

# Pricing of catastrophe insurance options written on a loss index with reestimation

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April 28, 2009

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## Abstract

We propose a valuation model for catastrophe insurance options written on a loss index. This kind of options distinguishes between a loss period  $[0, T_1]$ , during which the catastrophes may happen, and a development period  $[T_1, T_2]$ , during which losses entered before  $T_1$  are reestimated. Here we suppose that the underlying loss index is given by a time inhomogeneous compound Poisson process before  $T_1$  and that losses are reestimated by a common factor given by an exponential time inhomogeneous Lévy process after  $T_1$ . In this setting, using Fourier transform techniques, we are able to provide analytical pricing formulas for catastrophe options written on this kind of index.

**Key words:** Catastrophe insurance options, loss index, Fourier transform, option pricing formulas, heavy tails.

**JEL code:** C02.

**Subject Category and Insurance Branch Category:** IM10, IM11, IM54.

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# 1 Introduction

In the last 20 years natural catastrophes have been happening with increasing intensity and have been characterized by an amount of losses never reached before. In order to securitize the catastrophe risk, insurance companies have tried to take advantage of the vast potential of capital markets by introducing exchange-traded catastrophe insurance options. Exchange-traded insurance instruments present several advantages with respect to reinsurance. For example, they offer low transaction costs, because they are standardized, and include minimal credit risk because the obligations are guaranteed by the exchange. See [23] and [24] for the comparison of insurance securities. In particular, catastrophe options are standardized contracts based on an index of catastrophe losses, for example compiled by Property Claim Service (PCS), an internationally recognized market authority on property losses from catastrophes in the US.

The first index based catastrophe derivatives were CAT futures, which were introduced by the Chicago Board of Trade (CBOT) in 1992. Some models for the index underlying the CAT futures can be found for example in [1] and [6]. However, due to the structure of these products, there was only little trading activity on CAT futures in the market. A second version of catastrophe insurance derivatives were PCS options based on the index compiled by PCS. For the description of PCS catastrophe insurance options see for example [17], [22] or [23]. On its peak, the total capacity created by this version of insurance options amounted to 89 millions Dollars per year. Trading in PCS options slowed down in 1999, because of market illiquidity and lack of qualified personal (see e.g. [23]).

However, the record losses caused by the hurricanes Katrina, Rita and Wilma in 2005 have been a catalyst in creating new derivative instruments to trade catastrophe risks in capital markets. On March 2007, the New York Mercantile Exchange<sup>1</sup> (NYMEX) has begun the trading of catastrophe futures and options again. These new contracts have been designed to bring the transparency and liquidity of the capital markets to the insurance sector, providing effective ways of protecting against property catastrophe risk and providing the investors with the opportunity to trade a new asset class which has little or no correlation to other exchange traded asset classes. The NYMEX catastrophe options are settled against the Re-Ex loss index, which is created from the data supplied by PCS.

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<sup>1</sup>**Acknowledgement:** We wish to thank the New York Mercantile Exchange for the information provided concerning PCS options.

The structure of catastrophe options is described as follows. The option is written on an index that evolves over two periods, the *loss period* and the *development period*. During the contract specific *loss period*  $[0, T_1]$  the index measures catastrophic events that may occur. In addition to the loss period, option users choose a *development period*  $[T_1, T_2]$ . During the *development period* damages of catastrophes occurred in the loss period are reestimated and continue to affect the index. The contract expires at the end of the chosen development period.

Since the introduction of catastrophe insurance derivatives in 1992, the pricing of these products has been a problem. The underlying loss index is not traded and hence the market becomes incomplete. It is then an open question how the pricing measure should be determined. The next challenge is that even for fairly simple models the pricing problem becomes quite complicated.

So far, there have been several approaches in the literature to model a catastrophe index and to price catastrophe options written on this index. In [17], [18] and [19], the underlying catastrophe index has been represented as a compound Poisson process with nonnegative jumps. However, no distinction between loss and reestimation period has been made. In [5] and [16], the authors distinguish between a loss and a reestimation period and model the index as an exponential Lévy process over each period. While technicalities for pricing purposes are simplified in this setting, the assumption of an exponential model for accumulated losses during the loss period is rather unrealistic. For example, this implies that later catastrophes are more severe than earlier ones, and that the index starts in a positive value (instead of starting in 0). Yet another model is proposed in [21] where immediate reestimation is assumed and modelled through individual reestimation factors for each catastrophe. However, no efficient pricing methods are obtained in this model.

In this paper, we consider the distinction between loss and reestimation period as in [5] and [16], but propose a more realistic model for the loss index. We assume that the index is described by a time-inhomogeneous compound Poisson process during the loss period, and that during the reestimation period the index is reestimated by a factor which is given as an exponential time inhomogeneous Lévy process. In this framework we then consider the problem of pricing European catastrophe options written on the index. Interpreting the option as a payoff on a two-dimensional asset, we are able to obtain analytical pricing formulas by employing Fourier transform techniques. To this end we extend Fourier transform techniques for dampened payoff functions as introduced in [4] and [8] to the case of

a general payoff depending on two factors. We conclude by calculating explicitly the price of the most commonly traded catastrophe options in the market.

More precisely the paper is organized as follows. In Section 2 we specify our model for the loss index. In Section 3 we introduce a class of structure preserving pricing measures before we derive the price process of European style catastrophe options by using Fourier transform techniques. Finally, in Section 4 we compute explicitly the prices of the most common option types traded in the market, where in particular Subsection 4.1 is devoted to pricing with heavy-tailed losses.

## 2 Modeling of the loss index

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space. We consider a financial market endowed with a *risk-free asset* with *deterministic* interest rate  $r_t$ , and the *possibility of trading catastrophe insurance options*, written on a loss index. Following [5] and [16] we distinguish two time periods:

- a *loss period*  $[0, T_1]$ , where catastrophes may occur and losses are accumulated,
- a *development period*  $[T_1, T_2]$ ,  $T_1 < T_2 < \infty$ , where losses happened before  $T_1$  are reestimated.

Note that we assume that reestimation begins in  $T_1$  for all insurance claims that have occurred during the loss period. In reality the starting point of reestimation might differ from claim to claim. However, the approximation using one common starting point for reestimation is accepted among practitioners and can be found at several places in the literature (see for example [5] and [16]). Technically, as we will see in the next section, this assumption facilitates the derivation of explicit pricing formulas. For option pricing in a model with immediate reestimation of single loss occurrences we refer to [2], where we treat this more complex setting (see also [21]).

Precisely, we model the loss index by the stochastic process  $L = (L_t)_{0 \leq t \leq T_2}$  as follows:

- i) For  $t \in [0, T_1]$ ,

$$L_t = \sum_{j=1}^{N_t} Y_j \tag{1}$$

is a time inhomogeneous compound Poisson process, where

- $N_t$  is a time inhomogeneous Poisson process with deterministic intensity  $\lambda(t) > 0$ ,
- $Y_j, j = 1, 2, \dots$ , are positive i.i.d. random variables with distribution function  $G$ , independent of  $N_t$ .

Note that we allow for seasonal behavior of loss occurrence modeled by a time dependent intensity  $\lambda(t)$ .

ii) For  $t \in [T_1, T_2]$

$$L_t = L_{T_1+u} = L_{T_1} Z_u, \quad u = t - T_1 \in [0, T_2 - T_1], \quad (2)$$

where  $Z_u$  is a process that represents the reestimation factor with

- $Z_0 = 1$  a.s.,
- $(L_t)_{t \leq T_1}$  and  $(Z_u)_{0 \leq u \leq T_2 - T_1}$  are independent.

We suppose that all investors in the market observe the past evolution of the loss index including the current value. Therefore, the flow of information is given by the filtration  $(\mathcal{F}_t^0)_{0 \leq t \leq T_2}$  generated by the process  $L$ , which is of the form

- $\mathcal{F}_0^0 = \{\emptyset, \Omega\}$ ,
- $\mathcal{F}_t^0 := \sigma(L_u, u \leq t) = \sigma(\sum_{j=1}^{N_u} Y_j, u \leq t)$ , for  $t \in [0, T_1]$ ,
- $\mathcal{F}_t^0 := \sigma(L_u, u \leq t) = \sigma(L_s, s \leq T_1) \vee \sigma(Z_{u-T_1}, T_1 < u \leq t)$ , for  $t \in (T_1, T_2]$ ,
- $\mathcal{F}_{T_2}^0 \subseteq \mathcal{F}$ .

We assume that the filtration  $(\mathcal{F}_t^0)_{0 \leq t \leq T_2}$  is right-continuous. Let  $(\mathcal{F}_t)_{0 \leq t \leq T_2}$  be the completion of the filtration  $(\mathcal{F}_t^0)_{0 \leq t \leq T_2}$  with  $\mathbb{P}$ -null sets of  $\mathcal{F}$ .

