COMPUTING DELTAS WITHOUT DERIVATIVES

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Abstract. A well-known application of Malliavin calculus in Mathematical Finance is the probabilistic representation of option price sensitivities, the so-called Greeks, as expectation functionals that do not involve the derivative of the pay-off function. This allows for numerically tractable computation of the Greeks even for discontinuous pay-off functions. However, while the pay-off function is allowed to be irregular, the coefficients of the underlying diffusion are required to be smooth in the existing literature, which for example excludes already simple regime switching diffusion models. The aim of this article is to generalise this application of Malliavin calculus to Itô diffusions with irregular drift coefficients, whereat we here focus on the computation of the Delta, which is the option price sensitivity with respect to the initial value of the underlying. To this purpose we first show existence, Malliavin differentiability, and (Sobolev) differentiability in the initial condition of strong solutions of Itô diffusions with drift coefficients that can be decomposed into the sum of a bounded but merely measurable and a Lipschitz part. Furthermore, we give explicit expressions for the corresponding Malliavin and Sobolev derivative in terms of the local time of the diffusion, respectively. We then turn to the main objective of this article and analyse the existence and probabilistic representation of the corresponding Deltas for lookback and Asian type options. We conclude with a simulation study of several regime-switching examples.

Key words and phrases: Greeks, Delta, option sensitivities, Malliavin calculus, Bismut-Elworthy-Li formula, irregular diffusion coefficients, strong solutions of stochastic differential equations, relative L^2-compactness
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1. Introduction

Throughout this paper, let $T > 0$ be a given time horizon and $(\Omega, \mathcal{F}, P)$ a complete probability space equipped with a one-dimensional Brownian motion $\{B_t\}_{t\in[0,T]}$ and the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by $\{B_t\}_{t\in[0,T]}$ augmented by the P-null sets. Further, we will only deal with random variables that are Brownian functionals, i.e. we assume $\mathcal{F} := \mathcal{F}_T$.

One of the most prominent applications of Malliavin calculus in financial mathematics concerns the derivation of numerically tractable expressions for the so-called *Greeks*, which are important sensitivities of option prices with respect to involved parameters. The first paper to address this application was [15], which has consecutively triggered an active research interest in this topic, see e.g. [14], [4], [1]. See also [7], [11] and references therein for a related approach based on functional Itô calculus. Suppose the risk-neutral dynamics of the underlying asset of a European option is driven by a stochastic differential equation (for short SDE) of the form

$$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dB_t, \;\; X_0^x = x \in \mathbb{R},$$

where $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are some given drift and volatility coefficients, respectively. Let $\Phi : \mathbb{R} \to \mathbb{R}$ denote the pay-off function and the expectation $E[\Phi(X_T^x)]$ the risk-neutral price at time zero of the option with maturity $T > 0$. For notational simplicity we assume the discounting rate to be zero. In this paper we will focus on the *Delta*

$$\frac{\partial}{\partial x} E[\Phi(X_T^x)], \tag{1.1}$$

which is a measure for the sensitivity of the option price with respect to changes of the initial value of the underlying asset. As is well known, the Delta has a particular role among the Greeks as it determines the hedge portfolio in many complete market models. If the drift $b(\cdot)$, the volatility $\sigma(\cdot)$, and the pay-off $\Phi(\cdot)$ are "sufficiently regular" to allow for differentiation under the expectation, the Delta can be computed in a straight-forward manner as

$$E\left[\frac{\partial}{\partial x}\Phi(X_T^x)\right] = E[\Phi'(X_T^x)\, Z_T]\,, \tag{1.2}$$

where the *first variation process* $Z_t := \frac{\partial}{\partial x} X_t^x$ is given by

$$Z_t = \exp\left\{\int_0^t \left[b'(X_s^x) - \frac{1}{2}(\sigma'(X_s^x))^2\right] ds + \int_0^t \sigma'(X_s^x)\, dB_s\right\}\,, \tag{1.3}$$

and where Φ', b', σ' denote the derivatives of Φ, b, σ, respectively. For example, requiring that Φ, b, σ are continuously differentiable with bounded derivatives would allow (1.2) to hold (we refer to [18] for conditions on b and σ that guarantee the existence of the first variation process), and the expectation in (1.2) could be approximated e.g. by Monte Carlo methods. In most realistic situations, though, straight-forward computations as in (1.2) are not possible. In that case, one could combine numerical methods to approximate the derivative and the expectation in (1.1), respectively, to compute the Delta. However, in particular for discontinuous pay-offs Φ as is the case for a digital option this procedure might be numerically inefficient, see for example [15]. At that point, the following result for lookback options obtained with the help of Malliavin calculus appears to be useful, where the option pay-off is allowed to depend on the path of the underlying at finitely many time points.

Theorem 1.1 (Proposition 3.2 in [15])**.** *Let $b(\cdot)$ and $\sigma(\cdot)$ be continuously differentiable with bounded Lipschitz derivatives, $\sigma(\cdot) > \epsilon > 0$, and $\Phi : \mathbb{R}^m \to \mathbb{R}$ be such that the pay-off $\Phi(X_{T_1}^x, \ldots, X_{T_m}^x)$, $T_1, \ldots, T_m \in (0, T]$, of the corresponding lookback option is square integrable. Then the Delta exists and is given by*

$$\frac{\partial}{\partial x} E[\Phi(X_{T_1}^x, \ldots, X_{T_m}^x)] = E\left[\Phi(X_{T_1}^x, \ldots, X_{T_m}^x)\int_0^T a(t)\, \sigma^{-1}(X_t^x)\, Z_t\, dB_t\right]\,, \tag{1.4}$$

where Z_t is the first variation process given in (1.3) *and $a(t)$ is any square integrable deterministic function such that, for every $i = 1, \ldots, m$,*

$$\int_0^{T_i} a(s) ds = 1.$$

While for notational simplicity we present the above result for one-dimensional X^x we remark that in [15] the extension to multi-dimensional underlying asset and Brownian motion is considered. If the option is of European type, i.e. the pay-off $\Phi(X_T^x)$ depends only on the underlying at T, then (1.4) is the probabilistic representation of the space derivative of a solution to a Kolmogorov equation which is also referred to as *Bismuth-Elworthy-Li type formula* in the literature due to [13], [6]. The strength of (1.4) is that the Delta is expressed again as an expectation of the pay-off multiplied by the so-called Malliavin weight $\int_0^T a(t)\, \sigma^{-1}(X_t^x)\, Z_t\, dB_t$. Computing the Delta by Monte-Carlo via this reformulation then guarantees a convergence rate that is independent of the regularity of the pay-off function Φ and the dimensionality. Note that the Malliavin weight is independent of the option pay-off, and thus the same weight can be employed in the computations of the Deltas of different options. Also, in [14] and [3] the question of how to optimally choose the function $a(t)$ with respect to computational efficiency is considered.

While the representation (1.4) succeeds to handle irregular pay-offs by getting rid of the derivative of Φ, the regularity assumptions on the coefficients b and σ driving the dynamics of the underlying diffusion are rather strong. Consider for example an extended Black and Scholes model where the stock pays a dividend yield that switches to a higher level when the stock value passes a certain threshold. Then, again with the risk-free rate equal to zero for simplicity, the logarithm of the stock price is modelled by the following dynamics under the risk-neutral measure:

$$dX_t^x = b(X_t^x)dt + \sigma dB_t, \;\; X_0^x = x \in \mathbb{R}\,,$$

where $\sigma > 0$ is constant and the drift coefficient $b : \mathbb{R} \to \mathbb{R}$ is given by

$$b(x) := -\lambda_1 \mathbf{1}_{(-\infty,R)}(x) - \lambda_2 \mathbf{1}_{[R,\infty)}(x),$$

for dividend yields $\lambda_1, \lambda_2 \in \mathbb{R}_+$ and a given threshold $R \in \mathbb{R}$. In [9], a (more complex) irregular drift b is interpreted as state-dependent fees deducted by the insurer in the evolution of variable annuities instead of dividend yield. Already, this simple regime-switching model is not covered by the result in Theorem 1.1 since the drift coefficient is not continuously differentiable.

Or allow for state-dependent regime-switching of the mean reversion rate in an extended Ornstein-Uhlenbeck process:

$$dX_t^x = b(X_t^x)dt + \sigma dB_t, \;\; X_0^x = x \in \mathbb{R}\,,$$

where $\sigma > 0$ is constant and the drift coefficient $b : \mathbb{R} \to \mathbb{R}$ is given by

$$b(x) := -\lambda_1 x \mathbf{1}_{(-\infty,R)}(x) - \lambda_2 x \mathbf{1}_{[R,\infty)}(x)$$

for mean reversion rates $\lambda_1, \lambda_2 \in \mathbb{R}_+$ and a given threshold $R \in \mathbb{R}$ (here the mean reversion level is set equal to zero). This type of model captures well, for instance, the evolution of electricity spot prices, which switches between so-called spike regimes on high price levels with very fast mean reversion and base regimes on normal price levels with moderate speed of mean reversion, see e.g. [5], [17], [26] and references therein. Alternatively, an extended Ornstein-Uhlenbeck process with state-dependent regime-switching of the mean reversion level (low and high interest rate environments) is an interesting modification of the Vašíček short rate model. Note that in that case the Delta is rather a generalised Rho, i.e. a sensitivity measure with respect to the short end of the yield curve. We observe that also these two extended Ornstein-Uhlenbeck processes are not covered by the result in Theorem 1.1.

Motivated by these examples, this paper aims at deriving an analogous result to Theorem 1.1 when the underlying is driven by an SDE with irregular drift coefficient. More precisely, we will consider SDE's

$$dX_t^x = b(t, X_t^x)dt + dB_t, \;\; 0 \le t \le T, \;\; X_0^x = x \in \mathbb{R}\,, \tag{1.5}$$

where we allow for time-inhomogeneous drift coefficients $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ in the form

$$b(t,x) = \tilde{b}(t,x) + \hat{b}(t,x)\,, \quad (t,x) \in [0,T] \times \mathbb{R}\,, \tag{1.6}$$

for $\tilde{b}$ merely bounded and measurable, and $\hat{b}$ Lipschitz continuous and at most of linear growth in x uniformly in t, i.e. there exists a constant $C > 0$ such that

$$|\hat{b}(t,x) - \hat{b}(t,y)| \le C|x-y| \tag{1.7}$$

$$|\hat{b}(t,x)| \le C(1+|x|) \tag{1.8}$$

for $x, y \in \mathbb{R}$ and $t \in [0,T]$. Adding the Lipschitz component $\hat{b}(t,x)$ in (1.6) is motivated by the fact that many drift coefficients interesting for financial applications are of linear growths. At present we are not able to show our results for general measurable drift coefficients of linear growths, but only for those where the irregular behavior remains in a bounded spectrum. However, from an application point of view this class is very rhich already, and in particular it contains the regime switching examples from above. In (1.5) we consider a constant volatility coefficient $\sigma(t,x) := 1$, but we will see at the end of Section 3 (Theorem 3.8) that our results apply to many SDE's with more general volatility coefficients which can be reduced to SDE's of type (1.5) (which for example is possible for volatility coefficients as in Theorem 1.1).

In order to be able to apply Malliavin calculus to the underlying diffusion, the first thing we need to ensure is that the solution of SDE (1.5) is a Brownian functionals, i.e. we are interested in the existence of strong solutions of (1.5).

Definition 1.2. *A* strong solution *of SDE* (1.5) *is a continuous,* $\{\mathcal{F}_t\}_{t\in[0,T]}$*-adapted process* $\{X_t^x\}_{t\in[0,T]}$ *that solves equation* (1.5).

Remark 1.3. *Note that the usual definition of a strong solution requires the existence of a Brownian-adapted solution of* (1.5) *on* any *given stochastic basis. However, an* $\{\mathcal{F}_t\}_{t\in[0,T]}$*-adapted solution* $\{X_t^x\}_{t\in[0,T]}$ *on the given stochastic basis* $(\Omega, \mathcal{F}, P, B)$ *can be written in the form* $X_t^x = F_t(B_\cdot)$ *for some family of functionals* F_t*,* $t \in [0,T]$*, (see e.g.* [24] *for an explicit form of* F_t*). Then for any other stochastic basis* $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}, \hat{B})$ *one gets that* $X_t^x := F_t(\hat{B}_\cdot)$*,* $t \in [0,T]$*, is a* $\hat{B}$*-adapted solution to SDE (1.5). So once there is a Brownian-adapted solution of* (1.5) *on one given stochastic basis, it follows that there indeed exists a strong solution in the usual sense. This justifies our definition of a strong solution above.*

To pursue our objectives we proceed as follows in the remaining parts of the paper. In Section 2 we recall some fundamental concepts from Malliavin calculus and local time calculus which compose central mathematical tools in the following analysis.

We then analyse in Section 3 the existence and Malliavin differentiability of a unique strong solution of SDE's with irregular drift coefficients as in (1.5) (Theorem 3.1). It is well known that the SDE is Malliavin differentiable as soon as the coefficients are Lipschitz continuous (see e.g. [28]); for merely bounded and measurable drift coefficients Malliavin differentiability was shown only recently in [25], (see also [23]). Here, we extend ideas introduced for bounded coefficients in [25] to drift coefficients of type (1.6). Unlike in most of the existing literature on strong solutions of SDE's with irregular coefficients our approach does not rely on a pathwise uniqueness argument (Yamada-Watanabe Theorem). Instead, we employ a compactness criterium based on Malliavin calculus together with local time calculus to directly construct a strong solution which in addition is Malliavin differentiable. Also, we are able to give an explicit expression for the Malliavin derivative of the strong solution of (1.5) in terms of the integral of b (and not the derivative of b) with respect to local time of the strong solution (Proposition 3.2). We mention that while existence and Malliavin differentiability of strong solutions could be extended to analogue multi-dimensional SDE's as in [23], the explicit expression of the Malliavin derivative is in general only possible for one-dimensional SDE's as considered in this paper. Moreover, in this paper we replace arguments that are based on White Noise analysis in [25] and [23] by alternative proofs which might make the text more accessible for readers who are unfamiliar with concepts from White Noise analysis.

Next, we need to analyse the regularity of the dependence of the strong solution in its initial condition and to introduce the analogue of the first variation process (1.3) in case of irregular drift coefficients. Using the close connection between the Malliavin derivative and the first variation process, we find that the strong solution is Sobolev differentiable in its initial condition (Theorem 3.4). Again, we give an explicit expression for the corresponding (Sobolev) first variation process which does not include the derivative of b (Proposition 3.5).

In Section 4 we develop our main result (Theorem 4.2) which extends Theorem 1.1 to SDE's with irregular drift coefficients. To this end, one has to show in the first place that the Delta exists, i.e. that $E[\Phi(X_{T_1}^x, \ldots, X_{T_m}^x)]$ is continuously differentiable in x. At this point the explicit expressions for the Malliavin derivative and the first variation process are essential. In the final representation of the Delta we then have gotten rid of both the derivative of the pay-off Φ and the derivative of the drift coefficient b in the first variation process, whence the title "Computing Deltas without Derivatives" of the paper. In addition to Deltas of lookback options as in Theorem 1.1, we further consider Deltas of Asian options with pay-offs of the type $\Phi\left(\int_{T_1}^{T_2} X_u^x \, du\right)$ for $T_1, T_2 \in [0,T]$ and some function $\Phi : \mathbb{R} \to \mathbb{R}$. In case the starting point of the averaging period of the Asian pay-off lies in the future, i.e. $T_1 > 0$, we are able to give analogue results to the ones of lookback options. If the averaging period starts today, i.e. $T_1 = 0$, the Malliavin weight in the expression for the Delta would include a general Skorohod integral which is neither numerically nor mathematically tractable in our analysis (except for linear coefficients as in the Black and Scholes model where the Skorohod integral turns out to be an Itô integral). However, we are still able to state two approximation results for the Delta in this case.

In Section 5 we consider some examples and compute the Deltas in the concrete regime-switching models mentioned above. We do a small simulation study and compare the performance to a finite difference approximation of the Delta in the same spirit as in [15].

We conclude the paper by an appendix with some technical proofs from Section 3 which have been deferred to the end of the paper for better readability.

Notations: We summarise some of the most frequently used notations:

- $C^1(\mathbb{R})$ denotes the space of continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$.
- $C_0^\infty([0,T] \times \mathbb{R})$, respectively $C_0^\infty(\mathbb{R})$, denotes the space of infinitely many times differentiable functions on $[0,T] \times \mathbb{R}$, respectively $\mathbb{R}$, with compact support.
- For a measurable space $(S, \mathcal{G})$ equipped with a measure μ, we denote by $L^p(S, \mathcal{G})$ or $L^p(S)$ the Banach space of (equivalence classes of) functions on S integrable to some power p, $p \geq 1$.
- $L^p_{loc}(\mathbb{R})$ denotes the space of locally Lebesgue integrable functions to some power p, $p \geq 1$, i.e. $\int_U |f(x)|^p dx < \infty$ for every open bounded subset $U \subset \mathbb{R}$.
- $W^{1,p}_{loc}(\mathbb{R})$ denotes the subspace of $L^p_{loc}(\mathbb{R})$ of weakly (Sobolev) differentiable functions such that the weak derivative f' belongs to $L^p_{loc}(\mathbb{R})$, $p \geq 1$.
- For a progressive process Y. we denote the Doléans-Dade exponential of the corresponding Brownian integral (if well defined) by
$$\mathcal{E}\left(\int_0^t b(u, Y_u) dB_u\right) := \exp\left(\int_0^t b(u, Y_u) dB_u - \frac{1}{2}\int_0^t b^2(u, Y_u) du\right), \quad t \in [0,T]. \tag{1.9}$$
- For $Z \in L^2(\Omega, \mathcal{F}_T)$ we denote the *Wiener-transform* of Z in $f \in L^2([0,T])$ by
$$\mathcal{W}(Z)(f) := E\left[Z\mathcal{E}\left(\int_0^T f(s) dB_s\right)\right].$$
- We will use the symbol $\lesssim$ to denote *less or equal than* up to a positive real constant $C > 0$ not depending on the parameters of interest, i.e. if we have two mathematical expressions $E_1(\theta), E_2(\theta)$ depending on some parameter of interest θ then $E_1(\theta) \lesssim E_2(\theta)$ if, and only if, there is a positive real number $C > 0$ independent of θ such that $E_1(\theta) \leq CE_2(\theta)$.

2. Framework

Our main results centrally rely on tools from Malliavin calculus as well as integration with respect to local time both in time and space. We here provide a concise introduction to the main concepts in these two areas that will be employed in the following sections. For deeper information on Malliavin calculus the reader is referred to i.e. [28, 21, 22, 10]. As for theory on local time integration for Brownian motion we refer to i.e. [12, 29].

2.1. **Malliavin calculus.** Denote by $\mathcal{S}$ the set of simple random variables $F \in L^2(\Omega)$ in the form
$$F = f\left(\int_0^T h_1(s) dB_s, \dots, \int_0^T h_n(s) dB_s\right), \quad h_1, \dots, h_n \in L^2([0,T]), \quad f \in C_0^\infty(\mathbb{R}^n).$$
The Malliavin derivative operator D acting on such simple random variables is the process $DF = \{D_t F, t \in [0,T]\}$ in $L^2(\Omega \times [0,T])$ defined by
$$D_t F = \sum_{i=1}^n \partial_i f\left(\int_0^T h_1(s) dB_s, \dots, \int_0^T h_n(s) dB_s\right) h_i(t).$$
Define the following norm on $\mathcal{S}$:
$$\|F\|_{1,2} := \|F\|_{L^2(\Omega)} + \|DF\|_{L^2(\Omega; L^2([0,T]))} = E[|F|^2]^{1/2} + E\left[\int_0^T |D_t F|^2 dt\right]^{1/2}. \tag{2.1}$$

We denote by $\mathbb{D}^{1,2}$ the closure of the family of simple random variables $\mathcal{S}$ with respect to the norm given in (2.1), and we will refer to this space as the space of Malliavin differentiable random variables in $L^2(\Omega)$ with Malliavin derivative belonging to $L^2(\Omega)$.

In the derivation of the probabilistic representation for the Delta, the following chain rule for the Malliavin derivative will be essential:

Lemma 2.1. *Let $\varphi : \mathbb{R}^m \to \mathbb{R}$ be continuously differentiable with bounded partial derivatives. Further, suppose that $F = (F_1, \dots, F_m)$ is a random vector whose components are in $\mathbb{D}^{1,2}$. Then $\varphi(F) \in \mathbb{D}^{1,2}$ and*

$$D_t\varphi(F) = \sum_{i=1}^{m} \partial_i\varphi(F) D_t F_i, \quad P-a.s., \quad t \in [0,T].$$

The Malliavin derivative operator $D : \mathbb{D}^{1,2} \to L^2(\Omega \times [0,T])$ admits an adjoint operator $\delta = D^* : \mathrm{Dom}(\delta) \to L^2(\Omega)$ where the domain $\mathrm{Dom}(\delta)$ is characterised by all $u \in L^2(\Omega \times [0,T])$ such that for all $F \in \mathbb{D}^{1,2}$ we have

$$E\left[\int_0^T D_t F\, u_t dt\right] \le C\|F\|_{1,2},$$

where C is some constant depending on u.

For a stochastic process $u \in \mathrm{Dom}(\delta)$ (not necessarily adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$) we denote by

$$\delta(u) := \int_0^T u_t \delta B_t \tag{2.2}$$

the action of δ on u. The above expression (2.2) is known as the Skorokhod integral of u and it is an anticipative stochastic integral. It turns out that all $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted processes in $L^2(\Omega \times [0,T])$ are in the domain of δ and for such processes u_t we have

$$\delta(u) = \int_0^T u_t dB_t,$$

i.e.the Skorokhod and Itô integrals coincide. In this sense, the Skorokhod integral can be considered to be an extension of the Itô integral to non-adapted integrands.

The dual relation between the Malliavin derivative and the Skorokhod integral implies the following important formula:

Theorem 2.2 (Duality formula)**.** *Let $F \in \mathbb{D}^{1,2}$ and $u \in Dom(\delta)$. Then*

$$E\left[F\int_0^T u_t \delta B_t\right] = E\left[\int_0^T u_t D_t F dt\right]. \tag{2.3}$$

The next result, which is due to [8] and central in proving existence of strong solutions in the following, provides a compactness criterion for subsets of $L^2(\Omega)$ based on Malliavin calculus.

