Electricity futures price modeling with Lévy term structure models

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Abstract

In this paper we generalize the approach of Hinz and Wilhelm [2006] replacing in the dynamics of the asset prices the Brownian motion by a more general Lévy process, also taking into account the occurrence of spikes. In particular, we reduce the modeling of an electricity futures market to the modeling of a Lévy bond market with an additional risky asset. This allows to employ well established techniques from interest rate term structure modeling. We then examine Markovianity of the induced electricity spot price, an important property when it comes to option pricing. We show that the considered method combined with the Fourier transform techniques provides semi analytic pricing formulas for European electricity options. Finally we consider the pricing of path dependent derivatives such as electricity swing options.

Key words: Electricity futures market, interest rate term structure modeling, Lévy processes, Fourier transform techniques, electricity swing options.

1 Introduction

In the stochastic modeling of electricity markets, there are two main approaches in the literature. The first one starts with a stochastic model for the spot price, and from this derives the futures price dynamics by using arbitrage pricing theory. The second approach models directly the price dynamics of the complete curves of forward and futures contracts traded in electricity markets. We refer to Benth et al. [2008] and references therein for an overview of literature on electricity markets.

Spot price models have two major disadvantages that result from the non-storability of electricity. Since spot electricity is not a traded asset spot models

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induce highly incomplete markets. The usual buy-and-hold relationship between futures and their underlying spots does not hold anymore, and one has to identify the market price of risk, or equivalently the pricing measure, to determine futures prices (or prices of any other derivatives). However, to calibrate the pricing measure on market data seems to be a very challenging task on electricity markets. Further, the flexibility of spot models is rarely sufficient to generate a family of futures curves that is consistent with observed market curves.The second problem is to define the market information filtration, which is an essential ingredient in derivative pricing. The usual approach is here to assume that the information filtration is generated by the underlying electricity spot price. However, the assumption that all information available to the market is incorporated in the past evolution of the underlying might be acceptable for storable assets on classical financial markets, but for non-storable underlyings (like electricity or temperature) this supposition is fundamentally wrong. In contrast to storable assets, one cannot profit from forward looking information about non-storable assets by taking long or short positions today. Thus, forward looking information available to the market is not reflected in the past evolution of the non-storable underlying and is therefore not included in the filtration generated by the underlying.

A natural ansatz to cope with this information misspecification, which is carried out in Benth and Meyer-Brandis [2009], could be to enlarge the filtration by forward looking information. It seems, however, rather difficult to enlarge the filtration explicitly by all forward looking information available to the market. Further, from a mathematical point of view, one encounters the theory of enlargement of filtrations which restricts the type of included forward looking information by its analytic tractability.

In the light of the above mentioned problems it seems to be more promising to choose the second model approach, which is to model the complete curves of electricity futures and forwards. In particular, because futures are traded contracts it is acceptable to assume that the complete futures curve, and thus the filtration generated by the underlyings in this situation (futures), integrates all forward looking information about electricity available to the market. The electricity spot price is then read off in the short end of the curve and the evolution of the spot price will now also be governed by the forward looking information contained in futures prices.

In analogy to the HJM approach for forward interest rates, several authors have proposed a HJM-type model for electricity futures curves (see Benth and Koekebakker [2008] and Benth et al. [2008]). It seems though problematic to directly apply model approaches from interest theory to electricity futures modeling, which are two qualitatively very different phenomena. Also, a general HJM ansatz for electricity futures generically implies very complex non-Markovian dynamics for the spot price. This is in particular a problem when it comes to pricing of path dependent electricity products like, for example, swing options (see e.g., Hambly et al. [2009], Keppo [2004] and Wallin [2008]). Here, the Markovian property of the spot price is essential for the dynamic programming principle needed to find the solution of the constrained stochastic optimal control problem of maximizing the expected profit of the path dependent electricity product (see e.g. Wallin [2008]).

In Hinz and Wilhelm [2006] the authors respond to this challenge in that they introduce an approach which converts a given electricity market into a a money market. By a change of numeraire they establish a one-to-one correspondence between electricity markets and markets consisting of bonds and a risky asset. This then allows for the use of well-established theory and models from interest rate markets in pricing of electricity derivatives.

The aim of this paper is to generalize the approach of Hinz and Wilhelm [2006] replacing in the dynamics of the asset prices the Brownian motion by a more general Lévy process, also taking into account the occurrence of spikes (which are a very prominent feature of electricity prices). Advanced interest rate theory combined with change of numéraire techniques is used to develop a model setting for electricity futures curves. In the second part of the paper we then consider the pricing of electricity derivatives in our setting. We first provide semi-analytic pricing formulas for European electricity options employing Fourier transform techniques before we deal with the valuation of path dependent electricity swing options. For the pricing of the latter, one valuable feature of our model approach is that the induced dynamics of the spot price becomes multi-dimensional Markovian (see Section 4). We specify the stochastic optimal control problem associated to the pricing of electricity swing options as previously studied in Lund and Ollmar [2003], Keppo [2004], Hambly et al. [2009], and Wallin [2008]. In particular, we derive in our setting the Hamilton-Jacobi-Bellman equation associated to the pricing of swing options.

The remaining parts of the paper are organized as follows. In the next section we develop the correspondence between electricity and fixed-income markets. Then, in Section 3 we introduce an electricity market model derived from a Lévy term structure model for fixed income markets. As a special example we show that exponential Ornstein-Uhlenbeck processes, which is a commonly used model for spot prices, can be derived as spot price dynamics in our framework. Thereafter, in Section 4 we examine the Markov property of the spot price process in our framework. Finally, we apply the results of Sections 3 and 4 to valuation of electricity derivatives in Section 5.

2 Connection between electricity market and money market

Let $F(t,\tau)$, $0 \le t \le \tau$, be the futures price at time t of electricity and T be a finite time horizon, $\tau \le T$. Denote the set of chronological time pairs by

$$\mathcal{D} := \{(t,\tau) : 0 \le t \le \tau \le T\}.$$

We model the futures market starting by the following axioms:
C1: For every $\tau \in [0,T]$ the futures price evolution $(F(t,\tau))_{(t,\tau)\in\mathcal{D}}$ is a positive-valued adapted stochastic process realized on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$.
C2: There exists a martingale measure $\mathbb{Q}^F$ equivalent to $\mathbb{P}$ such that for all $\tau \in [0,T]$ the futures price process $(F(t,\tau))_{(t,\tau)\in\mathcal{D}}$ is a $\mathbb{Q}^F$-martingale.
C3: At $t = 0$ futures prices start at deterministic positive values $F(0,\tau)$, $\tau \in [0,T]$.
C4: Terminal prices form a *spot price process* $S_t := F(t,t)$, $t \in [0,T]$.

