Credit contagion in a long range dependent macroeconomic factor model

Francesca Biagini [*] Serena Fuschini [†] Claudia Klüppelberg [‡]

September 4, 2009

Abstract

We propose a new default contagion model, where default may originate from the performance of a specific firm itself, but can also be directly influenced by defaults of other firms. The default intensities of our model depend on smoothly varying macroeconomic variables, driven by a long range dependent process. In particular, we focus on the pricing of defaultable derivatives, whose default depends on the macroeconomic process and may be affected by the contagion effect. In our approach we are able to provide explicit formulas for prices of defaultable derivatives at any time $t \in [0, T]$. Finally we calculate some examples explicitly, where the macroeconomic factor process is given by a functional of the fractional Brownian motion with Hurst index $H > \frac{1}{2}$.

2000 MSC Subject Classifications: primary: 91B70
secondary: 91B28, 60G99

Keywords: credit risk, contagion modeling, credit intensity, latent process, macroeconomic variables process, long range dependence, fractional Brownian motion, pricing defaultable derivatives.

[*]Department of Mathematics, Ludwig-Maximilians Universität, D-80333 Munich, Germany, email: biagini@math.lmu.de

[†]Department of Mathematics, University of Bologna, I-40127 Bologna, Italy, email: fuschini@dm.unibo.it

[‡]Zentrum Mathematik, Technische Universität München, D-85747 Garching, Germany, email: cklu@ma.tum.de, url: www-m4.ma.tum.de

1 Introduction

The ongoing financial crisis has been triggered by the dramatic rise in mortgage delinquencies and foreclosures in the United States. This crisis has not only manifested the weaknesses in financial industry regulation, but also of the financial models used for pricing instruments of mortgage pools like MBSs and CDOs. In particular, the systemic risk has been desastrously underestimated. It has been industry standard to model contagion within a pool of credits by an intensity model, where the intensities of surviving credits may increase at default of some credit. This approach increases the probability for default of dependent credits, so has no direct effect. In a static model, Davis and Lo [8] suggested a direct contagion model, which is able to capture the immediate effect of one credit default to other credits in a pool.

We investigate a dynamic version of the direct contagion model of Davis and Lo [8], which is based on interacting intensities. Each default indicator process may be influenced by the default of other firms, which is modeled by an indicator variable representing the contagion possibility. In addition, we allow the default intensities to depend on smoothly varying macroeconomic variables (for example supply and demand, interest rates, the gross national product, or other measures of economics activities), which are often modeled by a Markov state vector leading to affine models; see e.g. Duffie [10] and Duffie, Filipovic and Schachermayer [11].

It is, however, well-known that many macroeconomic processes show a long-range dependence effect; see e.g. Henry and Zaffaroni [14]. Consequently, in this paper we model the latent macroeconomic process governing the default intensities by a long range dependent process, here exemplified by a one-dimensional process, which stands for instance for a weighted mean of a vector of macroeconomic variables.

In this paper we focus on the pricing of defaultable derivatives depending on the macroeconomic process and affected by the contagion effect. We remark that we are not assuming that the primary assets on the market are driven by a long range dependent process. Hence no arbitrage problem arises in the use of our model. For a discussion on this topic we refer to Björk and Hult [2] or Øksendal [15]. In our model the long range dependent macroeconomic process enters as a progressively measurable process into the default intensity. By usual no-arbitrage arguments the price of a contingent claim at time t is given by the conditional expectation under the pricing measure, which we suppose to be given by the market.

In this not at all standard model we are able to provide explicit formulas for the derivative price at any time $t \in [0, T]$. We discuss suitable long range dependent models for the macroeconomic process and calculate some examples, where the macroeconomic factor is given by a functional of the fractional Brownian motion with Hurst index $H > \frac{1}{2}$.

Our paper is organized as follows. In Section 2 we present our model and the contagion mechanism for instantaneous contagion, modeling the intensity as a function of the macroeconomic process. We explain the model in detail in Section 2 - it is an intensity-based model - and we present all assumptions here. We present a specific example in Section 3 and calculate its infinitesimal generators of the default indicator process and the default number process. Afterwards, we calculate a defaultable derivatives price in Section 4 at first conditionally on the latent process. We conclude the section with a specific example, calculating the prices of a defaultable bond under contagion. Finally, in Section 5, we introduce a general long range dependent fractional macroeconomic process as intensity process. We obtain an explicit formula, which can be evaluated numerically. In Section 5 we discuss some specific macroeconomic models and give an explicit financial example.

2 The credit model

2.1 The default model

We consider a portfolio of m firms indexed by $i \in \{1, \ldots, m\}$. Its default state is described by a *default indicator process*

$$Z_t = (Z_t(1), \ldots, Z_t(m)), \qquad t \geq 0,$$

with values in the set $\{0,1\}^m$. For every $i \in \{1, \ldots, m\}$ the random variable $Z_t(i)$ indicates, if the firm i has defaulted or not by time t, i.e. $Z_t(i) = 1$ if the firm i has defaulted by time t and $Z_t(i) = 0$ otherwise.

Aiming at an extension of the idea of Davis and Lo [7] as indicated in [8], Section 3, to a dynamic setting we distinguish between default *caused by itself* and default *caused by contagion*, based on the default of some other firms. To this purpose we introduce the *self-default indicator process*

$$Y_t = (Y_t(1), \ldots, Y_t(m)), \qquad t \geq 0,$$

with values in $\{0,1\}^m$, where again $Y_t(i) = 1$ if the firm i has defaulted by time t by itself and $Y_t(i) = 0$ otherwise. We denote by τ_i the default time of the i-th firm for $i \in \{1, \ldots, m\}$ and by $\mathcal{I}$ the indicator function, then

$$Y_t(i) = \mathcal{I}_{\{\tau_i \leq t\}}, \quad i = 1, \ldots, m.$$

Next we model contagion by using a *contagion matrix indicator process*: if firm i defaults by itself at some time t, then $C_t(i,j)$ determines whether infection of default from firm i to firm j takes place or not at time t.

We assume that, if default of firm i causes default of firm j, then this happens instantaneously resulting in $C_t(i,j) = 1$. More precisely, for any time $t \geq 0$,

$$C_t(i,j) = \begin{cases} 1 & \text{if the default of firm } i \text{ causes default of firm } j \text{ at time } t, \\ 0 & \text{otherwise}. \end{cases} \tag{2.1}$$

This results in a representation of the default indicator process of firm j

$$\begin{aligned} Z_t(j) &= Y_t(j) + (1 - Y_t(j))\Big(1 - \prod_{i \neq j}(1 - C_{t\wedge\tau_i}(i,j)Y_t(i))\Big) \\ &= Y_t(j) + (1 - Y_t(j))\Big(1 - \prod_{i \neq j}(1 - C_{\tau_i}(i,j)Y_t(i))\Big), \quad t \geq 0. \end{aligned} \tag{2.2}$$

Since firm j is influenced by itself, we define $C_t(j,j) \equiv 1$ for all $j \in \{1,\dots,m\}$ and $t \geq 0$. Then equation (2.2) can also be written as

$$Z_t(j) = 1 - \prod_{i=1}^{m} \left(1 - C_{\tau_i}(i,j) Y_t(i)\right) , \quad t \geq 0 . \tag{2.3}$$

The defaults of the portfolio, either by itself or by infection, are caused by fluctuations in the macroeconomic environment, which we model by a state variable process $\Psi = (\Psi_t)_{t\geq 0}$ with values in $\mathbb{R}^d$ for $d \in \mathbb{N}$, representing the evolution of macroeconomic variables such as supply and demand, interest rates, the gross national product, or other measures of economics activities. In the literature Ψ is usually taken to be Markovian, so that the overall model of the system, given by $(\Psi_t, Y_t, C_t)_{t\geq 0}$ is Markovian.

It is, however, well-known that many macroeconomic variables show a long-range dependence effect; see e.g. Henry and Zaffaroni [14]. Consequently, we model the macroeconomic environment by a long range dependent process $(\Psi_t)_{t\geq 0}$ to be specified later (see Section 5).

2.2 The probability space

The overall state of our system is described by the process $(\Psi_t, Y_t, C_t)_{t\geq 0}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the filtration

$$\mathcal{F}_t := \mathcal{F}_t^{\Psi} \vee \mathcal{F}_t^{Y} \vee \mathcal{F}_t^{C} , \quad t \geq 0 ,$$

where $(\mathcal{F}_t^{\Psi})_{t\geq 0}$, $(\mathcal{F}_t^{Y})_{t\geq 0}$ and $(\mathcal{F}_t^{C})_{t\geq 0}$ are the natural filtrations associated to the processes Ψ, Y and C, respectively. Here we assume that the agent on the market knows if a firm has defaulted by itself or not and the contagion structure among the firms. Moreover, we define the filtration

$$\mathcal{G}_t := \mathcal{F}_\infty^{\Psi} \vee \mathcal{F}_t^{Y} \vee \mathcal{F}_t^{C} , \quad t \geq 0 .$$

As explained in Frey and Backhaus [12] we assume that investors have access to $(\mathcal{F}_t)_{t\geq 0}$, whereas the larger filtration $(\mathcal{G}_t)_{t\geq 0}$, which contains information about the whole path $(\Psi_t)_{t\geq 0}$ serves mainly theoretical purposes. Finally, we assume that all filtrations satisfy the usual hypotheses of completeness and right-continuity.

From now on we work under the following assumptions.

Assumption 2.1. (1) We remain in the framework of most reduced-form credit risk models in the literature and assume that the dynamic of Ψ is not affected by the evolution of the default indicator process Z. This has the advantage that we first model the dynamic of Ψ and, in a second step, the conditional distribution of the default indicator process Z for a given realization of the macroeconomic factor process Ψ. In particular, we require that Ψ is not affected by the evolution of the default indicator process Y and the contagion matrix C. In mathematical terms this means that for every bounded $\mathcal{F}_\infty^{\Psi}$-measurable random variable η,

$$\mathbb{E}\left[\eta \mid \mathcal{F}_t\right] = \mathbb{E}\left[\eta \mid \mathcal{F}_t^{\Psi}\right] , \quad t \geq 0 .$$

(2) The processes $(Y_t(i))_{t\geq 0}$ for $i \in \{1,\ldots,m\}$, $(C_t(i,j))_{t\geq 0}$ for $i,j \in \{1,\ldots,m\}, i \neq j$, are conditionally independent with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$. This means that for every $\{i_1,\ldots,i_k\} \subseteq \{1,\ldots,m\}$ and for every choice $(\alpha_1,\beta_1),\ldots,(\alpha_l,\beta_l)$ in $\{(i,j) \in \{1,\ldots,m\}^2 \mid i \neq j\}$ we have for all $t_j \geq t, j = 1,\ldots,k$, and $s_n \geq t, n = 1,\ldots,l$

$$\begin{aligned}
&\mathbb{E}\left[\prod_{j=1}^{k}\prod_{n=1}^{l} f\left(Y_{t_j}(i_j)\right) g\left(C_{s_n}(\alpha_n,\beta_n)\right) \,\middle|\, \mathcal{G}_t\right] \\
= &\prod_{j=1}^{k} \mathbb{E}\left[f\left(Y_{t_j}(i_j)\right) \mid \mathcal{G}_t\right] \prod_{n=1}^{l} \mathbb{E}\left[g\left(C_{s_n}(\alpha_n,\beta_n)\right) \mid \mathcal{G}_t\right] \\
= &\prod_{j=1}^{k} \mathbb{E}\left[f\left(Y_{t_j}(i_j)\right) \mid \mathcal{F}^{\Psi}_{\infty} \vee \mathcal{F}^{Y(i_j)}_t\right] \prod_{n=1}^{l} \mathbb{E}\left[g\left(C_{s_n}(\alpha_n,\beta_n)\right) \mid \mathcal{F}^{\Psi}_{\infty} \vee \mathcal{F}^{C(\alpha_n,\beta_n)}_t\right]
\end{aligned}$$

for $f,g : \{0,1\} \to \mathbb{R}$, with $\mathcal{F}^{Y(i)}_t := \sigma(Y_u(i) : u \leq t)$ and $\mathcal{F}^{C(i,j)}_t := \sigma(C_u(i,j) : u \leq t)$, for every $i,j \in \{1,\ldots,m\}$, $i \neq j$.

