

# Trade duration risk in subdiffusive financial models\*

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March 5, 2019

## Abstract

Subdiffusive processes can be used in finance to explicitly accommodate the presence of random waiting times between trades or “duration”, which in turn allows the modelling of price staleness effects. Option pricing models based on subdiffusions are incomplete, as they naturally account for the presence of a market risk of trade duration. However, when it comes to pricing this risk matters are quite subtle, since the subdiffusive Lévy structure is not maintained under equivalent martingale measure changes unless the price of this risk is set to zero. We argue that this shortcoming can be resolved by introducing the broader class of *tempered subdiffusive* models. We highlight some additional features of tempered models that are consistent with economic stylized facts, and in particular explain the role of the stability and tempering parameters in capturing the time multiscale properties of equity prices. Finally, we show that option pricing can be performed using standard integral representations.

**Keywords:** Duration risk, subdiffusions, tempered subdiffusions, derivative pricing, inverse tempered stable subordinator, Lévy processes.

**JEL classification:** C65, G13

## 1 Introduction

Subdiffusive stochastic processes are used in science to model natural phenomena of slow particle displacement, typically in fluid-dynamics, hydrology, engineering and physics. Such processes allow explicit modelling of resting times between particle movements, that in financial asset models can be interpreted as extended random periods of zero returns between price revisions, that is, the presence of a trade duration.

Transition densities of subdiffusive models are characterized as solutions of Fokker-Planck equations of fractional order, and their stochastic representations are in the form of Lévy processes time-changed with an inverse stable subordinator (e. g. Baeumer and Meerschaert 2001). Subdiffusions have been introduced in mathematical finance by the pioneering work of Scalas et al. (2000), where price densities are assumed to follow a fractional diffusion in both space

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\*The author would like to thank Francesca Biagini for the useful comments. The remaining errors are this authors' alone.

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and time: evidence of the consistency of this model with real financial data has been found. Magdziarz (2009) proposes the Black-Scholes subdiffusive model, with the aim of adapting the Samuelson paradigm of normally-distributed log-returns to an illiquid market scenario.

On the other hand, systematic treatment of option pricing models based on subdiffusive price return models remain scarce in the literature: an example is Cartea and Meyer-Brandis (2010), who use the Mittag-Leffler distribution to incorporate random trade waiting times in the price evolution, provide analytical pricing formulae, and study the effect of duration on the volatility surface.

In this note we aim at illustrating how subdiffusive models provide a convenient way to account for the risk of trade duration in derivative securities valuation. Explicitly recognizing the random nature of waiting times during trades in a pricing model provides a way to embed in the option premia the price of risk of the duration market determinants, such as illiquidity or the returns impact of different levels of trading activity. This is of particular importance for instance in pricing ultra-short term options, products for which there has been a surge of interest in recent times. Another example is when the regulator enforces trading suspensions, a form of duration risk studied in Torricelli and Fries (2018) with techniques similar to those presented here.

We are interested here in studying the general model class of the type  $X_{H_t}$  when  $X_t$  is a sufficiently regular Lévy process and  $H_t$  an inverse-stable subordinator, which we term *subdiffusive Lévy models* (SL). Many SL financial models have been studied in detail by Cartea and del-Castillo-Negrete (2007) in terms of the solutions of their associated fractional integro-differential equations; more in general, the mathematics of subdiffusive Lévy processes are well understood. However, the classic analytic approach falls short of explaining certain dynamic martingale properties of exponential models based on subdiffusions, something to which the stochastic time changed representation we adopt is better suited. With a view towards a better understanding of the risk determinants of SL models, we clarify the role of measure changing in defining the risk-neutral dynamics of a subdiffusive Lévy model (SL), and deal with related question of market incompleteness. Incompleteness of subdiffusive models rests on the existence a whole family of possible equivalent measure transformations for the time change  $H_t$  under which SL models remain martingales. This makes the concept of *market price of duration risk* surface. The inverse stable subordinators family is parametric, so option premia will be reflected in the risk-neutral martingale density parameters.

However, as it turns out, risk-neutral specifications of an SL model which prices duration risk is not structure-preserving for the SL class, meaning that after an EMM change risk-neutral distributions may not come from a subdiffusion. This naturally leads to consider the extension of SL price processes to their *tempered* counterparts. The *tempered subdiffusive Lévy models* (TSL) are attained as risk-neutral versions of an SL model when an Esscher transformation for the time component is used. Furthermore, risk-neutral specifications of a physical TSL model remain of TSL form. The natural pricing framework where the price of duration risk can be fully accounted for is therefore that of the TSL model class. We explain the role of the stability and tempering parameter in the price evolution process, and illustrate how this model class naturally captures the fact that for any given asset different price idleness patterns are

detected at different time scales, a phenomenon which we refer to as *time multiscaling*. Finally, by exploiting transform results on the involved processes, we derive semi-analytic derivative valuation formulae of integral type.