Following the approach of Hinz and Wilhelm [2006] we now convert an electricity market into a *money market* consisting of zero bonds $(P(t,\tau))_{0\le t\le\tau}$ equipped with an additional risky asset $(N_t)_{t\in[0,T]}$ by using the following transformation:

$$P(t,\tau) := \frac{F(t,\tau)}{S_t}, \tag{1}$$

$$N_t := \frac{1}{S_t}. \tag{2}$$

We have that the money market defined by the currency change (1)–(2) satisfies the following axioms:

M1: $(N_t)_{t\in[0,T]}$ and $(P(t,\tau))_{(t,\tau)\in\mathcal{D}}$ are positive, adapted stochastic processes defined on $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$.

M2: There exists a positive-valued, adapted numéraire process $(C_t)_{t\in[0,T]}$ and there exists a martingale measure $\mathbb{Q}^M$ equivalent to $\mathbb{P}$, such that for all $\tau\in[0,T]$ the discounted price processes $\hat{P}(t,\tau) := \frac{P(t,\tau)}{C_t}$, $(t,\tau)\in\mathcal{D}$, and $\hat{N}_t := \frac{N_t}{C_t}$, $0\le t\le T$, are $\mathbb{Q}^M$-martingales.

M3: Prices start at deterministic values N_0 and $(P(0,\tau))_{\tau\in[0,T]}$.

M4: Bond prices finish at one, i.e. $P(t,t)=1$, for every $t\in[0,T]$.

We now need a slight generalization of Theorem 1 in Hinz and Wilhelm [2006].

Theorem 2.1. *i) Suppose that the commodity market $(F(t,\tau))_{(t,\tau)\in\mathcal{D}}$ fulfills C1 to C4 with an initial futures curve $(F(0,\tau))_{\tau\in[0,T]}$ and a martingale measure $\mathbb{Q}^F$. Then the transformation (1)– (2) provides a money market satisfying M1 to M4 with the deterministic initial values*

$$P(0,\tau) := \frac{F(0,\tau)}{S_0}, \quad \forall\,\tau\in[0,T], \quad \text{and} \quad N_0 := \frac{1}{S_0},$$

where the numéraire process and the martingale measure are given by

$$C_t = N_t, \quad t\in[0,T], \quad \text{and} \quad \mathbb{Q}^M = \mathbb{Q}^F. \tag{3}$$

ii) Suppose that the money market $(P(t,\tau))_{(t,\tau)\in\mathcal{D}}$, $(N_t)_{t\in[0,T]}$ fulfills M1 to M4 with initial values $(P(0,\tau))_{\tau\in[0,T]}$, N_0, a discounting process $(C_t)_{t\in[0,T]}$ and a martingale measure $\mathbb{Q}^M$. Then the transformation

$$F(t,\tau) := \frac{P(t,\tau)}{N_t}, \quad (t,\tau)\in\mathcal{D}, \tag{4}$$

gives an electricity market with the deterministic initial futures curve $F(0,\tau) := \frac{P(0,\tau)}{N_0}$, for all $\tau\in[0,T]$, and the martingale measure

$$d\mathbb{Q}^F = \frac{N_T}{C_T}\frac{C_0}{N_0}d\mathbb{Q}^M. \tag{5}$$

Note that in Theorem 1 of Hinz and Wilhelm [2006] all price processes were assumed continuous. In our proof we will only use the integrability properties of the processes involved.

Proof.

i) It is easy to see that the properties M1– M4 are obvious consequences of C1– C4 due to (1) and (2), if the discounting process and the martingale measure are given by (3).

ii) Define the futures price process $F(t,\tau)$ as in (4). The process $F(t,\tau)$ is then positive and adapted by Assumption M1. Consider the equivalent probability measure $\mathbb{Q}^F$ given by (5). $F(t,\tau)$ is integrable w.r.t. $\mathbb{Q}^F$, since

$$\begin{aligned} E^{\mathbb{Q}^F}[F(t,\tau)] &= E^{\mathbb{Q}^M}\left[\frac{P(t,\tau)}{N_t}\frac{d\mathbb{Q}^F}{dQ^M}\Big|_{\mathcal{F}_t}\right] = \frac{C_0}{N_0}E^{\mathbb{Q}^M}\left[\frac{P(t,\tau)}{N_t}\frac{N_t}{C_t}\right] \\ &= \frac{C_0}{N_0}E^{\mathbb{Q}^M}\left[\frac{P(t,\tau)}{C_t}\right] < \infty \end{aligned}$$

by Assumption M2. Furthermore, M2 yields

$$\begin{aligned} E^{\mathbb{Q}^F}[F(t,\tau)|\mathcal{F}_s] &= \frac{E^{\mathbb{Q}^M}[F(t,\tau)\frac{N_t}{C_t}|\mathcal{F}_s]}{\frac{N_s}{C_s}} = \frac{E^{\mathbb{Q}^M}[\frac{P(t,\tau)}{C_t}|\mathcal{F}_s]}{\frac{N_s}{C_s}} \\ &= \frac{P(s,\tau)}{N_s} = F(s,\tau), \quad \forall\ 0 \le s \le t \le \tau. \end{aligned}$$

Hence, $(F(t,\tau))_{0\le t\le\tau}$ is a $\mathbb{Q}^F$-martingale.

□

In the following sections we apply this approach and study electricity markets derived from term structure models driven by general Lévy processes, using the HJM approach.

3 Money market construction

We follow the HJM approach and specify the term structure by modeling the (instantaneous) forward rate $f(t,\tau)$, $(t,\tau) \in \mathcal{D}$. Let $P(t,\tau)$, $(t,\tau) \in \mathcal{D}$, be the market price at moment t of a bond paying 1 at the maturity time τ, $\tau \le T$. Given the forward rate curve $f(t,\tau)$ the bond prices are defined by

$$P(t,\tau) = \exp\{-\int_t^\tau f(t,s)ds\}, \tag{6}$$

while the instantaneous short rate r at time t is given by

$$r(t) := f(t,t). \tag{7}$$

A general introduction to fixed-income markets is given in Björk [1998].

Let $L = (L^1,\dots,L^n)$ be an n-dimensional Lévy process with independent components, defined on a probability space $(\Omega,\mathcal{F},\mathbb{Q}^M)$ endowed with the completed canonical filtration $(\mathcal{F}_t)_{t\in[0,T]}$ associated with L. We denote by (b_i,c_i,ν_i) the characteristic triplet of each component L^i, $i=1,\dots,n$.

We assume that

A1: we are given an $\mathbb{R}$-valued and $\mathbb{R}^n$-valued stochastic process $\alpha(t,\tau)$ and $\eta(t,\tau) = (\eta^1(t,\tau),\dots,\eta^n(t,\tau))$, $(t,\tau) \in \mathcal{D}$, respectively, such that $\alpha(t,\tau)$ and $\eta(t,\tau)$ are continuous and adapted.

A2: $\int_0^T \int_0^T E|\alpha(s,u)|dsdu < \infty$, $\int_0^T \int_0^T E\|\eta(s,u)\|^2 dsdu < \infty$.
A3: The initial forward curve is given by a deterministic and continuously differentiable function $\tau \to f(0,\tau)$ on the interval $[0,T]$.