(3) For every $i \in \{1,\ldots,m\}$ the self-default indicator process $(Y_t(i))_{t\geq 0}$ is a doubly stochastic indicator process with respect to the filtration $(\mathcal{F}^{\Psi}_{\infty} \vee \mathcal{F}^{Y}_t)_{t\geq 0}$ with stochastic intensity depending only on the path of $(\Psi_t)_{t\geq 0}$. In particular we assume that the stochastic intensity of firm i is of the form $\lambda^i(t,\Psi_t)$ for $\lambda^i : \mathbb{R}^2 \to \mathbb{R}_+$ a continuous function. This means that

$$\mathbb{E}\left[1 - Y_s(i) \mid \mathcal{G}_t\right] = (1 - Y_t(i)) e^{-\int_t^s \lambda^i(u,\Psi_u)du}, \quad s \geq t, \tag{2.4}$$

where the last equality holds by Corollary 5.1.5 of Bielecki and Rutkowski [5].

(4) The contagion processes $(C_t(i,j))_{t\geq 0}$ for $i \neq j$ are $\mathcal{F}^{\Psi}_{\infty}$–conditionally time-inhomogeneous Markov chains; i.e. for every function $f : \{0,1\} \to \mathbb{R}$,

$$\mathbb{E}\left[f(C_s(i,j)) \mid \mathcal{F}^{\Psi}_{\infty} \vee \mathcal{F}^{C(i,j)}_t\right] = \mathbb{E}\left[f(C_s(i,j)) \mid \mathcal{F}^{\Psi}_{\infty} \vee \sigma(C_t(i,j))\right], \qquad s \geq t.$$

For all $i,j \in \{1,\ldots,m\}$, $i \neq j$ and $k,h \in \{0,1\}$, we denote the conditional transition probabilities by

$$p^{ij}_{ts}(k,h) = \mathbb{P}\left[C_s(i,j) = h \mid \mathcal{F}^{\Psi}_{\infty} \vee \sigma(C_t(i,j) = k)\right],$$

and assume that $(p^{ij}_{ts}(k,h))_{s\in\mathbb{R}^+}$ is a continuous process for every $t \in \mathbb{R}^+$, $i,j \in \{1,\ldots,m\}$ and $k,h \in \{0,1\}$.

In the sequel we will use the fact that on $\{C_t(i,j) = k\}$

$$p^{ij}_{ts}(k,h) = \frac{\mathbb{P}\left[C_s(i,j) = h, C_t(i,j) = k \mid \mathcal{F}^{\Psi}_{\infty}\right]}{\mathbb{P}\left[C_t(i,j) = k \mid \mathcal{F}^{\Psi}_{\infty}\right]}.$$

Note that, unlike the default indicator processes, the processes $(C_t(i,j))_{t\geq 0}$ are allowed to change between 0 and 1 back and forth in time. They model the presence of a business relationship between firm i and firm j, which can be present at time 0, absent at some later time, and come in force again even later.

3 A portfolio with disjoint contagion classes

We want to discuss our model assumptions for the simple case of a credit portfolio with group structure. For simplicity we assume that the matrix C is *time-independent* and *deterministic.* This means that we can divide the credit portfolio of m firms into groups, which we can identify by the following assumptions.

Assumption 3.1. (1) Reflexivity: By definition $C(i,i) = 1$ for all $i \in \{1,\dots,m\}$.

(2) Symmetry: $C(i,j) = C(j,i)$ for all $i,j \in \{1,\dots,m\}$.
The influence of default is symmetric.

(3) Transitivity: $C(i,h)C(h,j) \leq C(i,j)$ for all $i,j,h \in \{1,\dots,m\}$.
If the default of firm i causes firm h to default, and firm h causes firm j to default, then also firm i causes the default of firm j.

Assumptions 3.1 define an equivalence relation on the credit portfolio, i.e. $i \sim j$ if and only if $C(i,j) = 1$. The equivalence relation subdivides the portfolio into disjoint equivalence classes, which we call *contagion classes* and denote by

$$[i] := \{j \in \{1,\dots,m\} \mid C(i,j) = 1\}\,.$$

We assume that the portfolio consists of k contagion classes $[i_1],\dots,[i_k]$, representing for instance business sections or local markets.

By definition (2.2) of the default indicator process we have:

$$Z_t(i) = \begin{cases} 1 \quad \text{if} \quad (Y_t(i) = 1) \quad \vee \quad (\exists j \neq i \quad C(i,j) = 1 \quad \text{s.t.} \quad Y_t(j) = 1) \\ 0 \quad \text{if} \quad (Y_t(i) = 0) \quad \wedge \quad (Y_t(j) = 0 \quad \forall j \neq i \quad \text{s.t.} \quad C(i,j) = 1)\,. \end{cases} \tag{3.1}$$

Given some $i \in \{1,\dots,m\}$, from the definition of the default indicator process in (2.2) and Assumption 3.1 we have

$$Z_t(i) = 0 \quad \Longleftrightarrow \quad Z_t(j) = 0 \quad \forall j \in [i].$$

This means that either all firms of the same contagion class default at the same time or all of them are alive. Here we see that our modeling is different from (and more drastic than) the usual credit risk contagion modeling, where the default of some firm within a group only increases the hazard of all other group members; for examples and further references see Schönbucher [17], Chapter 10.5.

Conditionally on the macroeconomic state variable process Ψ the default indicator process $(Z_t)_{t\geq 0}$ is Markovian. Since in this case C is supposed to be deterministic it is to be expected that the intensities of $(Z_t)_{t\geq 0}$ are inherited in a deterministic way by the default intensities of the self-indicator process $(Y_t)_{t\geq 0}$ as given by (2.4) of Assumption 2.1(3).

This allows us to calculate the conditional generator of the default indicator process as well as of the default number process.

3.1 Conditional infinitesimal generator of the default indicator process

We calculate the conditional infinitesimal generator of the default indicator process $(Z_t)_{t\geq 0}$, where we use Definition 2.2 of Yin and Zhang [20].

Theorem 3.2. *The infinitesimal generator $\mathcal{A}_t$ of the $\mathcal{F}^{\Psi}_{\infty}-$conditional time inhomogeneous Markov process $(Z_t)_{t\geq 0}$ is for any test function $f : \{0,1\}^m \to \mathbb{R}$ given by*

$$\mathcal{A}^Z_t f(z) = \prod_{j=1}^{m} \prod_{u\in[j]} (1 - |z_j - z_u|) \sum_{i=1}^{m} \left[f(z^{(i)}) - f(z) \right] (1 - z_i)\lambda^i(t, \Psi_t) \,, \quad z \in \{0,1\}^m \,, \tag{3.2}$$

where

$$z^{(i)} = (z_1 + (1 - z_1)C(i, 1), \dots, z_m + (1 - z_m)C(i, m)). \tag{3.3}$$

Proof. By Proposition 11.3.1 of [5] we obtain that the infinitesimal conditional generator of Z_t is given by

$$\mathcal{A}^Z_t f(z) = \sum_{w\neq z} [f(w) - f(z)] \, \lambda^Z_t(z, w)$$

for any $f : \{0,1\}^m \to \mathbb{R}$, where $\lambda^Z_t(z, w)$ denotes the $\mathcal{F}^{\Psi}_{\infty}-$conditional stochastic intensity of the process Z from state z to state w, given by

$$\lambda^Z_t(z, w) := \lim_{h\to 0} \frac{p^Z_{tt+h}(z, w) - p^Z_{tt}(z, w)}{h} \tag{3.4}$$

with conditional transition probabilities

$$p^Z_{tt+s}(z, w) := \frac{\mathbb{P}\left(Z_{t+s} = w, Z_t = z \mid \mathcal{F}^{\Psi}_{\infty} \right)}{\mathbb{P}\left(Z_t = z \mid \mathcal{F}^{\Psi}_{\infty} \right)} =: \mathbb{P}^{\Psi}\left(Z_{t+s} = w \mid Z_t = z \right), \quad t, s \geq 0.$$

and

$$p^Z_{tt}(z, w) = \delta_{z,w} := \begin{cases} 1 & \text{if } z = w, \\ 0 & \text{otherwise.} \end{cases}$$

Since the different contagion classes are independent, we factorize the transition probabilities as follows:

$$p^Z_{tt+s}(z, w) := \prod_{h=1}^{k} \mathbb{P}^{\Psi}\left(\cap_{i\in[i_h]} Z_{t+s}(i) = w_i \mid \cap_{i\in[i_h]} Z_t(i) = z_i \right). \tag{3.5}$$

Recall that in each factor in (3.5) the states $w_i, z_i \in \{0,1\}$ and that 1 is the absorbing state. Because of the deterministic contagion mechanism, at any time either the whole contagion class of firms has defaulted or has not, i.e.

$$\exists\, i \in [i_h] \text{ s.t. } \{Z_t(i) = 0\} \quad \Leftrightarrow \quad \cap_{i\in[i_h]}\{Z_t(i) = 0\}\,. \tag{3.6}$$

Moreover, by definition (3.1) we have that

$$\cap_{i\in[i_h]}\{Z_t(i) = 0\} \quad \Leftrightarrow \quad \cap_{i\in[i_h]}\{Y_t(i) = 0\}\,. \tag{3.7}$$

Setting $\overline{Z}_t(i_h) := \prod_{i\in[i_h]} Z_t(i)$, which also is a 0-1 random variable, we have

$$\cap_{i\in[i_h]}\{Z_t(i) = 0\} \quad \Leftrightarrow \quad \{\overline{Z}_t(i_h) = 0\}$$

and by (2.4), (3.6) and (3.7) we get

$$\mathbb{E}[1 - \overline{Z}_{t+s}(i_h) \mid \mathcal{G}_t] = (1 - \overline{Z}_t(i_h))e^{-\int_t^{t+s}\sum_{i\in[i_h]}\lambda^i(u,\Psi_u)du}. \tag{3.8}$$