Proposition 2.3. *Let $F_n \in \mathbb{D}^{1,2}$, $n = 1, 2...$, be a given sequence of Malliavin differentiable random variables. Assume that there exist constants $\alpha > 0$ and $C > 0$ such that*

$$\sup_n E[|F_n|^2] \le C,$$

$$\sup_n E\left[|D_t F_n - D_{t'} F_n|^2\right] \le C|t - t'|^\alpha$$

for $0 \le t' \le t \le T$, and

$$\sup_n \sup_{0\le t\le T} E\left[|D_t F_n|^2\right] \le C\,.$$

Then the sequence F_n, $n = 1, 2...$, is relatively compact in $L^2(\Omega)$.

We conclude this review on Malliavin calculus by stating a relation between the Malliavin derivative and the first variation process of the solution of an SDE with smooth coefficients that is essential in the derivation of Theorem 1.1. We give the result for the case when the volatility coefficient is equal to 1, but the analogue result is valid for more general smooth volatility coefficients. Assume the drift coefficient $b(t, x)$ in the SDE (1.5) fulfils the Lipschitz and linear growth conditions (1.7)-(1.8). Then it is well known that there exists a unique strong solution

X_t^x, $t \in [0,T]$, to equation (1.5) that is Malliavin differentiable, and that for all $0 \le s \le t \le T$ the Malliavin derivative $D_s X_t^x$ fulfils, see e.g. [28, Theorem 2.2.1]

$$D_s X_t^x = 1 + \int_s^t b'(u, X_u^x) D_s X_u^x du, \tag{2.4}$$

where b' denotes the (weak) derivative of b with respect to x.

Further, under these assumptions the strong solution is also differentiable in its initial condition, and the first variation process $\frac{\partial}{\partial x} X_t^x$, $t \in [0,T]$, fulfils (see e.g. [18] for differentiable coefficients and [2] for an extension to Lipschitz coefficients)

$$\frac{\partial}{\partial x} X_t^x = 1 + \int_0^t b'(u, X_u^x) \frac{\partial}{\partial x} X_u^x du. \tag{2.5}$$

Solving equations (2.4) and (2.5) thus yields the following proposition.

Proposition 2.4. *Let X_t^x, $t \in [0,T]$, be the unique strong solution to equation (1.5) when $b(t,x)$ fulfils the Lipschitz and linear growth condition* (1.7)-(1.8). *Then X_t^x is Malliavin differentiable and differentiable in its initial condition for all $t \in [0,T]$, and for all $s \le t \le T$ we have*

$$D_s X_t^x = \exp\left\{ \int_s^t b'(u, X_u^x) du \right\} \tag{2.6}$$

and

$$\frac{\partial}{\partial x} X_t^x = \exp\left\{ \int_0^t b'(u, X_u^x) du \right\}. \tag{2.7}$$

As a consequence,

$$\frac{\partial}{\partial x} X_t^x = D_s X_t^x \frac{\partial}{\partial x} X_s^x, \tag{2.8}$$

where all equalities hold P-a.s.

2.2. **Integration with respect to local-time.** Let now X^x be a given (strong) solution to SDE (1.5). In the sequel we need the concept of stochastic integration over the plane with respect to the local time $L^{X^x}(t,y)$ of X^x. For Brownian motion, the local time integration theory in time and space has been introduced in [12]. We extend this local time integration theory to more general diffusions of type (1.5) by resorting to the Brownian setting under an equivalent measure where X^x is a Brownian motion. To this end, we notice the following fact that is extensively used throughout the paper.

Remark 2.5. *The Radon-Nikodym density*

$$\frac{dQ}{dP} = \mathcal{E}\left(-\int_0^T b(s, X_s^x) dB_s \right)$$

defines a probability measure Q equivalent to P under which X^x is Brownian motion starting in x. Indeed, because b is of at most linear growth we obtain by Grönwall's inequality as in the proof of Lemma A.1 a constant $C_{t,x} > 0$ such that $|X_t^x| \le C_{t,x}(1 + |B_t|)$. One can thus find a equidistant partition $0 = t_0 < t_1 ... < t_m = T$ such that

$$E\left[\exp\left\{ \int_{t_i}^{t_{i+1}} b^2(s, X_s^x) ds \right\} \right] \le E\left[\exp\left\{ \int_{t_i}^{t_{i+1}} \left(C_1 + C_2|B_s| + C_3|B_s|^2 \right) ds \right\} < \infty \right]$$

for all $i = 0, ..., m-1$, where C_1, C_2 and C_3 are some positive constants. Then it is well-known, see e.g. [16, Corollary 5.16]*, that Q is an equivalent probability measure under which X^x is Brownian motion by Girsanov's theorem.*

We now define the feasible integrands for the local time-space integral with respect to $L^{X^x}(t,y)$ by the Banach space $(\mathcal{H}^x, \|\cdot\|)$ of functions $f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ with norm

$$\begin{aligned}\|f\|_x = & 2\left(\int_0^T \int_{\mathbb{R}} f^2(s,y) \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{|y-x|^2}{2s}\right) dyds\right)^{1/2} \\ & + \int_0^T \int_{\mathbb{R}} |y-x|\,|f(s,y)| \frac{1}{s\sqrt{2\pi s}} \exp\left(-\frac{|y-x|^2}{2s}\right) dyds.\end{aligned}$$

We remark that this space of integrands is the same as the one introduced in [12] for Brownian motion (i.e. the special case when the X^x is a Brownian motion), except that we have in a straight forward manner generalised the space in [12] to the situation when the Brownian motion has arbitrary initial value x.

We denote by $f_\Delta : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ a simple function in the form

$$f_\Delta(s,y) = \sum_{1\le i\le n-1, 1\le j\le m-1} f_{ij} \mathbf{1}_{(y_i,y_{i+1}]}(y) \mathbf{1}_{(s_j,s_{j+1}]}(s),$$

where $(s_j)_{1\le j\le m}$ is a partition of $[0,T]$ and $(y_i)_{1\le i\le n}$ and $(f_{ij})_{1\le i\le n, 1\le j\le m}$ are finite sequences of real numbers. It is readily checked that the space of simple functions is dense in $(\mathcal{H}^x, \|\cdot\|)$. The local time-space integral of an simple function f_Δ with respect to $L^{X^x}(dt,dy)$ is then defined by

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} f_\Delta(s,y) L^{X^x}(ds,dy) := \\ &:= \sum_{\substack{1\le i\le n-1\\ 1\le j\le m-1}} f_{ij}(L^{X^x}(s_{j+1},y_{i+1}) - L^{X^x}(s_j,y_{i+1}) - L^{X^x}(s_{j+1},y_i) + L^{X^x}(s_j,y_i)).\end{aligned}$$

Lemma 2.6. *For $f \in \mathcal{H}^x$ let $f_n, n \ge 1$, be a sequence of simple functions converging to f in $\mathcal{H}^x$. Then $\int_0^T \int_{\mathbb{R}} f_n(s,y) L^{X^x}(ds,dy)$, $n \ge 1$, converges in probability. Further, for any other approximating sequence of simple functions the limit remains the same.*

Proof. Define $F_n^{X^x} := \int_0^T \int_{\mathbb{R}} f_n(s,x) L^{X^x}(ds,dx)$. Now consider the equivalent measure Q from Remark 2.5 under which X^x is Brownian motion. Define $F^{X^x} := \int_0^T \int_{\mathbb{R}} f(s,x) L^{X^x}(ds,dx)$ to be the time-space integral of f with respect to the local time of Brownian motion X^x under Q, which exists as an $L^1(Q)$-limit of $F_n^{X^x}, n \ge 1$ by the Brownian local time integration theory introduced in [12] (since $f_n, n \ge 1$ converge to f in $\mathcal{H}^x$). We show that $F_n^{X^x}, n \ge 1$ converge in probability to F^{X^x} under P. Indeed,

$$\begin{aligned}E[1 \wedge |F^{X^x} - F_n^{X^x}|] &= E\left[\left(1 \wedge |F^{B^x} - F_n^{B^x}|\right) \mathcal{E}\left(\int_0^T b(s,B_s^x) dB_s\right)\right] \\ &\le E\left[\mathcal{E}\left(\int_0^T b(s,B_s^x) dB_s\right)^{1+\varepsilon}\right]^{1/(1+\varepsilon)} E\left[\left(1 \wedge |F^{B^x} - F_n^{B^x}|\right)^{\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{1+\varepsilon}} \\ &\le C_\varepsilon E[\left(1 \wedge |F^{B^x} - F_n^{B^x}|\right)]^{\frac{\varepsilon}{1+\varepsilon}} \overset{n\to\infty}{\longrightarrow} 0\,, \end{aligned} \tag{2.9}$$

where, in analogy to the notation F^{X^x} and $F_n^{X^x}$ above, the notation F^{B^x} and $F_n^{B^x}$ refers to the corresponding integrals with respect to local time of Brownian motion B^x under P, and where in the first equality we have used that $(F^{B^x}, F_n^{B^x})$ has the same law under P as $(F^{X^x}, F_n^{X^x})$ under Q. The inequalities follow by Lemma A.1 for some $\varepsilon > 0$ suitably small. Further, by [12] we know that $F_n^{B^x}, n \ge 1$ converge to F^{B^x} in $L^1(P)$, which implies the convergence in (2.9). Hence $F_n^{X^x}, n \ge 1$ converge to F^{X^x} in the Ky-Fan metric $d(X,Y) = E[1 \wedge |X - Y|]$, $X, Y \in L^0(\Omega)$, which characterises convergence in probability. Finally, again by [12], F^{X^x} is independent of the approximating sequence $f_n, n \ge 1$. □

Definition 2.7. *For $f \in \mathcal{H}^x$ the limit in Lemma 2.6 is called the time-space integral of f with respect to $L^{X^x}(dt, dx)$ and is denoted by $\int_0^T \int_{\mathbb{R}} f(s, y) L^{X^x}(ds, dy)$. Further, for any $t \in [0, T]$ we define $\int_0^t \int_{\mathbb{R}} f(s, y) L^{X^x}(ds, dy) := \int_0^T \int_{\mathbb{R}} f(s, y) I_{[0,t]}(s) L^{X^x}(ds, dy)$.*

Remark 2.8. *We notice that the drift coefficient $b(t, x)$ in (1.6), which is of linear growth in x uniformly in t, is in $\mathcal{H}^x$, and thus the local time integral of $b(t, x)$ with respect to $L^{X^x}(dt, dy)$ exists for any $x \in \mathbb{R}$.*

If X^x is a Brownian motion $B.$ we have the following decomposition due to [12] that we employ in the construction of strong solutions, and that also constitutes the foundation in the construction of the local time integral in [12].

Theorem 2.9. *Let $f \in \mathcal{H}^0$. Then*

$$\int_0^t \int_{\mathbb{R}} f(s, y) L^{B^x}(ds, dy) = \\ = - \int_0^t f(s, B_s^x) dB_s + \int_{T-t}^T f(T - s, \widehat{B}_s^x) dW_s - \int_{T-t}^T f(T - s, \widehat{B}_s^x) \frac{\widehat{B}_s}{T - s} ds, \tag{2.10}$$

where $\widehat{B}_t = B_{T-t}, 0 \le t \le T$ is time-reversed Brownian motion, and $W.$, defined by

$$\widehat{B}_t = B_T + W_t - \int_0^t \frac{\widehat{B}_s}{T - s} ds,$$

is a Brownian motion with respect to the filtration of $\widehat{B}.$.

We conclude this subsection by stating three further identities for the local time integral of a general diffusions X^x which will be useful later on.

Lemma 2.10. *Let $f \in \mathcal{H}^x$ be Lipschitz continuous in x. Then for all $t \in [0, T]$*

$$- \int_0^t \int_{\mathbb{R}} f(s, y) L^{X^x}(ds, dy) = \int_0^t f'(s, X_s^x) ds. \tag{2.11}$$

where f' denotes the (weak) derivative of $f(t, y)$ with respect to y.

If $f \in \mathcal{H}^x$ is time homogeneous (i.e. $f(t, y) = f(y)$ only depends on the space variable) and locally square integrable, then for any $t \in [0, T]$

$$\int_0^t \int_{\mathbb{R}} f(s, x) L^{X^x}(ds, dx) = -[f(\cdot, X^x), X^x]_t. \tag{2.12}$$

and

$$- \int_0^t \int_{\mathbb{R}} f(s, y) L^{X^x}(ds, dy) = 2F(X_t^x) - 2F(x) - 2 \int_0^t f(X_s^x) dX_s^x \tag{2.13}$$

where F is a primitive function of f and $[\tilde{b}(\cdot, X_\cdot^x), X_\cdot^x]_t$ is the generalised covariation process

$$[f(\cdot, X_\cdot^x), X_\cdot^x]_t := P - \lim_{m \to \infty} \sum_{k=1}^m \left(f(t_k^m, X_{t_k^m}^x) - f(t_{k-1}^m, X_{t_{k-1}^m}^x) \right) \left(X_{t_k}^x - X_{t_{k-1}}^x \right)$$

where for every m $\{t_k^m\}_{k=1}^m$ is a partition of $[0, t]$ such that $\lim_m \sup_{k=1,\dots,m} |t_k^m - t_{k-1}^m| = 0$. Note that (2.13) *can be considered as a generalised Itô formula.*

Proof. If $X^x = x + B$, then identities (2.11)-(2.13) are given in [12]. For general X^x, we consider the identities under the equivalent measure Q from Remark 2.5. Then, by the construction of the local time integral outlined in Lemma 2.6, the integrals in the identities are the ones with respect to Brownian motion X^x, for which we know the identities hold by [12] (where such identities are given in the case $x = 0$ but one can easily extend them to the case of the Brownian motion starting at an arbitrary $x \in \mathbb{R}$). $\square$

3. Existence, Malliavin, and Sobolev differentiability of strong solutions

In this section we prepare the necessary theoretical grounds to develop the probabilistic representation of Deltas. Being notationally and technically rather heavy, the proofs of this section are deferred to Appendix A for an improved flow and readability of the paper. We first study the existence and Malliavin differentiability of a unique strong solution of SDE (1.5) before we turn to the differentiability of the strong solution in its initial condition and the corresponding first variation process. We state the first main result of this section:

Theorem 3.1. *Suppose that the drift coefficient $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ is in the form (1.6). Then there exists a unique strong solution $\{X_t^x\}_{t\in[0,T]}$ to SDE (1.5). In addition, X_t^x is Malliavin differentiable for every $t \in [0,T]$.*

The proof of Theorem 3.1 employs several auxiliary results presented in Appendix A. The main steps are:

(1) First, we construct a weak solution X^x to (1.5) by means of Girsanov's theorem, that is we introduce a probability space $(\Omega, \mathcal{F}, P)$ that carries some Brownian motion B and a continuous process X^x such that (1.5) is fulfilled. However, a priori X^x is not adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by Brownian motion B.

(2) Next, we approximate the drift coefficient $b = \tilde{b} + \hat{b}$ by a sequence of functions (which always exists by standard approximation results)
$$b_n := \tilde{b}_n + \hat{b}, \quad n \geq 1, \tag{3.1}$$
such that $\{\tilde{b}_n\}_{n\geq 1} \subset C_0^\infty([0,T]\times\mathbb{R})$ with $\sup_{n\geq 1} \|\tilde{b}_n\|_\infty \leq C < \infty$ and $\tilde{b}_n \to \tilde{b}$ in $(t,x) \in [0,T]\times\mathbb{R}$ a.e. with respect to the Lebesgue measure. By standard results on SDE's, we know that for each smooth coefficient b_n, $n \geq 1$, there exists a unique strong solution $X^{n,x}$ to the SDE
$$dX_t^{n,x} = b_n(t, X_t^{n,x})dt + dB_t, \quad 0 \leq t \leq T, \quad X_0^{n,x} = x \in \mathbb{R}\,. \tag{3.2}$$
We then show that for each $t \in [0,T]$ the sequence $X_t^{n,x}$ converges weakly to the conditional expectation $E[X_t^x|\mathcal{F}_t]$ in the space $L^2(\Omega;\mathcal{F}_t)$ of square integrable, $\mathcal{F}_t$-measurable random variables.

(3) By Proposition 2.4 we know that for each $t \in [0,T]$ the strong solutions $X_t^{n,x}$, $n \geq 1$, are Malliavin differentiable with
$$D_s X_t^{n,x} = \exp\left\{\int_s^t b_n'(u, X_u^{n,x})du\right\}, \quad 0 \leq s \leq t \leq T, \quad n \geq 1, \tag{3.3}$$
where b_n' denotes the derivative of b_n with respect to x. We will use representation (3.3) to employ a compactness criterion based on Malliavin calculus to show that for every $t \in [0,T]$ the set of random variables $\{X_t^{n,x}\}_{n\geq 1}$ is relatively compact in $L^2(\Omega;\mathcal{F}_t)$, which then allows to conclude that $X_t^{n,x}$ converges strongly in $L^2(\Omega;\mathcal{F}_t)$ to $E[X_t^x|\mathcal{F}_t]$. Further we obtain that $E[X_t^x|\mathcal{F}_t]$ is Malliavin differentiable as a consequence of the compactness criterion.

(4) In the last step we show that $E[X_t^x|\mathcal{F}_t] = X_t^x$, which implies that X_t^x is $\mathcal{F}_t$-measurable and thus a strong solution. Moreover, we show that this solution is unique.

Notation: In the following we sometimes include the drift coefficient b into the sequence $\{b_n\}_{n\geq 0}$ by putting $b_0 := \tilde{b}_0 + \hat{b} := \tilde{b} + \hat{b} = b$.

The next important result is an explicit representation of the Malliavin derivative of the strong solution X_t^x, $t \in [0,T]$. For smooth coefficients b we can explicitly express the Malliavin derivative in terms of the derivative of b as stated in (3.3). For general, not necessarily differentiable coefficients b, we are still able to give an explicit formula which now only involves the coefficient b in a local time integral:

Proposition 3.2. *For $0 \le s \le t \le T$, the Malliavin derivative $D_s X_t^x$ of the unique strong solution X_t^x to equation (1.5) has the following explicit representation:*

$$D_s X_t^x = \exp\left\{ -\int_s^t \int_{\mathbb{R}} b(u,y) L^{X^x}(du,dy) \right\} \quad P\text{-a.s.}, \tag{3.4}$$

where $L^{X^x}(du,dy)$ denotes integration in space and time with respect to the local time of X^x, see Section 2.2 for definitions.

Next, we turn our attention to the study of the strong solution X_t^x as a function in its initial condition x for SDE's with possible irregular drift coefficients. The first result establishes Hölder continuity jointly in time and space.

Proposition 3.3. *Let X_t^x, $t \in [0,T]$ be the unique strong solution to the SDE* (1.5). *Then for all $s,t \in [0,T]$ and $x,y \in K$ for any arbitrary compact subset $K \subset \mathbb{R}$ there exists a constant $C = C(K, \|\tilde{b}\|_\infty, \|\hat{b}'\|_\infty) > 0$ such that*

$$E\left[|X_t^x - X_s^y|^2\right] \le C(|t-s| + |x-y|^2).$$

In particular, there exists a continuous version of the random field $(t,x) \mapsto X_t^x$ with Hölder continuous trajectories of order $\alpha < 1/2$ in $t \in [0,T]$ and $\alpha < 1$ in $x \in \mathbb{R}$.

If the drift coefficient b is regular, then we know by Proposition 2.4 that X_t^x is even differentiable as a function in x. The first variation process $\frac{\partial}{\partial x} X_\cdot^x$ is then given by (2.7) in terms of the derivative of the drift coefficient and is closely related to the Malliavin derivative by (2.8). In the following we will derive analogous results for irregular drift coefficients, where in general the first variation process will now exist in the Sobolev derivative sense. Let $U \subset \mathbb{R}$ be an open and bounded subset. The Sobolev space $W^{1,2}(U)$ is defined as the set of functions $u : \mathbb{R} \to \mathbb{R}$, $u \in L^2(U)$ such that its weak derivative belongs to $L^2(U)$. We endow this space with the norm

$$\|u\|_{1,2} = \|u\|_2 + \|u'\|_2$$

where u' stands for the *weak derivative* of $u \in W^{1,2}(U)$. We say that the solution X_t^x, $t \in [0,T]$, is *Sobolev differentiable* in U if for all $t \in [0,T]$, $X_t^\cdot$ belongs to $W^{1,2}(U)$, P-a.s. Observe that in general $X_t^\cdot$ is not in $W^{1,2}(\mathbb{R})$, e.g. take $b \equiv 0$.

Theorem 3.4. *Let $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ be as in (1.6). Let X_t^x , $t \in [0,T]$ be the unique strong solution to the SDE (1.5) and $U \subset \mathbb{R}$ an open, bounded set. Then for every $t \in [0,T]$ we have*

$$(x \mapsto X_t^x) \in L^2(\Omega, W^{1,2}(U)).$$

We remark that using analogue techniques as in [27] one could even establish that the strong solution gives rise to a flow of Sobolev diffeomorphisms. This, however, is beyond the scope of this paper.

Similarly as for the Malliavin derivative, we are able to give an explicit representation for the first variation process in the Sobolev sense that does not involve the derivative of the drift coefficient by employing local time integration.

Proposition 3.5. *Let $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ be as in (1.6). Then the first variation process (in the Sobolev sense) of the strong solution X_t^x, $t \in [0,T]$ to SDE (1.5) has the following explicit representation*

$$\frac{\partial}{\partial x} X_t^x = \exp\left\{ -\int_0^t \int_{\mathbb{R}} b(u,y) L^{X^x}(du,dy) \right\} \quad dt \otimes P - a.s. \tag{3.5}$$

As a direct consequence of Proposition 3.5 together with Proposition 3.2 we obtain the following relation between the Malliavin derivative and the first variation process, which is an extension of Proposition 2.4 to irregular drift coefficients and which is a key result in deriving the desired expression for the Delta.

Corollary 3.6. *Let X_t^x, $t \in [0,T]$, be the unique strong solution to (1.5). Then the following relationship between the spatial derivative and the Malliavin derivative of X_t^x holds:*

$$\frac{\partial}{\partial x} X_t^x = D_s X_t^x \frac{\partial}{\partial x} X_s^x \quad P-a.s. \tag{3.6}$$

for any $s,t \in [0,T]$, $s \leq t$.