It is reasonable to assume that the reestimation is not biased (see also [21]). Therefore, we suppose that  $(Z_t)_{0 \leq t \leq T_2 - T_1}$  is a positive martingale with respect to the filtration  $(\mathcal{F}_t)_{0 \leq t \leq T_2}$  of the form

$$Z_t = e^{X_t} \quad (3)$$

for a process  $X = (X_t)_{0 \leq t \leq T_2 - T_1}$  such that  $X_0 = 0$  a.s.. More precisely, in this paper we assume that  $X_t$  is a *time inhomogeneous Lévy process*.

**Definition 2.1.** *An adapted stochastic process  $(X_t)_{t \in [0, T]}$  with values in  $\mathbb{R}$  is a time inhomogeneous Lévy process or a process with independent increments and absolutely continuous characteristics, if the following conditions hold:*

1.  $X$  has independent increments, i.e.  $X_t - X_s$  is independent of  $\mathcal{F}_s$ ,  $0 \leq s \leq t \leq T$ .
2. For every  $t \in [0, T]$ , the law of  $X_t$  is characterized by the characteristic function

$$E[e^{iuX_t}] = \exp \left\{ \int_0^t \left( iub_s - \frac{1}{2}c_s u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iuxI_{\{|x| \leq 1\}}) F_s(dx) \right) ds \right\}$$

for deterministic functions

$$\begin{aligned} b &: [0, T] \rightarrow \mathbb{R}, \\ c &: [0, T] \rightarrow \mathbb{R}^+, \\ F &: [0, T] \rightarrow LM(\mathbb{R}), \end{aligned}$$

where  $LM(\mathbb{R})$  is the family of Lévy measures  $\nu(dx)$  on  $\mathbb{R}$ , i.e.

$$\int_{\mathbb{R}} (x^2 \wedge 1) \nu(dx) < \infty \quad \text{and} \quad \nu(\{0\}) = 0.$$

It is assumed that

$$\int_0^T \left( |b_s| + c_s + \int_{\mathbb{R}} (x^2 \wedge 1) F_s(dx) \right) ds < \infty.$$

The triplet  $(b, c, F) := (b_s, c_s, F_s)_{s \in [0, T]}$  is called the characteristics of  $X$ .

We assume the following exponential integrability condition.

(C1) There exists  $\epsilon > 0$  such that for all  $u \in [-(1 + \epsilon), 1 + \epsilon]$

$$E[e^{uX_t}] < \infty \quad \forall t \in [0, T].$$

By Lemma 1.6 of [12] this is equivalent to require the following integrability condition on  $F_s$ :

(C1') There exists  $\epsilon > 0$  such that for all  $u \in [-(1 + \epsilon), 1 + \epsilon]$

$$\int_0^T \int_{\{|x| > 1\}} e^{ux} F_s(dx) ds < \infty.$$

In particular, we have that  $E[Z_t] < \infty$  for all  $t \in [0, T]$ , if (C1) is in force. Furthermore we require the following condition on the characteristics

$$(C2) \quad 0 = \int_0^t b_s ds + \frac{1}{2} \int_0^t c_s ds + \int_0^t \int_{\mathbb{R}} (e^x - 1 - h(x)) F_s(dx) ds,$$

which implies (see e.g. [8], Remark 3.1, and [11], Lemma 4.4) that  $Z_t = e^{X_t}$  is a martingale.

Further, by [12] we obtain that  $X_t$  can be canonically represented as

$$X_t = \int_0^t b_s ds + \int_0^t \sqrt{c_s} dW_s + \int_0^t \int_{\mathbb{R}} x(\mu(ds, dx) - F_s(dx)ds), \quad (4)$$

where  $W_t$  is a standard Brownian motion and  $\mu$  is the integer-valued random measure associated with the jumps of  $X_t$ .

**Remark 2.2.** By assuming time-inhomogeneous Lévy process to model  $Z_t$ , we allow for time dependent reestimation behavior. For example, one could imagine that the reestimation frequency is higher in the beginning than later on.

## 3 Pricing of catastrophe insurance derivatives

### 3.1 Pricing measure

In the catastrophe insurance market the underlying index  $L$  is not traded. Hence the market is incomplete and there exist infinitely many equivalent martingale measures. We make here the usual assumption that under the pricing measure  $\mathbb{Q}$  the index process is described by the same kind of process as under  $\mathbb{P}$ . In particular, we assume that the class of pricing measures is determined by Radon-Nykodym derivatives of the following form:

$$\begin{aligned} \frac{d\mathbb{Q}}{d\mathbb{P}} = & \exp \left\{ \sum_{j=1}^{N_{T_1}} \beta(Y_j) - \int_0^{T_1} \lambda_s ds E \left[ e^{\beta(Y_1)} - 1 \right] \right\} \\ & \cdot \exp \left\{ \int_0^T \gamma(s) dW_s - \frac{1}{2} \int_0^T \gamma^2(s) ds \right\} \\ & \cdot \exp \left\{ \int_0^T \ln \phi(s, x) (\mu(ds, dx) - F_s(dx)ds) \right. \\ & \quad \left. - \int_0^T \int_{\mathbb{R}} (\phi(s, x) - 1 - \ln \phi(s, x)) F_s(dx) ds \right\} \end{aligned} \quad (5)$$

for some Borel function  $\beta$  such that  $E[e^{\beta(Y_1)}] < \infty$  and positive deterministic integrands  $\phi(t, x)$  and  $\gamma(t)$  such that  $E[\frac{d\mathbb{Q}}{d\mathbb{P}}] = 1$ .

By Girsanov's Theorem for Brownian motion and random measures (see [10]) this class of pricing measures preserves the structure of our model. In particular,

under the measure  $\mathbb{Q}$  the process  $L_t$ ,  $t \in [0, T_1]$ , is again a time inhomogeneous compound Poisson process with intensity

$$\lambda_t^{\mathbb{Q}} = \lambda_t E[e^{\beta(Y_1)}] \quad (6)$$

and distribution function of jumps

$$dG^{\mathbb{Q}}(y) = \frac{e^{\beta(y)}}{E[e^{\beta(Y_1)}]} dG(y). \quad (7)$$

Further, under  $\mathbb{Q}$  the process  $X$  is again a time inhomogeneous Lévy process independent of  $L_t$ ,  $t \in [0, T_1]$ , with characteristics  $(b^{\mathbb{Q}}, c^{\mathbb{Q}}, F^{\mathbb{Q}})$  given by

$$\begin{aligned} b_t^{\mathbb{Q}} &= b_t - \gamma_t \sqrt{c_t}, \\ c_t^{\mathbb{Q}} &= c_t, \\ F_t^{\mathbb{Q}}(dx) &= \phi(t, x) F_t(dx). \end{aligned}$$

In order to specify a pricing measure  $\mathbb{Q}$ , one possible method is now to calibrate  $\beta$ ,  $\phi$  and  $\gamma$  to observed market prices. For example, in [19] the pricing measure is calibrated on the prices of insurance portfolios (i.e. from the premiums) and the prices of catastrophe derivatives. Another approach to pick a pricing measure is chosen in [5], [16], [19] and [21], where the choice of the pricing measure for catastrophe insurance options is motivated through an equilibrium argument between the premium price and the price of an insurance derivative written on the same catastrophe losses. In [5] and in [16], the Esscher transform is used to compute the equivalent martingale measure, which is justified by looking at a representative investor maximizing her expected utility.

Here we do not discuss the problem of choosing  $\beta$ ,  $\phi$  and  $\gamma$ , but we assume to be given an equivalent martingale measure  $\mathbb{Q}$  of the form (5) and proceed to the risk neutral pricing under  $\mathbb{Q}$  of catastrophe options as described in the next section.