Given $z = (z_1, \ldots, z_m) \in \{0,1\}^m$ we define for $h \in \{1, \ldots, k\}$

$$z^{[i_h]} := (z_1 + (1 - z_1)C(i_h, 1), \ldots, z_m + (1 - z_m)C(i_h, m)),$$

representing the fact that only group $[i_h]$ can default and, if it does, then all other components of z remain the same. Then by (3.5) and (3.8), taking the limit in (3.4), we obtain for $z^{[i_h]} \neq z$

$$\lambda_t^Z(z, z^{[i_h]}) = \prod_{j\in[i_h]}(1 - z_j)\sum_{i\in[i_h]}\lambda^i(t, \Psi_t)$$

and $\lambda_t^Z(z, w) = 0$ for $w \neq z^{[i_h]}$ or $w = z$.
Then the infinitesimal generator for elements z such that $z_i = z_j$, if firms i, j are in the same contagion class, is given by

$$A_t^Z f(z) = \sum_{h=1}^{k}\Big[f(z^{[i_h]}) - f(z)\Big]\prod_{j\in[i_h]}(1 - z_j)\sum_{i\in[i_h]}\lambda^i(t, \Psi_t),$$

which can equivalently be represented as

$$A_t^Z f(z) = \sum_{i=1}^{m}\Big[f(z^{(i)}) - f(z)\Big](1 - z_i)\lambda^i(t, \Psi_t)$$

where $z^{(i)}$ is defined as in (3.3). To guarantee that at the same time only defaults in one contagion class take place, we multiply the right hand side by $\prod_{j=1}^{m}\prod_{u\in[j]}(1 - |z_j - z_u|)$, which means that the vector z can not have two different components which correspond to equivalent firms. This gives the form of the generator as in (3.2). $\square$

3.2 Conditional infinitesimal generator of the default number process

We invoke the previous result to calculate the generator of the default number process for the portfolio. To this end we split the group of all firms in l homogeneous groups $G_1, \ldots, G_l$, where each group contains all the firms with the same default intensity. We recall that firms belonging to the same equivalent class $[i]$ have a default intensity given by

$$\lambda_t^{[i]} = \sum_{j\in[i]}\lambda^j(t, \Psi_t).$$

It follows that each homogeneous group G_h is given by the union of a certain number s_h of contagion classes, i.e.

$$G_h = [j_1^h] \cup \cdots \cup [j_{s_h}^h].$$

For every $h \in \{1, \ldots, l\}$, we denote by n_i^h the cardinality of the class $[j_i^h]$ for $i = 1, \cdots, s_h$, and by $\lambda^{G_h}(t, \Psi_t)$ the intensity of every firm belonging to the group G_h. Let

$$M_t(h) := \frac{1}{s_h} \left[\sum_{i \in [j_1^h]} \frac{Z_t(i)}{n_1^h} + \cdots + \sum_{i \in [j_{s_h}^h]} \frac{Z_t(i)}{n_{s_h}^h} \right] \tag{3.9}$$

be the weighted average number of defaults in the group G_h.We now consider the process $M_t := (M_t(1), \ldots, M_t(l))$. Because of the conditional independence of contagion classes the components of this process are also conditionally independent. We calculate the conditional infinitesimal generator of $(M_t)_{t \geq 0}$.

Recall first from our calculations in the proof of Theorem 3.2 that we can not have simultaneous defaults for two different contagion classes, and that inside a contagion class all firms default at the same time. Hence the counting process $(M_t)_{t \geq 0}$ can jump from a state $u = (u_1, \ldots, u_l) = \left(\frac{v_1}{s_1}, \ldots, \frac{v_l}{s_l}\right)$, where $v_k \in \{0, \ldots, s_k\}$ (for $k = 1, \ldots, l$), only to a state of the form $u + \frac{1}{s_k} e_k$, where e_k is the k-th element of the canonical basis of $\mathbb{R}^l$. With an analogous proof as in Lemma 3.4 of Frey and Backhaus [12], we obtain that the transition intensity of M from u into the state $u + \frac{1}{s_k} e_k$ is given by

$$\lambda_t^M \left(u, u + \frac{1}{s_k} e_k \right) = s_k (1 - u_k) \lambda^{G_k}(s, \Psi_s) \, .$$

Then the infinitesimal conditional generator of $(M_t)_{t \geq 0}$ has the following form.

Theorem 3.3. *Let $M_t = (M_t(1), \ldots, M_t(l))$, $t \geq 0$, be the default number process with components defined in* (3.9)*. Under Assumptions 2.1 and 3.1 the infinitesimal generator of this $\mathcal{F}_\infty^\Psi$-conditional Markov process is for any test function $f : \{0, \frac{1}{s_1}, \cdots, 1\} \times \cdots \times \{0, \frac{1}{s_l}, \ldots, 1\} \to \mathbb{R}$ given by*

$$\mathcal{A}_t f(u) = \sum_{k=1}^{l} \left[f\left(u + \frac{1}{s_k} e_k \right) - f(u) \right] s_k (1 - u_k) \lambda^{G_k}(t, \Psi_t) \, .$$

4 The price of credit derivatives as a function of Ψ

We consider the problem of pricing derivatives, whose values are influenced by the contagion mechanism represented by the matrix C and the underlying macroeconomics factors Ψ as described in Section 2.1.

Assumption 4.1 (Market structure; cf. Frey and Backhaus [13], Ass. 3.1.)**.**

(1) The investor information at time t is given by the default history $\mathcal{F}_t$; i.e. the investor knows the latent process Ψ, the self-default indicator process Y and the contagion matrix C up to time t.

(2) The default-free interest rate is deterministic, so that we can w.l.o.g. set it equal to 0. This does not prevent us to include for instance the LIBOR rate as one of the macroeconomic variables processes.

(3) The risk neutral (martingale) pricing measure $\mathbb{P}$ exists and is known, such that the price in t of any $\mathcal{F}_T$-measurable claim $L \in L^1(\Omega, \mathbb{P})$ with maturity $T > 0$ is given by $L_t = \mathbb{E}\left[L \mid \mathcal{F}_t\right]$ for $0 \leq t \leq T$.

Without further specifying the macroeconomic process Ψ we can formulate the following result.

Theorem 4.2. *Let $f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ be a bounded measurable function. Let $\alpha = (\alpha_1, \dots, \alpha_m)$, $\beta = (\beta_1, \dots, \beta_m)$ and $z = (z_1, \dots, z_m)$ be in $\{0,1\}^m$ and $h^{(i)}, k^{(i)} \in \{0,1\}^{m-1}$ for $i = 1, \dots, m$. Set $h_{ii} = k_{ii} := 1$ for $i = 1, \dots, m$, $h_{ij} := [h^{(i)}]_j$ and $k_{ij} := [k^{(i)}]_j$ for $j \neq i$. Then for $t \in [0,T]$*

$$\mathbb{E}\left[f(\Psi_T, Z_T) \mid \mathcal{F}_t\right] = \sum_{z,\alpha,\beta \in \{0,1\}^m} (-1)^{\sum_{j=1}^m \alpha_j z_j} \prod_{j=1}^m z_j^{1-\alpha_j} \prod_{i=1}^m \left(Y_t(i) a_t(i)\right)^{1-\beta_i} \left(1 - Y_t(i)\right)^{\beta_i}$$
$$\times \mathbb{E}\left[f(\Psi_T, z) \prod_{i=1}^m b_{t,T}(i)^{\beta_i} \,\middle|\, \mathcal{F}_t^{\Psi} \right], \tag{4.1}$$

with

$$\begin{aligned} a_t(i) &= \sum_{h^{(i)} \in \{0,1\}^{m-1}} \mathcal{I}_{\{\tilde{h}_i(\alpha,h)=0\}} \mathcal{I}_{\{C^{(i)}_{\tau_i} = h^{(i)}\}} \\ b_{t,T}(i) &= \sum_{h^{(i)},k^{(i)} \in \{0,1\}^{m-1}} \mathcal{I}_{\{C^{(i)}_t = k^{(i)}\}} \left(\int_T^{\infty} \lambda^i(u, \Psi_u) e^{-\int_t^u \lambda^i(s,\Psi_s)ds} p_{t,u}(k^{(i)}, h^{(i)}) du \right. \\ &\quad \left. + \mathcal{I}_{\{\tilde{h}_i(\alpha,h)=0\}} \int_t^T \lambda^i(u, \Psi_u) e^{-\int_t^u \lambda^i(s,\Psi_s)ds} p_{t,u}(k^{(i)}, h^{(i)}) du \right), \end{aligned}$$

where

$$\tilde{h}_i(\alpha, h) := \begin{cases} 0 & \text{if } \sum_{j=1}^m \alpha_j h_{ij} = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{4.2}$$

and $p_{t,\tau_i}(k^{(i)}, h^{(i)}) := \prod_{j=1}^m p^{ij}_{t\tau_i}([k^{(i)}]_j, [h^{(i)}]_j)$ denotes the joint transition probabilities of the random vector $C^{(i)}_{\tau_i}$ from time t to time τ_i.

Our proof is based on the following lemma.

Lemma 4.3. *Assume the same notation as in Theorem 4.2. Then for all $z \in \{0,1\}^m$ and $t \in [0,T]$*

$$\begin{aligned} &\mathbb{E}\left[\mathcal{I}_{\{Z_T = z\}} \mid \mathcal{G}_t\right] \\ &= \sum_{\alpha \in \{0,1\}^m} (-1)^{\sum_{j=1}^m \alpha_j z_j} \prod_{j=1}^m z_j^{1-\alpha_j} \prod_{i=1}^m \left[Y_t(i) \sum_{h^{(i)} \in \{0,1\}^{m-1}} \mathcal{I}_{\{\tilde{h}_i(\alpha,h)=0\}} \mathcal{I}_{\{C^{(i)}_{\tau_i} = h^{(i)}\}} \right. \\ &\quad + (1 - Y_t(i)) \sum_{h^{(i)},k^{(i)} \in \{0,1\}^{m-1}} \mathcal{I}_{\{C^{(i)}_t = k^{(i)}\}} \left(\int_T^{+\infty} \lambda^i(u, \Psi_u) e^{-\int_t^u \lambda^i(s,\Psi_s)ds} p_{t,u}(k^{(i)}, h^{(i)}) du \right. \\ &\quad \left.\left. + \mathcal{I}_{\{\tilde{h}_i(\alpha,h)=0\}} \int_t^T \lambda^i(u, \Psi_u) e^{-\int_t^u \lambda^i(s,\Psi_s)ds} p_{t,u}(k^{(i)}, h^{(i)}) du \right) \right]. \end{aligned} \tag{4.3}$$

Proof. By (2.3) we have for $z_j \in \{0,1\}$

$$\mathcal{I}_{\{Z_T(j)=z_j\}} = z_j + (-1)^{z_j} \prod_{i=1}^{m} (1 - C_{\tau_i}(i,j) Y_T(i)) \,.$$

Then for $z \in \{0,1\}^m$,

$$\mathcal{I}_{\{Z_T=z\}} = \prod_{j=1}^{m} \left[z_j + (-1)^{z_j} \prod_{i=1}^{m} (1 - C_{\tau_i}(i,j) Y_T(i)) \right] . \tag{4.4}$$