For the forward rate we consider a generalized HJM model, i.e we assume that the forward rate process follows the dynamics

$$f(t,\tau) = f(0,\tau) + \int_0^t \alpha(s,\tau)ds + \sum_{i=1}^n \int_0^t \eta^i(s,\tau)dL_s^i, \quad t \leq \tau. \tag{8}$$

In terms of the short rate we can rewrite (8) and (7) as

$$r(t) = f(0,t) + \int_0^t \alpha(s,t)ds + \sum_{i=1}^n \int_0^t \eta^i(s,t)dL_s^i, \quad t \leq T. \tag{9}$$

Lévy term structure models of type (8)–(9) are frequently considered in the literature (see e.g. Eberlein and Raible [1999], Eberlein and Özkan [2003], Filipović and S. [2008], or Jakubowski and Zabczyk [2007]).

Putting (6) and (9) together and assuming that

$$\alpha(t,\tau) = \eta(t,\tau) = 0 \quad \text{a.s.} \quad \text{for} \quad t > \tau, \tag{10}$$

so that the forward rate (8) is defined for all $t, \tau \in [0,T]$, we can derive the following representation for the bond price given in Eberlein and Özkan [2003]:

$$\begin{aligned} P(t,\tau) = P(0,\tau) \exp\Big\{ &\int_0^t r(u)du \\ &- \int_0^t \int_0^\tau \alpha(s,u)duds - \sum_{i=1}^n \int_0^t \int_0^\tau \eta^i(s,u)dudL_s^i \Big\}. \end{aligned} \tag{11}$$

We now consider the bank account process as a discounting factor, i.e.

$$C_t = \exp\{\int_0^t r(s)ds\}. \tag{12}$$

In order to provide a condition which ensures that $\mathbb{Q}^M$ is a local martingale measure for

$$\hat{P}(t,\tau) := \frac{P(t,\tau)}{C_t}, \quad t \in [0,\tau], \tag{13}$$

we assume that there exist $a_i < 0$ and $d_i > 1$ such that the Lévy measures ν_i of L^i satisfy

$$\int_{\{|x|>1\}} e^{ux}\nu_i(dx) < \infty, \quad u \in [a_i, d_i], \quad i = 1, \ldots, n, \tag{14}$$

(see Eberlein and Raible [1999] or Filipović and S. [2008]). Condition (14) ensures the existence of the cumulant generating function

$$\Theta^i(u) := \log E[\exp(uL_1^i)] \tag{15}$$

at least on the set $\{u \in \mathbb{C} | \Re u \in [a_i, d_i]\}$, where $\Re u$ denotes the real part of $u \in \mathbb{C}$, $i = 1, \ldots, n$. By Lemma 26.4 in Sato [1999] Θ^i is continuously differentiable and has the representation:

$$\Theta^i(u) = b_i u + \frac{c_i}{2} u^2 + \int_{\mathbb{R}} (e^{ux} - 1 - ux)\nu_i(dx), \quad i = 1, \ldots, n. \tag{16}$$

As a consequence, the Lévy processes L^i, $i = 1, \ldots, n$, have finite moments of arbitrary order. Provided

$$-\int_0^\tau \eta^i(s,u)du \in (a_i, d_i) \quad \text{for} \quad i = 1, \ldots, n,$$

for any $\tau \leq T$, the HJM condition on the drift

$$\alpha(t,x) = \sum_{i=1}^n \frac{\partial}{\partial x} \Theta^i\Big(-\int_0^x \eta^i(t,u)du \Big) \quad \text{a.s.} \tag{17}$$

implies that $\mathbb{Q}^M$ is a local martingale measure. The drift condition (17) is derived in Eberlein and Özkan [2003] and Eberlein and Raible [1999], for an analogous drift condition in the infinite dimensional Lévy setting see Jakubowski and Zabczyk [2007] and Filipović and S. [2008].

Denoting

$$\sigma^i(t,\tau) := -\int_0^\tau \eta^i(t,u)du, \quad i = 1, \ldots, n, \tag{18}$$

we can rewrite the HJM drift condition (17) as

$$\begin{aligned} \int_0^\tau \alpha(s,u)du &= \sum_{i=1}^n \int_0^\tau \frac{\partial}{\partial u} \Theta^i(\sigma^i(s,u))du \\ &= \sum_{i=1}^n \Theta^i(\sigma^i(s,\tau)) \quad \text{a.s.} \end{aligned} \tag{19}$$

Substituting (19) into (11), we get the same representation for $P(t,\tau)$ as in Eberlein and Raible [1999]

$$\begin{aligned} P(t,\tau) = P(0,\tau) \exp\Big\{ \int_0^t r(u)du - \sum_{i=1}^n \int_0^t \Theta^i(\sigma^i(s,\tau))ds \\ + \sum_{i=1}^n \int_0^t \sigma^i(s,\tau)dL_s^i \Big\}. \end{aligned} \tag{20}$$

To complete the modeling of the arbitrage-free money market satisfying Assumptions M1–M4, we assume that the risky asset N_t is given by

$$N_t = \exp\{\int_0^t r(u)du - \sum_{i=1}^n \int_0^t \Theta^i(v^i(s))ds + \sum_{i=1}^n \int_0^t v^i(s)dL_s^i\}, \quad N_0 = 1, \tag{21}$$

where $v = (v^1, \ldots, v^n)$ is a continuous and adapted process, such that

$$\hat{N}_t = \frac{N_t}{C_t}$$

is a well-defined local martingale under $\mathbb{Q}^M$.

We consider the electricity price processes

$$F(t,\tau) = \frac{P(t,\tau)}{N_t} \quad \text{and} \quad S(t) = \frac{1}{N_t}, \quad \forall (t,\tau) \in \mathcal{D}, \tag{22}$$

where $P(t,\tau)$ and N_t are now given by (20) and (21).

According to Theorem 2.1 the transformation (22) gives an arbitrage-free commodity futures market with the deterministic initial futures curve $F(0,\tau) := P(0,\tau)/N_0 = P(0,\tau)$. By the same theorem,

$$\begin{aligned} d\mathbb{Q}^F &= \frac{N_T}{C_T}\frac{C_0}{N_0} d\mathbb{Q}^M \\ &= \exp\left\{ \sum_{i=1}^n \int_0^T v^i(s) dL_s^i - \sum_{i=1}^n \int_0^T \Theta^i(v^i(s)) ds \right\} d\mathbb{Q}^M \end{aligned} \tag{23}$$

is a martingale measure for $F(t,\tau)$, $(t,\tau) \in \mathcal{D}$.