### 3.2 Pricing via Fourier transform techniques

Consider a European derivative written on the loss index with maturity  $T_2$  and payoff

$$h(L_{T_2}) > 0$$

for a continuous payoff function  $h : \mathbb{R} \mapsto \mathbb{R}_+$ . Since we have assumed that the interest rate  $r$  is deterministic, without loss of generality, we can express the price process of the insurance derivative in discounted terms, i.e. we can set  $r \equiv 0$ . Let



a risk neutral pricing measure  $\mathbb{Q}$  be given. Then the consistent price process of the option is given by

$$\begin{aligned}\pi_t^{\mathbb{Q}} &= E^{\mathbb{Q}} [h(L_{T_2}) | \mathcal{F}_t] = E^{\mathbb{Q}} [h(L_{T_1} Z_{T_2-T_1}) | \mathcal{F}_t] \\ &= E^{\mathbb{Q}} [h(L_{T_1} e^{X_{T_2-T_1}}) | \mathcal{F}_t].\end{aligned}$$

Interpreting the claim as a payoff on two factors, we can rewrite the price process as

$$\pi_t^{\mathbb{Q}} = E^{\mathbb{Q}} [g(L_{T_1}, X_{T_2-T_1}) | \mathcal{F}_t], \quad (8)$$

where  $g : \mathbb{R}^2 \mapsto \mathbb{R}_+$  is the function such that

$$g(x_1, x_2) = h(x_1 e^{x_2}) \quad \text{for any } (x_1, x_2) \in \mathbb{R}^2. \quad (9)$$

In the following we will calculate the expected payoff in (8) by Fourier transform techniques. To this end we extend the approach of dampened payoffs on one dimensional assets of [8] (see also [4], [20]) to general payoffs on two dimensional assets. We impose the following conditions:

(A1) Assume that

$$I_1 := \{(\alpha, \beta) \in \mathbb{R}^2 \mid \int_{\mathbb{R}^2} e^{-\alpha x_1 - \beta x_2} g(x_1, x_2) dx_1 dx_2 < \infty\} \neq \emptyset.$$

(A2) Let

$$I_2 := \{(\alpha, \beta) \in \mathbb{R}^2 \mid \int_{\mathbb{R}^2} e^{\alpha x_1 + \beta x_2} G_{(L_{T_1}, X_{T_2-T_1})}^{\mathbb{Q}}(dx_1, dx_2) < \infty\},$$

where  $G_{(L_{T_1}, X_{T_2-T_1})}^{\mathbb{Q}}$  is the cumulative distribution function of  $(L_{T_1}, X_{T_2-T_1})$  under  $\mathbb{Q}$  and assume that  $I_1 \cap I_2 \neq \emptyset$ .

Note that, since  $L_{T_1}$  and  $X_{T_2-T_1}$  remain independent under  $\mathbb{Q}$ , it follows that

$$I_2 = \{(\alpha, \beta) \in \mathbb{R}^2 \mid E^{\mathbb{Q}}[e^{\alpha L_{T_1}}] < \infty \text{ and } E^{\mathbb{Q}}[e^{\beta X_{T_2-T_1}}] < \infty\}. \quad (10)$$

Now, the *dampened payoff function* is introduced as

$$f(x_1, x_2) = e^{-\alpha x_1 - \beta x_2} g(x_1, x_2) \quad \text{for } (\alpha, \beta) \in I_1 \cap I_2. \quad (11)$$

Note, that under Assumption (A1), we have that

$$f(\cdot) \in L^1(\mathbb{R}^2)$$

for  $(\alpha, \beta) \in I_1 \cap I_2$ . Hence the Fourier transform

$$\hat{f}(u_1, u_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(x_1 u_1 + x_2 u_2)} f(x_1, x_2) dx_1 dx_2 \quad (12)$$

is well defined for every  $u = (u_1, u_2) \in \mathbb{R}^2$ . Assuming also

$$(A3) \quad \hat{f}(\cdot) \in L^1(\mathbb{R}^2),$$

we get by the Inversion Theorem (cf. [15], Section 8.2) that

$$f(x_1, x_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i(x_1 u_1 + x_2 u_2)} \hat{f}(u_1, u_2) du_1 du_2. \quad (13)$$

**Remark 3.1.** Note that the equality in (13) is everywhere and not only almost everywhere because we have assumed a continuous payoff function  $h$ . If the probability distribution of  $L_{T_2}$  had a Lebesgue density, an almost everywhere equality in (13) would have been sufficient for the following computations. However, since the loss index is driven by a compound Poisson process, the distribution of  $L_{T_2}$  has atoms and we need an everywhere equality to guarantee (14) below.

Now, returning to the valuation problem (8), we obtain that

$$\begin{aligned} \pi_t^{\mathbb{Q}} &= E^{\mathbb{Q}} [g(L_{T_1}, X_{T_2-T_1}) | \mathcal{F}_t] = E^{\mathbb{Q}} \left[ e^{\alpha L_{T_1} + \beta X_{T_2-T_1}} f(L_{T_1}, X_{T_2-T_1}) | \mathcal{F}_t \right] \\ &= \frac{1}{2\pi} E^{\mathbb{Q}} \left[ e^{\alpha L_{T_1} + \beta X_{T_2-T_1}} \int_{\mathbb{R}^2} e^{-i(u_1 L_{T_1} + u_2 X_{T_2-T_1})} \hat{f}(u_1, u_2) du_1 du_2 \Big| \mathcal{F}_t \right] \quad (14) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2\pi} E^{\mathbb{Q}} \left[ \int_{\mathbb{R}^2} e^{-i\{(u_1 + i\alpha)L_{T_1} + (u_2 + i\beta)X_{T_2-T_1}\}} \hat{f}(u_1, u_2) du_1 du_2 \Big| \mathcal{F}_t \right] \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} E^{\mathbb{Q}} \left[ e^{-i\{(u_1 + i\alpha)L_{T_1} + (u_2 + i\beta)X_{T_2-T_1}\}} | \mathcal{F}_t \right] \hat{f}(u_1, u_2) du_1 du_2 \quad (15) \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2\pi} \int_{\mathbb{R}^2} E^{\mathbb{Q}} \left[ e^{-i(u_1 + i\alpha)L_{T_1}} | \mathcal{F}_t \right] E^{\mathbb{Q}} \left[ e^{-i(u_2 + i\beta)X_{T_2-T_1}} | \mathcal{F}_t \right] \\ &\quad \cdot \hat{f}(u_1, u_2) du_1 du_2, \quad (16) \end{aligned}$$

where in the equality (15) we could apply Fubini's theorem, because (A3) holds. The last equation holds by the independence of  $L_{T_1}$  and  $X_{T_2-T_1}$  and by Assumption (A2).

Since  $L$  is a time inhomogeneous compound Poisson process until  $T_1$  and  $X$  is a time inhomogeneous Lévy process independent of  $L_t$ ,  $t \in [0, T_1]$ , we can explicitly compute the conditional expectations by using the known form of the conditional characteristic functions:

1. If  $t < T_1$ , we have

$$\begin{aligned}
& E^{\mathbb{Q}} \left[ e^{-i(u_1+i\alpha)L_{T_1}} | \mathcal{F}_t \right] \\
&= e^{-i(u_1+i\alpha)L_t} E^{\mathbb{Q}} \left[ e^{-i(u_1+i\alpha)(L_{T_1}-L_t)} \right] \\
&= e^{-i(u_1+i\alpha)L_t} \exp \left\{ - \int_t^{T_1} \lambda_s^{\mathbb{Q}} ds \int_0^{\infty} (1 - e^{-i(u_1+i\alpha)x}) G^{\mathbb{Q}}(dx) \right\} \\
&= e^{-\int_t^{T_1} \lambda_s^{\mathbb{Q}} ds} e^{-i(u_1+i\alpha)L_t} \exp \left\{ \int_t^{T_1} \lambda_s^{\mathbb{Q}} ds \int_0^{\infty} e^{-i(u_1+i\alpha)x} G^{\mathbb{Q}}(dx) \right\},
\end{aligned}$$

and

$$\begin{aligned}
& E^{\mathbb{Q}} \left[ e^{-i(u_2+i\beta)X_{T_2-T_1}} | \mathcal{F}_t \right] = E^{\mathbb{Q}} \left[ e^{-i(u_2+i\beta)X_{T_2-T_1}} \right] \\
&= \exp \left\{ \int_0^{T_2-T_1} \left( i(u_2+i\beta)b_s^{\mathbb{Q}} - \frac{1}{2}c_s^{\mathbb{Q}}(u_2+i\beta)^2 \right) ds \right\} \\
&\cdot \exp \left\{ \int_0^{T_2-T_1} \int_{\mathbb{R}} (e^{i(u_2+i\beta)x} - 1 - i(u_2+i\beta)xI_{\{|x|\leq 1\}}) F_s^{\mathbb{Q}}(dx) ds \right\}.
\end{aligned}$$