## 2 Subdiffusive models and incompleteness

In order to discuss subdiffusive models we must first introduce inverse processes and time changing. For a process  $L_t$ , its inverse or first exit time process is:

$$H_t = \inf\{s > 0 : L_s > t\}. \quad (2.1)$$

The process  $H_t$  is an instance of a *time-change*, that is an increasing, right-continuous, almost surely locally bounded family of stopping times, diverging almost surely as  $t \rightarrow \infty$ . Note that if  $L_t$  is almost surely strictly increasing,  $H_t$  is almost surely continuous. The processes we will look at for the most part are of the form  $X_{H_t}$  for some given Lévy process  $X_t$  independent of  $H_t$ , when  $L_t$  is a standard  $\alpha$ -stable subordinator. A clear indication of the suitability for this process to model trade duration and price staleness is the “almost everywhere flat” nature of its paths. Since  $L_t$  jumps infinitely often in any time interval, it can be shown that  $H_t$  increases on a set of Lebesgue set measure zero, while remaining constant on its full-measure complement.

Exponential (stochastic or natural) models based on  $X_{H_t}$  are termed *subdiffusive Lévy* (SL) *models*. When  $X_t$  is a Brownian motion, the exponential stemming from  $X_{H_t}$  is the subdiffusive Black-Scholes model of Magdziarz (2009). When  $X_t$  is a compound Poisson process in view of the results of Meerschaert et al. (2011) we have an equivalent representation of the model of Cartea and Meyer-Brandis (2010); if  $X_t$  is a stable process we instead have the classic model in Scalas et al. (2000).

We recall that “market completeness” in finance is attained when at any given time the future price of every financial security can be replicated by using a fundamental set of traded instruments. In the mathematical theory of arbitrage completeness is synonym with the existence of a *unique* equivalent martingale measures for all the tradable assets. The market is thus *incomplete* when multiple such measures exist. Adding further traded products to the market may or may not resolve this non-uniqueness. Lack of completeness is generally due to the presence of sources of risk external to the price returns generating process, such as abrupt crashes (jump risk) or the presence of stochastic volatility or stochastic interest rates, or even jumps in volatility.

In order to illustrate some inadequacies surrounding the status of completeness of SL models we begin by discussing an incompleteness theorem for the subdiffusive Black-Scholes model, reported first in Magdziarz (2009).

**Theorem 1.** *On a filtered space  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ , let  $W_t$  a Brownian motion,  $H_t$  a standard  $\alpha$ -stable subordinator independent of  $W_t$  and  $\mu, \sigma > 0$ . Set*

$$S_t := \exp(\sigma W_{H_t} - H_t(\mu + \sigma^2/2)). \quad (2.2)$$

*Then:*

(i) fix  $T > 0$ ; for all  $\epsilon \geq 0$ , the process  $S_t$ ,  $0 \leq t \leq T$ , is a martingale with respect to the measure defined by

$$\mathbb{Q}_\epsilon(A) = c_\epsilon \int_A \exp\left(-\gamma W_{H_T} - \left(\epsilon + \frac{\gamma^2}{2}\right) H_T\right) d\mathbb{P} \quad (2.3)$$

for all  $A \in \mathcal{F}_t$ , where  $c_\epsilon^{-1} = \mathbb{E}\left[\exp\left(-\gamma W_{H_T} - \left(\epsilon + \frac{\gamma^2}{2}\right) H_T\right)\right]$  and  $\gamma = (\sigma^2/2 + \mu)/\sigma$ ;

(ii) the risk neutral measure  $\mathbb{Q}$  under which  $S_t$  is a martingale is not unique, and therefore the market consisting of  $S_t$  is incomplete.

The process  $S_t$  is the subdiffusive geometric Brownian motion with expected return rate  $\mu > 0$  and diffusion coefficient  $\sigma > 0$ . The original Theorem 3 in Magdziarz (2009) consists of statement (ii) alone, and uses part (i) in the proof. The claim (i) is recovered from Magdziarz and Schilling (2015), Theorem 1, once one takes  $f(x) = x^\alpha$ ,  $0 < \alpha < 1$ . Other similar claims appear in various sources. In particular, (i) can be seen as a particular case of the more general result of Cherny and Shiryaev (2002), Theorem 5.4.