In order to study the electricity market (22) under the martingale measure $\mathbb{Q}^F$ defined by (23) we need the distribution of L under $\mathbb{Q}^F$. By Girsanov's Theorem for semimartingales (cf. Theorem III.3.24 in Jacod and Shiryaev [2003]), L is a semimartingale under $\mathbb{Q}^F$. In particular, if the process $v(t)$ appearing in (21) and (23) is deterministic, we get from Girsanov's Theorem the following Proposition:

Proposition 3.1. *$L = (L^1, \ldots, L^n)$ is a (non-homogeneous) Lévy process with independent components under the measure $\mathbb{Q}^F$, where for every $j = 1, \ldots, n$, the* characteristic triplet *of L^j w.r.t. $\mathbb{Q}^F$ is given by*

$$b_j^{\mathbb{Q}^F}(t) := b_j + c_j v^j(t) + \int_{\mathbb{R}} (e^{v^j(t)x} - 1) x I_{|x|\le 1}(x) \nu_j(dx), \tag{24}$$

$$c_j^{\mathbb{Q}^F}(t) := c_j, \tag{25}$$

$$\nu_j^{\mathbb{Q}^F}(dt, dx) := e^{v^j(t)x} \nu_j(dx) dt. \tag{26}$$

For the definition of a non-homogeneous Lévy process we refer to Hambly et al. [2009].

Remark 3.2. *Note that if $v(t)$ is a constant function, then by Proposition 3.1 L is a time-homogeneous Lévy process under $\mathbb{Q}^F$.*

For the reminder of the paper we assume that $v(t)$ is a deterministic, continuous function.

Let us consider an example, which shows that our model for the electricity market contains the case, where the spot price process is an exponential Ornstein-Uhlenbeck process with seasonality effect. We refer also to Bregman [2008], Examples 3.4.4-3.4.5.

Example 3.3. Let $W_t^{\mathbb{Q}^F}$ be a standard Brownian motion under $\mathbb{Q}^F$ and a Lévy process L is given by

$$L_t = W_t^{\mathbb{Q}^F} + \int_0^t \int_{\mathbb{R}} x J_L(dx \times ds)$$

for some Poisson random measure J_L on $\mathbb{R} \times (0, \infty)$. Let X_t be an Ornstein-Uhlenbeck process driven by L, i.e.

$$dX_t = -X_t dt + dL_t, \quad X_0 = x_0 \in \mathbb{R}, \quad t \leq T.$$

Then

$$X_t e^{-(T-t)} = \underbrace{x_0 e^{-T}}_{=:k} + \int_0^t e^{-(T-s)} dL_s. \tag{27}$$

We assume also that

$$\int_{\mathbb{R}} e^x (1 + |x|) \nu^{\mathbb{Q}^F}(dx) < \infty, \tag{28}$$

where $\nu^{\mathbb{Q}^F}$ is the Lévy measure of L under $\mathbb{Q}^F$.

Our attention now is to specify a money market $N_t, P(t,T)$, $t \leq T$, such that the corresponding futures market induces an exponential Ornstein-Uhlenbeck model for the spot prices. This illustrates that our approach includes this very established class of spot models. Let us consider

$$N_t = e^{-(X_t + \theta(t))}, \quad t \leq T, \tag{29}$$

where $\theta(t) : [0, T] \to \mathbb{R}$ is a deterministic, differentiable function that characterize seasonality of the corresponding spot price. Let a bond price process $P(t,T)$, $t \in [0, T]$, be given by

$$\begin{aligned} P(t,T) = \exp\Big\{ & \theta(T) - \theta(t) + (e^{-(T-t)} - 1) X_t + \frac{1}{2} \int_t^T e^{-2(T-s)} ds \\ & + \int_t^T \int_{\mathbb{R}} \left(\exp\{e^{-(T-s)} x\} - 1 \right) \nu^{\mathbb{Q}^F}(dx) ds \Big\}. \end{aligned} \tag{30}$$

Then by (22)

$$S_t = \frac{1}{N_t} = e^{X_t + \theta(t)}, \quad t \leq T, \tag{31}$$

and by (30), and (31)

$$
\begin{aligned}
F(t,T) = S(t)P(t,T) = F(0,T)\exp\Big\{ &\int_0^t e^{-(T-s)} dL_s - \frac{1}{2}\int_0^t e^{-2(T-s)} ds \\
&- \int_0^t \int_{\mathbb{R}} \left(e^{e^{-(T-s)}x} - 1 \right) \nu^{\mathbb{Q}^F}(dx) ds \Big\} \\
= F(0,T)\exp\Big\{ &\int_0^t e^{-(T-s)} dW_s^{\mathbb{Q}^F} - \frac{1}{2}\int_0^t e^{-2(T-s)} ds \\
&+ \int_0^t \int_{\mathbb{R}} e^{-(T-s)} x dJ_L(dx \times ds) \\
&- \int_0^t \int_{\mathbb{R}} \left(e^{e^{-(T-s)}x} - 1 \right) \nu^{\mathbb{Q}^F}(dx) ds \Big\}, \quad t \le T, \qquad (32)
\end{aligned}
$$

where

$$
\begin{aligned}
F(0,T) = \exp\Big\{ &k + \theta(T) + \frac{1}{2}\int_0^T e^{-2(T-s)} ds \\
&+ \int_0^T \int_{\mathbb{R}} \left(\exp\{e^{-(T-s)}x\} - 1 \right) \nu^{\mathbb{Q}^F}(dx) ds \Big\},
\end{aligned}
$$

and $k \in \mathbb{R}$ is defined in (27). Note that by the exponential formula for Poisson random measures (see e.g. Cont and Tankov [2004], Proposition 3.6) the process $F(t,T)$ given in (32) is a martingale. Indeed, choosing $C_t = N_t$ as a discounting process we have obviously $\mathbb{Q}^F = \mathbb{Q}^M$ in this example.

We now derive the forward rate that gives us the bond $P(t,T)$ as in (30):

$$
\begin{aligned}
f(t,T) &= -\frac{\partial}{\partial T} \ln P(t,T) \\
&= e^{-(T-t)} X_t - \theta'(T) - \frac{1}{2} + \int_t^T \int_{\mathbb{R}} \exp\{e^{-(T-s)}x\} e^{-(T-s)} x \nu^{\mathbb{Q}^F}(dx) ds\} \\
&\quad - \int_{\mathbb{R}} (e^x - 1) \nu^{\mathbb{Q}^F}(dx) + \int_t^T e^{-2(T-s)} ds. \qquad (33)
\end{aligned}
$$

In particular, the corresponding short rate process is then given by

$$
r(t) = f(t,t) = X_t - \theta'(t) - \frac{1}{2} - \int_{\mathbb{R}} (e^x - 1) \nu^{\mathbb{Q}^F}(dx).
$$

Note that condition (28) guarantees that $f(t,T)$ in (33) and $P(t,T)$ in (30) are well-defined.

4 Markov property

In this section we examine the Markov property of the spot price process S given by (22). To begin with, applying Proposition 3.1, we compute the dynamics of

S under $\mathbb{Q}^F$ as follows.