2. If  $t \in [T_1, T_2]$ ,

$$E^{\mathbb{Q}} \left[ e^{-i(u_1+i\alpha)L_{T_1}} | \mathcal{F}_t \right] = e^{-i(u_1+i\alpha)L_{T_1}};$$

and

$$\begin{aligned}
& E^{\mathbb{Q}} \left[ e^{-i(u_2+i\beta)X_{T_2-T_1}} | \mathcal{F}_t \right] \\
&= e^{-i(u_2+i\beta)X_{t-T_1}} E^{\mathbb{Q}} \left[ e^{-i(u_2+i\beta)(X_{T_2-T_1}-X_{t-T_1})} \right] \\
&= e^{-i(u_2+i\beta)X_{t-T_1}} \exp \left\{ \int_{t-T_1}^{T_2-T_1} \left( i(u_2+i\beta)b_s^{\mathbb{Q}} - \frac{1}{2}c_s^{\mathbb{Q}}(u_2+i\beta)^2 \right) ds \right\} \\
&\cdot \exp \left\{ \int_{t-T_1}^{T_2-T_1} \int_{\mathbb{R}} (e^{i(u_2+i\beta)x} - 1 - i(u_2+i\beta)xI_{\{|x|\leq 1\}}) F_s^{\mathbb{Q}}(dx) ds \right\}.
\end{aligned}$$

Hence, in order to calculate the price process  $(\pi_t^{\mathbb{Q}})_{t \in [0, T_2]}$  the only remaining task is to compute the Fourier transform of the dampened payoff function  $f$ . We summarize our results in the following

**Theorem 3.2.** *Under the Hypotheses (A1)-(A3), the price process  $\pi_t^{\mathbb{Q}}$  of an catastrophe insurance option written on the loss index with maturity  $T_2$  and payoff  $h(L_{T_2}) > 0$  is given by*

1. for  $t \in [0, T_1]$  by

$$\begin{aligned} \pi_t^{\mathbb{Q}} &= \frac{1}{2\pi} e^{-\int_t^{T_1} \lambda_s^{\mathbb{Q}} ds} \int_{\mathbb{R}^2} \hat{f}(u_1, u_2) e^{-i(u_1+i\alpha)L_t} \\ &\quad \exp\left\{ \int_t^{T_1} \lambda_s^{\mathbb{Q}} ds \int_0^\infty e^{-i(u_1+i\alpha)x} G^{\mathbb{Q}}(dx) \right\} \\ &\quad \exp\left\{ \int_0^{T_2-T_1} \left( i(u_2+i\beta)b_s^{\mathbb{Q}} - \frac{1}{2}c_s^{\mathbb{Q}}(u_2+i\beta)^2 \right) ds \right\} \\ &\quad \exp\left\{ \int_0^{T_2-T_1} \int_{\mathbb{R}} \left( e^{i(u_2+i\beta)x} - 1 - i(u_2+i\beta)xI_{\{|x|\leq 1\}} \right) F_s^{\mathbb{Q}}(dx) \right\} ds du_1 du_2, \end{aligned}$$

and

2. for  $t > T_1$  by

$$\begin{aligned} \pi_t^{\mathbb{Q}} &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{f}(u_1, u_2) e^{-i(u_1+i\alpha)L_{T_1}} e^{-i(u_2+i\beta)X_{t-T_1}} \\ &\quad \exp\left\{ \int_{t-T_1}^{T_2-T_1} \left( i(u_2+i\beta)b_s^{\mathbb{Q}} - \frac{1}{2}c_s^{\mathbb{Q}}(u_2+i\beta)^2 \right) ds \right\} \\ &\quad \exp\left\{ \int_{t-T_1}^{T_2-T_1} \int_{\mathbb{R}} \left( e^{i(u_2+i\beta)x} - 1 - i(u_2+i\beta)xI_{\{|x|\leq 1\}} \right) F_s^{\mathbb{Q}}(dx) \right\} ds du_1 du_2. \end{aligned}$$

Here  $f$  is the dampened payoff as defined in (11) and  $\hat{f}$  its Fourier transform (12).

**Remark 3.3.** In order to estimate  $\pi_t^{\mathbb{Q}}$  numerically several methods are possible. One commonly used technique is the fast Fourier transform (FFT). In our case we need to apply FFT for a double integral which implies reduced speed of convergence. There exist various techniques to improve the convergence speed (see for example the “integration-along-cut” method suggested in [3]). However, speed becomes an issue only when one repeatedly needs to price a large number of options. For further discussion on this topic we refer to [7].

**Remark 3.4.** In this paper we have chosen to model  $Z_t$  with a time inhomogeneous Lévy process. This class of processes is very rich and flexible to model a wide range of phenomena, and at the same time it is analytically very tractable. Note, however, that all the calculations go through explicitly in the same way even for other choices of processes for  $Z_t$ , as long as the conditional characteristic function is known.

## 4 Application: call, put and spread catastrophe options

In this section we consider the most common catastrophe insurance options traded in the market: call, put, and spread options. By computing explicitly the Fourier transform corresponding to the payoff, we are able to provide pricing formulas for these options using Theorem 3.2.

**Example 4.1** (Call option).

Consider the payoff function of a catastrophe call option in the form

$$h_{\text{call}}(x) = (x - K)^+ \quad (17)$$

for some strike price  $K > 0$ . Then the corresponding payoff on a two dimensional asset as introduced in (9) is

$$g_{\text{call}}(x_1, x_2) = (x_1 e^{x_2} - K)^+ I_{\{x_1 > 0\}} = (x_1 e^{x_2} - K) I_{\{x_1 > 0, x_2 > \ln \frac{K}{x_1}\}}$$

and the dampened payoff function is

$$\begin{aligned} f_{\text{call}}(x_1, x_2) &= e^{-\alpha x_1 - \beta x_2} g_{\text{call}}(x_1, x_2) \\ &= e^{-\alpha x_1 - \beta x_2} (x_1 e^{x_2} - K) I_{\{x_1 > 0, x_2 > \ln \frac{K}{x_1}\}}, \end{aligned} \quad (18)$$

that belongs to  $L^1(\mathbb{R}^2)$  for all  $(\alpha, \beta) \in I_1 = (0, \infty) \times (1, \infty)$ . For the Fourier transform  $\hat{f}_{\text{call}}$  we obtain

$$\begin{aligned} \hat{f}_{\text{call}}(u_1, u_2) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(x_1 u_1 + x_2 u_2)} f_{\text{call}}(x_1, x_2) dx_1 dx_2 \\ &= \frac{1}{2\pi} \int_0^\infty \int_{\ln \frac{K}{x_1}}^\infty e^{-(\alpha - i u_1)x_1 - (\beta - i u_2)x_2} (x_1 e^{x_2} - K) dx_2 dx_1 \\ &= \frac{1}{2\pi} \left[ \int_0^\infty x_1 e^{-(\alpha - i u_1)x_1} \int_{\ln \frac{K}{x_1}}^\infty e^{-(\beta - 1 - i u_2)x_2} dx_2 dx_1 \right. \\ &\quad \left. - K \int_0^\infty e^{-(\alpha - i u_1)x_1} \int_{\ln \frac{K}{x_1}}^\infty e^{-(\beta - i u_2)x_2} dx_2 dx_1 \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \left[ \frac{1}{\beta - 1 - iu_2} \int_0^\infty x_1 e^{-(\alpha - iu_1)x_1} e^{-(\beta - 1 - iu_2) \ln K/x_1} dx_1 \right. \\
&\quad \left. - \frac{K}{\beta - iu_2} \int_0^\infty e^{-(\alpha - iu_1)x_1} e^{-(\beta - iu_2) \ln K/x_1} dx_1 \right] \\
&= \frac{1}{2\pi} \left[ \frac{1}{(\beta - 1 - iu_2)K^{(\beta - 1 - iu_2)}} \int_0^\infty x_1^{\beta - iu_2} e^{-(\alpha - iu_1)x_1} dx_1 \right. \\
&\quad \left. - \frac{1}{(\beta - iu_2)K^{(\beta - 1 - iu_2)}} \int_0^\infty x_1^{\beta - iu_2} e^{-(\alpha - iu_1)x_1} dx_1 \right] \\
&= \frac{1}{2\pi} \frac{1}{(\beta - 1 - iu_2)(\beta - iu_2)K^{(\beta - 1 - iu_2)}} \int_0^\infty x_1^{\beta - iu_2} e^{-(\alpha - iu_1)x_1} dx_1 \\
&= \frac{1}{2\pi} \frac{1}{(\beta - 1 - iu_2)(\beta - iu_2)(\alpha - iu_1)^{(\beta + 1 - iu_2)} K^{(\beta - 1 - iu_2)}} \Gamma(\beta + 1 - iu_2),
\end{aligned}$$

where  $\Gamma(\cdot)$  is the Gamma function.