Typically, changes of measures in a filtered probability space modeling an economy are attained through “state price densities”, that is processes acting as statistical likelihood-ratios between the physical distribution of asset prices and the risk-neutral one. A defining property of such processes is that they are themselves (exponential) martingales in the original measure, something which the process under the integral in the right hand side of (2.3), for  $\epsilon > 0$  fails to be. The main advantage of using martingale densities for generating equivalent measure changes is that this enables the use of Girsanov Theorem, which in turn gives a clear recipe to determine the risk-neutral parameters.

Of course, for practical purposes like model calibration, it is necessary to establish one such connection between the risk neutral and physical dynamics of the price process. Part (ii) of the Theorem should then be derived, if possible, by using arguments which make clear how the dynamics of *all* the involved processes transform under a martingale measure, something which is unclear from (i).

### 3 The duration market price of risk

As said, the fundamental issue of Theorem 1, (i) is that it does not make transparent what the risk-neutral dynamics of  $S_t$  are once the change of measure to  $\mathbb{Q}_\epsilon$  is operated. In particular, even by assuming that the part involving  $\gamma$  is responsible for the new drift of the Brownian motion, it is not clear what the  $\mathbb{Q}_\epsilon$ -dynamics of  $H_t$  should be. However, the construct in the proof of Theorem 1, ignores an important piece of information, that is, that  $H_t$  is generated as a first hitting time as in (2.1). It is clear that changing the distribution of  $H_t$  can be attained by considering the distribution of  $L_t$  in an equivalent measure, which can be easily characterized since  $L_t$  is a Lévy process. We show the implications of this in the theorem below.

In what follows we denote by  $\mathcal{E}(\cdot)$  the Doleans-Dade (stochastic) exponential of a process. When no filtration is specified, with “martingale” we mean martingale with respect to the own filtration. For a process  $X_t$ ,  $\kappa^X(z)$  is the Fourier cumulant process of  $X_t$ , that is the almost

surely unique predictable process such that  $\exp(izX_t)/\mathcal{E}(\kappa^X(z))$  is a local martingale. If  $Y_t$  is a Lévy process we denote by  $\psi_Y(z)$  its characteristic exponent, i.e. the complex-valued function such that  $e^{t\psi_Y(z)} = \mathbb{E}[e^{izY_t}]$ .

**Theorem 2.** *Let  $X_t$  be a semimartingale,  $L_t$  a strictly increasing process independent of  $X_t$  and  $H_t$  be given by (2.1). Consider the time-changed exponential model  $S_t = \exp(X_{H_t})$  and assume an EMM  $\mathbb{Q}^X \sim \mathbb{P}$  for the underlying model  $S_t^0 = \exp(X_t)$  exists, with associated martingale density  $\mathcal{X}_t$ . Assume further that there exists a martingale density  $\mathcal{H}_t$  inducing an equivalent change of measure  $\mathbb{Q}^H \sim \mathbb{P}$  with*

$$\frac{d\mathbb{Q}^H}{d\mathbb{P}} = \mathcal{H}_t \quad (3.1)$$

such that  $L_t$  remains strictly increasing under  $\mathbb{Q}^H$ . Then if  $\mathcal{X}_{H_t}$  is an  $\mathcal{F}_{H_t}$ -martingale, we have that

$$\mathcal{X}_t^{\mathcal{H}} = \mathcal{X}_{H_t} \mathcal{H}_t \quad (3.2)$$

is a martingale such that the measure  $\mathbb{Q}^{X,H}$  defined by

$$\frac{d\mathbb{Q}^{X,H}}{d\mathbb{P}} = \mathcal{X}_t^{\mathcal{H}} \quad (3.3)$$

is equivalent to  $\mathbb{P}$  on  $\mathcal{G}_t = \sigma(\mathcal{F}_{H_t} \cup \mathcal{F}_t)$ , and  $S_t$  is a martingale with respect to  $\mathbb{Q}^{X,H}$ .