Lemma 4.1. *The dynamics of S under $\mathbb{Q}^F$ is given by*

$$dS(t) = S(t)[-r(t) + \frac{1}{2}\sum_{i=1}^{n} c_i(v^i(t))^2 + \sum_{i=1}^{n}\Theta^i(v^i(t))]dt - S(t-)\sum_{i=1}^{n} v^i(t)dL_t^i$$
$$+ \int_{R^n} S(t-)(e^{\langle v(t-),x\rangle} - 1 + \langle v(t-), x\rangle)J_L(dx \times dt), \tag{34}$$

where J_L is the jump measure of L.

Proof. First, by (21)–(22)

$$S(t) = \exp\{-\int_0^t r(u)du + \sum_{i=1}^{n}\int_0^t \Theta^i(v^i(s))ds - \sum_{i=1}^{n}\int_0^t v^i(s)dL_s^i\}, \tag{35}$$

where $\Theta(t)$ and $v(t)$ are deterministic, continuous functions and L is a non-homogeneous Lévy process satisfying Proposition 3.1. The rest follows from the Itô formula. □

Hence, since v is deterministic, we get the following result:

Proposition 4.2. *Suppose the short rate process r is a Markov process. Then the vector process (S, r) is a Markov process.*

Proof. Since r is a Markov process and v is deterministic, (S, r) is a Markov process by (34). □

Remark 4.3. *Note that if the volatility η is deterministic, then also the drift α is deterministic by* (17) *and the short rate process r is a Markov process by* (9).

Next we elaborate the question, when the spot price S itself is a one-dimensional Markov process under $\mathbb{Q}^F$. For this purpose we assume from now on that the volatility $\eta = (\eta^1, \dots, \eta^n)$ appearing in (9) is deterministic.

Note that by (22), (21), and (20) we can factorize electricity price processes as follows

$$F(t, \tau) = F(0, \tau)\exp\{\sum_{i=1}^{n}\int_0^t \delta^i(s, \tau)dL_s^i - \sum_{i=1}^{n}\int_0^t \psi^i(s, \tau)ds\}, \tag{36}$$

where

$$\delta^i(s, \tau) := \sigma^i(s, \tau) - v^i(s), \quad \text{and} \tag{37}$$
$$\psi^i(s, \tau) := \Theta^i(\sigma^i(s, \tau)) - \Theta^i(v^i(s)). \tag{38}$$

Setting $\tau = t$ in (36) we obtain the electricity spot price process

$$S(t) = F(t, t) = F(0, t)\exp\Big\{\sum_{i=1}^{n}\int_0^t \delta^i(s, t)dL_s^i - \sum_{i=1}^{n}\int_0^t \psi^i(s, t)ds\Big\}. \tag{39}$$

Note that by assumption the coefficients $\delta = (\delta^1, \dots, \delta^n)$ and $\psi = (\psi^1, \dots, \psi^n)$ are deterministic, since $\sigma = (\sigma^1, \dots, \sigma^n)$ is deterministic. For the sake of simplicity we will only consider the one-dimensional case, i.e. we assume $n = 1$. However, all results of this subsection still hold in the case of multidimensional non-homogeneous Lévy process with independent components. In other words, we examine the Markov property of the spot price process S given by

$$S(t) = F(0,t) \exp\{\int_0^t \delta(s,t) dL_s - \int_0^t \psi(s,t) ds\}, \quad t \in [0,T], \tag{40}$$

under the futures martingale measure $\mathbb{Q}^F$ when $\delta(s,t)$ and $\psi(s,t)$ are deterministic and continuous. Because $F(0,t)$ is also deterministic by assumptions, S is a Markov process iff the process

$$Z_t = \int_0^t \delta(s,t) dL_s, \quad t \in [0,T], \tag{41}$$

is Markovian. Recall that L is a non-homogeneous Lévy process under $\mathbb{Q}^F$ by Proposition 3.1.

Proposition 4.4. *We assume that there are constants $\epsilon, \eta > 0$ and functions $c(t), \gamma(t) : [0,T] \to \mathbb{R}^+$, such that for all $t \in [0,T]$*

1. $\int_0^t c(s) ds < \infty$,
2. $\gamma(t) \geq \epsilon$,
3. $\Re\Phi_t(u) \leq c(t) - \gamma(t)|u|^\eta$, *for every $u \in \mathbb{R}$, where $\Phi_t(\cdot)$ is the characteristic exponent of L_t under $\mathbb{Q}^F$ defined by*

$$E^{\mathbb{Q}^F}[e^{iuL_t}] = e^{\Phi_t(u)}, \quad u \in \mathbb{R},$$

where $\Re\Phi_t(u)$ denotes the real part of $\Phi_t(u)$. Then the spot price process S is Markovian iff for all fixed w and u with $0 < w < u \leq T$ there exists a real constant $\xi = \xi_u^w$ (which may depend on w and u) such that

$$\delta(t,u) = \xi_u^w \delta(t,w), \quad \forall t \in [0,T],$$

where δ is the volatility structure of S in (40).

Corollary 4.5. *Under the hypotheses of Proposition 4.4 the spot price process S is Markovian iff its volatility structure δ admits the representation*

$$\delta(t,\tau) = \zeta(t)\rho(\tau), \quad \forall (t,\tau) \in \mathcal{D}, \tag{42}$$

where $\zeta, \rho : [0,T] \to \mathbb{R}$ are continuously differentiable functions.

The proofs of Proposition 4.4 and Corollary 4.5 are omitted, since they can be recovered from the ones given in Section 4 in Eberlein and Raible [1999] under minimal technical changes.

Now we consider two examples of the volatility function δ that satisfies (42).

Example 4.6 (Vasicek volatility structure)**.** Recall that

$$\delta(t, \tau) = \sigma(t, \tau) - v(t),$$

where σ is the volatility of the corresponding bond and v is a deterministic function. Let

$$\sigma(t, \tau) = \frac{\hat{\sigma}}{a}(1 - e^{-a(\tau - t)}) \quad \textit{(Vasicek volatility)},$$

where $\hat{\sigma} > 0$ and $a \neq 0$. Then by Corollary 4.5 the spot price process S is Markovian iff there exist continuously differentiable functions $\zeta, \rho : [0, T] \to \mathbb{R}$, such that

$$v(t) = \frac{\hat{\sigma}}{a}(1 - e^{-a(\tau - t)}) - \zeta(t)\rho(\tau).$$

Since v is constant in τ, by deriving we obtain

$$\zeta(t)\rho'(\tau) = \hat{\sigma}e^{at}e^{-a\tau},$$

and consequently

$$\zeta(t) = \lambda\hat{\sigma}e^{at},$$
$$\rho'(\tau) = \frac{1}{\lambda}e^{-a\tau}$$

for $(t, \tau) \in \mathcal{D}$ and some $\lambda \neq 0$. Then $\rho(\tau) = -\frac{1}{a\lambda}e^{-a\tau} + c$ for some $c \in \mathbb{R}$, $\lambda \neq 0$. Hence, in this example the spot price process S is Markovian iff $v(t)$ is of the form

$$v(t) = \frac{\hat{\sigma}}{a} - \hat{\sigma}ce^{at}$$

for some $c \in \mathbb{R}$.