To prove that the payoff function of a catastrophe call option (17) satisfies the conditions of Theorem 3.2, it remains to show that

$$\hat{f}_{\text{call}}(u_1, u_2) \in L^1(\mathbb{R}^2). \quad (19)$$

Note that to prove (19) it is sufficient to consider the asymptotics of  $|\hat{f}_{\text{call}}(u_1, u_2)|$  for  $|u_1|, |u_2| \rightarrow \infty$ . In fact, since

$$\lim_{|u_2| \rightarrow \infty} |\Gamma(\beta + 1 - iu_2)| e^{\frac{\pi}{2}|u_2|} |u_2|^{-\beta - \frac{1}{2}} = \sqrt{2\pi} \quad (20)$$

(see 8.328.1 in [9]), we get

$$\begin{aligned}
|\hat{f}_{\text{call}}(u_1, u_2)| &= \frac{1}{2\pi} \frac{1}{K^{\beta - 1} |e^{-iu_2 \ln K}|} \\
&\quad \cdot \frac{|\Gamma(\beta + 1 - iu_2)| |e^{iu_2(\ln |\alpha - iu_1| - i \arctan \frac{u_1}{\alpha})}|}{|(\beta - 1 - iu_2)(\beta - iu_2)(\alpha - iu_1)^{(\beta + 1)}|} \\
&= \frac{1}{2\pi} \frac{1}{K^{\beta - 1}} \frac{|\Gamma(\beta + 1 - iu_2)| e^{u_2 \arctan \frac{u_1}{\alpha}}}{|(\beta - 1 - iu_2)(\beta - iu_2)(\alpha - iu_1)^{(\beta + 1)}|} \\
&\sim \frac{1}{\sqrt{2\pi}} \frac{1}{K^{\beta - 1}} \frac{e^{-\frac{\pi}{2}|u_2|} |u_2|^{\beta - \frac{3}{2}} e^{u_2 \arctan \frac{u_1}{\alpha}}}{|u_1|^{\beta + 1}}, \quad (21)
\end{aligned}$$

where

$$f_1(u_1, u_2) \sim f_2(u_1, u_2) \quad :\Leftrightarrow \quad \lim_{|u_1|, |u_2| \rightarrow \infty} \frac{|f_1(u_1, u_2)|}{|f_2(u_1, u_2)|} = 1.$$

Now we distinguish the following cases:

1. If  $u_1 u_2 < 0$ , then (21) simplifies to

$$\begin{aligned} |\hat{f}_{\text{call}}(u_1, u_2)| &\sim \frac{1}{\sqrt{2\pi}} \frac{1}{K^{\beta-1}} \frac{e^{-\frac{\pi}{2}|u_2|} |u_2|^{\beta-\frac{3}{2}} e^{-|u_2 \arctan \frac{u_1}{\alpha}|}}{|u_1|^{\beta+1}} \\ &\sim \frac{1}{\sqrt{2\pi}} \frac{1}{K^{\beta-1}} \frac{e^{-\pi|u_2|} |u_2|^{\beta-\frac{3}{2}}}{|u_1|^{\beta+1}}, \end{aligned} \quad (22)$$

where the right hand side of (22) is integrable at infinity.

2. If  $u_1 u_2 > 0$ , then (21) is equivalent to

$$\begin{aligned} |\hat{f}_{\text{call}}(u_1, u_2)| &\sim \frac{1}{\sqrt{2\pi}} \frac{1}{K^{\beta-1}} \frac{e^{-\frac{\pi}{2}|u_2|} |u_2|^{\beta-\frac{3}{2}} e^{|u_2| \arctan \frac{|u_1|}{\alpha}}}{|u_1|^{\beta+1}} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{K^{\beta-1}} \frac{e^{-\frac{\pi}{2}|u_2|} |u_2|^{\beta-\frac{3}{2}} e^{|u_2|(\frac{\pi}{2} - \arctan \frac{\alpha}{|u_1|})}}{|u_1|^{\beta+1}} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{K^{\beta-1}} \frac{|u_2|^{\beta-\frac{3}{2}} e^{-|u_2| \arctan \frac{\alpha}{|u_1|}}}{|u_1|^{\beta+1}} \\ &\sim \frac{1}{\sqrt{2\pi}} \frac{1}{K^{\beta-1}} \frac{|u_2|^{\beta-\frac{3}{2}} e^{-|u_2| \frac{\alpha}{|u_1|}}}{|u_1|^{\beta+1}}. \end{aligned} \quad (23)$$

$$\sim \frac{1}{\sqrt{2\pi}} \frac{1}{K^{\beta-1}} \frac{|u_2|^{\beta-\frac{3}{2}} e^{-|u_2| \frac{\alpha}{|u_1|}}}{|u_1|^{\beta+1}}. \quad (24)$$

Since

$$\int_0^\infty \frac{u_2^{\beta-\frac{3}{2}} e^{-u_2 \frac{\alpha}{|u_1|}}}{|u_1|^{\beta+1}} du_2 = \alpha^{\frac{1}{2}-\beta} \Gamma(\beta - \frac{1}{2}) |u_1|^{-\frac{3}{2}}$$

is integrable at infinity, the right hand side of (24) is integrable as  $|u_1|, |u_2| \rightarrow \infty$ .

We can thus apply Theorem 3.2 and obtain an explicit price for the call option.

Once we know the price for call options, pricing of catastrophe insurance put and spread options can be reduced to the pricing of call options with standard arguments.

**Example 4.2** (Put option).

Let

$$h_{\text{put}}(x) = (K - x)^+$$

be the payoff of a catastrophe insurance put option. Then the payoffs of call and put options with the same strike  $K$  are related through the formula

$$h_{\text{put}}(x) = h_{\text{call}}(x) + K - L_{T_2}.$$

We can thus determine the price  $\pi_{\text{put}}^{\mathbb{Q}}(t)$  of the put option through computing the price  $\pi_{\text{call}}^{\mathbb{Q}}(t)$  of the call option and through the following call-put parity:

$$\begin{aligned} \pi_{\text{put}}^{\mathbb{Q}}(t) &= \pi_{\text{call}}^{\mathbb{Q}}(t) + K - E^{\mathbb{Q}}[L_{T_2} | \mathcal{F}_t] \\ &= \pi_{\text{call}}^{\mathbb{Q}}(t) + K - E^{\mathbb{Q}}[L_{T_1} Z_{T_2-T_1} | \mathcal{F}_t]. \end{aligned}$$

For the conditional expectation  $E^{\mathbb{Q}}[L_{T_1} Z_{T_2-T_1} | \mathcal{F}_t]$  we get by independence of  $(L_t)_{t \leq T_1}$  and  $(Z_{T_1+u})_{u \leq T_2-T_1}$  that

1. if  $t \leq T_1$

$$\begin{aligned} &E^{\mathbb{Q}}[L_{T_1} Z_{T_2-T_1} | \mathcal{F}_t] \\ &= E^{\mathbb{Q}}[L_{T_1} | \mathcal{F}_t] E^{\mathbb{Q}}[Z_{T_2-T_1} | \mathcal{F}_t] \\ &= (L_t + E^{\mathbb{Q}}[L_{T_1} - L_t]) E^{\mathbb{Q}}[e^{X_{T_2-T_1}}] \\ &= (L_t + E^{\mathbb{Q}}[Y_1] \int_t^{T_1} \lambda_s^{\mathbb{Q}} ds) \\ &\quad \cdot \exp\left\{ \int_0^{T_2-T_1} \left( b_s^{\mathbb{Q}} + \frac{1}{2} c_s^{\mathbb{Q}} + \int_{\mathbb{R}} (e^x - 1 + x I_{\{|x| \leq 1\}}) F_s^{\mathbb{Q}}(dx) \right) ds \right\}; \end{aligned}$$

2. if  $t \in [T_1, T_2]$

$$\begin{aligned} &E^{\mathbb{Q}}[L_{T_1} Z_{T_2-T_1} | \mathcal{F}_t] \\ &= E^{\mathbb{Q}}[L_{T_1} e^{X_{T_2-T_1}} | \mathcal{F}_t] = L_{T_1} Z_{t-T_1} E^{\mathbb{Q}}[\exp\{X_{T_2-T_1} - X_{t-T_1}\}] \\ &= L_{T_1} Z_{t-T_1} \\ &\quad \cdot \exp\left\{ \int_{t-T_1}^{T_2-T_1} \left( b_s^{\mathbb{Q}} + \frac{1}{2} c_s^{\mathbb{Q}} + \int_{\mathbb{R}} (e^x - 1 + x I_{\{|x| \leq 1\}}) F_s^{\mathbb{Q}}(dx) \right) ds \right\}. \end{aligned}$$



**Example 4.3** (Call and put spread option).

A call spread option is a capped call option which is a combination of buying a call option with strike price  $K_1$ , and selling at the same time a call option with the same maturity but with the strike price  $K_2 > K_1$ . This corresponds to a payoff function at maturity of the form

$$\begin{aligned} h_{\text{spread}}(x) &= (x - K_1)^+ - (x - K_2)^+ \\ &= \begin{cases} 0, & \text{if } 0 \leq x \leq K_1; \\ x - K_1, & \text{if } K_1 < x \leq K_2; \\ K_2 - K_1, & \text{if } x > K_2. \end{cases} \end{aligned}$$

The price of the catastrophe call option is thus the difference of the prices of the call options with strike prices  $K_1$  and  $K_2$  respectively. Analogously we can calculate the price of a put spread catastrophe option using the results in Example 4.2.