*Proof.* By independence, after operating the equivalent measure change  $\mathbb{Q}^H \sim \mathbb{P}$  the processes  $X_t$  and  $S_t^0$  under  $\mathbb{Q}^H$  are the same as under  $\mathbb{P}$ , and hence  $\mathcal{X}_t = d\mathbb{Q}^X/d\mathbb{P} = d\mathbb{Q}^X/d\mathbb{Q}^H$ . Also,  $H_t$  remains a continuous  $\mathbb{Q}^H$ -time change. Now let  $\mathbb{Q}^{X,H}$  be defined by

$$\frac{d\mathbb{Q}^{X,H}}{d\mathbb{Q}^H} = \mathcal{X}_{H_t} \quad (3.4)$$

As observed in Torricelli and Fries (2018), Lemma 5.1, under mild conditions time and measure change commute, meaning that  $S_t$  under  $\mathbb{Q}^{X,H}$  coincides with the process obtained by applying first the change of measure  $\mathbb{Q}^X \sim \mathbb{Q}^H$  to  $S_t^0$  and then time changing by  $H_t$ . Therefore, by Kallsen and Shiryaev (2002), Lemma 5,  $S_t = S_{H_t}^0 = \exp(X_{H_t})/\mathcal{E}(\kappa^{X_H}(-i))$  under  $\mathbb{Q}^{X,H}$ . That this process is a martingale is easily verified by taking the expectation and conditioning under independence. Finally

$$\frac{d\mathbb{Q}^{X,H}}{d\mathbb{P}} = \frac{d\mathbb{Q}^{X,H}}{d\mathbb{Q}^H} \frac{d\mathbb{Q}^H}{d\mathbb{P}} = \mathcal{X}_t^{\mathcal{H}}. \quad (3.5)$$

□

This theorem makes the presence of a *market price of duration risk* naturally emerge. To the best of this author's knowledge the concept of duration as random waiting time between trades has been introduced by Engle (2000) and Dufour and Engle (2000). The authors show that duration is inversely correlated to the volatility and price trade impact, which justifies its interpretation as a financial risk factor. Hence, once  $S_t$  is calibrated to liquid market prices, the market price of duration risk will be reflected in the parameters of the density  $\mathcal{H}_t$ . Every admissible parametrization of this density leads to a theoretically correct risk-neutral price: as a consequence, the market is incomplete. Remarkably, even if it exists only one EMM for  $X_t$ , such

as in the Brownian or pure Poisson cases, *several EMMs for  $X_{H_t}$  may exist*. Trade duration is therefore enough on its own to determine incompleteness, even in absence of unhedgeable jumps. The process  $\mathcal{H}_t$  is effectively the new state price density incorporating the duration risk born by the model  $X_{H_t}$ . What is more is that, analogously to the jump risk of Lévy process, duration risk cannot be fully hedged away, not even introducing a new set of traded products. This owes to the fact that the generator of  $L_t$  of  $H_t$  is a Lévy process: jumps of  $L_t$  cannot be “announced” beforehand, and can take a continuum of values<sup>1</sup>. This same features will be thus reflected in the (risk-adjusted) frequency and duration of trade pauses incorporated in  $H_t$ .

Theorem 2 can be applied to models of SL form, by the standard Esscher transform technique. If  $S_t = \exp(X_t)$  is a sufficiently regular Lévy model admitting a martingale density<sup>2</sup>  $\mathcal{X}_t$ , in Theorem 1 we can take

$$\mathcal{H}_t = \exp(\theta L_t - t\psi_L(-i\theta)) \quad (3.6)$$

for all  $\theta \in \mathbb{R}$  such that the right hand side exists. The measure change(s) entailed by  $\mathcal{H}_t$  is called the *Esscher transform* of  $L_t$ . It is easily checked that  $\mathcal{H}_t$  is a positive martingale, and thus  $\mathbb{Q}^H \sim \mathbb{P}$ . Moreover  $L_t$  satisfies Sato (1999), Theorem 33.1, so that  $L_t$  remains a strictly increasing Lévy subordinator under  $\mathbb{Q}^H$  with transformed Lévy measure  $e^{\theta x}\nu(dx)$ , where  $\nu(dx)$  is the positively-supported Lévy measure of  $L_t$  under  $\mathbb{P}$ .

## 4 The case for tempered subdiffusions

For practical purposes, most notably calibration, a desirable property for an option pricing model is that its structure is maintained after an EMM change, in the sense that after operating a martingale measure change, the resulting risk-neutral distributions remains in the same class of that of the original physical specification. For example an EMM  $\mathbb{Q}$  is a good candidate for a Lévy model, if after a measure change the model is still Lévy and belongs to the same class of the original specification, which is often the case for stochastic volatility models (e.g. Wong and Heyde 2006) and Lévy models (Hubalek and Sgarra, 2006).

