.17), (A.19), and (A.20) we get

$$\begin{aligned}
\lim\frac{1}{\varepsilon}&\left\{\theta_m\left(\frac{dQ^m_{\mathbf{X}+\varepsilon\mathbf{V}}}{d\mathbb{P}}\right)-\theta_m\left(\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}\right)\right\}\\
&=\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m,\overline{X}_m]-\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\overline{V}_m\right]+\mathbb{E}_{Q^m_{\mathbf{X}}}\left[\overline{V}_m\right]\\
&=\frac{1}{\beta_m}COV_{Q^m_{\mathbf{X}}}[\overline{V}_m,\overline{X}_m].
\end{aligned}$$

6. It follows by (A.14).

□

Remark A.11. In (A.16) we can apply the dominated convergence theorem because of the following. We have

$$
\begin{aligned}
&\left|\frac{1}{\epsilon}\left(\frac{dQ^m_{\mathbf{X}+\epsilon\mathbf{V}}}{d\mathbb{P}}-\frac{dQ^m_{\mathbf{X}}}{d\mathbb{P}}\right)\right| \\
&=\left|\frac{1}{\epsilon}\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}-\frac{\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}\right| \\
&=\left|\frac{1}{\epsilon}\frac{\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]-\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m+\epsilon\overline{V}_m}{\beta_m}\right)\right]\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}\right| \\
&\leq\left|\frac{1}{\epsilon}\frac{\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}\frac{\exp\left(-\frac{\epsilon\overline{V}_m}{\beta_m}\right)\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]-\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\exp\left(-\frac{\epsilon\overline{V}_m}{\beta_m}\right)\right]}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m-|\overline{V}_m|}{\beta_m}\right)\right]}\right| \\
&\leq f(\overline{X}_m,\overline{V}_m)\underbrace{\left\{\frac{|\overline{V}_m|}{\beta_m}\exp\left(\frac{|\overline{V}_m|}{\beta_m}\right)\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]+\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\frac{|\overline{V}_m|}{\beta_m}\exp\left(\frac{|\overline{V}_m|}{\beta_m}\right)\right]\right\}}_{:=Z_m}
\end{aligned}
$$

where we have set

$$
f(\overline{X}_m,\overline{V}_m):=\frac{\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}\frac{1}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m-|\overline{V}_m|}{\beta_m}\right)\right]},
$$

and used that

$$
\left|\frac{\exp(-\frac{\epsilon\overline{V}_m}{\beta_m})-1}{\epsilon}\right|\leq\frac{1}{\beta_m}|\overline{V}_m|\exp\left(\frac{|\overline{V}_m|}{\beta_m}\right).
$$

Note that in this case $M^{\phi_0}\subseteq L^2(\mathbb{R})$, hence

$$
f(\overline{X}_m,\overline{V}_m)=\frac{\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]}\frac{1}{\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m-|\overline{V}_m|}{\beta_m}\right)\right]}=K_1\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\in L^2(\mathbb{R})
$$

and

$$
\begin{aligned}
Z_m&\leq\exp\left(\frac{2|\overline{V}_m|}{\beta_m}\right)\mathbb{E}\left[\exp\left(-\frac{\overline{X}_m}{\beta_m}\right)\right]+\mathbb{E}\left[\exp\left(\frac{2|\overline{V}_m|-\overline{X}_m}{\beta_m}\right)\right] \\
&=K_2+K_3\exp\left(\frac{2|\overline{V}_m|}{\beta_m}\right)\in L^2(\mathbb{R})
\end{aligned}
$$

because $\overline{X}_m,\overline{V}_m\in M^{\phi_0}$. We can conclude that $f(\overline{X}_m,\overline{V}_m)Z_m$ is in $L^1(\mathbb{R})$.

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