We apply the following identity, which can be proved easily, for instance, by induction on m:

$$\prod_{j=1}^{m} (A_j + B_j) = \sum_{\alpha \in \{0,1\}^m} \prod_{j=1}^{m} A_j^{1-\alpha_j} B_j^{\alpha_j} , \tag{4.5}$$

where $\alpha_j \in \{0,1\}$, $j = 1, \dots, m$. Setting $0^0 := 1$ the formula holds also if there exists $j \in \{1, \dots, m\}$ such that $A_j = 0$ or $B_j = 0$. We apply this formula to

$$A_j \;:=\; z_j \quad \text{and} \quad B_j := (-1)^{z_j} \prod_{i=1}^{m} (1 - C_{\tau_i}(i,j) Y_T(i))$$

Then we obtain the following expression for the indicator function in (4.4):

$$\begin{aligned} \mathcal{I}_{\{Z_T=z\}} &= \sum_{\alpha \in \{0,1\}^m} \prod_{j=1}^{m} \left(z_j^{1-\alpha_j} \left[(-1)^{z_j} \prod_{i=1}^{m} (1 - C_{\tau_i}(i,j) Y_T(i)) \right]^{\alpha_j} \right) \qquad (4.6) \\ &= \sum_{\alpha \in \{0,1\}^m} (-1)^{\sum_{j=1}^{m} \alpha_j z_j} \Big(\prod_{j=1}^{m} z_j^{1-\alpha_j} \Big) \prod_{j=1}^{m} \prod_{i=1}^{m} (1 - C_{\tau_i}(i,j) Y_T(i))^{\alpha_j} \end{aligned}$$

Then by Assumption 2.1(2) we have that

$$\mathbb{E}\left[\mathcal{I}_{\{Z_T=z\}} \,\middle|\, \mathcal{G}_t \right] = \sum_{\alpha \in \{0,1\}^m} (-1)^{\sum_{j=1}^{m} \alpha_j z_j} \Big(\prod_{j=1}^{m} z_j^{1-\alpha_j} \Big) \prod_{i=1}^{m} \mathbb{E} \left[\prod_{j=1}^{m} (1 - C_{\tau_i}(i,j) Y_T(i))^{\alpha_j} \,\middle|\, \mathcal{G}_t \right] . \tag{4.7}$$

We focus now on the calculation of the conditional expectation in (4.7). The total probability theorem, by considering all the possible contagion structures for i-th row $C^{(i)}_{\tau_i}$ of the the random matrix C_{τ_i} (written in its vector representation and avoiding the element $C_{\tau_i}(i,i)$), yields

$$\begin{aligned} &\mathbb{E} \left[\prod_{j=1}^{m} (1 - C_{\tau_i}(i,j) Y_T(i))^{\alpha_j} \,\middle|\, \mathcal{G}_t \right] \\ &\quad = \sum_{h^{(i)} \in \{0,1\}^{m-1}} \mathbb{E} \left[\prod_{j=1}^{m} (1 - h_{ij} Y_T(i))^{\alpha_j} \mathcal{I}_{\{C^{(i)}_{\tau_i} = h^{(i)}\}} \,\middle|\, \mathcal{G}_t \right] \\ &\quad = \sum_{h^{(i)} \in \{0,1\}^{m-1}} \mathbb{E} \left[(1 - Y_T(i))^{\tilde{h}_i(\alpha,h)} \mathcal{I}_{\{C^{(i)}_{\tau_i} = h^{(i)}\}} \,\middle|\, \mathcal{G}_t \right] \qquad (4.8) \end{aligned}$$

where $h_{ii} := 1$ and $h_{ij} := [h^{(i)}]_j$ for $j \neq i$ and $\tilde{h}_i(\alpha, h)$ is as in (4.2). We now calculate

$$
\begin{aligned}
&\mathbb{E}\left[(1 - Y_T(i))^{\tilde{h}_i(\alpha,h)} \mathcal{I}_{\{C^{(i)}_{\tau_i} = h^{(i)}\}} \,\Big|\, \mathcal{G}_t \right] \\
&\quad = \mathbb{E}\left[(\mathcal{I}_{\{T<\tau_i\}})^{\tilde{h}_i(\alpha,h)} \mathcal{I}_{\{C^{(i)}_{\tau_i} = h^{(i)}\}} \,\Big|\, \mathcal{G}_t \right] \\
&\quad = \mathbb{E}\left[\mathbb{E}\left[(\mathcal{I}_{\{T<\tau_i\}})^{\tilde{h}_i(\alpha,h)} \mathcal{I}_{\{C^{(i)}_{\tau_i} = h^{(i)}\}} \,\Big|\, \mathcal{F}^{Y(i)}_T \vee \mathcal{G}_t \right] \Big|\, \mathcal{G}_t \right] \\
&\quad = \mathcal{I}_{\{\tau_i \le t\}} \mathcal{I}_{\{\tilde{h}_i(\alpha,h)=0\}} \mathcal{I}_{\{C^{(i)}_{\tau_i} = h^{(i)}\}} \\
&\qquad + \mathcal{I}_{\{\tilde{h}_i(\alpha,h)=0\}} \mathbb{E}\left[\mathcal{I}_{\{t<\tau_i\}} \sum_{k^{(i)} \in \{0,1\}^{m-1}} p_{t,\tau_i}(k^{(i)}, h^{(i)}) \mathcal{I}_{\{C^{(i)}_t = k^{(i)}\}} \,\Bigg|\, \mathcal{G}_t \right] \\
&\qquad + \mathcal{I}_{\{\tilde{h}_i(\alpha,h)\neq 0\}} \mathbb{E}\left[\mathcal{I}_{\{T<\tau_i\}} \sum_{k^{(i)} \in \{0,1\}^{m-1}} p_{t,\tau_i}(k^{(i)}, h^{(i)}) \mathcal{I}_{\{C^{(i)}_t = k^{(i)}\}} \,\Bigg|\, \mathcal{G}_t \right] \\
&\quad = Y_t(i) \mathcal{I}_{\{\tilde{h}_i(\alpha,h)=0\}} \mathcal{I}_{\{C^{(i)}_{\tau_i} = h^{(i)}\}} + \sum_{k^{(i)} \in \{0,1\}^{m-1}} \mathcal{I}_{\{C^{(i)}_t = k^{(i)}\}} \qquad (4.9) \\
&\qquad \times \left(\mathcal{I}_{\{\tilde{h}_i(\alpha,h)=0\}} \mathbb{E}\left[\mathcal{I}_{\{t<\tau_i\}} p_{t,\tau_i}(k^{(i)}, h^{(i)}) \,\Big|\, \mathcal{G}_t \right] + \mathcal{I}_{\{\tilde{h}_i(\alpha,h)\neq 0\}} \mathbb{E}\left[\mathcal{I}_{\{T<\tau_i\}} p_{t,\tau_i}(k^{(i)}, h^{(i)}) \,\Big|\, \mathcal{G}_t \right] \right),
\end{aligned}
$$

where by using Assumption 2.1(2) and (4) we have set $p_{t,\tau_i}(k^{(i)}, h^{(i)}) := \prod_{j=1}^m p^{ij}_{t\tau_i}([k^{(i)}]_j, [h^{(i)}]_j)$ to denote the joint transition probabilities of the random vector $C^{(i)}_{\tau_i}$ from time t to time τ_i under the convention that $[h^{(i)}]_i = [k^{(i)}]_i = p^{ii}_{t\tau_i}([k^{(i)}]_i, [h^{(i)}]_i) := 1$. Note that in the second term of (4.9) we have $\tau_i > t$.
Since by Assumption 2.1(4) $p_{t,\cdot}(k^{(i)}, h^{(i)})$ is a bounded continuous stochastic process, we can now apply Proposition 5.1.1(ii) and Corollary 5.1.1(ii) of Bielecki and Rutkowski [5] and obtain

$$
\begin{aligned}
&\mathcal{I}_{\{\tilde{h}_i(\alpha,h)=0\}} \mathbb{E}\left[\mathcal{I}_{\{t<\tau_i\}} p_{t,\tau_i}(k^{(i)}, h^{(i)}) \,\Big|\, \mathcal{G}_t \right] + \mathcal{I}_{\{\tilde{h}_i(\alpha,h)\neq 0\}} \mathbb{E}\left[\mathcal{I}_{\{T<\tau_i\}} p_{t,\tau_i}(k^{(i)}, h^{(i)}) \,\Big|\, \mathcal{G}_t \right] \\
&\quad = \mathcal{I}_{\{\tau_i > t\}} \Bigg(\mathcal{I}_{\{\tilde{h}_i(\alpha,h)=0\}} \mathbb{E}\left[\int_t^{+\infty} \lambda^i(u, \Psi_u) e^{-\int_t^u \lambda^i(s,\Psi_s) ds} p_{t,u}(k^{(i)}, h^{(i)}) du \,\Big|\, \mathcal{F}^{\Psi}_{\infty} \right] \\
&\qquad + \mathcal{I}_{\{\tilde{h}_i(\alpha,h)\neq 0\}} \mathbb{E}\left[\int_T^{+\infty} \lambda^i(u, \Psi_u) e^{-\int_t^u \lambda^i(s,\Psi_s) ds} p_{t,u}(k^{(i)}, h^{(i)}) du \,\Big|\, \mathcal{F}^{\Psi}_{\infty} \right] \Bigg) \\
&\quad = (1 - Y_t(i)) \Bigg(\int_T^{+\infty} \lambda^i(u, \Psi_u) e^{-\int_t^u \lambda^i(s,\Psi_s) ds} p_{t,u}(k^{(i)}, h^{(i)}) du \\
&\qquad + \mathcal{I}_{\{\tilde{h}_i(\alpha,h)=0\}} \int_t^T \lambda^i(u, \Psi_u) e^{-\int_t^u \lambda^i(s,\Psi_s) ds} p_{t,u}(k^{(i)}, h^{(i)}) du \Bigg), \qquad (4.10)
\end{aligned}
$$

where in the last equality we have used the fact that all terms in the conditional expectation are $\mathcal{F}^{\Psi}_{\infty}$-measurable (Assumption 2.1(4)).
By plugging now (4.9) and (4.10) in (4.8) and then in (4.7) we conclude the proof. $\square$

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. Iterating the conditional expectation we get

$$\mathbb{E}\left[f(\Psi_T, Z_T) \mid \mathcal{F}_t\right] \;=\; \mathbb{E}\left[\mathbb{E}\left[f(\Psi_T, Z_T) \mid \mathcal{G}_t\right] \mid \mathcal{F}_t\right].$$

In order to calculate the inner conditional expectation we use formula (4.3) of Lemma 4.3. For the sake of simplicity, we set

$$a_t(i) = \sum_{h^{(i)} \in \{0,1\}^{m-1}} \mathcal{I}_{\{\tilde{h}_i(\alpha,h)=0\}} \mathcal{I}_{\{C^{(i)}_{\tau_i} = h^{(i)}\}}$$

and

$$\begin{aligned} b_{t,T}(i) = \sum_{h^{(i)},k^{(i)} \in \{0,1\}^{m-1}} \mathcal{I}_{\{C^{(i)}_t = k^{(i)}\}} &\left(\int_T^{+\infty} \lambda^i(u, \Psi_u) e^{-\int_t^u \lambda^i(s,\Psi_s)ds} p_{t,u}(k^{(i)}, h^{(i)}) du \right. \\ &\left. + \mathcal{I}_{\{\tilde{h}_i(\alpha,h)=0\}} \int_t^T \lambda^i(u, \Psi_u) e^{-\int_t^u \lambda^i(s,\Psi_s)ds} p_{t,u}(k^{(i)}, h^{(i)}) du \right), \end{aligned}$$