When it comes to pricing the duration risk, the situation in the SL setup described is rather different. We begin by observing that after a non-trivial equivalent measure change an  $\alpha$ -stable subordinator cannot remain a stable subordinator. This can be illustrated as follows. Assume that a general process  $X_t$  is Lévy under  $\mathbb{P}$  and  $\mathbb{Q}$ . A necessary condition for  $\mathbb{Q} \sim \mathbb{P}$  is that the Lévy measures  $\nu^{\mathbb{P}}(dx)$  and  $\nu^{\mathbb{Q}}(dx)$  are equivalent and the *Hellinger distance*  $H(\nu^{\mathbb{P}}, \nu^{\mathbb{Q}})$  between them is finite, (again Sato 1999 Theorem 33.1) whose square is defined by:

$$H(\nu^{\mathbb{P}}, \nu^{\mathbb{Q}})^2 := \frac{1}{2} \int_{\mathbb{R}} \left( \sqrt{\frac{\nu^{\mathbb{Q}}(dx)}{\nu^{\mathbb{P}}(dx)}} - 1 \right)^2 \nu^{\mathbb{P}}(dx) < \infty. \quad (4.1)$$

Consider  $\mathbb{Q} = \mathbb{Q}^H$  from Theorem 2. For a standard  $\alpha$ -stable Lévy subordinator we have, with  $0 < \alpha < 1$ :

$$\nu^{\mathbb{P}}(dx) = \frac{\alpha}{\Gamma(1-\alpha)} x^{-(\alpha+1)} I_{\{x>0\}} dx. \quad (4.2)$$

<sup>1</sup>This suggests that hedging under duration risk should be performed using the mean-variance approach.

<sup>2</sup>Typically itself given by an Esscher transform, if we want to meet the minimum requirement that under the new measure  $Y_t$  is still a Lévy process.

If we let  $\nu^{\mathbb{Q}}(dx)$  be the Lévy density of a standard  $\beta$ -stable subordinator,  $\beta \neq \alpha$ , the integrand in (4.1) becomes

$$\left( \sqrt{\frac{\nu^{\mathbb{Q}}(dx)}{\nu^{\mathbb{P}}(dx)}} - 1 \right)^2 \frac{\nu^{\mathbb{P}}(dx)}{dx} = \left( c_{\beta} \sqrt{x^{-\beta-1}} - c_{\alpha} \sqrt{x^{-\alpha-1}} \right)^2 I_{\{x>0\}} \sim \frac{c_{\alpha} \nu \beta}{x^{\alpha \vee \beta + 1}} \quad (4.3)$$

for some  $c_{\alpha}, c_{\beta} > 0$ , and thus it diverges. In view of Theorem 2 this implicates that if for pricing purposes one wants to remain in the SL model class, the family of EMMs for  $X_{H_t}$  is in one-to-one correspondence with that of  $X_t$ . In other words, one is forced to set the market price of duration risk to zero.

Now let us instead fix  $X_t$  under  $\mathbb{P}$  and choose  $\mathbb{Q}^H$  using an Esscher transform (3.6) with  $\theta = -\lambda$ ,  $\lambda > 0$ ; this leads to:

$$\nu^{\mathbb{Q}}(dx) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{e^{-\lambda x}}{x^{\alpha+1}} I_{\{x>0\}} dx. \quad (4.4)$$

This time we have convergence around zero, since for  $c_{\alpha} > 0$

$$\left( \sqrt{\frac{\nu^{\mathbb{Q}}(dx)}{\nu^{\mathbb{P}}(dx)}} - 1 \right)^2 \frac{\nu^{\mathbb{P}}(dx)}{dx} = c_{\alpha} \frac{(e^{-\lambda x/2} - 1)^2}{x^{\alpha+1}} I_{\{x>0\}} \sim \frac{c_{\alpha} \lambda^2}{4} x^{1-\alpha}. \quad (4.5)$$

Convergence at infinity being clear, this is consistent with the equivalence of  $\mathbb{Q}^H$  and  $\mathbb{P}$  stated in the previous section.

A driftless Lévy subordinator having Lévy measure of the form (4.4) is called a *standard tempered stable subordinator* (TS), with stability parameter  $0 < \alpha < 1$  and tempering parameter  $\lambda > 0$ , and it is a member of the broader class of the tempered stable Lévy processes. The corresponding process  $H_t$  is called an *inverse tempered stable subordinator* (ITS). Its analytical properties are fully detailed in Kumar and Vellaisamy (2015) and Alrawashdeh et al. (2017).