Remark 3.7. *Note that by Lemma 2.10 the Malliavin derivative in* (3.4) *and the first variation process in* (3.5) *can be expressed in various alternative ways. Firstly, we observe that by formula (2.11) the local time integral of the regular part $\hat{b}$ in b can be separated and rewritten in the form*

$$-\int_s^t \int_{\mathbb{R}} b(u,y) L^{X^x}(du,dy) = -\int_s^t \int_{\mathbb{R}} \tilde{b}(u,y) L^{X^x}(du,dy) + \int_s^t \hat{b}'(u,X_u^x) du \ a.s. \tag{3.7}$$

If in addition $\tilde{b}(t,\cdot)$ is locally square integrable and continuous in t as a map from $[0,T]$ to $L^2_{loc}(\mathbb{R})$ or even time-homogeneous, then by Lemma 2.10 also the local time integral associated to the irregular part $\tilde{b}$ can be reformulated in terms of the generalised covariation process as in (2.12) *or in terms of the generalised Itô formula as in* (2.13)*, respectively. In particular, these reformulations appear to be useful for simulation purposes.*

We conclude this section by giving an extension of all the results seen so far to a class of SDE's with more general diffusion coefficients.

Theorem 3.8. *Consider the time-homogeneous SDE*

$$dX_t^x = b(X_t^x)dt + \sigma(X_t^x)dB_t, \quad X_0^x = x \in \mathbb{R}, \quad 0 \leq t \leq T, \tag{3.8}$$

where the coefficients $b : \mathbb{R} \longrightarrow \mathbb{R}$ and $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$ are Borel measurable. Require that there exists a twice continuously differentiable bijection $\Lambda : \mathbb{R} \longrightarrow \mathbb{R}$ with derivatives Λ' and Λ'' such that

$$\Lambda'(y)\sigma(y) = 1 \text{ for a.e. } y \in \mathbb{R},$$

as well as

$$\Lambda^{-1} \text{ is Lipschitz continuous.}$$

Suppose that the function $b_* : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$b_*(x) := \Lambda'\left(\Lambda^{-1}(x)\right) b(\Lambda^{-1}(x)) + \frac{1}{2}\Lambda''\left(\Lambda^{-1}(x)\right) \sigma(\Lambda^{-1}(x))^2$$

satisfies the conditions of Theorem 3.1. Then there exists a Malliavin differentiable strong solution $X_\cdot^x$ to (3.8) which is (locally) Sobolev differentiable in its initial condition.

Proof. The proof is obtained directly from Itô's formula. See [25]. □

4. Existence and derivative-free representations of the Delta

We now turn the attention to the study of option price sensitivities with respect to the initial value of an underlying asset with irregular drift coefficient. Notably, we will consider lookback options with path-dependent (discounted) pay-off in the form

$$\Phi(X_{T_1}^x, \ldots, X_{T_m}^x) \tag{4.1}$$

for time points $T_1, \ldots, T_m \in (0,T]$, some function $\Phi : \mathbb{R}^m \to \mathbb{R}$, and where the evolution of the underlying price process under the risk-neutral pricing measure is modelled by the strong solution X^x of SDE (1.5) with possibly irregular drift b as in (1.6). In particular, for $m = 1$ the pay-off (4.1) is associated to a European option with maturity T_1. Then the current option price is given by $E\left[\Phi(X_{T_1}^x, \ldots, X_{T_m}^x)\right]$ and the main objective of this section is to establish existence and a derivative-free, probabilistic representation of the Delta

$$\frac{\partial}{\partial x} E\left[\Phi(X_{T_1}^x, \ldots, X_{T_m}^x)\right].$$

After having analysed lookback options, we will also address the problem of computing Deltas of Asian options with (discounted) path-dependent pay-off in the form

$$\Phi\left(\int_{T_1}^{T_2} X_u^x\, du\right) \tag{4.2}$$

for $T_1, T_2 \in [0,T]$ and some function $\Phi : \mathbb{R} \to \mathbb{R}$.

We start with a preliminary result which shows that in case of a smooth pay-off function with compact support the Delta exists for a large class of path dependent options that includes both lookback as well as Asian options.

Lemma 4.1. *Let X_t^x, $t \in [0,T]$, be the strong solution to SDE (1.5) and $\{X_t^{n,x}\}_{n\geq 1}$ the corresponding approximating strong solutions of SDE (3.2). Let $\Phi \in C_0^\infty(\mathbb{R}^m)$ and consider the functions*

$$u_n(x) := E\left[\Phi\left(\int_0^T X_u^{n,x}\mu_1(du), \int_0^T X_u^{n,x}\mu_2(du), \dots, \int_0^T X_u^{n,x}\mu_m(du)\right)\right] \tag{4.3}$$

and

$$u(x) := E\left[\Phi\left(\int_0^T X_u^{x}\mu_1(du), \int_0^T X_u^{x}\mu_2(du), \dots, \int_0^T X_u^{x}\mu_m(du)\right)\right] \tag{4.4}$$

where $\mu_1, \dots, \mu_m$ are finite measures on $[0,T]$ independent of $x \in \mathbb{R}$. Consider also the function

$$\bar{u}(x) := E\left[\sum_{i=1}^m \partial_i\Phi\left(\int_0^T X_u^{x}\mu_1(du), \int_0^T X_u^{x}\mu_2(du), \dots, \int_0^T X_u^{x}\mu_m(du)\right)\int_0^T \frac{\partial}{\partial x}X_u^x\mu_i(du)\right] \tag{4.5}$$

where $\frac{\partial}{\partial x}X^x$ is the first variation process of X^x introduced in (3.5)*. Then*

$$u_n(x) \xrightarrow{n\to\infty} u(x) \quad \text{for all} \ \ x \in \mathbb{R},$$

and

$$u_n'(x) \xrightarrow{n\to\infty} \bar{u}(x)$$

uniformly on compact subsets $K \subset \mathbb{R}$, where u_n' denotes the derivative. As a result, we obtain that $u \in C^1(\mathbb{R})$ with $u' = \bar{u}$. In particular, we obtain the result for lookback options by taking $\mu_i = \delta_{t_i}$ the Dirac measure concentrated on t_i, $i = 1, \dots, m$, and for Asian options by taking $m = 1$ and $\mu_1 = du$.

Proof. First of all, observe that the expression in (4.5) is well-defined. This can be seen by using Cauchy-Schwarz inequality, the fact that $\Phi \in C_0^\infty(\mathbb{R}^m)$, and Corollary A.9.

It is readily checked that $u_n(x) \to u(x)$ for all $x \in \mathbb{R}$ since Φ is smooth by using the mean-value theorem and the fact that $X_t^{n,x} \to X_t^x$ in $L^2(\Omega)$ as $n \to \infty$ for every $t \in [0,T]$ (see Theorem A.6).

We introduce the following short-hand notation for the m-dimensional random vector associated to a process Y:

$$h(Y_{\cdot,T}) := \left(\int_0^T Y_u\mu_1(du)\ ,\ \int_0^T Y_u\mu_2(du)\ ,\ \dots\ ,\ \int_0^T Y_u\mu_m(du)\right).$$

For the smooth coefficients b_n we have $u_n \in C^1(\mathbb{R})$, $n \geq 1$, and since $\partial_i\Phi$ are bounded for all $i = 1, \dots, m$ and by dominated convergence we have

$$u_n'(x) = E\left[\sum_{i=1}^m \partial_i\Phi\left(h(X_{\cdot,T}^{n,x})\right)\int_0^T \frac{\partial}{\partial x}X_u^{n,x}\mu_i(du)\right].$$

Moreover, we can take integration with respect to $\mu_i(du)$, $i = 1, ...m$, outside the expectation. Thus

$$u_n'(x) = \sum_{i=1}^m \int_0^T E\left[\partial_i\Phi\left(h(X_{\cdot,T}^{n,x})\right)\frac{\partial}{\partial x}X_u^{n,x}\right]\mu_i(du).$$

Hence

$$
\begin{aligned}
|u_n'(x)-\bar{u}(x)| &= \sum_{i=1}^m \int_0^T E\left[\partial_i \Phi\left(h(X^{n,x}_{\cdot,T})\right)\frac{\partial}{\partial x}X_u^{n,x} - \partial_i\Phi\left(h(X^x_{\cdot,T})\right)\frac{\partial}{\partial x}X_u^x\right]\mu_i(du)\\
&=: \sum_{i=1}^m \int_0^T F_{n,i}(u,x)\mu_i(du)
\end{aligned}
$$

where $F_{n,i}(u,x)$ denotes the expectation in the integral. We will show that for any $i=1,\dots,m$ and compact subset $K\subset\mathbb{R}$,

$$
\sup_{(u,x)\in[0,T]\times K} |F_{n,i}(u,x)| \xrightarrow{n\to\infty} 0.
$$

Indeed, by plugging in expression (3.5) for the first variation process and Girsanov's theorem we get

$$
\begin{aligned}
|F_{n,i}(u,x)| \le & \Bigg| E\Big[\partial_i\Phi\left(h(B^x_{\cdot,T})\right)\exp\left\{-\int_0^u\int_{\mathbb{R}} b_n(v,y)L^{B^x}(dv,dy)\right\}\mathcal{E}\left(\int_0^T b_n(u,B_u^x)dB_u\right)\\
&\quad - \partial_i\Phi\left(h(B^x_{\cdot,T})\right)\exp\left\{-\int_0^u\int_{\mathbb{R}} b(v,y)L^{B^x}(dv,dy)\right\}\mathcal{E}\left(\int_0^T b(u,B_u^x)dB_u\right)\Big]\Bigg|\\
\le & \Bigg| E\Big[\partial_i\Phi\left(h(B^x_{\cdot,T})\right)\mathcal{E}\left(\int_0^T b(u,B_u^x)dB_u\right)\\
&\quad \times\left(\exp\left\{-\int_0^u\int_{\mathbb{R}} b_n(v,y)L^{B^x}(dv,dy)\right\} - \exp\left\{-\int_0^u\int_{\mathbb{R}} b(v,y)L^{B^x}(dv,dy)\right\}\right)\Big]\Bigg|\\
&+ \Bigg| E\Big[\partial_i\Phi\left(h(B^x_{\cdot,T})\right)\exp\left\{-\int_0^u\int_{\mathbb{R}} b_n(v,y)L^{B^x}(dv,dy)\right\}\\
&\quad \times\left(\mathcal{E}\left(\int_0^T b_n(u,B_u^x)dB_u\right) - \mathcal{E}\left(\int_0^T b(u,B_u^x)dB_u\right)\right)\Big]\Bigg|\\
:= & I_n + II_n
\end{aligned}
$$

Here, we will show estimates for II_n, for I_n the argument is analogous. Similarly to how we obtain the estimate $II_n^1+II_n^2$ in the proof of Lemma A.5, using inequality $|e^x-1|\le|x|(e^x+1)$ we get

$$
\begin{aligned}
II_n \lesssim & E\Big[|\partial_i\Phi\left(h(B^x_{\cdot,T})\right)||U_n|\exp\left\{-\int_0^u\int_{\mathbb{R}} b_n(v,y)L^{B^x}(dv,dy)\right\}\mathcal{E}\left(\int_0^T b_n(u,B_u^x)dB_u\right)\Big]\\
&+ E\Big[|\partial_i\Phi\left(h(B^x_{\cdot,T})\right)||U_n|\exp\left\{-\int_0^u\int_{\mathbb{R}} b_n(v,y)L^{B^x}(dv,dy)\right\}\mathcal{E}\left(\int_0^T b(u,B_u^x)dB_u\right)\Big]\\
=: & II_n^1 + II_n^2,
\end{aligned}
$$

where

$$
U_n := \int_0^T (\tilde{b}_n(u,B_u^x)-\tilde{b}(u,B_u^x))dB_u - \frac{1}{2}\int_0^T (b_n^2(u,B_u^x)-b^2(u,B_u^x))du.
$$

We will now show that $II_n^1\to 0$ as $n\to\infty$ uniformly in x on a compact subset $K\subset\mathbb{R}$. The convergence of II_n^2 then follows immediately, too. Denote $p=\frac{1+\varepsilon}{\varepsilon}$ with $\varepsilon>0$ from Lemma A.1 and use Hölder's inequality with exponent $1+\varepsilon$ on the Doléans-Dade exponential, then employ formula (2.11) on $\hat{b}$ in $b_n = \tilde{b}_n + \hat{b}$ and use Cauchy-Schwarz inequality successively. As a result,

$$II_n^1 \lesssim E\left[\mathcal{E}\left(\int_0^T b_n(u, B_u^x)dB_u\right)^{1+\varepsilon}\right]^{1/(1+\varepsilon)} E\left[\left|\partial_i\Phi\left(h(B^x_{\cdot,T^*})\right)\right|\right]^{1/(2p)} E[|U_n|^{8p}]^{1/(8p)}$$
$$\times E\left[\exp\left\{-4p\int_0^u\int_{\mathbb{R}} \tilde{b}_n(v,y)L^{B^x}(dv,dy)\right]^{1/(4p)}\right\} E\left[\exp\left\{8p\int_0^u \hat{b}'(v,B_v^x)dv\right\}\right]^{1/(8p)}.$$

The first and fourth factor are bounded uniformly in $n \geq 0$ and $x \in K$ by Remark A.2 and Lemma A.3, respectively. The second and and fifth factor can be controlled since $\partial_i\Phi$, $i = 1, \ldots, m$ and $\hat{b}'$ are uniformly bounded. It remains to show that

$$\sup_{x\in K} E[|U_n|^{8p}] \xrightarrow{n\to\infty} 0$$

for any compact subset $K \subset \mathbb{R}$.

Using Minkowski's inequality, Burkholder-Davis-Gundy's inequality and Hölder's inequality we can write

$$E[|U_n|^{8p}] \lesssim \int_0^T E[|\tilde{b}_n(u,B_u^x) - \tilde{b}(u,B_u^x)|^{8p}]du + \int_0^T E[|b_n^2(u,B_u^x) - b^2(u,B_u^x)|^{8p}]du. \tag{4.6}$$

Then write the integrand of the first term in (4.6) as

$$E[|\tilde{b}_n(u,B_u^x) - \tilde{b}(u,B_u^x)|^{8p}] = \frac{1}{\sqrt{2\pi u}}\int_{\mathbb{R}} |\tilde{b}_n(u,y) - \tilde{b}(u,y)|^{8p} e^{-\frac{(y-x)^2}{2u}}dy.$$

Using Cauchy-Schwarz inequality on $|\tilde{b}_n(u,y) - \tilde{b}(u,y)|^{8p}e^{\frac{-y^2}{4u}}$ we obtain

$$E[|\tilde{b}_n(u,B_u^x) - \tilde{b}(u,B_u^x)|^{8p}] \leq$$
$$\leq \frac{1}{\sqrt{2\pi u}}e^{-\frac{x^2}{2u}}\left(\int_{\mathbb{R}} |\tilde{b}_n(u,y) - \tilde{b}(u,y)|^{16p}e^{-\frac{y^2}{2u}}dy\right)^{1/2}\left(\int_{\mathbb{R}} e^{-\frac{y^2}{2u}+2\frac{xy}{u}}dy\right)^{1/2}.$$

Then for each $u \in [0,T]$, by taking the supremum over $x \in K$ and by dominated convergence, we get

$$\sup_{x\in K} E[|\tilde{b}_n(u,B_u^x) - \tilde{b}(u,B_u^x)|^{8p}] \xrightarrow{n\to\infty} 0\,,$$

and hence the result follows. Similarly, one can argue for the second term in (4.6).

In sum,

$$\sup_{(u,x)\in[0,t]\times K} |F_{n,i}(u,x)| \xrightarrow{n\to\infty} 0$$

for all $i = 1, \ldots, m$ and hence $u_n'(x) \xrightarrow{n\to\infty} \bar{u}(x)$ uniformly on compact sets $K \subset \mathbb{R}$, and thus $u \in C^1(\mathbb{R})$ with $u' = \bar{u}$. □

We come to the main result of this paper, which extends Theorem 1.1 to lookback options written on underlyings with irregular drift coefficients. In particular, when plugging in expression (3.5) for the first variation process, we see that the formula for the Delta in (4.8) below involves neither the derivative of the pay-off function Φ nor the derivative of the drift coefficient b. We obtain this result for pay-off functions $\Phi \in L_w^q(\mathbb{R}^m)$, where

$$L_w^q(\mathbb{R}^m) := \left\{f : \mathbb{R}^m \to \mathbb{R} \text{ measurable} : \int_{\mathbb{R}^m} |f(x)|^q\, w(x)dx < \infty\right\} \tag{4.7}$$

for the weight function w defined by $w(x) := \exp(-\frac{1}{2T}|x|^2)$, $x \in \mathbb{R}^m$, and where the exponent q depends on the drift b. Note that all pay-off functions of practical relevance are contained in these spaces.

Theorem 4.2. *Let X^x be the strong solution to SDE (1.5) and $\Phi : \mathbb{R}^m \to \mathbb{R}$ a function in $L_w^{4p}(\mathbb{R}^m)$, where $p > 1$ is the conjugate of $1+\varepsilon$ for $\varepsilon > 0$ in Lemma A.1. Then, for any $0 < T_1 \leq \cdots \leq T_m \leq T$, the price*

$$u(x) := E\left[\Phi(X_{T_1}^x, \ldots, X_{T_m}^x)\right]$$

of the associated lookback option is continuously differentiable in $x \in \mathbb{R}$, and its derivative, i.e. the Delta, takes the form

$$u'(x) = E\left[\Phi(X^x_{T_1}, \dots, X^x_{T_m}) \int_0^T a(s) \frac{\partial}{\partial x} X^x_s \, dB_s\right] \tag{4.8}$$

for any bounded measurable function $a : \mathbb{R} \to \mathbb{R}$ such that, for every $i = 1, \dots, m$,

$$\int_0^{T_i} a(s) ds = 1. \tag{4.9}$$

Proof. Assume first $\Phi \in C_0^\infty(\mathbb{R}^m)$. Then by Lemma 4.1 with $\mu_i = \delta_{t_i}$, $i = 1, \dots, m$, we know that $u(x) = E\left[\Phi(X^x_{T_1}, \dots, X^x_{T_m})\right]$ is continuously differentiable with derivative

$$u'(x) := \sum_{i=1}^m E\left[\partial_i \Phi(X^x_{T_1}, \dots, X^x_{T_m}) \frac{\partial}{\partial x} X^x_{T_i}\right].$$

Now, by Corollary 3.6, we have for any $i = 1, \dots, m$

$$\frac{\partial}{\partial x} X^x_{T_i} = D_s X^x_{T_i} \frac{\partial}{\partial x} X^x_s \text{ for all } s \le T_i\,. \tag{4.10}$$

Also recall that $D_s X^x_{T_i} = 0$ for $s \ge T_i$. So, for any function $a : \mathbb{R} \to \mathbb{R}$ satisfying (4.9) we have

$$\frac{\partial}{\partial x} X^x_{T_i} = \int_0^T a(s) D_s X^x_{T_i} \frac{\partial}{\partial x} X^x_s ds.$$

As a result,

$$\begin{aligned} u'(x) &= \sum_{i=1}^m E\left[\partial_i \Phi(X^x_{T_1}, \dots, X^x_{T_m}) \int_0^T a(s) D_s X^x_{T_i} \frac{\partial}{\partial x} X^x_s ds\right] \\ &= E\left[\int_0^T a(s) D_s \Phi(X^x_{T_1}, \dots, X^x_{T_m}) \frac{\partial}{\partial x} X^x_s ds\right], \end{aligned}$$

where in the last step we could use the chain rule for the Malliavin derivative backwards, see Lemma 2.1, since $\Phi(X^x_{T_1}, \dots, X^x_{T_m})$ is Malliavin differentiable due to Theorem 3.1. Then $a(s)\frac{\partial}{\partial x} X^x_s$ is an $\mathcal{F}_s$-adapted Skorokhod integrable process by Corollary A.9 with $p = 2$, so the duality formula for the Malliavin derivative (see Theorem 2.2) yields

$$u'(x) = E\left[\Phi(X^x_{T_1}, \dots, X^x_{T_m}) \int_0^T a(s) \frac{\partial}{\partial x} X^x_s dB_s\right].$$

Finally, we extend the result to a pay-off function $\Phi \in L^{4p}_w(\mathbb{R}^m)$. By standard arguments we can approximate Φ by a sequence of functions $\Phi_n \in C_0^\infty(\mathbb{R}^m)$, $n \ge 0$, such that $\Phi_n \to \Phi$ in $L^{4p}_w(\mathbb{R}^m)$ as $n \to \infty$. Now define $u_n(x) := E[\Phi_n(X^x_{T_1}, \dots, X^x_{T_m})]$ and $\bar{u}(x) := E[\Phi(X^x_{T_1}, \dots, X^x_{T_m}) \int_0^T a(s) \frac{\partial}{\partial x} X^x_s dB_s]$. Then

$$\begin{aligned} |u_n'(x) - \bar{u}(x)| &= \left| E\left[\left(\Phi_n(X^x_{T_1}, \dots, X^x_{T_m}) - \Phi(X^x_{T_1}, \dots, X^x_{T_m})\right) \int_0^T a(s) \frac{\partial}{\partial x} X^x_s dB_s\right]\right| \\ &\le E\left[\left|\Phi_n(X^x_{T_1}, \dots, X^x_{T_m}) - \Phi(X^x_{T_1}, \dots, X^x_{T_m})\right|^2\right]^{1/2} E\left[\int_0^T |a(s) \frac{\partial}{\partial x} X^x_s|^2 ds\right]^{1/2} \\ &\le CE\left[\left|\Phi_n(B^x_{T_1}, \dots, B^x_{T_m}) - \Phi(B^x_{T_1}, \dots, B^x_{T_m})\right|^2 \mathcal{E}\left(\int_0^T b(u, B^x_u) dB_u\right)\right]^{1/2}, \end{aligned}$$

where we have used Cauchy-Schwarz inequality, Itô's isometry, Corollary A.9 and Girsanov's theorem in this order. Then we apply Hölder's inequality with $1 + \varepsilon$ for a small enough $\varepsilon > 0$ and use Lemma A.1 to get

$|u_n'(x) - \bar{u}(x)| \leq$

$$\leq CE\left[\left|\Phi_n(B^x_{T_1},\dots,B^x_{T_m}) - \Phi(B^x_{T_1},\dots,B^x_{T_m})\right|^{2\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{2(1+\varepsilon)}} E\left[\mathcal{E}\left(\int_0^T b(u,B^x_u)dB_u\right)^{1+\varepsilon}\right]^{\frac{1}{2(1+\varepsilon)}}$$

$$\leq CE\left[\left|\Phi_n(B^x_{T_1},\dots,B^x_{T_m}) - \Phi(B^x_{T_1},\dots,B^x_{T_m})\right|^{2\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{2(1+\varepsilon)}}.$$