Then by the total probability theorem it follows that

$$\begin{aligned} \mathbb{E}\left[f(\Psi_T, Z_T) \mid \mathcal{F}_t\right] &= \mathbb{E}\left[\sum_{z \in \{0,1\}^m} f(\Psi_T, z) \mathbb{E}\left[\mathcal{I}_{\{Z_T = z\}} \,\middle|\, \mathcal{G}_t\right] \,\middle|\, \mathcal{F}_t \right] \qquad (4.11) \\ &= \sum_{\alpha, z \in \{0,1\}^m} (-1)^{\sum_{j=1}^m \alpha_j z_j} \prod_{j=1}^m z_j^{1-\alpha_j} \mathbb{E}\left[f(\Psi_T, z) \prod_{i=1}^m \left(Y_t(i) a_t(i) + (1 - Y_t(i)) b_{t,T}(i)\right) \,\middle|\, \mathcal{F}_t \right]. \end{aligned}$$

We now calculate the conditional expectation appearing in (4.11). By (4.5) we have that

$$\prod_{i=1}^m \left(Y_t(i) a_t(i) + (1 - Y_t(i)) b_{t,T}(i)\right) = \sum_{\beta \in \{0,1\}^m} \prod_{i=1}^m \left(Y_t(i) a_t(i)\right)^{1-\beta_i} \left((1 - Y_t(i)) b_{t,T}(i)\right)^{\beta_i}.$$

Hence

$$\begin{aligned} &\mathbb{E}\left[f(\Psi_T, z) \prod_{i=1}^m \left(Y_t(i) a_t(i) + (1 - Y_t(i)) b_{t,T}(i)\right) \,\middle|\, \mathcal{F}_t \right] \\ &= \mathbb{E}\left[f(\Psi_T, z) \sum_{\beta \in \{0,1\}^m} \prod_{i=1}^m \left(Y_t(i) a_t(i)\right)^{1-\beta_i} \left((1 - Y_t(i)) b_{t,T}(i)\right)^{\beta_i} \,\middle|\, \mathcal{F}_t \right] \\ &= \sum_{\beta \in \{0,1\}^m} \prod_{i=1}^m \left(Y_t(i) a_t(i)\right)^{1-\beta_i} \left(1 - Y_t(i)\right)^{\beta_i} \mathbb{E}\left[f(\Psi_T, z) \prod_{i=1}^m b_{t,T}(i)^{\beta_i} \,\middle|\, \mathcal{F}_t^{\Psi} \right]. \qquad (4.12) \end{aligned}$$

Plugging (4.12) in (4.11) concludes the proof. □

Equation (4.12) shows that the final pricing formula depends on the specification of the macroeconomic process Ψ and the dynamics of the contagion matrix C.

We first comment on the contagion matrix. Recall that it simply describes for two firms, if there is a business relation at time t or not. From our formulas it is clear that we only need to know C at the time of default. There is still room for more precise modeling of the contagion matrix. For the moment we assume a time-independent but possibly random contagion matrix given by

$$C_t(i,j) = C(i,j)Y_t(i), \quad t \geq 0, \tag{4.13}$$

where C has entries $[C]_{ij} = C_{ij}(\omega)$ given by iid random variables independent of the processes Y and Ψ. It follows that $\mathcal{F}_t = \mathcal{F}_t^{\Psi} \vee \mathcal{F}_t^{Y} \vee \sigma(C)$ for $t > 0$ and $\mathcal{F}_0 := \{\emptyset, \Omega\}$.

We study now this situation. Note that we still do not specify the macroeconomic process Ψ; this will only come in Section 5, where we price derivatives under the assumption of long range dependence for Ψ.

Theorem 4.4. *If the contagion matrix is of the form* (4.13)*, the pricing formula* (4.1) *is given for* $0 < t \leq T$ *by*

$$\begin{aligned}\mathbb{E}\left[f(\Psi_T, Z_T) \mid \mathcal{F}_t\right] = &\sum_{\alpha,z\in\{0,1\}^m} \sum_{h\in\{0,1\}^{m(m-1)}} (-1)^{\sum_{i=1}^m \alpha_i z_i} \prod_{i=1}^m z_i^{1-\alpha_i}(1-Y_t(i))^{\tilde{h}_i(\alpha,h)} \mathcal{I}_{\{C=h\}} \\ &\times \mathbb{E}\left[f(\Psi_T, z) e^{-\int_t^T \sum_{i=1}^m \tilde{h}_i(\alpha,h)\lambda^i(u,\Psi_u)du} \,\middle|\, \mathcal{F}_t^{\Psi}\right],\end{aligned} \tag{4.14}$$

and for $t = 0$ *by*

$$\begin{aligned}\mathbb{E}[f(\Psi_T, Z_T)] = &\sum_{\alpha,z\in\{0,1\}^m} \sum_{h\in\{0,1\}^{m(m-1)}} (-1)^{\sum_{i=1}^m \alpha_i z_i} \prod_{i=1}^m z_i^{1-\alpha_i} \mathbb{P}(C=h) \\ &\times \mathbb{E}\left[f(\Psi_T, z) e^{-\int_0^T \sum_{i=1}^m \tilde{h}_i(\alpha,h)\lambda^i(u,\Psi_u)du}\right],\end{aligned} \tag{4.15}$$

where $\tilde{h}_i(\alpha, h)$ *is as in* (4.2) *with* $h_{ii} := 1$ *for* $i = 1, \ldots, m$*, and* $h_{ij} := [h]_{ij}$ *for* $i \neq j$*.*

Proof. First we note that in this case $C_{\tau_i}(i,j)Y_t(i) = C(i,j)Y_t(i)$. The total probability theorem, by considering all possible contagion structures for the random matrix C (again written in its vector representation and avoiding the diagonal) yields with (4.6)

$$\begin{aligned}\mathbb{E}\left[\mathcal{I}_{\{Z_T=z\}} \mid \mathcal{G}_t\right] &= \sum_{h\in\{0,1\}^{m(m-1)}} \mathbb{E}\left[\mathcal{I}_{\{Z_T=z\}}\mathcal{I}_{\{C=h\}} \mid \mathcal{G}_t\right] \\ &= \sum_{h\in\{0,1\}^{m(m-1)}} \sum_{\alpha\in\{0,1\}^m} (-1)^{\sum_{j=1}^m \alpha_j z_j} \Big(\prod_{j=1}^m z_j^{1-\alpha_j}\Big) \mathbb{E}\left[\prod_{i=1}^m \prod_{j=1}^m (1 - h_{ij}Y_T(i))^{\alpha_j} \mathcal{I}_{\{C=h\}} \,\middle|\, \mathcal{G}_t\right].\end{aligned}$$

We now distinguish between $t = 0$ and $t > 0$. Since $\mathcal{I}_{\{C=h\}}$ is $\mathcal{F}_t$-measurable for every $t > 0$, by

Assumption 2.1(2) and by (4.8) we obtain that for $t > 0$

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{I}_{\{Z_T=z\}} \,\middle|\, \mathcal{G}_t\right] && (4.16)\\
&= \sum_{h\in\{0,1\}^{m(m-1)}} \sum_{\alpha\in\{0,1\}^m} (-1)^{\sum\limits_{j=1}^m \alpha_j z_j} \Big(\prod_{j=1}^m z_j^{1-\alpha_j}\Big)\mathcal{I}_{\{C=h\}} \prod_{i=1}^m \mathbb{E}\left[\prod_{j=1}^m (1-h_{ij}Y_T(i))^{\alpha_j} \,\middle|\, \mathcal{G}_t\right].\\
&= \sum_{h\in\{0,1\}^{m(m-1)}} \sum_{\alpha\in\{0,1\}^m} (-1)^{\sum\limits_{j=1}^m \alpha_j z_j} \Big(\prod_{j=1}^m z_j^{1-\alpha_j}\Big)\mathcal{I}_{\{C=h\}} \prod_{i=1}^m \mathbb{E}\left[(1-Y_T(i))^{\tilde{h}_i(\alpha,h)} \,\middle|\, \mathcal{G}_t\right],
\end{aligned}$$

where $\tilde{h}_i(\alpha, h)$ is as in (4.2) with $h_{ii} := 1$, $i = 1,\ldots,m$, and $h_{ij} := [h]_{ij}$, $i \neq j$. Since by (2.4)

$$\mathbb{E}\left[(1-Y_T(i))^{\tilde{h}_i(\alpha,h)} \,\middle|\, \mathcal{G}_t\right] = (1-Y_t(i))^{\tilde{h}_i(\alpha,h)} e^{-\int\limits_t^T \tilde{h}_i(\alpha,h)\lambda^i(u,\Psi_u)du},$$

we obtain that for $t > 0$

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{I}_{\{Z_T=z\}} \,\middle|\, \mathcal{G}_t\right] && (4.17)\\
&= \sum_{h\in\{0,1\}^{m(m-1)}} \sum_{\alpha\in\{0,1\}^m} (-1)^{\sum\limits_{j=1}^m \alpha_j z_j} \Big(\prod_{j=1}^m z_j^{1-\alpha_j}\Big)\mathcal{I}_{\{C=h\}} \prod_{i=1}^m (1-Y_t(i))^{\tilde{h}_i(\alpha,h)} e^{-\int\limits_t^T \tilde{h}_i(\alpha,h)\lambda^i(u,\Psi_u)du}.
\end{aligned}$$

To obtain the final pricing formula, we proceed analogously as in the proof of Theorem 4.2. By (4.17) and (4.11) we have for $t > 0$

$$\begin{aligned}
\mathbb{E}\left[f(\Psi_T, Z_T) \,|\, \mathcal{F}_t\right] &= \mathbb{E}\left[\sum_{z\in\{0,1\}^m} f(\Psi_T, z)\mathbb{E}\left[\mathcal{I}_{\{Z_T=z\}} \,\middle|\, \mathcal{G}_t\right] \,\middle|\, \mathcal{F}_t\right] \qquad (4.18)\\
&= \sum_{\alpha,z\in\{0,1\}^m} \sum_{h\in\{0,1\}^{m(m-1)}} (-1)^{\sum\limits_{j=1}^m \alpha_j z_j} \Big(\prod_{j=1}^m z_j^{1-\alpha_j}\Big)\mathcal{I}_{\{C=h\}}\\
&\quad\times \prod_{i=1}^m (1-Y_t(i))^{\tilde{h}_i(\alpha,h)} \mathbb{E}\left[f(\Psi_T, z) e^{-\int\limits_t^T \sum_{i=1}^m \tilde{h}_i(\alpha,h)\lambda^i(u,\Psi_u)du} \,\middle|\, \mathcal{F}_t^\Psi\right].
\end{aligned}$$

This proves equation (4.14). For $t = 0$ we obtain

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{I}_{\{Z_T=z\}} \,\middle|\, \mathcal{G}_0\right] && (4.19)\\
&= \sum_{h\in\{0,1\}^{m(m-1)}} \sum_{\alpha\in\{0,1\}^m} (-1)^{\sum\limits_{j=1}^m \alpha_j z_j} \Big(\prod_{j=1}^m z_j^{1-\alpha_j}\Big)\mathbb{P}(C=h) e^{-\int\limits_0^T \sum_{i=1}^m \tilde{h}_i(\alpha,h)\lambda^i(u,\Psi_u)du}.
\end{aligned}$$

Substituting (4.19) in (4.18) for $t = 0$, we obtain formula (4.15). $\square$

If the contagion matrix is deterministic, i.e. for every $i, j \in \{1,\ldots,m\}$ and all $t \geq 0$,

$$C_t(i,j)(\omega) = C_t(i,j) \in \{0,1\} \quad \forall \omega \in \Omega,$$

we have $\mathcal{F}_t^C = \{\emptyset, \Omega\}$ for every $t \in [0,T]$.