From the foregoing discussion it is also obvious that after applying an Esscher transform to a physical specification of an exponential model  $S_t = \exp(X_{H_t})$  where  $H_t$  is an inverse tempered stable subordinator with stability  $\alpha$  and tempering  $\lambda$ , we obtain a new inverse tempered stable subordinator with tempering parameter  $\lambda^* \neq \lambda$  but stability  $\alpha^* = \alpha$ . In other words, the Esscher transform is structure-preserving for the class of the *tempered subdiffusive Lévy models* (TSL), although it leaves unaltered the physical stability parameter. The specification when  $X_t$  is a Brownian motion is proposed in Magdziarz and Gajda (2012). Of course, Theorem 2 provides sufficient conditions under which the exponential of a TSL process is a viable no-arbitrage asset pricing model.

## 5 Aspects of tempered subdiffusive models

Besides preserving the model structure under equivalent measure changes, introducing a tempering parameter  $\lambda$  in a subdiffusive Lévy asset evolution can be directly related to some interesting financial stylized facts.

To start with, we need to briefly digress into the nature of the stable processes. As it is well-known a stable law does not have finite second (or even first) moment. The typical reason for tempering a stable process is then that of obtaining a new Lévy process with finite moments of all orders, which better adapts to observed natural phenomena. Tempering makes extreme events less likely to occur. For the tempered stable subordinator this can be easily checked. Integrating the Lévy measure (4.4) on  $\epsilon > 0$  one gets, as  $\lambda \rightarrow \infty$ :

$$\int_{\epsilon}^{\infty} \frac{\alpha}{\Gamma(1-\alpha)} \frac{e^{-\lambda x}}{x^{\alpha+1}} dx = \alpha \lambda^{\alpha} \frac{\Gamma(-\alpha, \lambda \epsilon)}{\Gamma(1-\alpha)} \sim \frac{\alpha}{\Gamma(1-\alpha)} \frac{e^{-\lambda \epsilon}}{\epsilon^{\alpha+1} \lambda} \rightarrow 0 \quad (5.1)$$

which means that the expected number of jumps of length greater than  $\epsilon$  decreases as  $\lambda$  increases (here  $\Gamma(\cdot, \cdot)$  is the upper incomplete Gamma function). Also, the scaling properties typical of stable laws extend after tempering, and transfer to the ITS subordinator. More precisely, as shown in Alrawashdeh et al. (2017), Proposition 5.1, indicating explicitly by  $H_t^{\lambda}$  the dependence on  $\lambda$  of the subordinator, we have the equality in distribution:

$$H_{ct}^{\lambda} = c^{\alpha} H_t^{c\lambda} \quad (5.2)$$

for all  $c > 0$ . From the above is also not difficult to show that for  $\lambda \rightarrow 0$ ,  $H_t^{\lambda}$  tends as a stochastic process to the plain inverse stable subordinator  $H_t^0$  of parameter  $\alpha$ .

These properties impact the behaviour of the ITS subordinator as follows. For any given  $t$ , a higher  $\lambda$  entails a lower incidence of large jumps in  $L_t$ . But jumps in  $L_t$  correspond to intervals in which  $H_t$  is constant, so that a large  $\lambda$  implicates a reduced impact of the trapping states in the time evolution. By the same token, in view (5.2), when  $\lambda$  is fixed and the time scale gets shorter, stickiness is reintroduced in the process (see Figure 1). In any case the limiting “high frequency” regime coincides with the purely subdiffusive SL case.

We can thus conclude that the tempered stable subordinator serves to model a time evolution that transitions from a stable behaviour at an early time to a linear clock at a later stage: the speed of this transition is dictated by the tempering parameter  $\lambda$ .

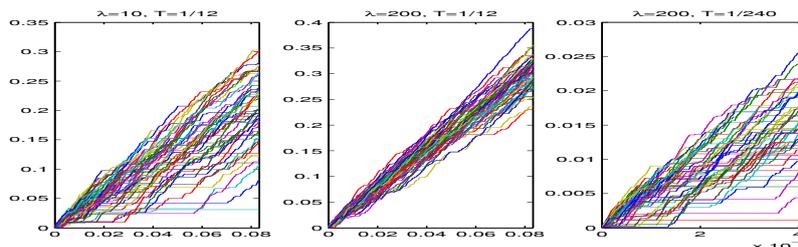


Figure 1: Time scaling property of TS subordinators, monthly to daily,  $\alpha = 0.8$ . We set  $T = 1/12$  and  $\lambda = 10$  in the first panel and changed  $\lambda$  as to solve (5.2) when  $c = 1/20$ .  $X_t$  is a Brownian motion with  $\sigma = 0.4$ .