For the last quantity, denote by $P_t(y) := \frac{1}{\sqrt{2\pi t}}e^{-y^2/(2t)}$, $y \in \mathbb{R}$ the density of B_t, and set $T_0 := 0$ and $y_0 := x$. Recall that $0 < T_1 \leq \cdots \leq T_m$. Using the independent increments of the Brownian motion we rewrite

$$E\left[\left|\Phi_n(B^x_{T_1},\dots,B^x_{T_m}) - \Phi(B^x_{T_1},\dots,B^x_{T_m})\right|^{2\frac{1+\varepsilon}{\varepsilon}}\right]$$
$$= \int_{\mathbb{R}^m} |\Phi_n(y_1,\dots,y_m) - \Phi(y_1,\dots,y_m)|^{2\frac{1+\varepsilon}{\varepsilon}} \prod_{i=1}^m P_{T_i - T_{i-1}}(y_i - y_{i-1})dy_1\cdots dy_m.$$

Furthermore, with $t^* := \min_{i=1,\dots,m-1}(t_{i+1}-t_i)$

$$E\left[\left|\Phi_n(B^x_{T_1},\dots,B^x_{T_m}) - \Phi(B^x_{T_1},\dots,B^x_{T_m})\right|^{2\frac{1+\varepsilon}{\varepsilon}}\right]$$
$$\leq (2\pi t^*)^{-m/2}\int_{\mathbb{R}^m} |\Phi_n(y_1,\dots,y_m) - \Phi(y_1,\dots,y_m)|^{2\frac{1+\varepsilon}{\varepsilon}} \prod_{i=1}^m e^{-\frac{y_i^2}{4(T_i-T_{i-1})}}$$
$$\times e^{-\frac{y_i^2}{4(T_i-T_{i-1})}+\frac{y_iy_{i-1}}{T_i-T_{i-1}}-\frac{y_{i-1}^2}{2(T_i-T_{i-1})}}dy_1\cdots dy_m.$$

By applying Cauchy-Schwarz inequality we obtain

$$E\left[\left|\Phi_n(B^x_{T_1},\dots,B^x_{T_m}) - \Phi(B^x_{T_1},\dots,B^x_{T_m})\right|^{2\frac{1+\varepsilon}{\varepsilon}}\right]$$
$$\leq (2\pi t^*)^{-m/2}\left(\int_{\mathbb{R}^m} |\Phi_n(y_1,\dots,y_m) - \Phi(y_1,\dots,y_m)|^{4\frac{1+\varepsilon}{\varepsilon}} e^{-\frac{|y|^2}{2T}}dy_1\cdots dy_m\right)^{1/2}$$
$$\times\left(\int_{\mathbb{R}^m}\prod_{i=1}^m e^{-\frac{y_i^2}{2(T_i-T_{i-1})}+\frac{2y_iy_{i-1}}{T_i-T_{i-1}}-\frac{y_{i-1}^2}{(T_i-T_{i-1})}}dy_1\cdots dy_m\right)^{1/2}$$
$$=: I_n \cdot II.$$

For the second factor we have

$$II \leq e^{-\frac{x^2}{T}}\left(\int_{\mathbb{R}^m} e^{-\frac{y_1}{2T}+\frac{xy_1}{T}}\prod_{i=2}^m e^{-\frac{(y_i-y_{i-1})^2}{2T}}dy_1\cdots dy_m\right)^{1/2}$$

and hence

$$\sup_{x\in K} II < \infty.$$

Thus, since factor I_n converges to 0 by assumption, we can approximate $\bar{u}$ uniformly in $x \in \mathbb{R}$ on compact sets by smooth pay-off functions. So $u \in C^1(\mathbb{R})$ and $u' = \bar{u}$. □

Next, we consider Asian options with pay-off given by (4.2). If $T_1 > 0$ we are able to give the analogous result to Theorem 4.2 by approximating the Asian pay-off with lookback pay-offs:

Corollary 4.3. *Let X^x be the strong solution to SDE* (1.5) *and $\Phi : \mathbb{R} \to \mathbb{R}$ a function in $L^{4p}_{\widetilde{w}}(\mathbb{R})$ where $\widetilde{w}$ is defined in* (4.15) *further below and where $p > 1$ is the conjugate of $1+\varepsilon$ for $\varepsilon > 0$ in*

Lemma A.1. Then for any T_1, $T_2 \in (0,T]$ with $T_1 < T_2$, the price

$$u(x) = E\left[\Phi\left(\int_{T_1}^{T_2} X_u^x du\right)\right]$$

of the associated Asian option is continuously differentiable in $x \in \mathbb{R}$, and its derivative, i.e. the Delta, takes the form

$$u'(x) = E\left[\Phi\left(\int_{T_1}^{T_2} X_s^x ds\right)\int_0^{T_1} a(s)\frac{\partial}{\partial x}X_s^x dB_s\right] \tag{4.11}$$

for any bounded measurable function $a : \mathbb{R} \to \mathbb{R}$ such that

$$\int_0^{T_1} a(s)ds = 1. \tag{4.12}$$

Proof. Assume first that $\Phi \in C^1(\mathbb{R})$, and consider a series of partitions of $[T_1, T_2]$ with vanishing mesh, i.e. let $\{T_1 = t_0^m < t_1^m < \ldots < t_m^m = T_2\}_{m=1}^\infty$ with $\lim_{m\to\infty}\sup_{i=1,\ldots,m}(t_i^m - t_{i-1}^m) = 0$. Then we may write the integral using Riemann sums as follows

$$\int_{T_1}^{T_2} X_s^x ds = \lim_{m\to\infty}\sum_{i=1,\ldots,m} X_{t_i^m}^x (t_i^m - t_{i-1}^m).$$

Then

$$\Phi\left(\int_{T_1}^{T_2} X_s^x ds\right) = \lim_{m\to\infty}\Phi\left(\sum_{i=1,\ldots,m} X_{t_i^m}^x (t_i^m - t_{i-1}^m)\right) =: \lim_{m\to\infty}\hat{\Phi}_m(X_{t_1^m}^x, \ldots, X_{t_m^m}^x).$$

By Theorem 4.2 we have

$$u'(x) = \lim_{m\to\infty} E\left[\hat{\Phi}_m(X_{t_1^m}^x, \ldots, X_{t_m^m}^x)\int_0^T a_m(s)\frac{\partial}{\partial x}X_s^x dB_s\right]$$

where a_m is a bounded measurable function such that $\int_0^{t_i^m} a_m(s)ds = 1$ for each $i = 1, \ldots, m$. Then

$$\begin{aligned} u'(x) &= \lim_{m\to\infty} E\left[\hat{\Phi}_m(X_{t_1^m}^x, \ldots, X_{t_m^m}^x)\int_0^T a_m(s)\frac{\partial}{\partial x}X_s^x dB_s\right] \\ &= E\left[\Phi\left(\int_{T_1}^{T_2} X_s^x ds\right)\int_0^{T_1} a(s)\frac{\partial}{\partial x}X_s^x dB_s\right], \end{aligned}$$

where a is a function such that $\int_0^{T_1} a(s)ds = 1$.

For a general pay-off Φ, we approximate Φ in $L_w^{4p}(\mathbb{R})$ by a sequence of functions $\{\Phi_n\}_{n\geq 0} \subset C_0^1(\mathbb{R})$ and define $u(x) := E[\Phi(\int_{T_1}^{T_2} X_s^x ds)]$ and $\bar{u}(x) := E\left[\Phi\left(\int_{T_1}^{T_2} X_s^x ds\right)\int_0^{T_1} a(s)\frac{\partial}{\partial x}X_s^x dB_s\right]$. Consider $u_n(x) = E[\Phi_n(\int_{T_1}^{T_2} X_s^x ds)]$. Finally, similarly as in Theorem 4.2 one has $u_n(x) \to u(x)$ as $n \to \infty$ for all $x \in \mathbb{R}$ and

$$|u_n'(x) - \bar{u}(x)| \lesssim E\left[\left|\Phi_n\left(\int_{T_1}^{T_2} B_s^x ds\right) - \Phi\left(\int_{T_1}^{T_2} B_s^x ds\right)\right|^{2p}\right]^{1/p},$$

which goes to zero uniformly in $x \in K$ on compact sets $K \subset \mathbb{R}$ as $n \to \infty$ by using the fact that $\int_{T_1}^{T_2} B_s^x ds$ has a Gaussian distribution with mean $x(T_2 - T_1)$ and variance $\frac{T_2^3 - T_1^3}{3} - (T_2 - T_1)T_1^2$ which explains the weight $\widetilde{w}$. □

Remark 4.4. *From the proof of Corollary 4.3 it follows that the Delta* (4.11) *of an Asian option can be approximated by the Delta*

$$E\left[\Phi\left(\sum_{i=1}^m X_{t_i}^x (t_i - t_{i-1})\right)\int_0^{T_2} a(s)\frac{\partial}{\partial x}X_s^x dB_s\right] \tag{4.13}$$

of a lookback option for a fine enough partition $T_1 = t_0 < t_1 < \cdots < t_m = T_2$*, where* $\int_0^{t_i} a(s)ds = 1$ *for each* $i = 1, \ldots, m$*.. From a numerical point of view, this might make a difference since the function* a *in* (4.13) *can be chosen to have support on the full segment* $[0, T_2]$*, while in* (4.11) *the function* a *can only have support on* $[0, T_1]$*.*

If the averaging period of the Asian option starts today, i.e. $T_1 = 0$, then formula (4.11) does not hold anymore. Instead, one can derive alternative closed-form expressions for the Asian delta for smooth diffusion coefficients, see e.g. [15] and [3], which potentially can be generalised to irregular drift coefficients. However, except for linear coefficients (Black & Scholes model), these expressions involve stochastic integrals in the Skorokhod sense which are, in general, hard to simulate. Instead, we here propose to enlarge the state space by one dimension and to consider a perturbed Asian pay-off. In that case we are able to derive a probabilistic representation for the corresponding Delta that only includes Itô integrals. More precisely, we consider the (strong) solution to the perturbed two-dimensional SDE

$$dX_t^x = b(t, X_t^x)dt + dB_t, \; X_0^x = x \in \mathbb{R},$$

$$dY_t^{\epsilon,x,y} = X_t^x dt + \epsilon dW_t \; , Y_0^{\epsilon,x,y} = y \in \mathbb{R}, \; 0 \le t \le T, \tag{4.14}$$

for $\epsilon > 0$, where W is a one-dimensional Brownian motion independent of B. The idea is now to consider the perturbed Asian pay-off with averaging period $[0, T_2]$, $T_2 \in (0, T]$ as a European pay-off on $Y_{T_2}^{\epsilon,x,y}$:

$$\Phi\left(\int_0^{T_2} X_s^x ds\right) \sim \Phi(Y_{T_2}^{\epsilon,x,0}) = \Phi\left(\int_0^{T_2} X_s^x ds + \epsilon W_{T_2}\right).$$

We then get the following result, where we now consider the slightly differently weighted pay-off function space

$$L^q_{\tilde{w}}(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} \text{ measurable: } \int_{\mathbb{R}} |f(x)|^q \tilde{w}(x) dx < \infty \right\}$$

for the weight function $\tilde{w}$ defined by

$$\tilde{w}(x) = \exp\left(-\frac{|x|^2}{2T_2\,(T_2^2/3 + 1)}\right), \quad x \in \mathbb{R}. \tag{4.15}$$

Theorem 4.5. *Let* $Y_\cdot^{\epsilon,x,y}$ *be the second component of the strong solution to* (4.14) *and* $\Phi \in L^{4p}_{\tilde{w}}(\mathbb{R})$*, where* $p > 1$ *is the conjugate of* $1+\varepsilon$ *for* $\varepsilon > 0$ *in Lemma A.1. For a given maturity time* $T_2 \in (0, T]$ *and* $0 < \epsilon \le 1$*, the price*

$$u_\epsilon(x) := E[\Phi(Y_{T_2}^{\epsilon,x,0})]$$

of the associated perturbed Asian option is continuously differentiable in $x \in \mathbb{R}$*, and its derivative, i.e. the Delta, takes the form*

$$u'_\epsilon(x) = E\left[\Phi(Y_{T_2}^{\epsilon,x,0})\left(\int_0^T a(s)\frac{\partial}{\partial x}X_s^x dB_s + \epsilon^{-1}\int_0^T a(s)\int_0^s \frac{\partial}{\partial x}X_u^x du \; dW_s\right)\right], \tag{4.16}$$

where $a : [0, T] \longrightarrow \mathbb{R}$ *is a bounded measurable function such that* $\int_0^T a(s)ds = 1$*.*

Proof. The proof is a straight forward generalization of the proof of Theorem 4.2 to the (particularly simple) two-dimensional extension (4.14) of the underlying SDE. Therefore, we here only give the main steps.

First observe that the strong solution $(X_t^x, Y_t^{\epsilon,x,y})$ is clearly differentiable in y, and by Theorem 3.4 also (weakly) differentiable in x, and we get

$$D_{x,y}\begin{pmatrix} X_t^x \\ Y_t^{\epsilon,x,y} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x}X_t^x & 0 \\ \int_0^t \frac{\partial}{\partial x}X_u^x du & 1 \end{pmatrix},$$

for all $t \in [0, T]$, where $D_{x,y}$ denotes the derivative.

Assume first $\Phi \in C_0^\infty(\mathbb{R})$. Then it follows from Lemma 4.1 that $E[\Phi(Y_{T_2}^{\epsilon,x,y})]$ is continuously differentiable in (x, y) with

$$D_{x,y}E[\Phi(Y_{T_2}^{\epsilon,x,y})] = E\left[\begin{pmatrix} 0 \\ \Phi'(Y_{T_2}^{\epsilon,x,y}) \end{pmatrix}^* D_{x,y}\begin{pmatrix} X_{T_2}^x \\ Y_{T_2}^{\epsilon,x,y} \end{pmatrix}\right],$$

where $*$ indicates the transposition of a matrix.

On the other hand, if we denote by D the Malliavin derivative in the direction of (B, W), it follows by means of the estimate in (A.12) and Corollary 3.6 that $Y_{T_2}^{\epsilon,x,y}$ is Malliavin differentiable and that for $0 \le s \le T$

$$D_s\begin{pmatrix} X_{T_2}^x \\ Y_{T_2}^{\epsilon,x,y} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}^{-1} D_{x,y}\begin{pmatrix} X_s^x \\ Y_s^{\epsilon,x,y} \end{pmatrix} = D_{x,y}\begin{pmatrix} X_{T_2}^x \\ Y_{T_2}^{\epsilon,x,y} \end{pmatrix} \tag{4.17}$$

$dx \otimes ds \otimes P-$a.e. Then, using (4.17), the chain rule from Lemma 2.1 and the duality relation for the Malliavin derivative, we see that

$$\begin{aligned} D_{x,y}E[\Phi(Y_{T_2}^{\epsilon,x,y})] &= E\left[\begin{pmatrix} 0 \\ \Phi'(Y_{T_2}^{\epsilon,x,y}) \end{pmatrix}^* \int_0^T a(s) D_s\begin{pmatrix} X_{T_2}^x \\ Y_{T_2}^{\epsilon,x,y} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}^{-1} D_{x,y}\begin{pmatrix} X_s^x \\ Y_s^{\epsilon,x,y} \end{pmatrix} ds\right] \\ &= E\left[\Phi(Y_{T_2}^{\epsilon,x,y}) \int_0^T a(s)\left(\begin{pmatrix} 1 & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} D_{x,y}\begin{pmatrix} X_s^x \\ Y_s^{\epsilon,x,y} \end{pmatrix}\right)^* d\begin{pmatrix} B_s \\ W_s \end{pmatrix}\right]. \end{aligned}$$

Thus

$$\frac{\partial}{\partial x}E[\Phi(Y_{T_2}^{\epsilon,x,y})] = E\left[\Phi(Y_{T_2}^{\epsilon,x,y})\left(\int_0^T a(s)\frac{\partial}{\partial x}X_s^x dB_s + \epsilon^{-1}\int_0^T a(s)\int_0^s \frac{\partial}{\partial x}X_u^x du\, dW_s\right)\right]$$

for all $x, y, \epsilon > 0$.

For general $\Phi \in L_{\widetilde{w}}^{4p}(\mathbb{R})$ one pursues an approximation argument analogously to the one in the proof of Theorem 4.2, where we now use the Gaussian distribution of $\int_0^{T_2} B_s^x ds + \epsilon W_{T_2}$ with mean xT_2 and variance $T_2^3/3 + \epsilon^2 T_2$, which explains the weight (4.15) for $0 < \epsilon \le 1$. □

Finally, we address the question whether both (4.11) for $T_1 \to 0$ as well as (4.16) for $\epsilon \to 0$ are indeed approximations for the Delta of the Asian option with averaging period starting in 0. We here give an affirmative answer for a class of pay-off functions Φ in spaces of the type

$$W_{\widetilde{w}}^{1,q}(\mathbb{R}) := \left\{ f \in W_{loc}^{1,q}(\mathbb{R});\ \int_{\mathbb{R}} |f(x)|^q \widetilde{w}(x)dx + \int_{\mathbb{R}} |f'(x)|^q \widetilde{w}(x)dx < \infty \right\}$$

for some $q > 1$, where f' denotes the weak derivative of f and the weight function $\widetilde{w}$ is defined in (4.15). See [**?**] for more information on weighted Sobolev spaces.

Theorem 4.6. *Let X^x be the strong solution to SDE* (1.5) *and $\Phi \in W_{\widetilde{w}}^{1,4p}(\mathbb{R})$, where $p > 1$ is the conjugate of $1+\varepsilon$ for $\varepsilon > 0$ in Lemma A.1. Further, require that the points of discontinuity of the distributional derivative Φ' are contained in a Lebesgue null set and that the following conditions are satisfied*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \sup_{\epsilon>0} |\Phi(y) - \Phi(y - \epsilon z)|^{2p}\, \widetilde{w}(y) P_T(z) dydz < \infty \tag{4.18}$$

and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \sup_{\epsilon>0} |\Phi'(y) - \Phi'(y - \epsilon z)|^{4p}\, \widetilde{w}(y) P_T(z) dydz < \infty, \tag{4.19}$$

where $P_t(z) = \frac{1}{\sqrt{2\pi t}}\exp(-\frac{1}{2t}z^2)$, $t > 0$, $z \in \mathbb{R}$ is the Gaussian kernel. Then

$$u(x) := E\left[\Phi\left(\int_0^{T_2} X_s^x ds\right)\right]$$

is continuously differentiable in $x \in \mathbb{R}$, and

$$u'(x) = \lim_{\epsilon \to 0} E\left[\Phi\left(Y_{T_2}^{\epsilon,x,0}\right)\left(\int_0^T a(s)\frac{\partial}{\partial x}X_s^x dB_s + \epsilon^{-1}\int_0^T a(s)\int_0^s \frac{\partial}{\partial x}X_u^x du\ dW_s\right)\right], \tag{4.20}$$

as well as

$$u'(x) = \lim_{T_1 \to 0} E\left[\Phi\left(\int_{T_1}^{T_2} X_u^x du\right)\int_0^{T_1} a(u)\frac{\partial}{\partial x}X_u^x dB_u\right]. \tag{4.21}$$

Proof. By Theorem 4.5 we have that $u_\epsilon \in C^1(\mathbb{R})$ for all $\epsilon > 0$. Hence,

$$\frac{\partial}{\partial x}E[\Phi(Y_{T_2}^{\epsilon,x,0})] = E\left[\Phi^{'}(Y_{T_2}^{\epsilon,x,0})\frac{\partial}{\partial x}Y_{T_2}^{\epsilon,x,0}\right]$$

for all $\epsilon > 0$, dx-a.e. Let $J \subset \mathbb{R}$ be a compact set. Then, using the same line of reasoning just as in the proof of Theorem 4.2, using Cauchy-Schwarz inequality, Girsanov's theorem, and Lemma A.1 we find the estimates

$$\sup_{x\in J}\left|\frac{\partial}{\partial x}E[\Phi(Y_{T_2}^{\epsilon,x,0})] - \frac{\partial}{\partial x}E[\Phi(Y_{T_2}^{0,x,0})]\right| \le C\left(E\left[\int_{\mathbb{R}}\left|\Phi^{'}(y) - \Phi^{'}(y-\epsilon W_{T_2})\right|^{4p}\tilde{w}(y)dy\right]\right)^{1/(4p)}$$

and

$$\sup_{x\in J}\left|E[\Phi(Y_{T_2}^{\epsilon,x,0})] - E[\Phi(Y_{T_2}^{0,x,0})]\right| \le K\left(E\left[\int_{\mathbb{R}}|\Phi(y) - \Phi(y-\epsilon W_{T_2})|^{2p}\tilde{w}(y)dy\right]\right)^{1/(2p)}$$

for constants C, K depending only on T_2, J, p (and not on ϵ).

Finally, using dominated convergence in connection with (4.18) and (4.19), the proof follows.

