

Pricing of catastrophe insurance options under immediate loss reestimation

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Abstract

We specify a model for a catastrophe loss index, where the initial estimate of each catastrophe loss is reestimated immediately by a positive martingale starting from the random time of loss occurrence. We consider the pricing of catastrophe insurance options written on the loss index and obtain option pricing formulas by applying Fourier transform techniques. An important advantage is that our methodology works for loss distributions with heavy tails, which is the appropriate tail behavior for catastrophe modeling. We also discuss the case when the reestimation factors are given by positive affine martingales and provide a characterization of positive affine local martingales.

Key words: Catastrophe insurance options, loss index, Fourier transform, option pricing formulas, heavy tails, positive affine martingales.

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1 Introduction

Over the past decades the rise in insured losses has exploded from USD 2.5 billions per year to an average value of the aggregated insurance losses

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of USD 30.4 billions per year (in prices of 2006), see [20] for more details. Table 1 gives a summary of the ten most expensive natural catastrophes for the last 30 years.

In order to securitize the catastrophe risk, insurance companies have tried to take advantage of the vast potential of capital markets by introducing exchange-traded catastrophe insurance options. Exchange-traded insurance instruments present several advantages with respect to reinsurance, for example they offer low transaction costs, because they are standardized, and include minimal credit risk, because the obligations are guaranteed by the exchange. See [18] and [19] for the comparison of insurance securities. In particular, catastrophe options are standardized contracts based on an index of catastrophe losses, for example compiled by Property Claim Service (PCS), an internationally recognized market authority on property losses from catastrophes in the US.

The first index based catastrophe derivatives were introduced at the Chicago Board of Trade (CBOT) in 1992, but there was only little activity in the market. A second version of the index, compiled by PCS, was introduced in 1995. At the peak the total capacity created by this version of PCS options amounted to 89 millions. Trading in these options has slowed in 1999. In a separate initiative, the Bermuda Commodities Exchange (BCE) was launched in 1997 to trade property catastrophe-linked option contracts. The BCE suspended its operations in 1997. Trading in PCS options slowed down in 1999, because of the lack of market liquidity and of qualified personal (see e.g. [18]).

However, the record losses caused by the hurricanes Katrina, Rita and Wilma in 2005 have been a catalyst in creating new derivative instruments to trade the catastrophe risks in the capital markets. On March 2007, the New York Mercantile Exchange (NYMEX) has begun the trading of catastrophe futures and options again. These new contracts have been designed to bring the transparency and liquidity of the capital markets to the insurance sector, providing effective ways of protecting against property catastrophe risk and providing the investors with the opportunity to trade a new asset class which has little or no correlation to other exchange traded asset classes. The NYMEX catastrophe options are settled against the Re-Ex loss index, which is created from the data supplied by PCS.

Following the description in [12], [17] and [18], the structure of catastrophe insurance options is described as follows. The options are written on a loss index that evolves over two periods, the *loss period* $[0, T_1]$ and the consecutive *development period* $[T_1, T_2]$. During the contract specific loss period the index measures catastrophic events that may occur. However, at

Insured Loss (USD Billions)	Year	Event	Country
66.3	2005	Hurricane Katrina; floods, dams burst, damage to oil rigs	U.S., Gulf of Mexico, Bahamas, North Atlantic
23.0	1992	Hurricane Andrew; flooding	U.S., Bahamas
21.4	2001	Terrorist attack on World Trade Center, Pentagon and other buildings	U.S.
19.0	1994	Northridge earthquake	U.S.
13.7	2004	Hurricane Ivan; damage to oil rigs	U.S., Caribbean
13.0	2005	Hurricane Wilma; torrential rain, floods	U.S., Mexico, Jamaica, Haiti
10.4	2005	Hurricane Rita; floods, damage to oil rigs	U.S., Gulf of Mexico, Cuba
8.6	2004	Hurricane Charley	U.S., Caribbean
8.4	1991	Typhoon Mireille	Japan
7.4	1989	Hurricane Hugo	U.S., Puerto Rico
7.2	1990	Winter storm Daria	France, U.K.

Table 1: Top 10 Insured Catastrophe Losses (Source: Swiss Re, Sigma Nr. 2/2007).

time of catastrophe occurrence the reported losses are only estimates of the true losses, and these estimates are consecutively reestimated until the end T_2 of the development period. The loss index provides thus at any $t \in [0, T_2]$ an *estimation* of the accumulation of the final time- T_2 -amounts of catastrophe losses that have occurred during the loss period. Let $N_t, t \in [0, T_1]$ denote the number of catastrophes up to time t , and $U_i, i = 1, \dots, N_t$ the corresponding final amounts of the losses at time T_2 (which are unknown at time $0 \leq t < T_2$). Then the value L_t of the loss index can be expressed as

$$L_t = \sum_{i=1}^{N_t \wedge T_1} E[U_i | \mathcal{F}_t], \quad t \in [0, T_2], \quad (1)$$

where the filtration $\{\mathcal{F}_t, t \in [0, T_2]\}$ represents the information available. If the number N_t of catastrophes is assumed to follow a Poisson process, the structure of the index is thus a compound Poisson sum with martingales as summands.