**Remark 4.4.** Note that for the above computations the damping parameter  $\alpha$  in (18) has to be strictly bigger than zero. By (10) this implies that the distribution  $G^{\mathbb{Q}}$  of the claim sizes  $Y_i$ ,  $i = 1, 2, \dots$ , has to fulfill

$$\int_{\mathbb{R}_+} e^{\alpha x} G^{\mathbb{Q}}(dy) < \infty, \quad \text{for some } \alpha > 0. \quad (25)$$

Typical examples of the distributions satisfying (25) are the exponential, Gamma, and truncated normal distributions. An important class of distribution functions which also satisfy (25) is the class of convolution equivalent distribution functions  $\mathcal{S}(\alpha)$  for  $\alpha > 0$ , which is convenient for the modelling of the claim sizes. See [13] for the definition and property and see [14] for the application of the convolution equivalent distributions. The generalized inverse Gaussian distribution is one of the most important example among the convolution equivalent distributions.

On the other hand, distributions  $G^{\mathbb{Q}}$  with heavy tails do not fulfill (25) (they would require  $\alpha \leq 0$ ). Because, however, the class of heavy tailed distributions is very relevant for catastrophe claim size modeling, we will in the next subsection specify a framework, in which we can also price catastrophe options with heavy tailed claims.

## 4.1 Pricing with heavy-tailed losses

In order to treat heavy-tailed losses, i.e. to be able to take a damping parameter  $\alpha = 0$  in (10), we make the assumption that the distribution function  $G^{\mathbb{Q}}$  of

$Y_i, i = 1, 2, \dots$ , has support on  $(\epsilon, \infty)$  for some  $\epsilon > 0$ . In other words, we assume that if a catastrophe occurs then the corresponding loss amount is greater than some arbitrarily small  $\epsilon > 0$ . This assumption is obviously no serious restriction, especially in the light of the fact that PCS defines a catastrophe as a single incident or a series of related incidents (man-made or natural disasters) that causes insured property losses of at least \$25 million. Note that this implies

$$\{L_{T_1} > 0\} = \{L_{T_1} > \epsilon\}, \quad (26)$$

since  $L$  is a time inhomogeneous compound Poisson process until  $T_1$ .

In this framework we now want to apply the Fourier technique of Section 3.2 to price a catastrophe put option. To this end we first perform the following transformations. The price process of a catastrophe put option is given by

$$\pi_t^{\mathbb{Q}} = E^{\mathbb{Q}} [(K - L_{T_1} e^{X_{T_2-T_1}})^+ | \mathcal{F}_t]. \quad (27)$$

Since  $L$  is a time inhomogeneous compound Poisson process until  $T_1$  under  $\mathbb{Q}$ , we can rewrite (27) as

$$\begin{aligned} \pi_t^{\mathbb{Q}} &= E^{\mathbb{Q}} \left[ (K - L_{T_1} e^{X_{T_2-T_1}})^+ I_{\{N_{T_1}=0\}} | \mathcal{F}_t \right] \\ &\quad + E^{\mathbb{Q}} \left[ (K - L_{T_1} e^{X_{T_2-T_1}})^+ I_{\{N_{T_1}>0\}} | \mathcal{F}_t \right] \\ &= K \mathbb{Q}(N_{T_1} = 0 | \mathcal{F}_t) + E^{\mathbb{Q}} \left[ (K - L_{T_1} e^{X_{T_2-T_1}})^+ I_{\{L_{T_1}>0\}} | \mathcal{F}_t \right], \end{aligned} \quad (28)$$

where we have used that  $L_{T_1} I_{\{N_{T_1}=0\}} = 0$ .

Let  $\bar{L}_{T_1} := L_{T_1} - \epsilon$ . Then by (26)

$$\{L_{T_1} > 0\} = \{L_{T_1} > \epsilon\} = \{\bar{L}_{T_1} + \epsilon > \epsilon\} = \{\bar{L}_{T_1} > 0\}.$$

Hence we obtain

$$\begin{aligned} &E^{\mathbb{Q}} \left[ (K - L_{T_1} e^{X_{T_2-T_1}})^+ I_{\{L_{T_1}>0\}} | \mathcal{F}_t \right] \\ &= E^{\mathbb{Q}} \left[ (K - (\bar{L}_{T_1} + \epsilon) e^{X_{T_2-T_1}})^+ I_{\{\bar{L}_{T_1}>0\}} | \mathcal{F}_t \right]. \end{aligned} \quad (29)$$

Define the pay off function  $g$  by

$$g(x_1, x_2) = (K - (x_1 + \epsilon) e^{x_2})^+ I_{\{x_1 > 0\}}.$$

In order to apply the Fourier method of Theorem 3.2, we continuously extend  $g$  from  $\mathbb{R}_+ \times \mathbb{R}$  to  $\mathbb{R}^2$  as

$$\bar{g}(x_1, x_2) = (K - (|x_1| + \epsilon) e^{x_2})^+.$$

Then we have

$$\begin{aligned} E^{\mathbb{Q}} [\bar{g}(\bar{L}_{T_1}, X_{T_2-T_1}) | \mathcal{F}_t] &= E^{\mathbb{Q}} \left[ (K - (\bar{L}_{T_1} + \epsilon) e^{X_{T_2-T_1}})^+ I_{\{\bar{L}_{T_1} > 0\}} | \mathcal{F}_t \right] \\ &\quad + E^{\mathbb{Q}} \left[ (K - (|\bar{L}_{T_1}| + \epsilon) e^{X_{T_2-T_1}})^+ I_{\{\bar{L}_{T_1} \leq 0\}} | \mathcal{F}_t \right]. \end{aligned} \quad (30)$$

Since  $\{\bar{L}_{T_1} \leq 0\} = \{L_{T_1} = 0\} = \{\bar{L}_{T_1} = -\epsilon\}$ , the second term on the right-hand side of (30) is

$$\begin{aligned} &E^{\mathbb{Q}} \left[ (K - (|\bar{L}_{T_1}| + \epsilon) e^{X_{T_2-T_1}})^+ I_{\{\bar{L}_{T_1} \leq 0\}} | \mathcal{F}_t \right] \\ &= E^{\mathbb{Q}} \left[ (K - 2\epsilon e^{X_{T_2-T_1}})^+ I_{\{\bar{L}_{T_1} = -\epsilon\}} | \mathcal{F}_t \right] \\ &= E^{\mathbb{Q}} \left[ (K - 2\epsilon e^{X_{T_2-T_1}})^+ | \mathcal{F}_t \right] \mathbb{Q}(\bar{L}_{T_1} = -\epsilon | \mathcal{F}_t) \\ &= E^{\mathbb{Q}} \left[ (K - 2\epsilon e^{X_{T_2-T_1}})^+ | \mathcal{F}_t \right] \mathbb{Q}(L_{T_1} = 0 | \mathcal{F}_t) \\ &= E^{\mathbb{Q}} \left[ (K - 2\epsilon e^{X_{T_2-T_1}})^+ | \mathcal{F}_t \right] \mathbb{Q}(N_{T_1} = 0 | \mathcal{F}_t). \end{aligned} \quad (31)$$

Together, equations (28)–(31) then lead to the following expression for the price process of a put option.