To prove (4.21) define $u_{T_1}(x) := E[\Phi(\int_{T_1}^{T_2} X_u^x du)]$. Since $\Phi \in L^{4p}_{\tilde{w}}(\mathbb{R})$, we have by Corollary 4.3 that $u_{T_1} \in C^1(\mathbb{R})$ for every $T_1 > 0$. Moreover, since $\Phi \in W^{1,4p}_{\tilde{w}}(\mathbb{R})$ we have

$$u_{T_1}'(x) = E\left[\Phi'\left(\int_{T_1}^{T_2} X_u^x du\right)\int_{T_1}^{T_2}\frac{\partial}{\partial x}X_u^x du\right].$$

Consequently, for every compact $J \subset \mathbb{R}$ we have

$$\begin{aligned}
&\sup_{x\in J}\left|\frac{\partial}{\partial x}E\left[\Phi\left(\int_{T_1}^{T_2} X_u^x du\right)\right] - \frac{\partial}{\partial x}E\left[\Phi\left(\int_0^{T_2} X_u^x du\right)\right]\right| \\
&\le \sup_{x\in J}\left|E\left[\left(\Phi'\left(\int_{T_1}^{T_2} X_u^x du\right) - \Phi'\left(\int_0^{T_2} X_u^x du\right)\right)\int_{T_1}^{T_2}\frac{\partial}{\partial x}X_u^x du\right]\right| \\
&\quad + \sup_{x\in J}\left|E\left[\Phi'\left(\int_0^{T_2} X_u^x du\right)\int_0^{T_1}\frac{\partial}{\partial x}X_u^x du\right]\right| \\
&=: A_1 + A_2
\end{aligned}$$

where A_1 and A_2 denote the respective summands. It is clear that A_2 goes to 0 uniformly in x on J as $T_1 \to 0$. To show the corresponding convergence for A_1, similar computations as in the beginning of the proof, using Cauchy-Schwarz inequality, Girsanov's theorem, Lemma A.1, and that $\Phi' \in L^{4p}_{\tilde{w}}(\mathbb{R})$, give for some constant $C_\varepsilon > 0$

$$A_1 = C_\varepsilon \sup_{x\in J} E\left[\left|\Phi'\left(\int_0^{T_1} B_u^x du\right)\right|^{4p}\right]^{1/(4p)} \le C_\varepsilon \|\Phi'\|_{L^{4p}_{\tilde{w}}(\mathbb{R})}\int_{\mathbb{R}} e^{-\frac{z^2}{2T_1(T_1^2/3+1)}}dz \xrightarrow{T_1\to 0} 0.$$

Hence (4.21) follows. □

Example 4.7. *We conclude this section by verifying the conditions in Theorem 4.6 for a pay-off function that is used in the next section. Consider the function $\Phi : \mathbb{R} \longrightarrow [0,\infty)$ given by*

$$\Phi(y) = \exp(-y)(C\exp(y) - K)_+,$$

where $C,\ K > 0$ are constants and $(x)_+ := \max(x,0)$ for $x \in \mathbb{R}$. We immediately see that $\Phi \in W^{1,4p}_{loc}(\mathbb{R}) \cap L^{4p}_{\tilde{w}}(\mathbb{R})$ and that

$$\Phi^{'}(y) = -\exp(-y)(C\exp(y) - K)_+ + C\mathbf{1}_{[\log(K/C),\infty)}(y)\ dx - a.e.$$

On the other hand we have that

$$\begin{aligned}\sup_{\epsilon>0}\left|\Phi^{'}(y) - \Phi^{'}(y-\epsilon z)\right|^{4p} &\leq M(\left|\Phi^{'}(y)\right|^{4p} + \sup_{\epsilon>0}\left|\Phi^{'}(y-\epsilon z)\right|^{4p} \\ &\leq M((2C + K\exp(|y|))^{4p} + (2C + K\exp(|y|+|z|))^{4p}).\end{aligned}$$

So condition (4.19) is fulfilled. In the same way one verifies condition (4.18). Hence Φ satisfies the assumptions of the previous theorem.

5. Examples and Simulations

We complete this paper by applying the results from Section 4 to the computation of the Deltas in the regime-switching examples mentioned in the Introduction. More complex examples of state-dependent drift coefficients (see e.g. [9]) can be treated following the same principles. To implement the methodology, we first employ Remark 3.7 and observe that all drift coefficients from the regime switching examples in the Introduction can be written in the form $b(t,x) = \tilde{b}(x) + \hat{b}(x)$ as in (1.6) such that identity (2.11) holds for $\hat{b}(x)$ and identity (2.13) holds for $\tilde{b}(x)$. We thus get the following rewriting of the first variation process (3.5):

$$\frac{\partial}{\partial x}X^x_t = \exp\left\{2\tilde{\beta}(X^x_t) - 2\tilde{\beta}(x) - 2\int_0^t \tilde{b}(X^x_s)\,dX^x_s + \int_0^t \hat{b}'(X^x_u)du\right\}, \tag{5.1}$$

where $\tilde{\beta}(\cdot) := \tilde{b}(0) + \int_0^{\cdot} \tilde{b}(y)\,dy$ is a primitive of $\tilde{b}(\cdot)$. This form is convenient for simulation purposes.

5.1. **Black & Scholes model with regime-switching dividend yield.** Consider an extended Black & Scholes model where the stock pays a dividend yield that switches to a higher level when the stock value passes a certain threshold $\overline{R} \in \mathbb{R}_+$. That is, under the risk-neutral measure the stock price S is given by the SDE

$$S^{s_0}_t = s_0 + \int_0^t \overline{b}(S^{s_0}_u)S^{s_0}_u du + \int_0^t \sigma S^{s_0}_u dB_u\,,$$

where $\sigma > 0$ is constant and the drift coefficient $\overline{b} : \mathbb{R} \to \mathbb{R}$ is given by

$$\overline{b}(x) := -\overline{\lambda}_1\mathbf{1}_{(-\infty,\overline{R})}(x) - \overline{\lambda}_2\mathbf{1}_{[\overline{R},\infty)}(x),$$

for dividend yields $\overline{\lambda}_1, \overline{\lambda}_2 \in \mathbb{R}_+$. We are interested in computing the Delta of a European option written on the stock with given pay-off function $\overline{\Phi} : \mathbb{R} \to \mathbb{R}$ and maturity T:

$$\frac{\partial}{\partial s_0}E[\overline{\Phi}(S^{s_0}_T)]\,.$$

In order to fit the computation of the Delta in our framework, we rewrite the stock price with the help of Itô's formula as

$$S^{s_0}_T = e^{\sigma X_T^{\ln(s_0)/\sigma}}\,,$$

where X^x_t is the solution of the SDE

$$X^x_t = x + \int_0^t b(X^x_u)du + B_t\,, \tag{5.2}$$

with

$$b(x) := -\lambda_1\mathbf{1}_{(-\infty,R)}(x) - \lambda_2\mathbf{1}_{[R,\infty)}(x) - \frac{\sigma}{2},$$

and $\lambda_1 := \frac{\overline{\lambda}_1}{\sigma}$, $\lambda_2 := \frac{\overline{\lambda}_2}{\sigma}$, $R := \frac{\ln(\overline{R})}{\sigma}$. We see that SDE (5.2) is in the required form (1.5) with $\tilde{b}(t,x) = -(\lambda_2 - \lambda_1)\mathbf{1}_{[R,\infty)}(x)$ and $\hat{b}(t,x) = -\lambda_1 - \frac{\sigma}{2}$. With $\Phi := \overline{\Phi} \circ \exp \circ\, \sigma\cdot$ we thus get by the chain rule

$$\frac{\partial}{\partial s_0}E[\overline{\Phi}(S^{s_0}_T)] = \frac{\partial}{\partial s_0}E[\Phi(X_T^{\ln(s_0)/\sigma})] = \frac{1}{s_0\sigma}\cdot\frac{\partial}{\partial x}E[\Phi(X^x_T)]\,\big|_{x=\frac{\ln(s_0)}{\sigma}}\,.$$

If $\Phi \in L^{4p}_w(\mathbb{R})$ we know by Theorem 4.2 that the Delta exists, and we can compute $\frac{\partial}{\partial x}E[\Phi(X^x_T)]$ by (4.8) to obtain

$$\frac{\partial}{\partial s_0}E[\overline{\Phi}(S^{s_0}_T)] = E\left[\overline{\Phi}(S^{s_0}_T)\int_0^T \frac{a(s)}{s_0\sigma}\frac{\partial}{\partial x}X^{\ln(s_0)/\sigma}_s dB_s\right] \tag{5.3}$$

for any bounded measurable function $a : \mathbb{R} \to \mathbb{R}$ such that $\int_0^T a(s)ds = 1$, and where $\frac{\partial}{\partial x}X^x_s$ is given by (5.1) with $\hat{b}' = 0$ and

$$\tilde{\beta}(x) := \int_0^x \tilde{b}(y)\,dy = -(\lambda_2 - \lambda_1)(x - R)\mathbf{1}_{[R,\infty)}(x).$$

We now consider the Delta for a call option, i.e. $\overline{\Phi}(x) := (x - K)_+$, and for a digital option, i.e. $\overline{\Phi}(x) := \mathbf{1}_{\{x \geq K\}}$, for some strike price $K > 0$. It is easily seen that in both cases $\Phi \in L^{4p}_w(\mathbb{R})$. To compute (5.3) by Monte Carlo, X^x is approximated by an Euler scheme (see [30], Theorem 3.1 on the Euler scheme approximation for coefficients b which are non-Lipschitz due to a set of discontinuity points with Lebesgue measure zero). As in [15] we compare the performance of (5.3) to the approximation of the Delta by a finite difference scheme combined with Monte Carlo:

$$\frac{\partial}{\partial s_0}E[\overline{\Phi}(S^{s_0}_T)] \sim \frac{E[\overline{\Phi}(S^{s_0+\epsilon}_T)] - E[\overline{\Phi}(S^{s_0-\epsilon}_T)]}{2\epsilon}\,, \tag{5.4}$$

for ϵ sufficiently small. We set the parameters $T = 1$, $s_0 = 100$, $\bar{\lambda}_1 = 0.05$, $\bar{\lambda}_2 = 0.15$, $\bar{R} = 108$, $\sigma = 0.1$ and $K = 94$. Our findings are analogue to the ones in [15]: for the continuous call option pay-off function the approximation (5.4) seems to be more efficient (see Figure 1), whereas for the discontinuous pay-off function of a digital option, the approximation (5.3) via the Malliavin weight exhibits considerably better convergence (see Figure 2).

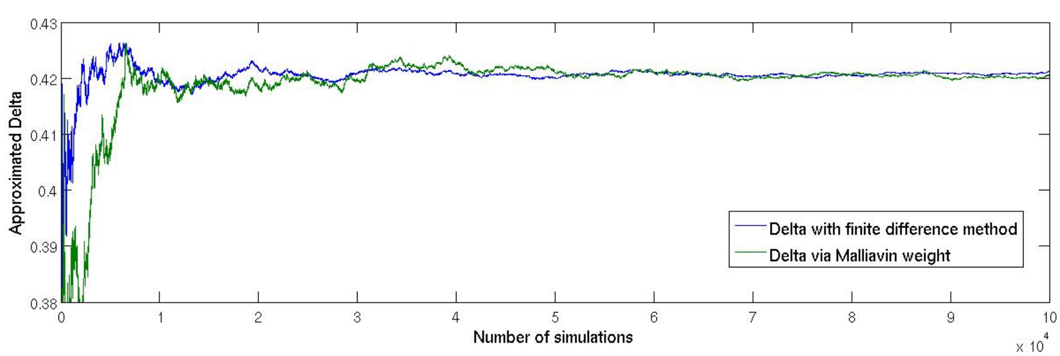

FIGURE 1. Delta of a European Call Option Black & Scholes model with regime-switching dividend yield.

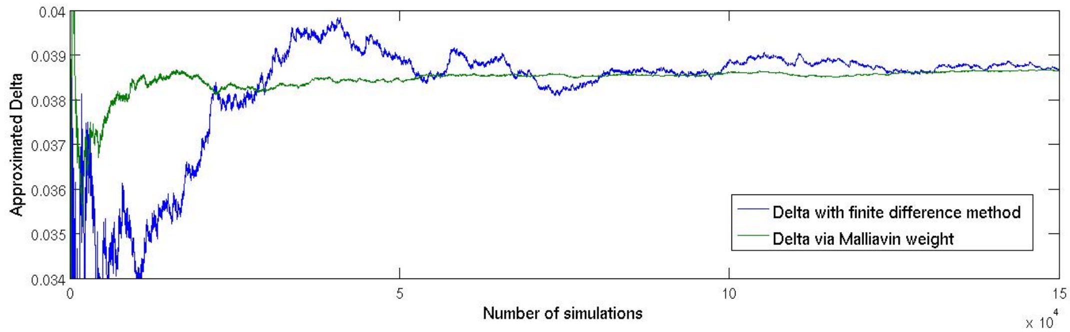

Figure 2. Delta of a European Digital Option under the Black & Scholes model with regime-switching dividend yield.

5.2. **Electricity spot price model with regime-switching mean-reversion rate.** Typically, electricity spot prices exhibit a mean-reverting behaviour with at least two different regimes of mean-reversion: a spike regime with very strong mean-reversion on exceptionally high price levels and a base regime with moderate mean-reversion on regular price levels. These features can be captured by modelling the electricity spot price S (under a risk-neutral pricing measure) by an extended Ornstein-Uhlenbeck process with regime-switching mean-reversion rate:

$$S_t^{s_0} = s_0 + \int_0^t \overline{b}(S_u^{s_0})du + \sigma B_t \,, \tag{5.5}$$

where the drift coefficient is given by

$$\overline{b}(x) := -\overline{\lambda}_1 x \mathbf{1}_{(-\infty,\overline{R})}(x) - \overline{\lambda_2} x \mathbf{1}_{[\overline{R},\infty)}(x) \tag{5.6}$$

for mean reversion rates $\overline{\lambda}_1, \overline{\lambda}_2 \in \mathbb{R}_+$, a given spike price threshold $\overline{R} \in \mathbb{R}$, and $\sigma > 0$. In order to guarantee positive prices, one could alternatively model the log-price by (5.5), or one could introduce another regime with high mean-reversion as soon as the price falls below zero (we recall that short periods of negative electricity prices have been observed).

Since electricity is a flow commodity, derivatives on spot electricity are written on the average price of the delivery of 1 kWh over a future period $[T_1, T_2]$, i.e. the underlying is of the type $\int_{T_1}^{T_2} S_t^{s_0}\, dt$ for $T_1 > 0$. The most liquidly traded electricity derivatives are futures and forwards. In that case the pay-off is linear and the computation of the Delta can be reduced to the computation of the Deltas of European type options:

$$\frac{\partial}{\partial s_0} E\left[\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} S_t^{s_0}\, dt\right] = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \frac{\partial}{\partial s_0} E[S_t^{s_0}]\, dt \,.$$

For derivatives with non-linear pay-off Φ, the Delta

$$\frac{\partial}{\partial s_0} E\left[\Phi\left(\int_{T_1}^{T_2} S_t^{s_0}\, dt\right)\right]$$

is of Asian type.

Again, in order to fit the computation of the Delta in our framework we rewrite the stock price with the help of Itô's formula as

$$S_t^{s_0} = \sigma X_t^{s_0/\sigma},$$

where X^x is the solution of the SDE

$$X_t^x = x + \int_0^t b(X_u^x)du + B_t, \tag{5.7}$$

with

$$b(x) := -\left(\lambda_1 \mathbf{1}_{(-\infty,R)}(x) + \lambda_2 \mathbf{1}_{[R,\infty)}(x)\right) x, \tag{5.8}$$

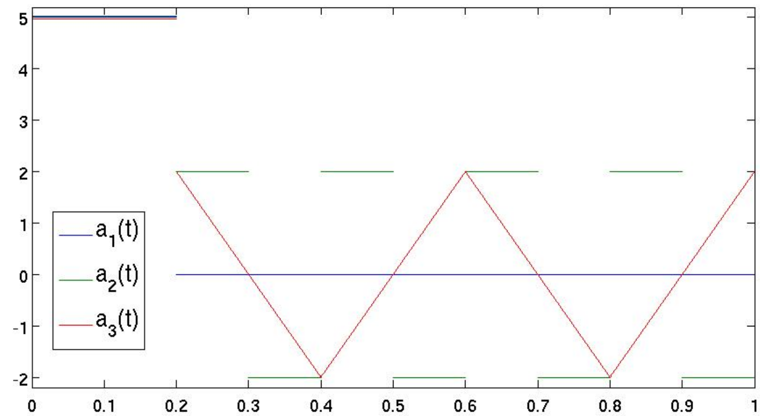

FIGURE 3. Three versions of the functions for $a(s)$ from Remark 4.4.

where $R = \overline{R}/\sigma$. We see that the SDE (5.7) is in the required form (1.5) with $\tilde{b}(x) = -(\lambda_2 - \lambda_1) R\mathbf{1}_{[R,\infty)}(x)$ and $\hat{b}(x) = b(x) - \tilde{b}(x)$. As in the previous example, by the chain rule we get that

$$\frac{\partial}{\partial s_0} E\left[\Phi\left(\int_{T_1}^{T_2} S_t^{s_0} dt\right)\right] = \frac{\partial}{\partial s_0} E\left[\overline{\Phi}\left(\int_{T_1}^{T_2} X_t^{s_0/\sigma} dt\right)\right] = \frac{1}{\sigma}\frac{\partial}{\partial x} E\left[\overline{\Phi}\left(\int_{T_1}^{T_2} X_t^{x} dt\right)\right]\bigg|_{x=\frac{s_0}{\sigma}} \quad (5.9)$$

with $\overline{\Phi} := \Phi \circ \cdot\sigma$. Note that in this example the first variation process $\frac{\partial}{\partial x} X_s^x$ is given by (5.1) with

$$\tilde{\beta}(x) := \int_0^x \tilde{b}(y)\, dy = -(\lambda_2 - \lambda_1) R(x - R)\mathbf{1}_{[R,\infty)}(x)$$

and

$$\int_0^t \hat{b}'(u, X_u^x) du = -\lambda_1 \int_0^t \mathbf{1}_{(-\infty,R)}(X_u^x) du - \lambda_2 \int_0^t \mathbf{1}_{[R,\infty)}(X_u^x) du.$$

We compare the performance of the formula for the Asian Delta in Corollary 4.3 with the approximation presented in Remark 4.4 and with a finite difference approximation analogous to (5.4) when Φ is a call option pay-off and a digital option pay-off, respectively. Obviously, in both cases the pay-off in terms of $X_\cdot^x$ fulfils the assumptions in Theorem 4.2. In the approximation presented in Remark 4.4 an optimal (in the sense that it minimises the variance of the Malliavin weight) choice for $a(s)$ could improve the convergence rate of the method. In the simulations we compared the following possible choices for $a(s)$:

$$a_1(s) := \begin{cases} \frac{1}{t_1} & \text{if} \quad 0 \le s \le t_1 \\ 0 & \text{if} \quad t_1 < s \le T_2 \end{cases}$$

$$a_2(s) := \begin{cases} \frac{1}{t_1} & \text{if} \quad 0 \le s \le t_1 \\ k & \text{if} \quad \left\lfloor \frac{s-T_1}{T_2-T_1} \cdot 2m \right\rfloor \equiv 0 \mod 2 \ \text{ and } t_1 < s \le T_2 \\ -k & \text{if} \quad \left\lfloor \frac{s-T_1}{T_2-T_1} \cdot 2m \right\rfloor \equiv 1 \mod 2 \ \text{ and } t_1 < s \le T_2 \end{cases}$$

$$a_3(s) := \begin{cases} \frac{1}{t_1} & \text{if} \quad 0 \le s \le t_1 \\ \left| \frac{s-T_1}{T_2-T_1} \cdot \frac{m}{2} - 1 - \left\lfloor \frac{s-T_1}{T_2-T_1} \cdot \frac{m}{2} - \frac{1}{2} \right\rfloor \right| - k & \text{if} \quad t_1 < s \le T_2 \end{cases},$$

see Figure 3. However, the different choices of function a above did not produce relevant differences in the results. Note, that implementing the approximation from Remark 4.4 with function $a_1(s)$ is essentially the same as the implementing the Delta from Corollary 4.3. We thus only compare the Delta from Corollary 4.3 with a finite difference scheme for parameters: $T_1 = 0.4$, $T_2 = 1$, $s_0 = 100$, $\overline{\lambda}_1 = 0.2$, $\overline{\lambda}_2 = 0.4$, $\overline{R} = 101$, $\sigma = 5$ and $K = 87$. We remark that if T_1 approaches zero,

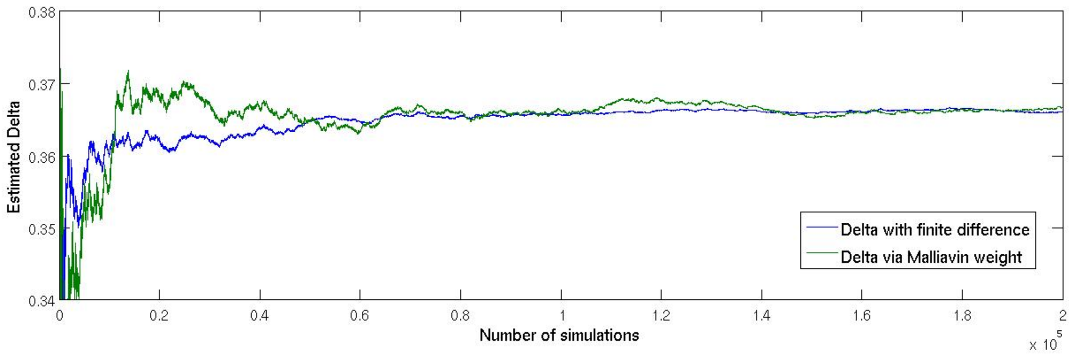

FIGURE 4. Delta of an Asian Call Option under the Electricity spot price model with regime-switching mean-reversion rate.

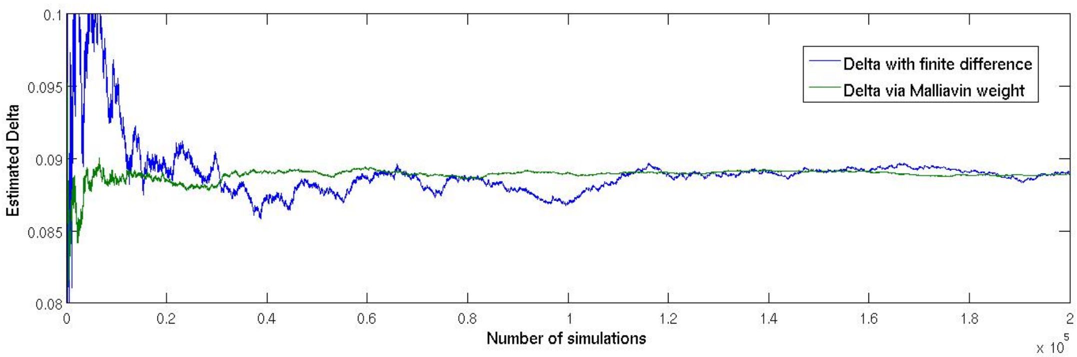

FIGURE 5. Delta of an Asian Digital Option under the Electricity spot price model with regime-switching mean-reversion rate.

the variance of the Malliavin weight increases, and thereby the Monte Carlo method becomes less effective. As for the European option in Subsection 5.1, also for these Asian type options the finite difference method seems to be more efficient for the continuous call option pay-off, see Figure 4, whereas for the digital option pay-off, the approximation through the Malliavin weight provides better convergence, see Figure 5.