Corollary 4.5. *Assuming that the contagion matrix is deterministic, the pricing formula* (4.1) *simplifies to*

$$\mathbb{E}\left[f(\Psi_T, Z_T) \mid \mathcal{F}_t \right] = \sum_{\alpha, z \in \{0,1\}^m} (-1)^{\sum_{i=1}^m \alpha_i z_i} \prod_{i=1}^m \left(z_i^{1-\alpha_i} (1 - Y_t(i))^{\tilde{h}_i(\alpha)} \right) \times \mathbb{E}\left[f(\Psi_T, z)\, e^{-\int_t^T \sum_{i=1}^m \tilde{h}_i(\alpha) \lambda^i(u, \Psi_u) du} \,\middle|\, \mathcal{F}_t^{\Psi} \right], \tag{4.20}$$

where

$$\tilde{h}_i(\alpha) := \begin{cases} 0 & \textit{if } \sum_{j=1}^m \alpha_j C_T(i,j) = 0, \\ 1 & \textit{otherwise.} \end{cases} \tag{4.21}$$

In the following simple example we investigate the effect of the contagion mechanism.

Example 4.6. We assume the group model of section 3 in its simplest form of a portfolio consisting of two classes, taking 5 firms in one group and 10 firms in the second group. We work with a deterministic contagion matrix and consider different contagion scenarios for C:

$$C = \begin{pmatrix} C_{5\times 5} & C_{5\times 10} \\ C_{10\times 5} & C_{10\times 10} \end{pmatrix}$$

We consider the following six scenarios, where I_d denotes the identity matrix in $\mathbb{R}^d$, $\mathbf{0}_{d\times k}$ the matrix with only entries 0, and $\mathbf{1}_{d\times k}$ the matrix with only entries 1.

$$C_1 = \mathrm{I}_{15} \qquad C_2 = \begin{pmatrix} \mathbf{1}_{5\times 5} & \mathbf{0}_{5\times 10} \\ \mathbf{0}_{10\times 5} & \mathrm{I}_{10} \end{pmatrix} \qquad C_3 = \begin{pmatrix} \mathrm{I}_5 & \mathbf{0}_{5\times 10} \\ \mathbf{0}_{10\times 5} & \mathbf{1}_{10\times 10} \end{pmatrix}$$

$$C_4 = \begin{pmatrix} \mathbf{1}_{5\times 5} & \mathbf{1}_{5\times 10} \\ \mathbf{0}_{10\times 5} & \mathbf{1}_{10\times 10} \end{pmatrix} \qquad C_5 = \begin{pmatrix} \mathbf{1}_{5\times 5} & \mathbf{0}_{5\times 10} \\ \mathbf{1}_{10\times 5} & \mathrm{I}_{10} \end{pmatrix} \qquad C_6 = \mathbf{1}_{15\times 15}$$

Obviously, C_1 corresponds to no contagion and will serve as reference scenario. C_2 models contagion within the first group, no contagion in the second and no contagion between firms of the two groups. C_3 models the complementary situation. Contagion matrix C_4 models contagion in the first group, but also the spill-over of default of group 1 firms into the second group. C_5 models contagion within both groups, and contagion from firms in the second group to the first group. Finally, C_6 models contagion between all 15 firms.

These scenarios determine the vectors $(\tilde{h}_i(\alpha), i = 1, \ldots, 15)$ for all $\alpha \in \{0,1\}^m$. We also assume that all firms in the same group have the same intensity of default, i.e. $\lambda^i = \lambda^{[1]}$ for all $i \in \{1, \ldots, 5\}$ and $\lambda^i = \lambda^{[2]}$ for all $i \in \{6, \ldots, 15\}$. Furthermore, we assume that $\lambda^{[2]} = 2\lambda^{[1]}$, and that both groups are exposed to the same realization of the macroeconomic process $\lambda^{[1]}$.

Now to understand precisely, what the effect of the contagion is, we take as simplest example one bond of one firm of the two groups at one time. For a defaultable bond of a firm in group $i \in \{1, 2\}$ with pricing formula (4.20) we obtain we have to calculate

$$V_0^{[i]} = E\left[(1 - Z_T^{[i]}) \exp\{- \int_0^T \sum_{i=1}^m \tilde{h}_i(\alpha) \lambda^{[i]}(u, \psi_u) du\} \right].$$

Note that the zeros in $\tilde{h}_i(\alpha)$ correspond to no default of all firms in group 1 and the second part of the vector to arbitrary values in the second group.

It remains to specify $\lambda^{[1]}$ and we take the CIR model, such that the intensities are positive a.s., i.e. $\lambda^{[1]}(t, B_t) = \lambda_t^{[1]}$ is the solution to

$$d\lambda_t^{[1]} = a(b - \lambda_t^{[1]}) + \sigma\sqrt{\lambda_t^{[1]}}dB_t\,, \quad t \geq 0,,$$

where $(B_t)_{t\geq 0}$ is standard Brownian motion, and we take the parameters $a = 2.0$, $b = 0.05$, $\sigma = 0.4$ and initial value $\lambda^{[1]}(0) = 0.03$. Obviously, prices should decrease for higher contagion scenarios and for bonds with higher maturity.

	Bond of firm	in group 1	Bond of firm	in group 2
	$T = 1$	$T = 2$	$T = 1$	$T = 2$
C_1	0.966936	0.923076	0.935458	0.853588
C_2	0.849446	0.681479	0.935458	0.853588
C_3	0.966936	0.923076	0.550128	0.258520
C_4	0.482438	0.195017	0.935458	0.853588
C_5	0.849446	0.681479	0.482438	0.195017
C_6	0.482438	0.195017	0.482438	0.195017

Table 4.1: Bond prices $V_0^{[i]}$ for maturities $T = 1$ and $T = 2$ and the different scenarios.

5 Pricing contingent claim depending on the macroeconomic process with credit risk contagion

5.1 Modeling the macroeconomic process

Now we turn to the macroeconomic process Ψ. There are many examples, which consider the intensity as a function of a state vector of Markov processes; see e.g. Frey and Backhaus [12] or Schönbucher [17], Chapter 7. Gaussian processes and processes driven by Brownian motion are the most prominent ones. Here we focus on the case, where $\Psi = \Psi^H$ is given by a long range dependent process with Hurst index $H > \frac{1}{2}$. This choice is motivated by the fact that macroeconomic variables like demand and supply, interest rates, or other economic activity measures often exhibit long range dependence. In the context of fractional processes examples include fractional geometric Brownian motion or other processes driven by fractional Brownian motion with non-negativity guaranteed.

We recall here the definition of fractional Brownian motion.

Definition 5.1. *A fractional Brownian motion (fBm) $B^H = (B_t^H)_{t\geq 0}$ with Hurst index $H \in$*

$(0,1)$ *is a continuous centered Gaussian process with covariance function*

$$cov(B^H_t, B^H_s) := R^H(t,s) := \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \qquad t,s \in \mathbb{R}^+.$$

In this section we focus on the case where the macroeconomic process is given by a suitable function ψ of a stochastic integral of a deterministic function with respect to fBm with Hurst index $H > \frac{1}{2}$. For examples see Buchmann and Klüppelberg [6] and for more details concerning fractional Brownian motion and the relevant stochastic calculus we refer to Biagini et al. [4]. Then we will compute the pricing formula (4.1) of Theorem 4.2 under the macroeconomic variables model

$$\Psi^H_t := \psi\left(I^H_t\right), \qquad I^H_t := \int_0^t g(s) dB^H_s\,, \quad t \in [0,T], \tag{5.1}$$

where ψ is an invertible continuous function and g is a deterministic function in $L^2_H([0,T])$. We recall that $L^2_H([0,T])$ is the completion of the Schwartz-space $\mathcal{S}([0,T])$ equipped with the inner product

$$\langle f,g\rangle_H := H(2H-1)\int_0^T\int_0^T f(s)g(t)|s-t|^{2H-2}dsdt < \infty \qquad f,g \in \mathcal{S}([0,T])\,.$$

In particular, in (5.1) we focus on deterministic integrands $g \in H^{\mu}([0,T])$ (which is a subset of $L^2_H([0,T])$), the space of the Hölder continuous functions on $[0,T]$ of order $\mu > 1-H$, and such that $1/g(s)$ is defined for all $s \in [0,T]$.

Remark 5.2. Under the above condition on ψ and g we get the following.
(i) The stochastic integral in the formula (5.1) can be understood pathwise in the Riemann-Stieltjes sense (see Section 5.1 in [4]).
(ii) By Theorem 4.4.2 of [21] we have also that $B^H(t) = \int\limits_0^t \frac{1}{g(s)} dI^H_s$, where this integral can again be interpreted in the Riemann-Stieltjes sense. This implies that the processes I^H and B^H generate the same filtration and that $\mathcal{F}^{\Psi^H}_t = \mathcal{F}^{B^H}_t$ (because ψ is invertible and measurable).

Although a long range dependent macroeconomic process may be more realistic than a Markovian one, it is clear that the calculations and the resulting pricing formulas become much more complicated. In this paper we shall restrict ourselves to the case, where for all $i \in \{1,\ldots,m\}$ the default intensities of the self-default processes $(Y_t(i))_{t\geq 0}$ are stochastic and of the form

$$\lambda^i(u, \Psi^H_u) = \beta^i(u) I^H_u + \gamma^i(u)\,, \qquad u \geq 0\,, \tag{5.2}$$

where β^i and γ^i are continuous functions.

Recall that intensities are supposed to be positive. Now, because the integral I^H has fBm as integrator, obviously it can happen with positive probability that the intensity becomes negative. By the affine transformation, however, we can at least control that this probability remains small. Of course, the same problem arises, when working with affine models driven by Brownian motion as, for instance, for the Ornstein-Uhlenbeck model (see Schönbucher [17], Section 7.1).