As a consequence, asset prices obtained as time changes with respect to the tempered stable subordinator show at small time scales the typical “flatness” of microstructural models and reverts to a frictionless diffusive behaviour in the long term. The TSL dynamics capture the *time multiscale* property of equity prices: at small time scales, the observed price pattern is

coarse and irregular, whereas it becomes more fluid at large ones. Crucially, this transition might have different (physical or risk-neutral) velocity for different assets, as regulated by  $\lambda$ . We illustrate this in Figure 2. For a given  $\alpha$ , at daily lag (left panel) there is not much difference between the price evolution pattern of a TSL model with  $\lambda = 1$  and one with  $\lambda = 20$ : both exhibit the typical granularity of intradaily tick-by-tick charts. However, as times goes by (right panel, 6-months horizon) we see that the model with higher  $\lambda$  resembles to a diffusive regime, while the one with low  $\lambda$  still suffers from price staleness effects.

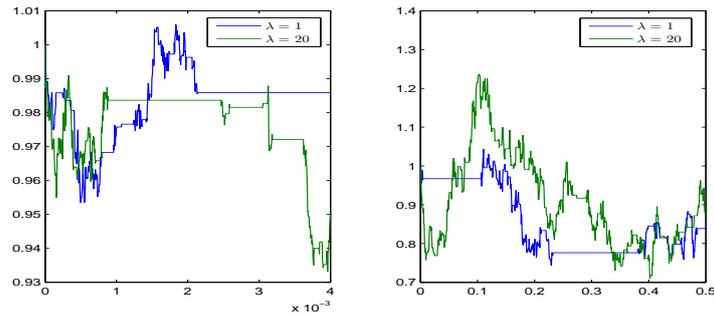


Figure 2: TSL model sample paths,  $\alpha = 0.7$ : left  $T = 1/250$ , right  $T = 1/2$ .  $X_t$  is a Brownian motion with  $\sigma = 0.4$ .

In the volatility surface analysis, faster or slower reversion to a Lévy model (i.e. higher or lower  $\lambda$ ) associates with a slower or faster rate of flattening of the skew with maturity. In Figure 3 we compare the volatility surfaces of a Variance Gamma Lévy model, with an SL and TSL model with parent variance gamma noise  $X_t$  of same parameters. We observe that in the TSL model Lévy the smile flattens with maturity to an asymptotic level, in a way analogous to the benchmark Lévy model, consistently with the TSL returns distribution approaching one of a Lévy process. In contrast, the purely subdiffusive SL case generates a vanishing term structure: here, the implied volatility must tend to zero at large maturity to compensate for the option price asymptotics of the SL model being slower than Black-Scholes (see Section 6 further on).

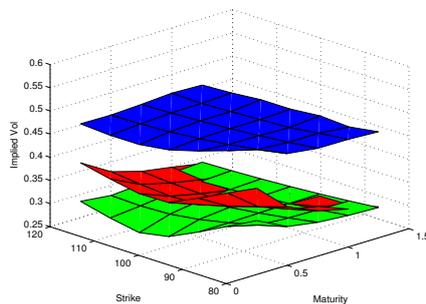


Figure 3: Volatility surface comparison.  $X_t$  is a Variance Gamma process,  $\sigma = \kappa = 0.3$ ,  $\theta = -0.2$ . In green the Lévy model, in red the pure SL model with  $\alpha = 0.7$  and blue the TSL model with  $\alpha = 0.7$ ,  $\lambda = 5$ . We used the pricing formulae in Section 6.

In light of these remarks, the TSL duration market price of risk can be further decomposed

in two factors: the stability parameter  $\alpha$ , that captures the degree of high frequency price staleness, and the parameter  $\lambda$  which expresses belief on the speed at which the granular price evolution of the asset will revert to a fully “liquid” state, well approximated by a standard Lévy-driven diffusion. We could then say that  $\alpha$  captures the *absolute trade duration risk* and  $\lambda$  the *risk of latency to liquidity*. In a tempered model, the latter has a market price, whereas the former remains a statistical quantity. Moreover, if we think of the basic subdiffusive SL model ( $\lambda = 0$ ) as a regime where prices exhibit maximum staleness at all temporal scales, we can interpret  $\lambda$  as an interpolating factor between a short term pure SL regime and a long term Lévy one. Therefore, TSL models offer a unified framework in which ultra short term option pricing accounts for the microstructural duration effect, whereas options at typical maturities are priced according to the usual Lévy paradigm: the cut-off between the two states is encoded in  $\lambda$ .