5.3. **Generalised Black & Scholes model with regime-switching short rate.** Consider a generalised Black & Scholes model where under the risk-neutral measure the stock price $S_\cdot^{s_0}$ is given by

$$S_t^{s_0} = s_0 + \int_0^t r_u^{r_0} S_u^{s_0} du + \int_0^t \sigma S_u^{s_0} dB_u \,, \tag{5.10}$$

and the stochastic short rate $r_\cdot^{r_0}$ is given by an extended Vašíček model where the mean-reversion level switches between a high interest rate regime and a low interest rate regime when the short rate passes a certain threshold $R \in \mathbb{R}$:

$$r_t^{r_0} = r_0 + \int_0^t b(r_u^{r_0}) du + B_t^* \,, \tag{5.11}$$

where $B_t^* = \rho \widetilde{B}_t + \sqrt{1-\rho^2} B_t$ and the drift coefficient is given by

$$b(x) := -\lambda(x - m_1 \mathbf{1}_{(-\infty,R)}(x) - m_2 \mathbf{1}_{[R,\infty)}(x)) \tag{5.12}$$

for a mean-reversion rate $\lambda \in \mathbb{R}_+$ and mean-reversion levels $m_1, m_2 \in \mathbb{R}$, and where $\widetilde{B}$ is a Brownian motion independent of B, i.e. we allow for a correlation coefficient $0 \leq \sqrt{1-\rho^2} < 1$ with the stock price. Note that we set the volatility coefficient in (5.11) equal to one for notational simplicity. We see that the drift of the SDE (5.11) is in the required form (1.5) with

$\tilde{b}(x) = -\lambda\,(m_1 - m_2)\,\mathbf{1}_{[R,\infty)}(x)$ and $\hat{b}(x) = -\lambda\,(x - m_1)$. Further, we mention that the SDE (5.11) has a Malliavin differentiable unique strong solution with respect to the filtration $\mathcal{F}_t, 0 \le t \le T$, generated by the Brownian motions $\widetilde{B}_\cdot$ and $B_\cdot$. Moreover, there exists an Ω^* with probability mass 1 such that for all $\omega \in \Omega^*$ and $0 \le t \le T$: $(x \longmapsto r^x(t,\omega)) \in \cap_{p>0} W^{1,p}_{loc}(\mathbb{R})$. The proofs of these properties are essentially the same as in Section 3. For example, Girsanov's theorem in the previous proofs is applied to the Brownian motion $B^*_t := \rho \widetilde{B}_t + \sqrt{1-\rho^2} B_t, 0 \le t \le T$.

Now consider the price of a European option with pay-off function Φ written on the stock at maturity T:

$$E\left[e^{-\int_0^T r_s^{r_0} ds}\Phi\left(s_0 e^{\int_0^T r_s^{r_0} ds + \sigma B_T - \frac{1}{2}\sigma^2 T}\right)\right].$$

In this example we are interested in computing the *generalised Rho*

$$\frac{\partial}{\partial r_0} E\left[e^{-\int_0^T r_s^{r_0} ds}\Phi\left(s_0 e^{\int_0^T r_s^{r_0} ds + \sigma B_T - \frac{1}{2}\sigma^2 T}\right)\right], \tag{5.13}$$

that is, the sensitivity of the option with respect to the initial value r_0 of the short rate (i.e. a sensitivity with respect to movements of the short end of the yield curve). We see that (5.13) has the form of a Delta with respect to an Asian pay-off in the short rate $r_\cdot^{r_0}$ which, however, additionally depends on the factor B_T.

Although the extension of the results in Section 4 is straight forward to this simple two-dimensional setting, we can still remain in the one-dimensional setting from Section 4 by considering the Malliavin derivative $\widetilde{D}_s$ only with respect to Brownian motion $\widetilde{B}_\cdot$ and by applying relation (3.6) from Corollary 3.6 in the form

$$\frac{\partial}{\partial r_0} r_t^{r_0} = \frac{1}{\rho}\widetilde{D}_s r_t^{r_0} \frac{\partial}{\partial r_0} r_s^{r_0} \text{ for all } s \le t\,. \tag{5.14}$$

We here intend to analyse the performance of the approximation (4.20) from Theorem 4.6 for an Asian Delta. Under the corresponding assumptions from Theorem 4.6 for the pay-off function

$$\bar{\Phi}\left(\int_0^T r_t^{r_0} dt, B_T\right) := \exp\left\{-\int_0^T r_t^{r_0} dt\right\}\Phi\left(s_0 \exp\left\{\int_0^T r_t^{r_0} dt + \sigma B_T - \frac{1}{2}\sigma^2 T\right\}\right),$$

and by following the argument in the proof of Theorem 4.6 we then obtain that the function

$$u(r_0) := E\left[\bar{\Phi}\left(\int_0^T r_t^{r_0} dt, B_T\right)\right]$$

is continuously differentiable in $r_0 \in \mathbb{R}$, and that

$$\frac{\partial}{\partial r_0} u(r_0) = \lim_{n\to\infty} E\left[\bar{\Phi}\left(\int_0^T r_s^{r_0} ds + n^{-1} W_T, B_T\right)\right.$$
$$\left.\left(\int_0^T \frac{a(s)}{\rho}\frac{\partial}{\partial r_0} r_s^{r_0} d\widetilde{B}_s + n\int_0^T a(s)\left(\int_0^s \frac{\partial}{\partial r_0} r_u^{r_0} du\right) dW_s\right)\right] \tag{5.15}$$

where $a : \mathbb{R} \to \mathbb{R}$ is as in Theorem 4.6. Note that in this example the first variation process $\frac{\partial}{\partial r_0} r_s^{r_0}$ is given by (5.1) with

$$\tilde{\beta}(x) := \int_0^x \tilde{b}(y)\,dy = -\lambda\,(m_1 - m_2)\,(x - R)\,\mathbf{1}_{[R,\infty)}(x)$$

and

$$\int_0^t \hat{b}'(u, X_u^x) du = -\lambda t.$$

We compare the performance of the approximation of the generalised Rho $\frac{\partial}{\partial r_0} u$ presented in (5.15) with a finite difference approximation analogous to (5.4) when Φ is a call option pay-off, see Figure 6. The parameters are $T = 1$, $s = 2$, $\sigma = 0.1$, $\lambda = 0.3$, $m_1 = 0.5$, $m_2 = 1.2$, $R = 1.4$ and $K = \exp(0.4)$ and we choose $a(s) = 1/T$. Note that for a call option pay-off Φ we know from Example 4.7 that the assumptions in Theorem 4.6 are fulfilled. Further, we also compute the Delta of a digital pay-off, see Figure 7, even though the conditions of Theorem 4.6 are not satisfied. Our conjecture is that the result of Theorem 4.6 also holds for discontinuous pay-offs, and the simulation reinforces that. As n from Theorem 4.6 increases $\bar{\Phi}\left(\int_0^T r^{r_0}(s)ds + n^{-1}W_T, B_T\right)$ becomes a better approximation of $\bar{\Phi}\left(\int_0^T r^{r_0}(s)ds, B_T\right)$ but at the same time the variance of the Malliavin weight increases, thus, the convergence of the Monte Carlo simulation becomes slower. The experience of several simulations is that $n \sim 20$ gives the best balance between these two opposite impacts. However, we can see that in both cases the finite difference method seems considerably more efficient.

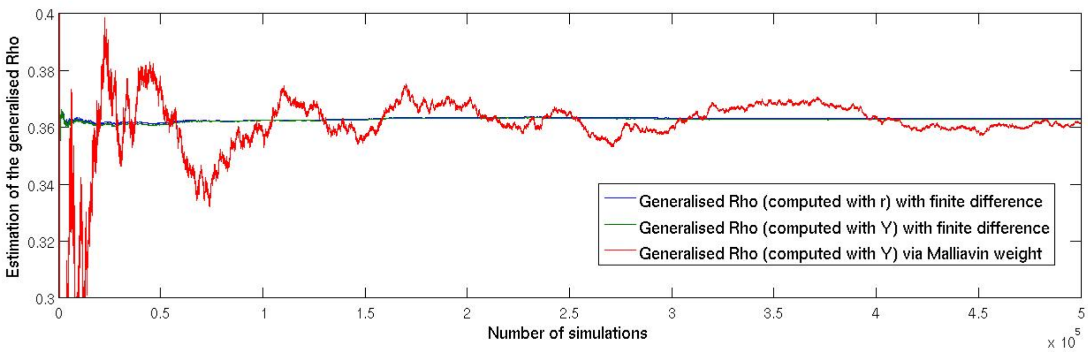

Figure 6. Approximation: Generalised Rho of a European Call Option under the Generalised Black & Scholes model with regime-switching short rate.

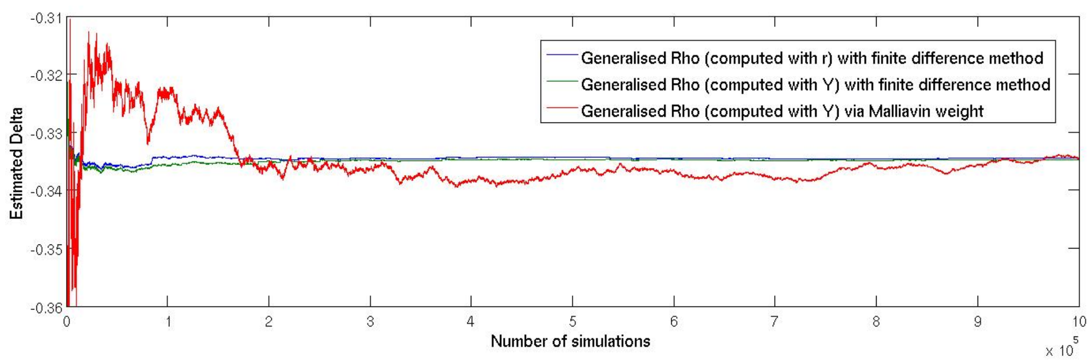

Figure 7. Approximation: Generalised Rho of a European Digital Option under the Generalised Black & Scholes model with regime-switching short rate.

Appendix A. Proofs of results in Section 3

In this appendix we recollect the proofs of the results in Section 3.

A.1. **Some auxiliary results.** We start by giving some auxiliary technical lemmata which provide relevant estimates that will be progressively used throughout some proofs in the sequel.

Lemma A.1. *Let $b : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a function of at most linear growth, i.e. $|b(t, x)| \leq C(1 + |x|)$ for some $C > 0$, all $x \in \mathbb{R}$ and $t \in [0, T]$. Then for any compact subset $K \subset \mathbb{R}$ there exists an $\varepsilon > 0$ such that*

$$\sup_{x \in K} E\left[\mathcal{E}\left(\int_0^T b(u, B_u^x)dB_u\right)^{1+\varepsilon}\right] < \infty \tag{A.1}$$

where $B_t^x := x + B_t$.

Proof. Indeed, write

$$
\begin{aligned}
E\left[\mathcal{E}\left(\int_0^T b(u,B_u^x)dB_u\right)^{1+\varepsilon}\right] &= E\left[\exp\left\{\int_0^T (1+\varepsilon)b(u,B_u^x)dB_u - \frac{1}{2}\int_0^T (1+\varepsilon)b^2(u,B_u^x)du\right\}\right] \\
&= E\Bigg[\exp\Bigg\{\int_0^T (1+\varepsilon)b(u,B_u^x)dB_u - \frac{1}{2}\int_0^T (1+\varepsilon)^2 b^2(u,B_u^x)du \\
&\quad + \frac{1}{2}\int_0^T \varepsilon(1+\varepsilon)b^2(u,B_u^x)du\Bigg\}\Bigg] \\
&= E\left[\exp\left\{\frac{1}{2}\int_0^T \varepsilon(1+\varepsilon)b^2(u,X_u^{\varepsilon,x})du\right\}\right]
\end{aligned}
$$

where in the last step $X^{\varepsilon,x}$ denotes a weak solution of the SDE

$$
\begin{cases} dX_t^{\varepsilon,x} = (1+\varepsilon)b(t,X_t^{\varepsilon,x})dt + dB_t, \quad t\in[0,T] \\ X_0^{\varepsilon,x} = x, \end{cases}
$$

which is obtained from Girsanov's theorem in the same way as in the first step of Subsection A.2 in equation (A.8). Observe that, since b has at most linear growth, we have

$$
|X_t^{\varepsilon,x}| \leq |x| + C(1+\varepsilon)\int_0^t (1+|X_u^{\varepsilon,x}|)du + |B_t|
$$

for every $t\in[0,T]$. Then Grönwall's inequality gives

$$
|X_t^{\varepsilon,x}| \leq (|x| + C(1+\varepsilon)T + |B_t|)\, e^{C(1+\varepsilon)T}, \tag{A.2}
$$

and due to the sublinearity of b and the estimate (A.2) we can find a constant $C_{\varepsilon,T}$ depending only on ε, T such that $\lim_{\varepsilon\searrow 0} C_{\varepsilon,T} < \infty$ and

$$
|b(u,X_u^{\varepsilon,x})| \leq C_{\varepsilon,T}\,(1+|x|+|B_t|)\,.
$$

As a result,

$$
\begin{aligned}
E\Big[\exp\Big\{\varepsilon(1+\varepsilon)\int_0^T b^2(u,X_u^{\varepsilon,x})du\Big\}\Big] &\leq E\left[\exp\left\{\varepsilon(1+\varepsilon)C_{\varepsilon,T}^2\int_0^T (1+|x|+|B_u|)^2\,du\right\}\right] \\
&\leq e^{\tilde{C}_{\varepsilon,T}T(1+|x|)^2} E\left[\exp\left\{2\tilde{C}_{\varepsilon,T}(1+|x|)\int_0^T |B_u|du + \tilde{C}_{\varepsilon,T}\int_0^T |B_u|^2 du\right\}\right]
\end{aligned}
$$

where $\tilde{C}_{\varepsilon,T} := \varepsilon(1+\varepsilon)C_{\varepsilon,T}^2 > 0$ is a constant such that $\lim_{\varepsilon\searrow 0}\tilde{C}_{\varepsilon,T} = 0$. Clearly, from the above expression we can see that for every compact set $K\subset\mathbb{R}$ we can choose $\varepsilon>0$ small enough such that

$$
\sup_{x\in K} E\Big[\exp\Big\{\varepsilon(1+\varepsilon)\int_0^T b^2(u,X_u^{\varepsilon,x})du\Big\}\Big] < \infty.
$$

□

Remark A.2. *From Lemma A.1 it follows immediately that if the approximating functions b_n, $n\geq 1$ are as in (3.1) then for any compact subset $K\subset\mathbb{R}$, one can find an $\varepsilon>0$ such that*

$$
\sup_{x\in K}\sup_{n\geq 0} E\left[\mathcal{E}\left(\int_0^T b_n(u,B_u^x)dB_u\right)^{1+\varepsilon}\right] < \infty, \tag{A.3}
$$

where we recall that $b_0 := b$.

Lemma A.3. *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a bounded measurable function. Then for every $t\in[0,T]$, $\lambda\in\mathbb{R}$ and compact subset $K\subset\mathbb{R}$ we have*

$$\sup_{x\in K} E\left[\exp\left\{\lambda\int_0^t\int_{\mathbb{R}} f(s,y)L^{B^x}(ds,dy)\right\}\right]<\infty \tag{A.4}$$

where $L^{B^x}(ds,dy)$ denotes integration with respect to local-time of the Brownian motion $B_t^x := B_t+x$ in both time and space, see Section 2 or [12] for more information on local-time integration.

Proof. By virtue of decomposition (2.10) from the Section 2 and Cauchy-Schwarz inequality twice we have

$$\begin{aligned}
E\left[\exp\left\{\lambda\int_0^t\int_{\mathbb{R}} f(s,y)L^{B^x}(ds,dy)\right\}\right] &\leq E\left[\exp\left\{-2\lambda\int_0^t f(s,B_s^x)dB_s\right\}\right]^{1/2}\\
&\quad\times E\left[\exp\left\{4\lambda\int_{T-t}^T f(T-s,B_{T-s}^x)dW_s\right\}\right]^{1/4}\\
&\quad\times E\left[\exp\left\{-4\lambda\int_{T-t}^T f(T-s,B_{T-s}^x)\frac{B_{T-s}}{T-s}ds\right\}\right]^{1/4}\\
&=: I\cdot II\cdot III.
\end{aligned}$$

where $W_t := \int_0^t \frac{B_{T-s}}{T-s}ds + B_{T-t}-B_T$ is a Brownian motion with respect to the filtration generated by $\hat{B}_\cdot$. For factor I, Hölder's inequality gives

$$\begin{aligned}
&E\bigg[\exp\bigg\{-2\lambda\int_0^t f(s,B_s^x)dB_s\bigg\}\bigg]\leq\\
&\qquad\leq E\left[\mathcal{E}\left(\int_0^t(-4\lambda f(s,B_s^x))dB_s\right)\right]^{1/2} E\left[\exp\left\{\int_0^t(8\lambda^2 f^2(s,B_s^x))ds\right\}\right]^{1/2}\\
&\qquad= E\left[\exp\left\{\int_0^t(8\lambda^2 f^2(s,B_s^x))ds\right\}\right]^{1/2}\\
&\qquad\leq C,
\end{aligned}$$

where $C>0$ is independent of x since f is bounded. Analogously, we obtain a bound for II. Finally, III follows from

$$E\left[\exp\left\{k\int_0^T\frac{|B_s|}{s}ds\right\}\right]<\infty \tag{A.5}$$

for any $k\in\mathbb{R}$, see Lemma A.4 below. □

Lemma A.4. *Let B be a one-dimensional Brownian motion on $[0,T]$. Then for any integer $p\geq 1$ and $0\leq\varepsilon<1/(4p)$*

$$E\left[\left|\int_0^T\frac{|B_u|^{1+\varepsilon}}{u^{1+\varepsilon}}du\right|^p\right]<\infty. \tag{A.6}$$

Proof. Indeed,

$$
\begin{aligned}
E\left[\left|\int_0^T \frac{|B_u|^{1+\varepsilon}}{u^{1+\varepsilon}} du\right|^p\right] &\leq E\left[\left(\sup_{u\in[0,T]}|B_u|^{\varepsilon}\right)^p \left|\int_0^T \frac{|B_u|}{u^{1+\varepsilon}} du\right|^p\right] \\
&\leq E\left[\sup_{u\in[0,T]}|B_u|^{2p\varepsilon}\right]^{1/2} E\left[\left|\int_0^T \frac{|B_u|}{u^{1+\varepsilon}} du\right|^{2p}\right]^{1/2} \\
&\leq CE\left[\left|\int_0^T \frac{|B_u|}{u^{1+\varepsilon}} du\right|^{2p}\right]^{1/2}
\end{aligned}
$$

for a positive constant $C > 0$. Now, set $d := 2p$ then we may write

$$
\begin{aligned}
E\left[\left|\int_0^T \frac{|B_u|}{u^{1+\varepsilon}} du\right|^{2p}\right] &= \int_0^T \overset{d)}{\cdots} \int_0^T \frac{E\left[|B_{u_1}|\cdots|B_{u_d}|\right]}{u_1^{1+\varepsilon}\cdots u_d^{1+\varepsilon}} du_1\cdots du_d \\
&= d! \int_{0<u_1<\cdots<u_d<T} \frac{E\left[|B_{u_1}|\cdots|B_{u_d}|\right]}{u_1^{1+\varepsilon}\cdots u_d^{1+\varepsilon}} du_1\cdots du_d \qquad \text{(A.7)}
\end{aligned}
$$

where the last equality follows from the fact that the integrand is a symmetric function.

Then for a centered random Gaussian vector $(Z_1, \ldots, Z_d)$ with covariances $Cov(Z_i, Z_j) = \sigma_{i,j}$, $i, j = 1, \ldots, d$ we have the following estimate that can be found in [20, Theorem 1]

$$
E[|Z_1\cdots Z_d|] \leq \left(\sum_{\pi\in S_d}\prod_{j=1}^d \sigma_{j,\pi(j)}\right)^{1/2}
$$

where S_d denotes the set of permutations of $(1, \ldots, d)$. Applying the above inequality to the integral in (A.7)

$$
\int_{0<u_1<\cdots<u_d<T} \frac{E\left[|B_{u_1}|\cdots|B_{u_d}|\right]}{u_1^{1+\varepsilon}\cdots u_d^{1+\varepsilon}} du_1\cdots du_d \leq \sum_{\pi\in S_d}\int_{0<u_1<\cdots<u_d<T}\prod_{j=1}^d\left(\frac{u_j\wedge u_{\pi(j)}}{u_j^{1+\varepsilon}u_{\pi(j)}^{1+\varepsilon}}\right)^{1/2} du_1\cdots du_d.
$$

Given a permutation $\pi \in S_d$ we have that, if $0 < u_1 < u_2 < \cdots u_d < T$ then

$$
\prod_{j=1}^d\left(\frac{u_j\wedge u_{\pi(j)}}{u_j^{1+\varepsilon}u_{\pi(j)}^{1+\varepsilon}}\right)^{1/2} = \frac{u_1^{\alpha_1/2}\cdots u_d^{\alpha_d/2}}{u_1^{1+\varepsilon}\cdots u_d^{1+\varepsilon}}
$$

where the α_i's, depend on π and have the property that $\sum_{i=1}^d \alpha_i = d$ and $\alpha_i \in \{0, 1, 2\}$ for all $i = 1, \ldots, d$. Moreover, observe that $\alpha_1 \geq 1$ independently of π since $u_1 \wedge u_{\pi(1)} = u_1$ for all $\pi \in S_d$. So, if we now integrate iteratively we obtain

$$
\int_{0<u_1<\cdots<u_d<T} \frac{E\left[|B_{u_1}|\cdots|B_{u_d}|\right]}{u_1^{1+\varepsilon}\cdots u_d^{1+\varepsilon}} du_1\cdots du_d \leq \sum_{\pi\in S_d}\frac{1}{\prod_{j=1}^d\left(\frac{1}{2}\sum_{i=1}^j\alpha_i - j\varepsilon\right)} T^{d(\frac{1}{2}-\varepsilon)}
$$

if, and only if $\frac{1}{2}\sum_{i=1}^j \alpha_i - j\varepsilon > 0$ for all $j = 1, \ldots, d$ which holds by just observing that

$$
\frac{1}{2}\sum_{i=1}^j \alpha_i > \frac{\alpha_1}{2} \geq d\frac{1}{2d} \geq j\frac{1}{2d}
$$

for every $j = 1, \ldots, d$ where we used $\alpha_1 \geq 1$. So it suffices to take $\varepsilon \geq 0$ such that $\varepsilon < \frac{1}{2d}$. $\square$

A.2. **Proof of Theorem 3.1.** We now develop the proof of Theorem 3.1 according to the four-step scheme outlined in Section 3. In order to construct a weak solution of (1.5) in the first step, let $(\Omega,\mathcal{F},\widetilde{P})$ be some given probability space which carries a Brownian motion $\widetilde{B}$, and put $X_t^x := \widetilde{B}_t+x$, $t\in[0,T]$. As we already noted in Remark 2.5, it is well-known, see e.g. [16, Corollary 5.16], that for sublinear coefficients b the Radon-Nikodym derivative $\frac{dP}{d\widetilde{P}} := \mathcal{E}\left(\int_0^T b(u,X_u^x)d\widetilde{B}_u\right)$ defines an equivalent probability measure P under which the process

$$B_t := X_t^x - x - \int_0^t b(s,X_s^x)ds,\ t\in[0,T], \tag{A.8}$$

is a Brownian motion on $(\Omega,\mathcal{F},P)$. Hence, because of (A.8), the pair (X^x,B) is a weak solution of (1.5) on $(\Omega,\mathcal{F},P)$. The stochastic basis that we operate on in the following is now given by the filtered probability space $(\Omega,\mathcal{F},P,\{\mathcal{F}_t\}_{t\in[0,T]})$, which carries the weak solution (X^x,B) of (1.5), where $\{\mathcal{F}_t\}_{t\in[0,T]}$ denotes the filtration generated by B_t, $t\in[0,T]$, augmented by the P-null sets.