Example 4.7 (Ho-Lee volatility structure)**.** In case the bond volatility structure σ satisfies

$$\sigma(t, \tau) = \hat{\sigma}(\tau - t) \quad \text{with} \quad \hat{\sigma} > 0 \quad \textit{(Ho-Lee volatility)},$$

Corollary 4.5 yields that the spot price S is a Markov process iff $v(t)$ is of the form $v(t) = \hat{\sigma}(c - t)$ for some $c \in \mathbb{R}$.

We follow the approach of Hinz and Wilhelm [2006] and, applying Corollary 4.5, characterize the class of stationary volatility structures δ that lead to Markovian spot price process S.

Proposition 4.8. *Suppose the volatility structure δ is stationary, that means, there exists a twice continuously differentiable function $\tilde{\delta} : [0, T] \to \mathbb{R}^+$ such that $\delta(t, \tau) = \tilde{\delta}(\tau - t)$ for all $(t, \tau) \in \mathcal{D}$. Then, under the hypotheses of Proposition 4.4, S is a Markov process iff δ is of the form*

$$\delta(t, \tau) = \hat{\delta}e^{a(\tau - t)} \tag{43}$$

with $a \in \mathbb{R}$ and $\hat{\delta} > 0$.

Proof. If δ is of the form (43), then S is a Markov process by Corollary 4.5. Assume now that S is Markovian. As $\delta(t,\tau)$ is stationary by assumption, the partial derivatives satisfy

$$\frac{\partial}{\partial \tau}\delta(t,\tau) = \tilde{\delta}'(\tau - t) = -\frac{\partial}{\partial t}\delta(t,\tau).$$

Corollary 4.5 yields then

$$\zeta'(t)\rho(\tau) = -\zeta(t)\rho'(\tau),$$

i.e.

$$(\log \rho)'(\tau) = -(\log \zeta')(t)$$

for all $(t,\tau) \in \mathcal{D}$. Since t and τ are independent variables, neither of the last equality sides can actually depend on t or τ. Hence both sides are constant. Denoting their common value by a, we obtain

$$\rho(\tau) = e^{a\tau + K_1} \quad \text{and} \quad \zeta(t) = e^{-at+K_2}$$

with two real constants K_1 and K_2, and hence

$$\delta(t,\tau) = e^{K_1+K_2}e^{a(\tau-t)}.$$

Defining $\hat{\delta} := e^{K_1+K_2}$, we get (43). $\square$

The volatility structure (43) picks up the maturity effect for $a < 0$: the volatility increases when a future contract comes to delivery, since temperature forecasts, outages and other specifics about the delivery period become more and more precise. However, the model (43) does not include seasonality: futures during winter months show higher prices than comparable contracts during the summer. See Benth and Koekebakker [2008], Klüppelberg et al. [2010], and Kiesel et al. [2009] for a description of electricity futures and options markets. In order to include the seasonality we can use, for example, the volatility model suggested in Fackler and Tian [1999]:

$$\delta(t,\tau) = a(t)e^{-b(\tau-t)}, \quad b \geq 0.$$

The seasonal part $a(t)$ can be modeled, for example, as a truncated Fourier series

$$a(t) = a + \sum_{j=1}^{J}(d_j \sin(2\pi j t) - f_j \cos(2\pi j t)),$$

where $a \geq 0$, $d_j, f_j \in \mathbb{R}$, and t is measured in years. See Fackler and Tian [1999] and Benth and Koekebakker [2008] for more details on the modeling of volatility.

5 Valuation of options

We recall that we consider the case of deterministic volatility η.

5.1 Pricing of European options

For the valuation of the European options on the spot price we use Fourier transform method applied to the dampened payoff. For an overview of this method see Papapantoleon [2006]. We here consider the example of pricing an electricity floor contract. Electricity calls, puts and caps can be priced similarly. See also Hinz and Wilhelm [2006] for the pricing of European options on the electricity spot price under the assumption of continuous futures and spot price processes.

A floor is a European type contract that protects against low commodity prices within $[\tau_1, \tau_2]$. It ensures a cash flow at intensity $((K - S(s))^+)_{s \in [\tau_1, \tau_2]}$ with strike price $K > 0$ at any time $s \in [\tau_1, \tau_2]$ of the contract.

In the remainder of this paper we suppose that the riskless interest rate r is constant. The fair price at time t of the floor option with strike price $K > 0$ is equal to

$$\text{Floor}(t, K) = E^{\mathbb{Q}^F} \Big[\int_{t \vee \tau_1}^{\tau_2} e^{-r(\tau - t)} (K - S(\tau))^+ d\tau \Big| \mathcal{F}_t \Big].$$

By Fubini's Theorem we get

$$\text{Floor}(t, K) = \int_{t \vee \tau_1}^{\tau_2} e^{-r(\tau - t)} E^{\mathbb{Q}^F} \Big[(K - S(\tau))^+ \Big| \mathcal{F}_t \Big] d\tau. \tag{44}$$

To simplify the notation we only consider the one-dimensional case under assumption of the deterministic coefficients, i.e. we assume the spot price process $S(t)$ to be given by (40), where δ and ψ are deterministic.

Recall that by (36)-(39) we can also factorize the spot price process S in the one-dimensional case as follows

$$S(\tau) = F(t, \tau) \exp\{ \int_t^\tau \delta(s, \tau) dL_s - \int_t^\tau \psi(s, \tau) ds \} =: F(t, \tau) U_t^\tau, \tag{45}$$

where $F(t, \tau)$, for $0 \le t \le \tau$, is a $\mathbb{Q}^F$-martingale, and L is a non-homogeneous Lévy process. Since $F(t, \tau)$ is $\mathcal{F}_t$-measurable and U_t^τ is independent of $\mathcal{F}_t$, by substituting (45) into (44) we obtain

$$\begin{aligned} \text{Floor}(t, K) &= \int_{t \vee \tau_1}^{\tau_2} e^{-r(\tau - t)} E^{\mathbb{Q}^F} \Big[(K - F(t, \tau) U_t^\tau)^+ d\tau \Big| \mathcal{F}_t \Big] d\tau \\ &= \int_{t \vee \tau_1}^{\tau_2} e^{-r(\tau - t)} F(t, \tau) e^{-\int_t^\tau \psi(s, \tau) ds} E^{\mathbb{Q}^F} \Big[(K(f) - e^{\int_t^\tau \delta(s, \tau) dL_s})^+ \Big] \Big|_{f := F(t, \tau)} d\tau, \end{aligned} \tag{46}$$

where $K(f) := \frac{K}{f} \exp\{ \int_t^\tau \psi(s, \tau) ds \}$, $f > 0$. In order to compute the expectation in (46), consider the integrable dampened pay-off function

$$g(x) := e^x (K(f) - e^x)^+ \in L^1(\mathbb{R}).$$

Denote by $\hat{g}$ its Fourier transform:

$$\hat{g}(u) := \int_{\mathbb{R}} e^{iux} g(x) dx = K(f)^{2+iu} \frac{1}{(1 + iu)(2 + iu)} \in L^1(\mathbb{R}). \tag{47}$$