In the literature, a few models have been proposed in order to model the catastrophe loss index and to price catastrophe options written on this index. In [12], [13] and [14], the underlying catastrophe index has been represented as a compound Poisson process with nonnegative jumps. In this model, reestimation is not taken into account at all. In [2] and [11], the authors distinguish between a loss and a reestimation period and model the index as an exponential Lévy process over each period. However, reestimation is assumed to start exactly in T_1 by a common reestimation factor. This assumption is not realistic because loss reestimation happens individually for each catastrophe and begins almost immediately after the catastrophic event. In addition, the assumption of an exponential model for accumulated losses during the loss period is rather unrealistic. For example, this implies that later catastrophes are more severe than earlier ones, and that the index starts in a positive value (instead of starting in 0). In [1] a more realistic model for the loss index is proposed and analytical catastrophe option pricing formulas are developed, but reestimation is also done by a common factor over the development period only. In [16], a model including immediate reestimation is assumed, where reestimation is modeled through individual reestimation factors given by geometric Brownian motion. However, no efficient pricing methods are obtained.

In this paper we specify a realistic model for the loss index that is consistent with (1). As a particular case it comprises the model proposed in [16]. We assume catastrophe occurrence is modeled by a Poisson process, and consider individual reestimation for each catastrophe where the initial esti-

mate of the i -th catastrophe loss is reestimated immediately by a positive martingale starting from the random time of loss occurrence. We then consider the pricing of catastrophe options written on the index. The main contribution of this paper is to employ Fourier transform techniques in order to obtain option pricing formulas. To this end we manage to reduce the calculation of the characteristic function of the index to the computation of an expectation of the characteristic function of the reestimation factor evaluated in two independent random variables. We mention in particular, that our methodology also works for loss distributions with heavy tails, which is the appropriate tail behavior for catastrophe modeling. We then proceed to discuss the case when the reestimation factors are given by positive affine martingales. In this situation, we provide a characterization of positive affine (local) martingales.

We believe that the use of exchange traded insurance derivatives will play a crucial role in the securitization of the increasing catastrophe risk in the future. For this purpose, one essential task is to develop quantitative tools that help to establish liquid trading of these instruments. We hope that this paper contributes to this aim in that it sets new insights in the pricing of catastrophe options.

The remaining parts of the paper are organized as follows. In Section 2 we present the model for the loss index. In Section 3 we consider the pricing of European options in the general model by using Fourier transform techniques, before we discuss the special case of positive affine martingales as reestimation factors in Section 4. We conclude with Section 5 where we explicitly compute prices for spread options, which are the typical instruments in the market.

2 Modeling the loss index

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We consider a financial market endowed with a *risk-free asset* with *deterministic* interest rate r_t , and the possibility of trading *catastrophe insurance options*, written on a loss index. Following the description in the introduction, we distinguish two time periods:

- a *loss period* $[0, T_1]$;
- a *development period* $[T_1, T_2]$, $T_1 < T_2 < \infty$.

During the contract specific *loss period* $[0, T_1]$, the catastrophic events occur. After the loss period, option users can choose either a six-month or a twelve-

month *development period* $[T_1, T_2]$, where the reestimates of catastrophe losses that occurred during the loss period continue to affect the index. The option contract matures at the end of the chosen development period T_2 .

Motivated by the index structure (1) elaborated in the introduction, we model the stochastic process $L = (L_t)_{0 \leq t \leq T_2}$ representing the loss index as follows:

$$L_t = \sum_{j=1}^{N_t \wedge T_1} Y_j A_{t-\tau_j}^j, \quad t \in [0, T_2], \quad (2)$$

where

(H1) $N_s, s \in [0, T_2]$, is a Poisson process with intensity $\lambda > 0$ and jump times τ_j , that models the number of catastrophes occurring during the loss period.

(H2) $Y_j, j = 1, 2, \dots$, are positive i.i.d. random variables with distribution function F^Y , that represent the first loss estimation at the time the j -th catastrophe occurs.

(H3) $A_s^j, s \in [0, T_2], j = 1, 2, \dots$, are positive i.i.d. martingales such that

$$A_0^j = 1, \quad \forall j = 1, 2, \dots$$

(H4) $A^j, Y_j, j = 1, 2, \dots$, and N are independent.

In the sequel we will drop the index j and simply write Y and A in some formulas, when the only matter is the probability distribution of the objects.

The martingales A_t^j represent the *unbiased reestimation factors*. Here we assume that reestimation begins *immediately* after the occurrence of the j -th catastrophe with initial loss estimate Y_j , individually for each catastrophe.