Next, we prove that for given $t\in[0,T]$ the sequence of strong solutions $\{X_t^{n,x}\}_{n\geq 1}$ of the SDE's (3.2) with regular coefficients b_n from (3.1) converges weakly in $L^2(\Omega;\mathcal{F}_t)$ to $E[X_t^x|\mathcal{F}_t]$.

Lemma A.5. *Let $b_n:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a sequence of functions approximating b a.e. as in (3.1) and $X_t^{n,x}$ the corresponding strong solutions to* (3.2), *$n\geq 1$. Then for every $t\in[0,T]$ and function $\varphi\in L_w^{2p}(\mathbb{R})$ where the space $L_w^{2p}(\mathbb{R})$ is defined as in* (4.7) *with p being the conjugate exponent of $1+\varepsilon$, $\varepsilon>0$ from Lemma A.1, we have*

$$\varphi(X_t^{n,x}) \xrightarrow{n\to\infty} E[\varphi(X_t^x)|\mathcal{F}_t]$$

weakly in $L^2(\Omega;\mathcal{F}_t)$.

Proof. First of all, we shall see that $\varphi(X_t^{n,x}), E[\varphi(X_t^x)|\mathcal{F}_t]\in L^2(\Omega;\mathcal{F}_t)$, $n\geq 0$. Indeed, Girsanov's theorem, Remark A.2 and the fact that $\varphi\in L_w^{2p}(\mathbb{R})$ imply that for some constant $C_\varepsilon>0$ with $\varepsilon>0$ small enough we have

$$\sup_{n\geq 0} E[|\varphi(X_t^{n,x})|^2] \leq C_\varepsilon E[|\varphi(x+B_t)|^{2\frac{1+\varepsilon}{\varepsilon}}]^{\frac{\varepsilon}{1+\varepsilon}} = C_\varepsilon \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} |\varphi(x+z)|^{2\frac{1+\varepsilon}{\varepsilon}} e^{-\frac{|z|^2}{2T}}dz < \infty. \tag{A.9}$$

To show that

$$E\left[\varphi(X_t^{n,x})Z\right] \xrightarrow{n\to\infty} E[E[\varphi(X_t^x)|\mathcal{F}_t]Z]$$

for any $Z\in L^2(\Omega;\mathcal{F}_t)$ it suffices to show

$$\mathcal{W}(X_t^{n,x})(f) \xrightarrow{n\to\infty} \mathcal{W}(E[X_t^x|\mathcal{F}_t])(f)$$

for every $f\in L^2([0,T])$

Indeed, by Girsanov's theorem we can write

$$\begin{aligned}
E&\left[\Big(\varphi(X_t^{n,x}) - E[\varphi(X_t^x)|\mathcal{F}_t]\Big)\mathcal{E}\left(\int_0^T f(u)dB_u\right)\right] = \\
&= E\left[\varphi(B_t^x)\left(\mathcal{E}\left(\int_0^T (b_n(u,B_u^x)+f(u))dB_u\right) - \mathcal{E}\left(\int_0^T (b(u,B_u^x)+f(u))dB_u\right)\right)\right] \\
&= E\left[\varphi(B_t^x)\mathcal{E}\left(\int_0^T (b(u,B_u^x)+f(u))dB_u\right)\right. \\
&\quad\left.\times\left(\mathcal{E}\left(\int_0^T (b_n(u,B_u^x)+f(u))dB_u\right)\Big/\mathcal{E}\left(\int_0^T (b(u,B_u^x)+f(u))dB_u\right)-1\right)\right]
\end{aligned}$$

Then, using inequality $|e^x-1|\le |x|(e^x+1)$ we have

$$
\begin{aligned}
E\Bigg[\Big(\varphi(X_t^{n,x})-E[\varphi(X_t^x)|\mathcal{F}_t]\Big)\mathcal{E}\left(\int_0^T f(u)dB_u\right)\Bigg] \\
&\le E\left[|\varphi(B_t^x)|\,|U_n|\,\mathcal{E}\left(\int_0^T (b_n(u,B_u^x)+f(u))dB_u\right)\right] \\
&\quad + E\left[|\varphi(B_t^x)|\,|U_n|\,\mathcal{E}\left(\int_0^T (b(u,B_u^x)+f(u))dB_u\right)\right] \\
&:= I_n + II_n
\end{aligned}
$$

where

$$
U_n := \int_0^T (b_n(u,B_u^x)-b(u,B_u^x))dB_u - \frac{1}{2}\int_0^T [(b_n(u,B_u^x)+f(u))^2-(b(u,B_u^x)+f(u))^2]du.
$$

For the term I_n, Hölder's inequality with exponents $p=\frac{1+\varepsilon}{\varepsilon}$ and $q=1+\varepsilon$ and then again for $p=q=2$ yields

$$
\begin{aligned}
I_n &\le E\left[|\varphi(B_t^x)U_n|^{\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{1+\varepsilon}} E\left[\mathcal{E}\left(\int_0^T (b_n(u,B_u^x)+f(u))dB_u\right)^{1+\varepsilon}\right]^{\frac{1}{1+\varepsilon}} \\
&\le E\left[|\varphi(B_t^x)|^{2\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{2(1+\varepsilon)}} E\left[|U_n|^{2\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{2(1+\varepsilon)}} E\left[\mathcal{E}\left(\int_0^T (b_n(u,B_u^x)+f(u))\,dB_u\right)^{1+\varepsilon}\right]^{\frac{1}{1+\varepsilon}} \\
&=: I^1\cdot I_n^2\cdot I_n^3,
\end{aligned}
$$

where I^1, I_n^2 and I_n^3 are the respective factors above and $\varepsilon>0$ is such that I_n^3 is bounded uniformly in $n\ge 0$ (see Remark A.2). We can then control the first factor I^1 due to the fact that $\varphi\in L_w^{2p}(\mathbb{R})$ as it is shown in (A.9).

Finally, for the second factor I_n^2 define $p:=2\frac{1+\varepsilon}{\varepsilon}$. Then using Minkowski's inequality, Burkholder-Davis-Gundy's inequality and Hölder's inequality we can write

$$
\begin{aligned}
(I_n^2)^p &= E\left[\left|\int_0^T (b_n(u,B_u^x)-b(u,B_u^x))dB_u - \frac{1}{2}\int_0^T [(b_n(u,B_u^x)+f(u))^2-(b(u,B_u^x)+f(u))^2]du\right|^p\right] \\
&\le 2^{p-1}E\left[\left|\int_0^T (b_n(u,B_u^x)-b(u,B_u^x))dB_u\right|^p\right] \\
&\quad + 2^{p-2}E\left[\left|\int_0^T [(b_n(u,B_u^x)+f(u))^2-(b(u,B_u^x)+f(u))^2]du\right|^p\right] \\
&\lesssim 2^{p-1}E\left[\left(\int_0^T |b_n(u,B_u^x)-b(u,B_u^x)|^2\,du\right)^{p/2}\right] \\
&\quad + 2^{p-2}T^{p-1}\int_0^T E\left[\left|(b_n(u,B_u^x)+f(u))^2-(b(u,B_u^x)+f(u))^2\right|^{2p}\right]du \\
&\lesssim 2^{p-1}T^{p/2-1}\int_0^T E\left[|b_n(u,B_u^x)-b(u,B_u^x)|^p\right]du \\
&\quad + 2^{p-2}T^{p-1}\int_0^T E\left[\left|(b_n(u,B_u^x)+f(u))^2-(b(u,B_u^x)+f(u))^2\right|^{2p}\right]du
\end{aligned}
$$

and by dominated convergence we obtain $I_n^2\to 0$ as $n\to\infty$. Similarly, we obtain the result for II_n. $\square$

We now turn to the third step of our scheme to prove Theorem 3.1. The next theorem gives the $L^2(\Omega;\mathcal{F}_t)$-convergence of the sequence of strong solutions $X_t^{n,x}$ to the limit $E[X_t^x|\mathcal{F}_t]$ which, in addition, is Malliavin differentiable. The technique used in this result is the compactness criterion given in Proposition 2.3 due to [8].

Theorem A.6. *Let $b_n : [0,T]\times\mathbb{R}\to\mathbb{R}$, $n\geq 1$, be as in (3.1) and $X_\cdot^{n,x}$ the corresponding strong solutions to (3.2). Then, for each $t\in[0,T]$*

$$X_t^{n,x} \xrightarrow{L^2(\Omega;\mathcal{F}_t)} E[X_t^x|\mathcal{F}_t] \tag{A.10}$$

as $n\to\infty$. Moreover, the right-hand side of (A.10) is Malliavin differentiable.

Proof. The main step is to show relative compactness of $\{X_t^{n,x}\}_{n\geq1}$ by applying Proposition 2.3. Let $t\in[0,T]$, $0\leq s\leq s'\leq t$ and a compact set $K\subset\mathbb{R}$ be given. Using the explicit representation introduced in (3.3), Girsanov's theorem, the mean-value theorem, Hölder's inequality with exponent $1+\varepsilon$ for a sufficiently small $\varepsilon>0$ and Cauchy-Schwarz inequality successively we obtain

$$\begin{aligned}
&E\big[(D_sX_t^{n,x}-D_{s'}X_t^{n,x})^2\big]=\\
&\quad= E\left[\exp\left\{2\int_{s'}^t b_n'(u,B_u^x)du\right\}\left(\exp\left\{\int_s^{s'} b_n'(u,B_u^x)du\right\}-1\right)^2\mathcal{E}\left(\int_0^T b_n(u,B_u^x)dB_u\right)\right]\\
&\quad\leq E\Bigg[\exp\left\{2\int_{s'}^t b_n'(u,B_u^x)du\right\}\left(\sup_{0\leq\alpha\leq1}\exp\left\{\alpha\int_s^{s'} b_n'(u,B_u^x)du\right\}\right)^2\\
&\qquad\times\left(\int_s^{s'} b_n'(u,B_u^x)du\right)^2\mathcal{E}\left(\int_0^T b_n(u,B_u^x)dB_u\right)\Bigg]\\
&\quad\leq E\Bigg[\exp\left\{2\frac{1+\varepsilon}{\varepsilon}\int_{s'}^t b_n'(u,B_u^x)du\right\}\sup_{0\leq\alpha\leq1}\exp\left\{2\frac{1+\varepsilon}{\varepsilon}\alpha\int_s^{s'} b_n'(u,B_u^x)du\right\}\\
&\qquad\times\left|\int_s^{s'} b_n'(u,B_u^x)du\right|^{2\frac{1+\varepsilon}{\varepsilon}}\Bigg]^{\frac{\varepsilon}{1+\varepsilon}} E\left[\mathcal{E}\left(\int_0^T b_n(u,B_u^x)dB_u\right)^{1+\varepsilon}\right]^{\frac{1}{1+\varepsilon}}\\
&\quad\leq E\left[\exp\left\{4\frac{1+\varepsilon}{\varepsilon}\int_{s'}^t(\tilde b_n'(u,B_u^x)+\hat b'(u,B_u^x))du\right\}\right]^{\frac{\varepsilon}{2(1+\varepsilon)}}\\
&\qquad\times E\left[\sup_{0\leq\alpha\leq1}\exp\left\{8\frac{1+\varepsilon}{\varepsilon}\alpha\int_s^{s'}(\tilde b_n'(u,B_u^x)+\hat b'(u,B_u^x))du\right\}\right]^{\frac{\varepsilon}{4(1+\varepsilon)}}\\
&\qquad\times E\left[\left|\int_s^{s'}(\tilde b_n'(u,B_u^x)+\hat b'(u,B_u^x))du\right|^{8\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{4(1+\varepsilon)}} E\left[\mathcal{E}\left(\int_0^T b_n(u,B_u^x)dB_u\right)^{1+\varepsilon}\right]^{\frac{1}{1+\varepsilon}}\\
&\quad=: I_n^1\cdot I_n^2\cdot I_n^3\cdot I_n^4,
\end{aligned}$$

where I_n^1, I_n^2, I_n^3 and I_n^4 denote the respective factors shown above.

Here, by Remark A.2, $\varepsilon>0$ is chosen such that

$$\sup_{x\in K}\sup_{n\geq0} I_n^4<\infty.$$

For I_n^1 and I_n^2 we use Cauchy-Schwarz inequality and the fact that $\hat b'$ is bounded and get

$$I_n^1\lesssim E\left[\exp\left\{4\frac{1+\varepsilon}{\varepsilon}\int_{s'}^t \tilde b_n'(u,B_u^x)du\right\}\right]^{\frac{\varepsilon}{2(1+\varepsilon)}}=: II_n^1$$

and

$$I_n^2 \lesssim E\left[\sup_{0\le\alpha\le 1}\exp\left\{8\frac{1+\varepsilon}{\varepsilon}\alpha\int_s^{s'}\tilde{b}_n'(u,B_u^x)du\right\}\right]^{\frac{\varepsilon}{4(1+\varepsilon)}} =: II_n^2.$$

For I_n^3, Minkowski's inequality and the boundedness of $\hat{b}'$ give

$$\begin{aligned}
I_n^3 &\le E\left[\left|\int_s^{s'}\tilde{b}_n'(u,B_u^x)du\right|^{8\frac{1+\varepsilon}{\varepsilon}} + \left|\int_s^{s'}\hat{b}'(u,B_u^x)du\right|^{8\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{4(1+\varepsilon)}} \\
&\lesssim E\left[\left|\int_s^{s'}\tilde{b}_n'(u,B_u^x)du\right|^{8\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{4(1+\varepsilon)}} + \|\hat{b}'\|_\infty^2 T|s'-s| \\
&\le II_n^3 + \|\hat{b}'\|_\infty^2 T|s'-s|.
\end{aligned}$$

Now we want to get rid of the derivatives $\tilde{b}_n'$ in II_n^1, II_n^2 and II_n^3. In order to do so, we use integration with respect to the local time of the Brownian motion, see Theorem 2.9 in the Section 2 or e.g. [12] for more information about local-time integration. We obtain

$$\begin{aligned}
E\big[(D_sX_t^{n,x} - D_{s'}X_t^{n,x})^2\big] \lesssim\; & E\left[\exp\left\{-4\frac{1+\varepsilon}{\varepsilon}\int_{s'}^t\int_{\mathbb{R}}\tilde{b}_n(u,y)L^{B^x}(du,dy)\right\}\right]^{\frac{\varepsilon}{2(1+\varepsilon)}} \\
&\times E\left[\sup_{0\le\alpha\le 1}\exp\left\{-8\frac{1+\varepsilon}{\varepsilon}\alpha\int_s^{s'}\int_{\mathbb{R}}\tilde{b}_n(u,x)L^{B^x}(du,dy)\right\}\right]^{\frac{\varepsilon}{4(1+\varepsilon)}} \\
&\times\left(E\left[\left|\int_s^{s'}\int_{\mathbb{R}}\tilde{b}_n(u,x)L^{B^x}(du,dy)\right|^{8\frac{1+\varepsilon}{\varepsilon}}\right]^{\frac{\varepsilon}{4(1+\varepsilon)}} + \|\hat{b}'\||s'-s|\right).
\end{aligned}$$

Observe that factors II_n^1 and II_n^2 can be controlled uniformly in $n \ge 1$ and $x \in K$ by virtue of Lemma A.3. Now, denote $p_\varepsilon := 4\frac{1+\varepsilon}{\varepsilon}$. Then for factor II_n^3 we use representation (2.11) from Theorem 2.9 in connection with (2.10) in Section 2 and apply Minkowski's inequality, Burkholder-Davis-Gundy's inequality and Hölder's inequality with exponent $(\varepsilon'+2)/\varepsilon'$ for a suitable $\varepsilon' > 0$ in order to obtain

$$\begin{aligned}
II_n^3 \le\; & E\left[\left|\int_s^{s'}\tilde{b}_n(u,B_u^x)dB_u - \int_{T-s'}^{T-s}\tilde{b}_n(T-u,\hat{B}_u^x)dW_u + \int_{T-s'}^{T-s}\tilde{b}_n(T-u,\hat{B}_u^x)\frac{\hat{B}_u}{T-u}du\right|^{2p_\varepsilon}\right]^{1/p_\varepsilon} \\
\lesssim\; & E\left[\left(\int_s^{s'}|\tilde{b}_n(u,B_u^x)|^2du\right)^{p_\varepsilon}\right]^{1/p_\varepsilon} + E\left[\left(\int_{T-s'}^{T-s}|\tilde{b}_n(T-u,\hat{B}_u^x)|^2du\right)^{p_\varepsilon}\right]^{1/p_\varepsilon} \\
&+ E\left[\left|\int_{T-s'}^{T-s}\tilde{b}_n(T-u,\hat{B}_u^x)\frac{\hat{B}_u}{T-u}du\right|^{2p_\varepsilon}\right]^{1/p_\varepsilon} \\
\lesssim\; & |s'-s|^{\varepsilon'/(\varepsilon'+2)}E\left[\left(\int_s^{s'}|\tilde{b}_n(u,B_u^x)|^{\varepsilon'+2}du\right)^{\frac{2p_\varepsilon}{\varepsilon'+2}}\right]^{1/p_\varepsilon} \\
&+ |s'-s|^{\varepsilon'/(\varepsilon'+2)}E\left[\left(\int_{T-s'}^{T-s}|\tilde{b}_n(T-u,\hat{B}_u^x)|^{\varepsilon'+2}du\right)^{\frac{2p_\varepsilon}{\varepsilon'+2}}\right]^{1/p_\varepsilon} \\
&+ |s'-s|^{2\varepsilon'/(\varepsilon'+2)}E\left[\left|\int_{T-s'}^{T-s}\left|\tilde{b}_n(T-u,\hat{B}_u^x)\frac{\hat{B}_u}{T-u}\right|^{(\varepsilon'+2)/2}du\right|^{\frac{4p_\varepsilon}{\varepsilon'+2}}\right]^{1/p_\varepsilon}.
\end{aligned}$$

The last expectation is bounded by taking $\varepsilon' < \frac{2}{8p_\varepsilon - 1}$ and applying Lemma A.4.

Altogether, we can find a constant $C > 0$ such that

$$\sup_{x \in K} \sup_{n \geq 1} E\left[(D_{s'} X_t^{n,x} - D_s X_t^{n,x})^2\right] \leq C|s' - s|^{\varepsilon'/(\varepsilon'+2)} \tag{A.11}$$

for $0 \leq s' \leq s \leq t$ where $0 < \varepsilon'/(\varepsilon'+2) < 1$.

Similarly, one also obtains

$$\sup_{x \in K} \sup_{0 \leq s \leq t} \sup_{n \geq 1} E\left[(D_s X_t^{n,x})^2\right] \leq C \tag{A.12}$$

for a constant $C > 0$.

Then (A.9) with $\varphi = id$, (A.11), (A.12) together with Proposition 2.3 imply that the set $\{X_t^{n,x}\}_{n \geq 1}$ is relatively compact in $L^2(\Omega; \mathcal{F}_t)$. Since the sequence of solutions $X_t^{n,x}$ also converges weakly to $E[X_t^x|\mathcal{F}_t]$ due to Lemma A.5 with $\varphi = id$, by uniqueness of the limit we have that

$$X_t^{n_k,x} \xrightarrow{L^2(\Omega;\mathcal{F}_t)} E[X_t^x|\mathcal{F}_t]$$

for a subsequence n_k, $k \geq 0$.

In fact, one observes that the $L^2(\Omega; \mathcal{F}_t)$-convergence holds for the whole sequence. Indeed, assume by contradiction, that there exists a subsequence n_j, $j \geq 0$, such that there is an $\varepsilon > 0$ with $E[|X_t^{n_j,x} - X_t^x|^2] > \varepsilon$ for all $j \geq 0$. Then $\{b_{n_j}\}_{j \geq 0}$ is a sequence of approximating coefficients as required in (3.1). Thus, by the previous results there exists a subsequence n_{j_m}, $m \geq 0$, such that $X^{n_{j_m},x} \to X^x$ in $L^2(\Omega; \mathcal{F}_t)$, which gives rise to a contradiction.

Moreover, since the sequence of Malliavin derivatives $\{D_s X_t^{n,x}\}_{n \geq 1}$ is bounded uniformly in n in the $L^2([0,T] \times \Omega)$-norm because of (A.12), we also have that the limit $E[X_t^x|\mathcal{F}_t]$ is Malliavin differentiable, see for instance [28, Lemma 1.2.3]. □

Remark A.7. *Note that we have proved the estimates (A.11) and (A.12) uniformly in $x \in K$ for a compact set K even though this is not needed to apply Proposition 2.3. We will, however, use this uniform bounds later on in the proofs of Lemma A.8 and Theorem 3.4.*

We are now ready to complete the proof of Theorem 3.1 by use of the previous steps.