Another important feature of SL and TSL processes is that these processes are *non-Markovian*, because their transition densities are solutions of fractional equations entailing a non-local time operator. Economically, that duration entails non-Markovianity is already implicit in the mentioned findings of Engle (2000); Dufour and Engle (2000): past activity affects price innovations. However, it is proved in Meerschaert and Straka (2014), that TSL and SL processes possess a Markovian embedding once the state space is augmented as to include the *idle time process*  $t - L_{H_t-}$ , which keeps track of the time currently elapsed from the last price innovation. Therefore, at any given instant, the law of the next price revision depends only on the current price and the time went by since such a price was first recorded. This is a realistic feature, as there is no reason to believe that future returns are impacted by price staleness levels observed far back in the trading history.

## 6 Pricing formula

Of particular relevance is also that option pricing in the TSL model can be performed in a semi-analytical way, which allows fast model calibration to vanilla option prices. Given a TSL exponential martingale model, we start from the classic Parseval-Plancharel integral representation of the price  $V_0$  of a regular contingent claim  $F$  maturing at  $T$  (see Lewis 2001):

$$V_0 = \frac{1}{2\pi} \int_{i\gamma-\infty}^{i\gamma+\infty} \mathbb{E}^{\mathbb{Q}^{X,H}} [e^{-izX_{HT}}] \hat{F}(z) dz \quad (6.1)$$

where  $\hat{\cdot}$  indicates the Fourier transformation, and  $\gamma$  is chosen such that the integration line is in the strip of holomorphy of both functions. In Meerschaert and Scheffler (2008), the formula for the Fourier-Laplace transform for an inverse-subordinated Lévy process  $X_{H_t}$  is given as:

$$\mathcal{L}(\mathbb{E}^{\mathbb{Q}^{X,H}} [e^{izX_{H_t}}], s) = \frac{1}{s} \frac{\psi_L(s)}{\psi_L(z) - \psi_X(-z)}. \quad (6.2)$$

Therefore, when  $L_t$  is a TS subordinator, we have  $\psi_L(s) = (\lambda + s)^\alpha - \lambda^\alpha$  and (6.2) becomes

$$\mathcal{L}(\mathbb{E}^{\mathbb{Q}^{X,H}} [e^{izX_{H_t}}], s) = \frac{1}{s} \frac{(\lambda + s)^\alpha - \lambda^\alpha}{(\lambda + s)^\alpha - \lambda^\alpha - \psi_X(-z)}. \quad (6.3)$$

Inverting the Laplace transform and substituting in (6.1) yields:

$$V_0 = \frac{1}{4\pi^2 i} \int_{i\gamma-\infty}^{i\gamma+\infty} \left( \int_{\xi-i\infty}^{\xi+i\infty} \hat{F}(z) \frac{e^{sT}}{s} \frac{(\lambda+s)^\alpha - \lambda^\alpha}{(\lambda+s)^\alpha - \lambda^\alpha - \psi_X(-z)} ds \right) dz. \quad (6.4)$$

To calculate (6.4) one can either use a two-dimensional integration routine (provided  $\xi$  can be chosen independently of  $z$ ), or one of the many available Laplace inversion quadrature methods and then integrate in  $dz$ . An alternative pricing formula can be derived using a conditioning argument and the integral representations of the ITS subordinator characteristic functions in Kumar and Vellaisamy (2015) and Alrawashdeh et al. (2017)

In the SL case ( $\lambda = 0$ ) equation (6.3) is well-known to be the Laplace transform of a Mittag-Leffler function. More precisely, in this case after the Laplace inversion we would have that:

$$\mathbb{E}^{\mathbb{Q}^{X,H}} [e^{izX_{Ht}}] = E_\alpha((T\psi_X(-z))^\alpha) \quad (6.5)$$

where  $E_\alpha$  is the one-parameter Mittag-Leffler function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \quad (6.6)$$

The expression above is a generalized exponential with thicker tails, so as  $T$  increases the option prices tend to the spot value slower than in a standard model. This property is precisely what generates the vanishing term structure of the SL model highlighted earlier. Since fast numerical methods are available for  $E_\alpha$ , pricing in the SL model is computationally less expensive than in the TSL model. Although elementary, the general integral option pricing formula for SL models following from (6.1)-(6.5) seems not to have appeared before. When  $X_t$  is a CPP the formula above has been derived directly in Cartea and Meyer-Brandis (2010).

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