5.2 Pricing contingent claims with a long range dependent Ψ

In the setting outlined in Section 5.1 we focus on the pricing of contingent claims written on the long range dependent macroeconomic index Ψ^H and affected by credit risk contagion. For the sake of simplicity we consider the case, where the contagion matrix C_t is deterministic for all $0 \leq t \leq T$. Referring to Corollary 4.5, the problem is now to calculate a term of the form

$$\begin{aligned}
&\mathbb{E}\left[f\left(\Psi_T^H, z\right) e^{-\int_t^T \sum_{i=1}^m \tilde{h}_i(\alpha)\lambda^i(u,\Psi_u^H)du} \,\middle|\, \mathcal{F}_t \right] \\
&\quad = \mathbb{E}\left[f^\psi\left(I_T^H, z\right) e^{-\int_t^T \sum_{i=1}^m \tilde{h}_i(\alpha)\left(\beta^i(u)I_u^H+\gamma^i(u)\right)du} \,\middle|\, \mathcal{F}_t \right] \\
&\quad = e^{-\int_t^T \sum_{i=1}^m \tilde{h}_i(\alpha)\gamma^i(u)du}\, e^{\int_0^t \sum_{i=1}^m \tilde{h}_i(\alpha)\beta^i(u)I_u^H du} \\
&\qquad \times \mathbb{E}\left[f^\psi\left(I_T^H, z\right) e^{-\int_0^T \sum_{i=1}^m \tilde{h}_i(\alpha)\beta^i(u)I_u^H du} \,\middle|\, \mathcal{F}_t^{\Psi^H} \right]
\end{aligned} \tag{5.3}$$

for fixed $z \in \{0,1\}^m$, where we have set $f \circ \psi := f^\psi$. Note that in (5.3) the last equality holds by Assumption 2.1(1).

For simplicity we omit in the sequel the index z and write simply $f(\Psi_T^H)$ and $f^\psi(I_T^H)$ instead of $f(\Psi_T^H, z)$ and $f^\psi(I_T^H, z)$, respectively.

We now proceed as follows. Define for $a \in \mathbb{R}$ the function $f_a^\psi(x) := e^{-ax}f^\psi(x)$ for $x \in \mathbb{R}$ and its Fourier transform by $\widehat{f_a^\psi}(\xi) := \int_{\mathbb{R}} e^{-i\xi x} f_a^\psi(x)dx$ for $\xi \in \mathbb{R}$. We assume that f and ψ are such that

$$A := \left\{ a \in \mathbb{R} \,|\, f_a^\Psi(\cdot) \in L^1(\mathbb{R}) \text{ and } \widehat{f_a^\Psi}(\cdot) \in L^1(\mathbb{R}) \right\} \neq \emptyset\,.$$

Then by Theorem 9.1 of Rudin [16] the following inversion formula holds:

$$f_a^\psi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \widehat{f_a^\psi}(\xi)d\xi\,, \quad x \in \mathbb{R}\,. \tag{5.4}$$

We collect some useful results in the following lemma.

Lemma 5.3. *With the same notation and assumptions as above, we have*

$$\mathbb{E}\left[f(\Psi_T^H) e^{-\int_0^T \sum_{i=1}^m \tilde{h}_i(\alpha)\beta^i(u)I_u^H du} \,\middle|\, \mathcal{F}_t^{\Psi^H} \right] = \frac{1}{2\pi} \int_{\mathbb{R}} \mathbb{E}\left[e^{\int_0^T \eta(s,\xi)dB_s^H} \,\middle|\, \mathcal{F}_t^{\Psi^H} \right] \widehat{f_a^\psi}(\xi)d\xi \tag{5.5}$$

where

$$\eta(s,\xi) := g(s)\left(a + i\xi - \int_s^T \sum_{i=1}^m \tilde{h}_i(\alpha)\beta^i(u)du \right), \; s \in [0,T] \tag{5.6}$$

where $\tilde{h}_i(\alpha)$ *for* $i \in \{1,\ldots,m\}$ *are defined in* (4.21).

Furthermore,

$$
\begin{aligned}
&\mathbb{E}\left[e^{\int_0^T \eta(s,\xi)dB_s^H} \,\Big|\, \mathcal{F}_t^{\Psi^H}\right] \qquad (5.7)\\
&= \exp\left\{\frac{1}{2}\int_t^T\int_0^t \eta(s,\xi)\eta(u,\xi)|u-s|^{2H-2}dsdu\right\}\\
&\quad\times\exp\left\{\int_0^t \eta(s,\xi)dB_s^H\right\}\\
&\quad\times\exp\left\{\int_0^t \left(I_{t^-}^{-\left(H-\frac{1}{2}\right)}\left(I_{T^-}^{-\left(H-\frac{1}{2}\right)}\left(\left(\eta(s,\xi)\mathcal{I}_{[t,T]}(s)\right)^{H-\frac{1}{2}}\right)\right)\right)^{H-\frac{1}{2}} dB_s^H\right\}
\end{aligned}
$$

where $\|f\|_H^2 := \langle f, f\rangle_H$ *for* $f \in L_H^2([0,T])$*, and for* $\alpha = H - \frac{1}{2} \in (0, 1/2)$ *we define*

$$
\left(I_{t^-}^{-\alpha}\eta\right)(s) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{ds}\left(\int_s^t \eta(r)(r-s)^{\alpha-1}dr\right). \qquad (5.8)
$$

Proof. We first prove (5.5). We introduce the notation $E_T := \exp\left[-\int_0^T \sum_{i=1}^m \tilde{h}_i(\alpha)\beta^i(u)I_u^H du\right]$. By Theorem 6.4 of Sottinen [18] follows that we can exchange the order of integration and obtain

$$
E_T = e^{-\int_0^T g(s)\left(\int_s^T \sum_{i=1}^m \tilde{h}_i(\alpha)\beta^i(u)du\right)dB_s^H}.
$$

Then by using the definition of f_a^ψ and the Fourier inversion formula (5.4) we get

$$
\begin{aligned}
\mathbb{E}\left[f(\Psi_T^H)E_T \,\big|\, \mathcal{F}_t^{\Psi^H}\right] &= \mathbb{E}\left[\frac{1}{2\pi}\int_{\mathbb{R}} e^{\int_0^T [a+i\xi-\int_s^T \sum_{i=1}^m \tilde{h}_i(\alpha)\beta^i(u)du]g(s)dB_s^H}\widehat{f_a^\psi}(\xi)d\xi \,\Big|\, \mathcal{F}_t^{\Psi^H}\right]\\
&= \frac{1}{2\pi}\mathbb{E}\left[\int_{\mathbb{R}} e^{\int_0^T \eta(s,\xi)dB_s^H} \,\Big|\, \mathcal{F}_t^{\Psi^H}\right]\widehat{f_a^\psi}(\xi)d\xi\,, \qquad (5.9)
\end{aligned}
$$

where

$$
\eta(s,\xi) := g(s)\left(a + i\xi - \int_s^T \sum_{i=1}^m \tilde{h}_i(\alpha)\beta^i(u)du\right).
$$

Since $\mathbb{E}\left[e^{\int_0^T \eta(s,\xi)dB_s^H}\right] < \infty$, we can exchange the order of integration in (5.9) and obtain (5.5). Next we prove (5.7). By Remark 5.2(ii) we have

$$
\begin{aligned}
&\mathbb{E}\left[e^{\int_0^T \eta(s,\xi)dB_s^H} \,\Big|\, \mathcal{F}_t^{\Psi^H}\right] = \mathbb{E}\left[e^{\int_0^T \eta(s,\xi)dB_s^H} \,\Big|\, \mathcal{F}_t^{B^H}\right]\\
&= e^{\frac{1}{2}\|\eta(\cdot,\xi)\mathcal{I}_{[0,T]}(\cdot)\|_H^2}\mathbb{E}\left[\exp\left\{\int_0^T \eta(s,\xi)dB_s^H - \frac{1}{2}\|\eta(\cdot,\xi)\mathcal{I}_{[0,T]}(\cdot)\|_H^2\right\} \,\Bigg|\, \mathcal{F}_t^{B^H}\right],
\end{aligned}
$$

which is equal to (5.7) by Proposition 2 of Duncan [9]. □

Example 5.4. For the special case where $g \equiv 1$ (that gives $I_t^H = B_t^H$) and $\beta^i \equiv 0$ for all $i \in \{1, \ldots, m\}$, formula (5.7) can be calculated by Theorem 3.2 of Valkeila [19] as follows

$$
\begin{aligned}
&\mathbb{E}\left[e^{(a+i\xi)B_T^H} \,\Big|\, \mathcal{F}_t^{B^H}\right]\\
&= \exp\left\{\frac{1}{2}(a+i\xi)^2\left(T^{2H} - \langle M^H\rangle_t\right) + (\alpha+i\xi)\left(B_t^H + \int_0^t \Phi_T(t,s)dB_s^H\right)\right\}
\end{aligned}
$$

where

$$\langle M^H\rangle_t = \int_0^t z^H(T,s)^2 ds$$

with

$$\begin{aligned} z^H(T,s) &:= \Big(H-\frac{1}{2}\Big) c_H s^{\frac{1}{2}-H} \int_s^T u^{H-\frac{3}{2}}(u-s)^{H-\frac{1}{2}} du \\ c_H &:= \Big(\frac{2H\Gamma\left(\frac{3}{2}-H\right)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}\Big)^{\frac{1}{2}} \end{aligned}$$

and

$$\Phi_T(t,s) := \frac{1}{\pi}\sin\Big(\pi\big(H-\frac{1}{2}\big)\Big)\, s^{\frac{1}{2}-H}(t-1)^{\frac{1}{2}-H}\int_t^T \frac{u^{H-\frac{1}{2}}(u-t)^{H-\frac{1}{2}}}{u-s} du\,.$$

We are now able to provide a pricing formula for a long range dependent macroeconomic state variable process.

Theorem 5.5. *Assume that the contagion matrix C is deterministic and that for all $i \in \{1,\dots,m\}$ the intensities of the self-default processes $Y_i = (Y_t(i))_{t\geq 0}$ are of the form*

$$\lambda^i(t,\Psi_t^H) := \beta^i(t) I_t^H + \gamma^i(t),\ t \geq 0,$$

where β^i and γ^i are continuous functions. Consider

$$I_t^H := \int_0^t g(s)dB^H(s)\,,\quad t \geq 0\,,$$

for $g \in H^\mu([0,T])$ with $\mu > 1-H$ and such that $\frac{1}{g}$ is well-defined. Let $f(\cdot,z)$ and $\psi(\cdot)$ be deterministic continuous functions and denote for all $z \in \{0,1\}^m$

$$f^\psi(x,z) := f(\psi(x),z), \qquad x \in \mathbb{R},$$

and

$$f_\alpha^\psi(x,z) := e^{-\alpha x} f^\psi(x,z), \qquad \alpha, x \in \mathbb{R}.$$

Assume that there exists some $a \in \mathbb{R}$ such that $f_a^\Psi(\cdot,z)$ and its Fourier transform $\hat{f}_a^\Psi(\cdot,z)$ belong to $L^1(\mathbb{R})$ for all $z \in \{0,1\}^m$.
Finally let ψ be invertible and set

$$\Psi_t^H := \psi\left(\int_0^t g(s)dB^H(s)\right).$$

Then the price (4.20) at time $t \in [0,T]$ is given by the following formula

$$
\begin{aligned}
&\mathbb{E}\left[f\left(\Psi_T, Z_T\right) \mid \mathcal{F}_t\right] \\
&= \sum_{\alpha, z\in\{0,1\}^m} (-1)^{\sum_{i=1}^{m}\alpha_i z_i} \prod_{i=1}^{m}\left(z_i^{1-\alpha_i}(1-Y_t(i))^{\tilde{h}_i(\alpha)}\right) e^{-\int_t^T \sum_{i=1}^{m}\tilde{h}_i(\alpha)\gamma^i(u)du} \\
&\quad \times e^{\int_0^t \sum_{i=1}^{m}\tilde{h}_i(\alpha)\beta^i(u)I_u^H du} \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left\{\frac{1}{2}\int_t^T\int_0^t \eta(s,\xi)\eta(u,\xi)|u-s|^{2H-2}dsdu\right\} \\
&\quad \times \exp\left\{\int_0^t \eta(s,\xi)dB_s^H\right\} \\
&\quad \times \exp\left\{\int_0^t \left(I_{t^-}^{-(H-\frac{1}{2})}\left(I_{T^-}^{-(H-\frac{1}{2})}\left(\left(\eta(s,\xi)\mathcal{I}_{[t,T](s)}\right)^{H-\frac{1}{2}}\right)\right)\right)^{H-\frac{1}{2}} dB_s^H\right\} \widehat{f_a^{\psi}}(\xi,z)d\xi
\end{aligned}
$$

where $\tilde{h}_i(\alpha)$ is given in (4.21), η in (5.6) and $I_{t^-}^{-\alpha}$ in (5.8).