Proof of Theorem 3.1. It remains to prove that X_t^x is $\mathcal{F}_t$-measurable for every $t \in [0,T]$ and by Remark 1.3 it then follows that there exists a strong solution in the usual sense that is Malliavin differentiable. Indeed, let φ be a continuous bounded function, then by Theorem A.6 we have, for a subsequence n_k, $k \geq 0$, that

$$\varphi(X_t^{n_k,x}) \to \varphi(E[X_t^x|\mathcal{F}_t]), \quad P-a.s.$$

as $k \to \infty$.

On the other side, by Lemma A.5 we also have

$$\varphi(X_t^{n,x}) \to E\left[\varphi(X_t^x)|\mathcal{F}_t\right]$$

weakly in $L^2(\Omega; \mathcal{F}_t)$. By the uniqueness of the limit we immediately have

$$\varphi\left(E[X_t^x|\mathcal{F}_t]\right) = E\left[\varphi(X_t^x)|\mathcal{F}_t\right], \quad P-a.s.$$

for all continuous, bounded functions φ, which implies that X_t^x is $\mathcal{F}_t$-measurable for every $t \in [0,T]$.

To show uniqueness, assume that we have two strong solutions X^x and Y^x to the SDE (1.5). Then using the Cameron-Martin formula shows that

$$\mathcal{W}(X_t^x)(h) = E[X_t^x(h)],$$

for $h \in L^2([0,T])$ where we recall that $\mathcal{W}(X_t^x)(h)$ denotes the Wiener transform, and the process $X_t^x(h), 0 \leq t \leq T$ satisfies the SDE

$$dX_t^x(h) = (b(t, X_t^x(h)) + h(t))dt + d\widehat{B}_t, \; X_0^x(h) = x \tag{A.13}$$

for a Brownian motion $\widehat{B}_t, 0 \le t \le T$. In the same way, the process $Y_t^x(h), 0 \le t \le T$ solves (A.13). On the other hand, it follows from the linear growth of the drift coefficient b that $X_t^x(h)$ and $Y_t^x(h), 0 \le t \le T$, are unique in law (see e.g. Proposition 3.10 in [16]). Hence

$$\mathcal{W}(X_t^x)(h) = \mathcal{W}(Y_t^x)(h)$$

for all t, h. Thus X^x and Y^x are indistinguishable. □

A.3. **Proof of Proposition 3.2:** By equation (3.3) and formula (2.11), we can write for regular coefficients b_n

$$D_s X_t^{n,x} = \exp\left\{-\int_s^t \int_{\mathbb{R}} b_n(u,y) L^{X^{n,x}}(du,dy)\right\}.$$

Then, since $X_t^{n,x}$, $n \ge 0$ is relatively compact in $L^2(\Omega; \mathcal{F}_t)$ and $\|D_s X_t^{n,x}\|_{L^2([0,T]\times\Omega)}$ is bounded uniformly in $n \ge 0$ due to the proof of Theorem A.6 we know that the sequence $D_s X_t^{n,x}$, $n \ge 0$ converges weakly to $D_s X_t^x$ in $L^2([0,T] \times \Omega)$, see [28, Lemma 1.2.3]. Therefore, it is enough to check that our candidate is the weak limit. So we must prove that

$$\left\langle \mathcal{W}\Big(\exp\left\{-\textstyle\int_\cdot^t \int_{\mathbb{R}} b_n(u,y) L^{X^{n,x}}(du,dy)\right\} - \exp\left\{-\textstyle\int_\cdot^t \int_{\mathbb{R}} b(u,y) L^{X^x}(du,dy)\right\} \Big)(f), g \right\rangle_{L^2([0,T])} \to 0$$

as $n \to \infty$ for every $f \in L^2([0,T])$ and $g \in C_0^\infty([0,T])$. It suffices to show that the Wiener transform goes to zero.

Then, as we did for Lemma A.5, using Girsanov's theorem we have

$$\begin{aligned}
&\left| E\left[\mathcal{E}\left(\textstyle\int_0^T f(u) dB_u\right) \left(\exp\left\{-\textstyle\int_s^t \int_{\mathbb{R}} b_n(u,y) L^{X^{n,x}}(du,dy)\right\} - \exp\left\{-\textstyle\int_s^t \int_{\mathbb{R}} b(u,y) L^{X^x}(du,dy)\right\}\right)\right] \right| \\
&= \Bigg| E\Bigg[\exp\left\{-\int_s^t \int_{\mathbb{R}} b_n(u,y) L^{B^x}(du,dy)\right\} \mathcal{E}\left(\int_0^T (b_n(u,B_u^x) + f(u)) dB_u\right) \\
&\quad - \exp\left\{-\int_s^t \int_{\mathbb{R}} b(u,y) L^{B^x}(du,dy)\right\} \mathcal{E}\left(\int_0^T (b(u,B_u^x) + f(u)) dB_u\right)\Bigg]\Bigg| \\
&\le \Bigg| E\Bigg[\left(\exp\left\{-\int_s^t \int_{\mathbb{R}} \tilde{b}_n(u,y) L^{B^x}(du,dy)\right\} - \exp\left\{-\int_s^t \int_{\mathbb{R}} \tilde{b}(u,y) L^{B^x}(du,dy)\right\}\right) \\
&\quad \times \exp\left\{\int_s^t \hat{b}'(u,B_u^x) du\right\} \mathcal{E}\left(\int_0^T (b(u,B_u^x) + f(u)) dB_u\right)\Bigg]\Bigg| \\
&\quad + \Bigg| E\Bigg[\left(\mathcal{E}\left(\int_0^T (b_n(u,B_u^x) + f(u)) dB_u\right) - \mathcal{E}\left(\int_0^T (b(u,B_u^x) + f(u)) dB_u\right)\right) \\
&\quad \times \exp\left\{-\int_s^t \int_{\mathbb{R}} \tilde{b}_n(u,y) L^{B^x}(du,dy)\right\} \exp\left\{\int_s^t \hat{b}'(u,B_u^x) du\right\}\Bigg]\Bigg| \\
&=: I_n + II_n.
\end{aligned}$$

For term I_n we define $p := \frac{1+\varepsilon}{\varepsilon}$ for a suitable $\varepsilon > 0$ and then apply Hölder's inequality with exponent $1+\varepsilon$ on the stochastic exponential. Then we apply Cauchy-Schwarz inequality and bound the factor with $\|\hat{b}'\|_\infty$, and finally we use inequality $|e^x - 1| \le |x|(e^x + 1)$. As a result we obtain

$$
\begin{aligned}
I_n =& \bigg| E\bigg[\exp\left\{ -\int_s^t \int_{\mathbb{R}} \tilde{b}(u,y) L^{B^x}(du,dy) \right\} \left(\exp\left\{ -\int_s^t \int_{\mathbb{R}} (\tilde{b}_n(u,y) - \tilde{b}(u,y)) L^{B^x}(du,dy) \right\} - 1 \right) \\
&\times \exp\left\{ \int_s^t \hat{b}'(u, B_u^x) du \right\} \mathcal{E}\left(\int_0^T (b(u, B_u^x) + f(u)) dB_u \right) \bigg] \bigg| \\
\lesssim& E\bigg[\exp\left\{ -2p \int_s^t \int_{\mathbb{R}} \tilde{b}(u,y) L^{B^x}(du,dy) \right\} \\
&\times \bigg| \left(\exp\left\{ -\int_s^t \int_{\mathbb{R}} (\tilde{b}_n(u,y) - \tilde{b}(u,y)) L^{B^x}(du,dy) \right\} - 1 \right)^{2p} \bigg| \bigg]^{1/(2p)} \\
&\times E\bigg[\mathcal{E}\left(\int_0^T (b(u, B_u^x) + f(u)) dB_u \right)^{1+\varepsilon} \bigg]^{1/(1+\varepsilon)} \\
\lesssim& E\bigg[\left| \int_s^t \int_{\mathbb{R}} (\tilde{b}_n(u,y) - \tilde{b}(u,y)) L^{B^x}(du,dy) \right|^{2p} \bigg(\exp\left\{ -\int_s^t \int_{\mathbb{R}} \tilde{b}_n(u,y) L^{B^x}(du,dy) \right\} \\
&+ \exp\left\{ -\int_s^t \int_{\mathbb{R}} \tilde{b}(u,y) L^{B^x}(du,dy) \right\} \bigg)^{2p} \bigg]^{1/(2p)}
\end{aligned}
$$

where in the last inequality we choose $\varepsilon > 0$ small enough so that the stochastic exponential is bounded due to Lemma A.1. Then Minkowski's inequality gives

$$
\begin{aligned}
(I_n)^{2p} \lesssim& E\left[|V_n|^{2p} \exp\left\{ -2p \int_s^t \int_{\mathbb{R}} \tilde{b}_n(u,y) L^{B^x}(du,dy) \right\} \right] \\
&+ E\left[|V_n|^{2p} \exp\left\{ -2p \int_s^t \int_{\mathbb{R}} \tilde{b}(u,y) L^{B^x}(du,dy) \right\} \right]
\end{aligned} \tag{A.14}
$$

where

$$
V_n := \int_s^t \int_{\mathbb{R}} (\tilde{b}_n(u,y) - \tilde{b}(u,y)) L^{B^x}(du,dy).
$$

Then Cauchy-Schwarz inequality and Lemma A.3 give

$$
\begin{aligned}
E\bigg[|V_n|^{2p} \exp\bigg\{ -2p \int_s^t \int_{\mathbb{R}} \tilde{b}_n(u,y) L^{B^x}(du,dy) \bigg\} \bigg] \leq& \\
\leq E\left[|V_n|^{4p} \right]^{1/2} E\left[\exp\left\{ -4p \int_s^t \int_{\mathbb{R}} \tilde{b}_n(u,y) L^{B^x}(du,dy) \right\} \right]^{1/2}& \\
\lesssim E\left[|V_n|^{4p} \right]^{1/2}.&
\end{aligned} \tag{A.15}
$$

Finally, using representation (2.10) in the Section 2, Minkowski's inequality, Burkholder-Davis-Gundy's inequality in the first two terms and Hölder's inequality in the last term we obtain

$$\begin{aligned}
E\big[|V_n|^p\big] &= E\Bigg[\bigg|\int_s^t (\tilde{b}_n(u,B_u^x)-\tilde{b}(u,B_u^x))dB_u + \int_{T-t}^{T-s} (\tilde{b}_n(T-u,\hat{B}_u^x)-\tilde{b}(T-u,\hat{B}_u^x))dW_u \\
&\quad - \int_{T-t}^{T-s} (\tilde{b}_n(T-u,\hat{B}_u^x)-\tilde{b}(T-u,\hat{B}_u^x))\frac{\hat{B}_u}{T-u}du\bigg|^p\Bigg] \\
&\le E\left[\left|\int_s^t (\tilde{b}_n(u,B_u^x)-\tilde{b}(u,B_u^x))dB_u\right|^p + E\left[\left|\int_{T-t}^{T-s} (\tilde{b}_n(T-u,\hat{B}_u^x)-\tilde{b}(T-u,\hat{B}_u^x))dW_u\right|^p\right]\right. \\
&\quad + E\left[\left|\int_{T-t}^{T-s} (\tilde{b}_n(T-u,\hat{B}_u^x)-\tilde{b}(T-u,\hat{B}_u^x))\frac{\hat{B}_u}{T-u}du\right|^p\right] \\
&\le E\left[\left[\int_s^t |\tilde{b}_n(u,B_u^x)-\tilde{b}(u,B_u^x)|^2du\right]^{p/2} + E\left[\int_{T-t}^{T-s} |\tilde{b}_n(T-u,\hat{B}_u^x)-\tilde{b}(T-u,\hat{B}_u^x)|^2du\right]^{p/2}\right] \\
&\quad + E\left[\left|\int_{T-t}^{T-s} (\tilde{b}_n(T-u,\hat{B}_u^x)-\tilde{b}(T-u,\hat{B}_u^x))\frac{\hat{B}_u}{T-u}du\right|^p\right].
\end{aligned}$$

By dominated convergence, all terms converge to zero as $n\to\infty$. In order to justify that the third term also converges to 0 one needs to use the estimate in Lemma A.4. The second term in (A.14) is estimated in the same way. Similarly, one can also bound II_n.

Lemma A.8. *Let $b_n:[0,T]\times\mathbb{R}\to\mathbb{R}$, $n\ge 0$ be as in (3.1) and $X_t^{n,x}$ the corresponding strong solutions with drift coefficients b_n. Then, for any compact subset $K\subset\mathbb{R}$ and $p\ge 1$*

$$\sup_{n\ge1}\sup_{x\in K}\sup_{t\in[0,T]} E\left[\left(\frac{\partial}{\partial x}X_t^{n,x}\right)^p\right]\le C_{K,p}$$

for a constant $C_{K,p}>0$ depending on K and p. Here, $\frac{\partial}{\partial x}X_t^{n,x}$ is the first variation process of $X_t^{n,x}$, $n\ge 1$ (see Proposition 2.4).

Proof. The proof of this result relies on the proof of (A.12) in Theorem A.6 by observing that $\frac{\partial}{\partial x}X_t^{n,x}=D_0X_t^{n,x}$ by Proposition 2.4. Then following exactly the same steps as in Theorem A.6 we see that all computations can be done for an arbitrary power $p\ge1$. Finally, from the term II_n^1 in the proof of Theorem A.6 one can see that $\sup_{n\ge1}\sup_{x\in K}\sup_{t\in[0,T]}E\left[\left(\frac{\partial}{\partial x}X_t^{n,x}\right)^p\right]<\infty$. □

A.4. **Proof of Proposition 3.3:** First, start observing that, for any given $p\ge1$, we have

$$\begin{aligned}
E\left[|X_t^{n,x}|^p\right] &\lesssim |x|^p + \int_0^t E\left[|\tilde{b}_n(u,X_u^{n,x})|^p\right]du + \int_0^t E\left[|\hat{b}(u,X_u^{n,x})|^p\right]du + E\left[|B_t|^p\right] \\
&\lesssim |x|^p+|t|^p + C\int_0^t E\left[|X_u^{n,x}|^p\right]du
\end{aligned}$$

due to the uniform boundedness of $\tilde{b}_n$, the continuity of $\hat{b}$ and Hölder continuity of the Brownian motion. Then, Grönwall's inequality gives

$$\sup_{n\ge1}E\left[|X_t^{n,x}|^p\right]\le C. \tag{A.16}$$

Now, assume that $0\le s<t\le T$. Then

$$\begin{aligned}
X_t^{n,x}-X_s^{n,y} &= x-y+\int_0^t b_n(u,X_u^{n,x})du - \int_0^s b_n(u,X_u^{n,y})du + B_t - B_s \\
&= x-y+\int_s^t b_n(u,X_u^{n,x})du + \int_0^s (b_n(u,X_u^{n,x})-b_n(u,X_u^{n,y}))du + B_t - B_s.
\end{aligned}$$

Now since b_n has linear growth together with (A.16), the uniform boundedness of $\tilde{b}_n$ and Hölder continuity of the Brownian motion yield

$$E\left[|X_t^{n,x} - X_s^{n,y}|^2\right] \lesssim |x-y|^2 + |t-s| + E\left[\left|\int_0^s (b_n(u, X_u^{n,x}) - b_n(u, X_u^{n,y}))du\right|^2\right].$$

Then we use the fact that $X_t^{n,s,\cdot}$ is a stochastic flow of diffeomorphisms (see e.g. [18]), the mean value theorem and Lemma A.8 in order to obtain

$$\begin{aligned}
&E\left[\left|\int_0^s (b_n(u, X_u^{n,x}) - b_n(u, X_u^{n,y}))du\right|^2\right] \\
&= |x-y|^2 E\left[\left|\int_0^s \int_0^1 b_n'(u, X_u^{n,x+\tau(y-x)}) \frac{\partial}{\partial x} X_u^{n,x+\tau(y-x)}) d\tau du\right|^2\right] \\
&\leq C|x-y|^2 \int_0^1 E\left[\left|\int_0^s b_n'(u, X_u^{n,x+\tau(y-x)}) \frac{\partial}{\partial x} X_u^{n,x+\tau(y-x)}) du\right|^2\right] d\tau \\
&= C|x-y|^2 \int_0^1 E\left[\left|\frac{\partial}{\partial x} X_s^{n,x+\tau(y-x)} - (1-\tau)\right|^2\right] d\tau \\
&\leq C|x-y|^2 \sup_{\substack{s\in[0,T] \\ x\in K}} E\left[\left|\frac{\partial}{\partial x} X_s^{n,x}\right|^2\right] \\
&\leq C|x-y|^2.
\end{aligned}$$

Altogether

$$E\left[\left|X_t^{n,x} - X_s^{n,y}\right|^2\right] \leq C\left(|t-s| + |x-y|^2\right)$$

for a finite constant $C > 0$ independent of n.

To conclude, we use Fatou's lemma applied to a subsequence and the fact that $X_t^{n,x} \to X_t^x$ in $L^2(\Omega)$ as $n \to \infty$ due to Theorem A.6.

A.5. **Proof of Theorem 3.4.** First of all, observe that for any smooth function with compact support $\varphi \in C_0^\infty(U, \mathbb{R})$ and $t \in [0,T]$, the sequence of random variables

$$\langle X_t^n, \varphi \rangle := \int_U X_t^{n,x} \varphi(x) dx$$

converges weakly in $L^2(\Omega)$ to $\langle X_t, \varphi \rangle$ by using the Wiener transform following the same steps as in Lemma A.5.

Then for all measurable sets $A \in \mathcal{F}$, $\varphi \in C_0^\infty(\mathbb{R})$ and using Cauchy-Schwarz inequality we have

$$E[\mathbf{1}_A \langle X_t^{n_k,x} - X_t^x, \varphi' \rangle] \leq \|\varphi'\|_{L^2(U)} |U|^{1/2} \left(\sup_{x\in \text{supp}(U)} E\left[\mathbf{1}_A (X_t^{n_k,x} - X_t^x)^2\right]\right)^{1/2} < \infty$$

where the last quantity is finite by Proposition 3.3. Then by Theorem A.6 we see that

$$\lim_{k\to\infty} E[\mathbf{1}_A \langle X_t^{n_k,x} - X_t^x, \varphi' \rangle] = 0.$$

In addition, by virtue of Lemma A.8 we have that

$$\sup_{n\geq 0} E\|X_t^{n,x}\|^2_{W^{1,2}(U)} < \infty,$$

that is $x \mapsto X_t^{n,x}$ is bounded in $L^2(\Omega, W^{1,2}(U))$. As a result, the sequence $X_t^{n,x}$ is weakly relatively compact in $L^2(\Omega, W^{1,2}(U))$, see e.g. [19, Theorem 10.44], and therefore there exists a subsequence n_k, $k \geq 0$ such that $X_t^{n_k,x}$ converges weakly to some element $Y_t \in L^2(\Omega, W^{1,2}(U))$ as $k \to \infty$. Let us denote by Y_t' the weak derivative of Y_t.

Then

$$E[\mathbf{1}_A \langle X_t^x, \varphi' \rangle] = \lim_{k\to\infty} E[\mathbf{1}_A \langle X_t^{n_k,x}, \varphi' \rangle] = -\lim_{k\to\infty} E[\mathbf{1}_A \langle \frac{\partial}{\partial x} X_t^{n_k,x}, \varphi \rangle] = -E[\mathbf{1}_A \langle Y_t', \varphi \rangle].$$

So

$$\langle X_t, \varphi' \rangle = -\langle Y_t', \varphi \rangle, \quad P-a.s. \tag{A.17}$$

Finally, we need to show that there exists a measurable set $\Omega_0 \subset \Omega$ with full measure such that $X_t^{\cdot}$ has a weak derivative on this subset. To this end choose a sequence $\{\varphi_n\}$ in $C^\infty(\mathbb{R})$ dense in $W^{1,2}(U)$. Choose a measurable subset Ω_n of Ω with full measure such that (A.17) holds on Ω_n with φ replaced by φ_n. Then $\Omega_0 := \cap_{n\geq 1}\Omega_n$ satisfies the desired property.

Corollary A.9. *Let $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ as in (1.6) and X_t^x the corresponding strong solution of* (1.5). *Then, for any compact subset $K \subset \mathbb{R}$ and $p \geq 1$*

$$\sup_{x\in K} \sup_{t\in[0,T]} E\left[\left(\frac{\partial}{\partial x} X_t^x\right)^p\right] \leq C_{K,p}$$

for a constant $C_{K,p} > 0$ depending on K and p. Here, $\frac{\partial}{\partial x} X_t^x$ is the first variation process of X_t^x, (see Proposition 3.5).

Proof. This is a direct consequence of Lemma A.8 in connection with Fatou's lemma. □

A.6. **Proof of Proposition 3.5:** By Theorem 3.4 we know that the sequence $\{X_t^{n,x}\}_{n\geq 0}$ converges weakly to X_t^x in $L^2(\Omega, W^{1,2}(U))$. Therefore, it is enough to check that our candidate is the limit of $\frac{\partial}{\partial x} X_t^{n,x}$ in the weak topology of $L^2(U \times \Omega)$ for any open bounded $U \subset \mathbb{R}$, i.e.

$$\int_U \mathcal{W}\left(\exp\left\{ -\int_0^t \int_{\mathbb{R}} b_n(u,y) L^{X^{n,x}}(du,dy) \right\} - \exp\left\{ -\int_0^t \int_{\mathbb{R}} b(u,y) L^{X^x}(du,dy) \right\} \right)(f) g(x) dx$$

converges to 0 as $n \to \infty$ for every $f \in L^2([0,T])$ and $g \in C_0^\infty(U)$. This can be shown following exactly the same steps as in Proposition 3.2 by integrating I_n and II_n against $g(x)$ over $x \in U$. The only difference here is that we need all bounds to be uniformly in $x \in U$. At the end, one needs to show that

$$\sup_{n\geq 0} \sup_{x\in \mathrm{supp}(U)} E\left[|V_n|^{2p} e^{-2p\int_0^t \int_{\mathbb{R}} \tilde{b}_n(u,y) L^{B^x}(du,dy)} \right] < \infty$$

where

$$V_n := \int_0^t \int_{\mathbb{R}} (\tilde{b}_n(u,y) - \tilde{b}(u,y)) L^{B^x}(du,dy)$$

which holds by Lemma A.3 and the fact that $\tilde{b}_n$, $n \geq 0$, is uniformly bounded. For II_n one can follow similar steps and use Remark A.2.

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