**Proposition 4.5.** *The price process of a catastrophe put option is given by*

$$\pi_t^{\mathbb{Q}} = KP_t^0 + P_t^1 P_t^0 + P_t^2,$$

where

$$\begin{aligned} P_t^0 &= e^{-\int_t^{T_1} \lambda^{\mathbb{Q}}(s) ds} I_{\{N_t=0\}}, \\ P_t^1 &= E^{\mathbb{Q}} \left[ (K - 2\epsilon e^{X_{T_2-T_1}})^+ | \mathcal{F}_t \right], \\ P_t^2 &= E^{\mathbb{Q}} \left[ (K - (|\bar{L}_{T_1}| + \epsilon) e^{X_{T_2-T_1}})^+ | \mathcal{F}_t \right]. \end{aligned}$$

**Proof.** Given equations (28)–(31), it only remains to validate the expression for  $P_t^0$ . Since  $N_t$  is a time inhomogeneous Poisson process with deterministic intensity  $\lambda^{\mathbb{Q}}(t) > 0$  under  $\mathbb{Q}$ , we have

$$\begin{aligned} \mathbb{Q}(N_{T_1} = 0 | \mathcal{F}_t) &= \mathbb{Q}((N_{T_1} - N_t) + N_t = 0 | \mathcal{F}_t) \\ &= \mathbb{Q}((N_{T_1} - N_t) + n = 0 | \mathcal{F}_t) |_{n=N_t} \\ &= e^{-\int_t^{T_1} \lambda^{\mathbb{Q}}(s) ds} I_{\{N_t=0\}}. \end{aligned}$$

□

Note that  $P_t^1$  is the price process of a regular put option written on a one dimensional asset that is given by an exponential Lévy process. This price can be

obtained by Fourier transform techniques or any other favorite method. To use in one dimension the Fourier transform methods of this paper, one computes that the dampened pay off

$$f_2(x_2) := (K - 2\epsilon e^{x_2})^+ e^{\beta x_2} \quad \text{for } \beta > 1,$$

has Fourier transform

$$\begin{aligned} \hat{f}_2(u) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\ln \frac{K}{2\epsilon}} e^{iu x_2} e^{\beta x_2} (K - 2\epsilon e^{x_2}) dx_2 \\ &= \frac{K}{\sqrt{2\pi}} \left( \frac{K}{2\epsilon} \right)^{\beta+iu} \frac{1}{(\beta + iu)(\beta + 1 + iu)} \in L^1(\mathbb{R}). \end{aligned}$$

In order to calculate the last term  $P_t^2$  of the put price process  $\pi_t^{\mathbb{Q}}$  we can now use Theorem 3.2 with damping parameter  $\alpha = 0$  (which then allows for heavy tailed loss distributions by Remark 4.4). For this purpose we check that Assumptions (A1)-(A3) hold true. First we consider the dampened function

$$f_1(x_1, x_2) := e^{\beta x_2} \bar{g}(x_1, x_2) = e^{\beta x_2} (K - (|x_1| + \epsilon) e^{x_2})^+ \quad \text{for } \beta > 1.$$

Since  $f_1 \in L^1(\mathbb{R}^2)$ , we have  $(0, -\beta) \in I_1$  for all  $\beta > 1$ . Hence Assumption (A1) is satisfied for  $\beta > 1$  and  $\alpha = 0$ . We assume that  $E^{\mathbb{Q}}[e^{\beta X_{T_2 - T_1}}] < \infty$  for some  $\beta > 1$ . Then by (10), we have  $(0, \beta) \in I_2 \cap I_1$ . Thus (A2) is also satisfied.

**Remark 4.6.** *Note that we can now admit heavy-tailed loss distributions, because we don't need to dampen in  $x_1$  anymore, since  $\alpha = 0$ .*

To prove (A3) we consider the Fourier transform of  $f_1$ :

$$\begin{aligned} \hat{f}_1(u_1, u_2) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(x_1 u_1 + x_2 u_2)} f_1(x_1, x_2) dx_1 dx_2 \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{i(x_1 u_1 + x_2 u_2)} e^{\beta x_2} (K - (|x_1| + \epsilon) e^{x_2}) I_{\{|x_1| \leq K e^{-x_2} - \epsilon, x_2 \leq \ln \frac{K}{\epsilon}\}} dx_1 dx_2 \\ &= \frac{1}{2\pi} \int_{-\infty}^{\ln \frac{K}{\epsilon}} \int_{-K e^{-x_2} + \epsilon}^{K e^{-x_2} - \epsilon} e^{i(u_1 x_1 + u_2 x_2)} e^{\beta x_2} (K - (|x_1| + \epsilon) e^{x_2}) dx_1 dx_2 \\ &= \frac{1}{2\pi} \int_{-\infty}^{\ln \frac{K}{\epsilon}} e^{iu_2 x_2} e^{(\beta+1)x_2} \frac{1 - \cos u_1 (K e^{-x_2} - \epsilon)}{u_1^2} dx_2. \end{aligned}$$

**Lemma 4.7.** *There exists  $C > 0$  such that*

$$|\hat{f}_1(u_1, u_2)|(1 + u_2^2|u_1|^{\beta-1} + u_1^2 + u_2^2) \leq C \quad \text{for all } u_1, u_2 \in \mathbb{R}. \quad (32)$$

**Proof.** See Appendix.

**Corollary 4.8.** *The Fourier transform  $\hat{f}_1$  belongs to  $L^1(\mathbb{R}^2)$ , i.e. (A3) is satisfied.*

**Proof.** By Lemma 4.7 we have

$$\begin{aligned} \int_{\mathbb{R}^2} |\hat{f}_1(u_1, u_2)| du_1 du_2 &\leq C \int_{\mathbb{R}^2} \frac{1}{1 + u_2^2(1 + |u_1|^{\beta-1}) + u_1^2} du_2 du_1 \\ &= 2\pi C \int_0^\infty \frac{1}{\sqrt{(1 + u_1^2)(1 + u_1^{\beta-1})}} du_1 < \infty, \end{aligned}$$

since  $\beta > 1$ . □

Hence all assumptions necessary to apply Theorem 3.2 to calculate  $P_t^2$  with a damping parameter  $\alpha = 0$  are satisfied, and we can compute prices of put options including heavy tailed distributed catastrophe losses. Pricing of catastrophe call and spread options can be obtained by using call-put parity arguments as in Examples 4.2–4.3.

## Appendix

**Proof of Lemma 4.7.** We prove Lemma 4.7 in four steps:

1. Since  $f_1 \in L^1(\mathbb{R}^2)$ ,  $\hat{f}_1$  is bounded, i.e. there exists  $0 < C_1 < \infty$  such that

$$|\hat{f}_1(u_1, u_2)| \leq C_1 \quad \text{for all } u_1, u_2 \in \mathbb{R}.$$

2. Then we have

$$|\hat{f}_1(u_1, u_2)|u_1^2 \leq \frac{1}{2\pi} \int_{-\infty}^{\ln \frac{K}{\epsilon}} 2e^{(\beta+1)x_2} dx_2 = \frac{1}{\pi} \frac{1}{\beta+1} \left(\frac{K}{\epsilon}\right)^{\beta+1} =: C_2 < \infty.$$

3. By integration by parts we obtain

$$\begin{aligned}
|\hat{f}_1(u_1, u_2)|u_2^2 &= \frac{1}{2\pi u_1^2} \left| \int_{-\infty}^{\ln \frac{K}{\epsilon}} \frac{\partial^2}{\partial x_2^2} (e^{iu_2 x_2}) \cdot e^{(\beta+1)x_2} (1 - \cos u_1 (K e^{-x_2} - \epsilon)) dx_2 \right| \\
&= \frac{1}{2\pi u_1^2} \left| \int_{-\infty}^{\ln \frac{K}{\epsilon}} \frac{\partial}{\partial x_2} (e^{iu_2 x_2}) \cdot e^{(\beta+1)x_2} ((\beta+1)(1 - \cos u_1 (K e^{-x_2} - \epsilon)) \right. \\
&\quad \left. - \sin u_1 (K e^{-x_2} - \epsilon) u_1 K e^{-x_2}) dx_2 \right| \\
&= \frac{1}{2\pi u_1^2} \left| \int_{-\infty}^{\ln \frac{K}{\epsilon}} e^{iu_2 x_2} \left\{ (\beta+1)^2 e^{(\beta+1)x_2} (1 - \cos u_1 (K e^{-x_2} - \epsilon)) \right. \right. \\
&\quad \left. \left. - 2(\beta+1) e^{\beta x_2} u_1 K \sin u_1 (K e^{-x_2} - \epsilon) + e^{\beta x_2} u_1 K \left( \sin u_1 (K e^{-x_2} - \epsilon) \right. \right. \right. \\
&\quad \left. \left. \left. + u_1 K e^{-x_2} \cos u_1 (K e^{-x_2} - \epsilon) \right) \right\} dx_2 \right| \\
&\leq \frac{1}{2\pi u_1^2} \int_{-\infty}^{\ln \frac{K}{\epsilon}} \left| (\beta+1)^2 e^{(\beta+1)x_2} (1 - \cos u_1 (K e^{-x_2} - \epsilon)) \right. \\
&\quad \left. - (2\beta+1) e^{\beta x_2} u_1 K \sin u_1 (K e^{-x_2} - \epsilon) + u_1^2 K^2 e^{(\beta-1)x_2} \cos u_1 (K e^{-x_2} - \epsilon) \right| dx_2 \\
&\leq \frac{1}{2\pi} \int_{-\infty}^{\ln \frac{K}{\epsilon}} \left( (\beta+1)^2 e^{(\beta+1)x_2} \frac{u_1^2 (K e^{-x_2} - \epsilon)^2}{2u_1^2} + (2\beta+1) e^{\beta x_2} \left| \frac{\sin u_1 (K e^{-x_2} - \epsilon)}{u_1} \right| \right. \\
&\quad \left. + K^2 e^{(\beta-1)x_2} |\cos u_1 (K e^{-x_2} - \epsilon)| \right) dx_2 \\
&\leq \frac{1}{2\pi} \int_{-\infty}^{\ln \frac{K}{\epsilon}} \left( (\beta+1)^2 e^{(\beta+1)x_2} \frac{K^2 e^{-2x_2} + \epsilon^2}{2} + (2\beta+1) e^{\beta x_2} |K e^{-x_2} - \epsilon| \right. \\
&\quad \left. + K^2 e^{(\beta-1)x_2} \right) dx_2 =: C_3 < \infty.
\end{aligned}$$