Using the Inversion Theorem for Fourier transform (cf. Königsberger [1993], Section 8.2) we get

$$
\begin{aligned}
&E^{\mathbb{Q}^F}\Big[\big(K(f) - e^{\int_t^\tau \delta(s,\tau)dL_s}\big)^+\Big] = E^{\mathbb{Q}^F}\Big[e^{-\int_t^\tau \delta(s,\tau)dL_s} g(\int_t^\tau \delta(s,\tau)dL_s)\Big] \\
&= E^{\mathbb{Q}^F}\Big[e^{-\int_t^\tau \delta(s,\tau)dL_s}\frac{1}{2\pi}\int_{\mathbb{R}} e^{-iu\int_t^\tau \delta(s,\tau)dL_s}\hat{g}(u)du\Big] \\
&= \frac{1}{2\pi}E^{\mathbb{Q}^F}\Big[\int_{\mathbb{R}} e^{-(1+iu)\int_t^\tau \delta(s,\tau)dL_s}\hat{g}(u)du\Big] \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} E^{\mathbb{Q}^F}\Big[e^{-(1+iu)\int_t^\tau \delta(s,\tau)dL_s}\Big]\hat{g}(u)du, \qquad (48)
\end{aligned}
$$

where (47) allows to apply Fubini's Theorem in the last equality. By Proposition 3.1 and Proposition 1.9 in Kluge [2005]

$$
E^{\mathbb{Q}^F}[e^{-\int_t^\tau (1+iu)\delta(s,\tau)dL_s}] = \exp\{\int_t^\tau \Theta_s^{\mathbb{Q}^F}(-(1+iu)\delta(s,\tau))ds\}, \qquad (49)
$$

where $\Theta_s^{\mathbb{Q}^F}$ is given by

$$
\Theta_s^{\mathbb{Q}^F}(z) = zb_s^{\mathbb{Q}^F} + \frac{z^2}{2}c_s^{\mathbb{Q}^F} + \int_{\mathbb{R}}(e^{zx} - 1 - zx)e^{v(s)x}\nu(dx), \quad s \le T.
$$

Substituting (48), (47), and (49) into (46), we obtain the following pricing formula

$$
\begin{aligned}
\mathrm{Floor}(t,K) = &\int_{t\vee\tau_1}^{\tau_2} e^{-r(\tau-t)}F(t,\tau)e^{-\int_t^\tau \psi(s,\tau)ds} \\
&\times \int_{\mathbb{R}} \exp\{\int_t^\tau \Theta_s^{\mathbb{Q}^F}(-(1+ix)\delta(s,\tau))ds\} \\
&\times \left(\frac{K}{F(t,\tau)}e^{\int_t^\tau \psi(s,\tau)ds}\right)^{2+ix}\frac{1}{(1+ix)(2+ix)}dxd\tau \\
= K^2e^{rt}&\int_{t\vee\tau_1}^{\tau_2} e^{-r\tau}\int_{\mathbb{R}}\exp\{\int_t^\tau \Theta_s^{\mathbb{Q}^F}(-(1+ix)\delta(s,\tau))ds\} \\
&\times \left(\frac{e^{\int_t^\tau \psi(s,\tau)ds}}{F(t,\tau)}\right)^{1+ix}\frac{K^{ix}}{(1+ix)(2+ix)}dxd\tau.
\end{aligned}
$$

5.2 Pricing of swing options

In this section we illustrate how the spot price model (34) can be used to valuate electricity swing options. Electricity swing options are spot path dependent derivatives, which can be used to hedge the electricity spot price risk as well as the risk in the option owner's electricity consumption process. The expression "swing options" comes from the constraint on the electricity consumption process which "swings" between the lower and upper boundaries.

For the sake of simplicity we consider a special case, where the process L is a one-dimensional Lévy process under $\mathbb{Q}^M$ and $v \in \mathbb{R}$ is a constant. Then by

Remark 3.2 L is also a Lévy process under $\mathbb{Q}^F$. Moreover, by Assumption (14) and Proposition 3.1 we have that L admits the canonical representation:

$$L_t = b^{\mathbb{Q}^F} t + \sqrt{c} W_t^{\mathbb{Q}^F} + \int_0^t \int_{\mathbb{R}} x \bar{J}_L^{\mathbb{Q}^F}(dx \times ds), \tag{50}$$

where $W^{\mathbb{Q}^F}$ is a standard Brownian motion and $\bar{J}_L^{\mathbb{Q}^F}$ is the compensated random measure of jumps under $\mathbb{Q}^F$. Recall that by (9) and (50) the short rate process r follows the dynamics

$$\begin{aligned} dr(t) &= \alpha(t,t)dt + \eta(t,t)dL_t \\ &= (b^{\mathbb{Q}^F}\eta(t,t) + \alpha(t,t))dt + \sqrt{c}\eta(t,t)dW_t^{\mathbb{Q}^F} \\ &\quad + \int_{\mathbb{R}} x\eta(t,t)\bar{J}_L^{\mathbb{Q}^F}(dx \times dt), \quad t \in [0,T]. \end{aligned} \tag{51}$$

Now we assume that the volatility η of the short rate process r is deterministic, and hence r is a Markov process.

Recall that, since r is Markovian, by Proposition 4.2 (S, r) is also a Markov process. Furthermore, by (34) and (50) the dynamics of the electricity spot price S is given by

$$\begin{aligned} dS(t) &= S(t)[-r(t) + \frac{1}{2}cv^2 + \Theta(v) - vb^{\mathbb{Q}^F}]dt - S(t)v\sqrt{c}dW_t^{\mathbb{Q}^F} \\ &\quad - S(t-)v\int_{\mathbb{R}} x\bar{J}_L^{\mathbb{Q}^F}(dx \times dt) + S(t-)\int_{\mathbb{R}}(e^{vx} - 1 + vx)J_L(dx \times dt) \\ &= S(t)[-r(t) + \frac{1}{2}cv^2 + \Theta(v) - vb^{\mathbb{Q}^F} + \int_{\mathbb{R}}(e^{vx} - 1 + vx)\nu^{\mathbb{Q}^F}(dx)]dt \\ &\quad - S(t)v\sqrt{c}dW_t^{\mathbb{Q}^F} + S(t-)\int_{\mathbb{R}}(e^{vx} - 1)\bar{J}_L^{\mathbb{Q}^F}(dx \times dt) \\ &= -S(t)(r(t) - \beta)dt - S(t)v\sqrt{c}dW_t^{\mathbb{Q}^F} \\ &\quad + S(t-)\int_{\mathbb{R}}(e^{vx} - 1)\bar{J}_L^{\mathbb{Q}^F}(dx \times dt), \quad t \in [0,T], \end{aligned} \tag{52}$$

where, by (16), (24), and (26) for β we get

$$\begin{aligned} \beta &:= \int_{\mathbb{R}}(e^{vx} - 1 - vx)\nu(dx) - v\int_{|x|\le 1}(e^{vx} - 1)x\nu(dx) \\ &\quad + \int_{\mathbb{R}}(e^{vx} - 1 + vx)e^{vx}\nu(dx) \\ &= \int_{\mathbb{R}}(e^{vx} - 1 - vxI_{|x|>1})\nu(dx) + \int_{\mathbb{R}}(e^{vx} - 1 + vxI_{|x|>1})e^{vx}\nu(dx). \end{aligned}$$