A basic structual analysis for pricing formulas with long range dependent hazard function models will be presented in Biagini, Fink and Klüppelberg [3] together with an extended numerical analysis study.

Example 5.6. (**Inflation-linked caps and floors**).
To illustrate how to compute $\hat{f}_a^{\Psi}$, we introduce some examples of inflation-linked derivatives, such as inflation-linked caps and floors, that we allow to be also exposed to contagion risk. By using the notation of Theorem 5.5, we consider a payoff of the form

$$f(\Psi_T, Z_T) := (\Psi_T - k)^+ b(Z_T),$$

where Ψ_T represents here the inflation index and $b(\cdot)$ is a positive measurable function, that describes the contagion effects. By Theorem 5.5 we have to find some $a \in \mathbb{R}$ such that

$$(f_{call})_a^{\psi} \in L^1(\mathbb{R}) \quad \text{and} \quad \widehat{(f_{call})_a^{\psi}} \in L^1(\mathbb{R}) \tag{5.10}$$

hold. We show (5.10) for $g \equiv 1$ and the two special cases $\psi(x) = x$ and $\psi(x) = e^x$, corresponding to fractional Brownian motion and geometric fractional Brownian motion, respectively.

(A) Let $\psi(x) = x$.
It follows immediately that (5.10) holds for $(f_{call})_a^{\psi}$, for all $a > 0$. We compute now the Fourier transform of $(f_{call})_a^{\psi}$ for $a > 0$.

$$\widehat{(f_{call})_a^{\psi}}(u) = \int_K^{\infty} e^{-x(a+iu)}(x-K)dx = \frac{e^{-K(a+iu)}}{(a+iu)^2}.$$

Since

$$\left|\frac{e^{-K(a+iu)}}{(a+iu)^2}\right| = \frac{e^{-Ka}}{u^2} = O\left(\frac{1}{u^2}\right), \quad u \to \infty,$$

we have that (5.10) holds also for the Fourier transform of $(f_{call})_a^{\psi}$ for all $a > 0$.

(B) Let $\psi(x) = e^x$.

It follows from the calculations in (A) that (5.10) holds for $(f_{call})_a^\psi$ for all $a > 1$. We compute now the Fourier transform of $(f_{call})_a^\psi$ for $a > 1$:

$$\widehat{(f_{call})_a^\psi}(u) \;=\; \int_{\ln K}^{\infty} e^{-x(a+iu)}(e^x - K)dx \;=\; \frac{e^{-(a-1+iu)\ln K}}{(a+iu)(a-1+iu)}.$$

Since

$$\left| \frac{e^{-(a-1+iu)\ln K}}{(a+iu)(a-1+iu)} \right| = \frac{e^{-(a-1)\ln K}}{|a(a-1) - u^2 + iu(2a-1)|} = O\left(\frac{1}{u^2}\right), \quad u \to \infty,$$

condition (5.10) holds also for the Fourier transform of $(f_{call})_a^\psi$ for all $a > 1$.

5.3 Comparison with Markovian Ψ

Theorem 5.7. *Assume that* (5.1) *holds for standard Brownian motion as integrator and assume also* (5.2). *Then*

$$\mathbb{E}\left[f(\Psi_T, Z_T) \mid \mathcal{F}_t\right] = \sum_{\alpha, z \in \{0,1\}^m} (-1)^{\sum_{i=1}^m \alpha_i z_i} \prod_{i=1}^m \left(z_i^{1-\alpha_i} (1 - Y_t(i))^{\tilde{h}_i(\alpha)} \right) e^{-\int_t^T \sum_{i=1}^m \tilde{h}_i(\alpha)\gamma^i(u)du}$$
$$\times e^{\int_0^t \sum_{i=1}^m \tilde{h}_i(\alpha)\beta^i(u) I_u du} \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left\{ \frac{1}{2} \int_t^T \eta^2(s,\xi)ds + \int_0^t \eta(s,\xi)dB_s \right\} \widehat{f_a^\psi}(\xi, z)d\xi$$

where $\tilde{h}_i(\alpha)$ is given in (4.21) *and η in* (5.6).

Proof. By Theorem 4.4, (5.3) and Lemma 5.3 we obtain that calculating the price $\mathbb{E}\left[f(\Psi_T, Z_T) \mid \mathcal{F}_t\right]$ boils down to compute the term

$$\mathbb{E}\left[e^{\int_0^T \eta(s,\xi)dB_s} \;\middle|\; \mathcal{F}_t^\Psi \right]$$

with $\eta(s,\xi) = g(s)\left(a + i\xi - \int_s^T \sum_{i=1}^m \tilde{h}_i(\alpha)\beta^i(u)du\right)$.
Since $\exp\left\{\frac{1}{2}\int_0^T \eta^2(s,\xi)ds\right\} < \infty$ for every fixed $\xi \in \mathbb{R}$, i.e. the Novikov condition is satisfied, we have that

$$\mathbb{E}\left[e^{\int_0^T \eta(s,\xi)dB_s} \;\middle|\; \mathcal{F}_t^\Psi \right] = \exp\left\{ \frac{1}{2}\int_t^T \eta^2(s,\xi)ds + \int_0^t \eta(s,\xi)dB_s \right\}.$$

□

Since the integrand g in (5.1) is in $L^2([0,T]) \subset L_H^2([0,T])$ (see [1] for the proof), we now compare the prices in $t = 0$ for the standard and the long range dependent case.

(i) For the standard Brownian motion case, the price V_0 in $t = 0$ is equal to

$$V_0 = \sum_{\alpha, z \in \{0,1\}^m} (-1)^{\sum_{i=1}^m \alpha_i z_i} \prod_{i=1}^m z_i^{1-\alpha_i} e^{-\int_0^T \sum_{i=1}^m \tilde{h}_i(\alpha)\gamma^i(u)du}$$
$$\times \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left\{ \frac{1}{2}\int_0^T \eta^2(s,\xi)ds \right\} \widehat{f_a^\psi}(\xi, z)d\xi;$$

(ii) For the fractional Brownian motion case, the price V_0^H in $t = 0$ is equal to

$$V_0^H = \sum_{\alpha, z \in \{0,1\}^m} (-1)^{\sum_{i=1}^m \alpha_i z_i} \prod_{i=1}^m z_i^{1-\alpha_i} e^{-\int_0^T \sum_{i=1}^m \tilde{h}_i(\alpha)\gamma^i(u)du} \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f_a^\psi}(\xi, z) d\xi.$$

The difference is indeed due to the fact that for Brownian motion we use the Itô integral and obtain consequently an Itô term, whereas for fBm integration is pathwise. Anyway, we see that $V_0 > V_0^H$ for every $H > \frac{1}{2}$. Of course, the long range dependence effect takes effect for prices V_t for $t > 0$, but then numerical calculations are called for.

Acknowledgment

We thank Damir Filipović for interesting discussions and remarks. We gratefully acknowledge the numerical support of Vincenzo Ferrazzano, who calculated the prices for the different scenarios of Example 4.6. Finally, we thank Peter Hepperger for careful reading of the paper, whose comments improved the paper considerably.

References

[1] E. Alòs and D. Nualart. (2003) Stochastic integration with respect to the fractional Brownian motion, *Stochastics and Stochastics Reports* **75**, 129–152.

[2] Björk, T. and Hult, H. (2005) A note on Wick products and the fractional Black Scholes model. *Finance Stoch.* **9**, 197–209.

[3] Biagini, F., Fink, H. and Klüppelberg, C. (2009) Credit contagion in a long rande dependen hazard model. In preparation.

[4] Biagini, F., Hu, Y., Øksendal, B. and Zhang, T. (2008) *Stochastic Calculus for Fractional Brownian Motion and Applications.* Springer, London.

[5] Bielecki, T.R. and Rutkowski, M. (2002) *Credit Risk: Modeling, Valuation and Hedging.* Springer, Berlin.

[6] Buchmann, B. and Klüppelberg, C. (2006) Fractional integral equations and state space transforms. *Bernoulli*, **12**(3), 431-456.

[7] Davis, M. and Lo, V. (2001a) Infectious defaults. *Quantitative Finance* **1**, 382-387.

[8] Davis, M. and Lo, V. (2001b) Modeling default correlation in bond portfolios. In: Alexander, C. (Ed.) *Mastering Risk, Vol. 2: Applications.* Financial Times Prentice Hall, pp. 141-151.

[9] Duncan, T.E. (2006) Prediction for some processes related to a fractional Brownian motion. *Statistics & Probability Letters* **76**, 128-134.

[10] Duffie, D. (2004) Credit risk modeling with affine processes. Scuola Normale Superiore, Pisa.

[11] Duffie, D., Filipovic, D. and Schachermayer (2003) Affine processes and applications in finance. *Ann. Appl. Prob.* **13**, 984-1053.

[12] Frey, R. and Backhaus, J. (2004) Portfolio credit risk models with interacting default intensities: a Markovian approach. Preprint, University of Leipzig.

[13] Frey, R. and Backhaus, J. (2008) Pricing and hedging of portfolio credit derivatives with interacting default intensities. *International Journal of Theoretical and Applied Finance* **11** (6), 611-634.

[14] Henry, M. and Zaffaroni, P. (2003) The long-range dependence paradigm for macroeconomics and finance. In: Doukhan, P., Oppenheim, G. and Taqqu, M. (Eds.) *Long-Range Dependence,* pp. 417-438. Birkhäuser, Boston.

[15] Øksendal, B. (2007) Fractional Brownian motion in finance. In: Jensen, B.S. and Palokangas, T. (Eds.) *Stochastic Economic Dynamics*, pp. 11-56. Copenhagen Business School Press, Frederiksberg.

[16] Rudin, W. (1987) *Real and Complex Analysis.* McGraw-Hill, New York.

[17] Schönbucher, P.J. (2003) *Credit Derivatives Pricing Models.* Wiley, Chichester.

[18] Sottinen, T. (2003) *Fractional Brownian Motion in Finance and Queueing.* PhD thesis, Helsinki University.

[19] Valkeila, E. (1999) On some properties of geometric fractional Brownian motion. Preprint number 224. Department of Mathematics, University of Helsinki,

[20] Yin, G.G. and Zhang, Q. (1998) *Continuous-time Markov Chains and Applications: a Singular Perturbation Approach.* Springer, New York.

[21] Zähle, M. (1998) Integration with respect to fractal functions and stochastic calculus I. *Probability Theory and Related Fields* **111**, 333-374.