4. Further we consider  $|\hat{f}_1(u_1, u_2)|u_2^2|u_1|^{1-\beta}$ . Since for  $0 < |u_1| < 1$  we have

$$|\hat{f}_1(u_1, u_2)|u_2^2|u_1|^{1-\beta} \leq |\hat{f}_1(u_1, u_2)|u_2^2 \leq C_3,$$

we can assume that  $|u_1| > 1$ . As above we get

$$\begin{aligned}
|\hat{f}_1(u_1, u_2)|u_2^2|u_1|^{1-\beta} &\leq \frac{|u_1|^{1-\beta}}{2\pi u_1^2} \int_{-\infty}^{\ln \frac{K}{\epsilon}} \left| (\beta+1)^2 e^{(\beta+1)x_2} \cdot (1 - \cos u_1 (K e^{-x_2} - \epsilon)) \right. \\
&\quad \left. - (2\beta+1) e^{\beta x_2} u_1 K \sin u_1 (K e^{-x_2} - \epsilon) \right. \\
&\quad \left. + u_1^2 K^2 e^{(\beta-1)x_2} \cos u_1 (K e^{-x_2} - \epsilon) \right| dx_2 =: G(u_1).
\end{aligned}$$

Note that  $G(-u_1) = G(u_1)$  and hence it is enough to show that  $G(u_1)$  is bounded for  $u_1 > 0$ . Substituting  $s := u_1 (K e^{-x_2} - \epsilon)$  we rewrite  $G(u_1)$  for

$u_1 > 0$  as

$$\begin{aligned}
G(u_1) &= \frac{u_1^{-1-\beta}}{2\pi} \int_0^\infty \left| (\beta+1)^2 \left( \frac{Ku_1}{u_1\epsilon+s} \right)^{\beta+1} (1-\cos s) \right. \\
&\quad \left. - (2\beta+1)u_1K \left( \frac{Ku_1}{u_1\epsilon+s} \right)^\beta \sin s \right. \\
&\quad \left. + u_1^2K^2 \left( \frac{Ku_1}{u_1\epsilon+s} \right)^{\beta-1} \cos s \right| \frac{ds}{u_1\epsilon+s} \\
&= \frac{1}{2\pi} \int_0^\infty \left| (\beta+1)^2 \left( \frac{K}{u_1\epsilon+s} \right)^{\beta+1} (1-\cos s) \right. \\
&\quad \left. - (2\beta+1)K \left( \frac{K}{u_1\epsilon+s} \right)^\beta \sin s + K^2 \left( \frac{K}{u_1\epsilon+s} \right)^{\beta-1} \cos s \right| \frac{ds}{u_1\epsilon+s} \\
&< \frac{1}{2\pi} \int_0^\infty \left( 2(\beta+1)^2 \left( \frac{K}{\epsilon+s} \right)^{\beta+1} + (2\beta+1)K \left( \frac{K}{\epsilon+s} \right)^\beta \right. \\
&\quad \left. + K^2 \left( \frac{K}{\epsilon+s} \right)^{\beta-1} \right) \frac{ds}{\epsilon+s} \\
&= \frac{K^{\beta+1}}{2\pi} \left( \frac{2(\beta+1)}{\epsilon^{\beta+1}} + \frac{2\beta+1}{\beta\epsilon^\beta} + \frac{1}{(\beta-1)\epsilon^{\beta-1}} \right) =: C_4 < \infty.
\end{aligned}$$

Now (32) holds with  $C := \sum_{i=1}^4 C_i$ . □

## References

- [1] Aase K. (2001) *A Markov model for the pricing of catastrophe insurance futures and spreads*. J. of Risk and Insurance, 68(1), 25-50.
- [2] Biagini F., Bregman Y. and Meyer-Brandis T. (2008) *Pricing of catastrophe insurance options under immediate loss reestimation*. Preprint University of Munich and CMA, University of Oslo.
- [3] Boyarchenko S. and Levendorskii S. (2002) *Perpetual American options under Lévy processes*, SIAM J. Control. Optim., 40, pp. 1663-1696.
- [4] Carr, M., Madan, D.B. (1999). *Option valuation using the fast Fourier transform.*, J. Comp. Finance, 2(4), 61-73.
- [5] Christensen C.V. (1999) *A new model for pricing catastrophe insurance derivatives*. CAF Working Paper Series No. 28.

- [6] Christensen C.V., Schmidli H. (2000) *Pricing catastrophe insurance products based on actually reported claims*. Insurance Math. Econom. 27, 189-200.
- [7] Cont R., Tankov P. (2004) *Financial Modelling With Jump Processes*. Chapman & Hall.
- [8] Eberlein E., Papantoleon A. and Shiryaev A.N. (2006) *On the duality principle in option pricing: Semimartingale setting*. FDM Preprint Nr. 92, University of Freiburg.
- [9] Gradshteyn, I. S. and Ryzhik, I. M. (2000) *Table of integrals, series, and products*. Academic Press Inc.
- [10] Jacod J. and Shiryaev A.N. (2003) *Limit theorems for stochastic processes*. Springer, Berlin.
- [11] Kallsen J. (2000) *Optimal portfolios for exponential Lévy processes*. Math. Meth. Oper. Res. 51, 357-374.
- [12] Kluge W. (2005) *Time-inhomogeneous Lévy processes in interest rate and credit risk models*. Dissertation zur Erlangung des Doktorgrades, Albert-Ludwigs-Universität Freiburg.
- [13] Klüppelberg C. (1989) *Subexponential distributions and characterizations of related classes*. Probab. Theory Related Fields, 82, 259-269.
- [14] Klüppelberg C., Kyprianou A. and Maller R. (2004) *Ruin probabilities and overshoots for general Lévy insurance risk processes*. Ann. of Appl. Probab., 14(4), 1766-1801.
- [15] Königsberger K. (1993) *Analysis 2*. Springer-Lehrbuch, Berlin.
- [16] Möller M. (1996) *Pricing PCS-Options with the Use of Esscher-Transforms*. Afir Colloquium Nürnberg, Oct.1-3, 1996. <http://www.actuaries.org/AFIR/colloquia/Nuernberg/papers.cfm>.
- [17] Mürmann A. (2001) *Pricing Catastrophe Insurance Derivatives*. Financial Markets Group Discussion Paper No. 400. Available at SSRN: <http://ssrn.com/abstract=310521> or DOI: 10.2139/ssrn.310521.
- [18] Mürmann A. (2003) *Actuarially Consistent Valuation of Catastrophe Derivatives*. Working paper. Published in Knowledge@Wharton, <http://knowledge.wharton.upenn.edu/muermann/>.



- [19] Mürmann A. (2006) *Market Price of Insurance Risk Implied by Catastrophe Derivatives*. <http://irm.wharton.upenn.edu/muermann/>.
- [20] Papapantoleon A. (2006) *Applications of semimartingales and Lévy processes in finance: duality and valuation*. Dissertation zur Erlangung des Doktorgrades, Albert-Ludwigs-Universität Freiburg.
- [21] Schmidli H. (2001) *Modelling PCS options via individual indices*. Working paper 187, Laboratory of Actuarial Mathematics, University of Copenhagen.
- [22] Schradin H.R. (1996) *PCS Catastrophe Insurance Options - A new Instrument for Managing Catastrophe Risk*. Contribution to the 6th AFIR International Colloquium.
- [23] Sigma Nr. 3 (2001) *Capital market innovation in the insurance industry*. Swiss Re, Zürich.
- [24] Sigma Nr. 7 (2006) *Securitization - new opportunities for insurers and investors*. Swiss Re, Zürich.