Moreover, we assume that there exists a unique solution $(S^u(t), r^u(t))$ of the system (52) – (51) satisfying the initial condition $(S^u(u), r^u(u)) = (s, r) \in \mathbb{R}^2$, and such that

$$E^{\mathbb{Q}^F}[(S^u(t))^2] < \infty \quad \text{for all } t \in [0,T].$$

For instance, if $u = 0$ then $(s, r) = (1, r(0)) = (1, f(0,0))$.

Let us consider a swing option on the spot price process (52). A swing option is an agreement to purchase energy at a certain fixed price over a specified time interval. Following Wallin [2008] we define the payoff of a swing option settled at time T as

$$\int_u^T \nu(t)(S^u(t) - K)dt, \tag{53}$$

where $\nu(t)$ is the production intensity, S is the electricity spot price and $K > 0$ is the strike price of the contract. The holder of the contract has the right (within specified limits), to control the intensity of electricity production at any moment. The goal of the option holder is to maximize the value of the contract by selecting the optimal intensity process ν among the processes that are limited by contract specific lower and upper bounds:

$$\nu_{low} \leq \nu(t) \leq \nu^{up} \quad \text{a.e. } t \in [u, T],$$

under the constraint that the optimal intensity process ν is such that the total volume produced

$$C^\nu(t) = c + \int_u^t \nu(x)dx, \quad u \leq t \leq T, \tag{54}$$

does not exceed the maximum amount $\bar{C}$ that can be produced during the contract life time.

Hence the option holder tries to maximize the expected profit, i.e. to find

$$V(u, s, r, c) := \sup_{\nu \in N} E^{\mathbb{Q}^F} \left[\int_u^{T \wedge \tau_{\bar{C}}} \nu(t)(S^u(t) - K)dt \right] \tag{55}$$

$$= E^{\mathbb{Q}^F} \left[\int_u^{T \wedge \tau_{\bar{C}}} \nu^*(t)(S^u(t) - K)dt \right], \tag{56}$$

where

$$N := \{\nu \text{ progressively measurable: } \nu(t) \in [\nu_{low}, \nu^{up}] \text{ for a.e. } t \in [u, T]\}$$

is the control set, and

$$\tau_{\bar{C}} := \inf\{t > u |\ C^\nu(t) = \bar{C}\}$$

is the first time when all of production rights are used up. Note that the value function V satisfies the boundary conditions

$$V(T, s, r, c) = 0 \quad \text{and} \quad V(u, s, r, \bar{C}) = 0. \tag{57}$$

If we assume that the value function V is sufficiently smooth, then by Itô formula and by (52), and (51) we get

$$\begin{aligned}
0 &= V(T \wedge \tau_{\bar{C}}, S^u(T \wedge \tau_{\bar{C}}), r^u(T \wedge \tau_{\bar{C}}), C^\nu(T \wedge \tau_{\bar{C}})) \\
&= V(u, s, r, c) + \int_u^{T \wedge \tau_{\bar{C}}} \mathcal{A}^\nu V(t, S^u(t), r^u(t), C^\nu(t))dt \\
&\quad - \sqrt{c} \int_u^{T \wedge \tau_{\bar{C}}} (\partial_s V S^u(t) v(t) + \partial_r V \eta(t,t)) dW_t^{\mathbb{Q}^F} \\
&\quad + \int_u^{T \wedge \tau_{\bar{C}}} \int_{\mathbb{R}} (\partial_r V \eta(t,t)x + S^u(t-)(e^{vx} - 1)\partial_s V) \bar{J}_L^{\mathbb{Q}^F}(dx \times dt),
\end{aligned} \tag{58}$$

where

$$\begin{aligned}
&\mathcal{A}^{\nu}V(t,S^u(t),r^u(t),C^{\nu}(t)) := \partial_t V + \partial_c V\nu(t) - \partial_s V S^u(t)(r^u(t)-\beta) \\
&+ \partial_r V(b^{\mathbb{Q}^F}\eta(t,t)+\alpha(t,t)) + \frac{c}{2}\Big((S^u(t))^2 v^2(t)\partial^2_{ss}V - 2S^u(t)v(t)\eta(t,t)\partial^2_{sr}V \\
&+ \eta^2(t,t)\partial^2_{rr}V\Big) + \int_{\mathbb{R}}\Big(V(t,S^u(t)e^{vx},r^u(t)+x\eta(t,t),C^{\nu}(t)) \\
&- V(t,S^u(t),r^u(t),C^{\nu}(t)) - \partial_s V S^u(t)(e^{vx}-1) - \partial_r V x\eta(t,t)\Big)e^{vx}\nu(dx).
\end{aligned} \tag{59}$$

Applying Dynkin formula (Theorem 1.24 in Øksendal and Sulem [2005]) we can now formulate a verification theorem for the optimal control problem (55) analogous to the classical result for the Hamilton-Jacobi-Bellman equation for jump diffusions (see Theorem 3.1 in Øksendal and Sulem [2005]):

Proposition 5.1. *Let $\mathcal{S} = [u,T)\times\mathbb{R}^2_+\times[0,\bar{C})$. Assume that there exist $\hat{V}\in C^2(\mathcal{S})\cap C(\bar{\mathcal{S}})$ and $\hat{\nu}\in N$, such that $(\hat{\nu},\hat{V})$ is a solution of the Hamilton-Jacobi-Bellman equation*

$$\mathcal{A}^{\hat{\nu}}V(t,s,r,c) + \hat{\nu}(s-K) = 0 \quad \textit{for each } (t,s,r,c)\in\mathcal{S}, \tag{60}$$

satisfying

$$E^{\mathbb{Q}^F}\Big[\int_u^{T\wedge\tau_{\bar{C}}} |A^{\hat{\nu}}\hat{V}(t,S^u(t),r^u(t),C^{\hat{\nu}}(t))|dt\Big] < \infty. \tag{61}$$

Moreover, suppose that $\hat{V}$ fulfills the terminal and boundary conditions (57)*. Then $\hat{V}$ is the value function of the swing option defined in* (55)*.*

Note that the Markov property of the process (S,r) is essential for the proof of Proposition 5.1. We refer to Øksendal and Sulem [2005] for more details on stochastic optimal control problems.

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