We here suppose that market participants observe the evolution of the individual catastrophe losses. Note that observing the market quotes of the catastrophe index L alone is in general not sufficient for the knowledge of the single reestimation factors A . However, it might not be unrealistic to assume that market participants are able to obtain additional information about the evolution of individual catastrophes. Therefore, we assume the market information filtration $(\mathcal{F}_t)_{0 \leq t \leq T_2}$ to be the right continuous completion of the filtration generated by the catastrophe occurrences N , the corresponding initial loss estimates Y_j , and the corresponding reestimation factors A^j .

3 Pricing of insurance derivatives

We consider now the problem of pricing a European option with payoff depending on the value L_{T_2} of the loss index at maturity T_2 . In the catastrophe insurance market the underlying index L is not traded. Hence the market is highly incomplete and the choice of the pricing measure is not clear.

We here suppose the common approach that the risk neutral pricing measure is structure preserving for the model, except for the fact that the pricing measure might introduce a drift into the reestimation martingales A^j , $j = 1, 2, \dots$. Here we don't discuss further the choice of the pricing measure, and without loss of generality, perform pricing with the model specification given under \mathbb{P} , where we substitute the hypothesis (H3) with

(H3') $A_s^j, s \in [0, T_2], j = 1, 2, \dots$, are positive i.i.d. semimartingales such that

$$A_0^j = 1, \quad \forall j = 1, 2, \dots$$

Consider a European derivative written on the loss index with maturity T_2 and payoff

$$h(L_{T_2}) > 0$$

for a payoff function $h : \mathbb{R} \mapsto \mathbb{R}_+$. Since we have assumed that the interest rate r is deterministic, without loss of generality, we can express the price process of the insurance derivative in discounted terms, i.e. we can set $r \equiv 0$. Then the price process of the option is given by

$$\pi_t = E[h(L_{T_2}) | \mathcal{F}_t], \quad t \in [0, T_2]. \quad (3)$$

In the following we will calculate the expected payoff in (3) by using Fourier transform techniques. To this end, we impose the following conditions:

(C1) The payoff function $h(\cdot)$ is continuous.

(C2) $h(\cdot) - k \in L^2(\mathbb{R}) = \left\{ g : \mathbb{R} \rightarrow \mathbb{C} \text{ measurable} \mid \int_{-\infty}^{\infty} |g(x)|^2 dx < \infty \right\}$
for some $k \in \mathbb{R}$.

Remark 3.1. In [1] we could consider more general options that did not necessarily fulfill (C2) by considering dampened payoffs. However, the cost of this approach is that treating heavy tailed distributions of claim sizes Y becomes more complicated. The approach in this paper allows for general claim size modeling, including distributions with heavy tails. Further, as we

will see in Section 5, assumption (C2) is for example satisfied by call and put spread catastrophe insurance options, the typical options traded in the market.

Let now

$$\hat{h}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuz} (h(z) - k) dz, \quad \forall u \in \mathbb{R},$$

be the Fourier transform of $h(\cdot) - k$ and assume that

$$(C3) \quad \hat{h}(\cdot) \in L^1(\mathbb{R}).$$

Note that condition (C2) does not necessarily imply (C3). Since (C2) and (C3) are in force, the following inversion formula holds (see cf. [9], Section 8.2)

$$h(x) - k = \int_{-\infty}^{\infty} e^{iux} \hat{h}(u) du. \quad (4)$$

Remark 3.2. Note that the equality in (4) is everywhere and not only almost everywhere because of (C1). If the probability distribution of L_{T_2} had a Lebesgue density, an almost everywhere equality in (4) would have been sufficient for the following computations. However, since the loss index is driven by a compound Poisson process, the distribution of L_{T_2} has atoms and we need an everywhere equality to guarantee (5) below.

By (4) and (C3) we obtain

$$\begin{aligned} \pi_t &= E[h(L_{T_2})|\mathcal{F}_t] = E[h(L_{T_2}) - k|\mathcal{F}_t] + k \\ &= E \left[\int_{-\infty}^{\infty} e^{iuL_{T_2}} \hat{h}(u) du | \mathcal{F}_t \right] + k \end{aligned} \quad (5)$$

$$= \int_{-\infty}^{\infty} \underbrace{E[e^{iuL_{T_2}}|\mathcal{F}_t]}_{:=c_t(u)} \hat{h}(u) du + k, \quad (6)$$

where in the last equation we could apply Fubini's theorem, because (C3) holds.

Hence, in order to calculate the price process $(\pi_t)_{t \in [0, T_2]}$ in (6), the essential task is to compute the conditional characteristic function of L_{T_2}

$$c_t(u) = E[e^{iuL_{T_2}}|\mathcal{F}_t] = E \left[\exp \left\{ iu \sum_{j=1}^{N_{T_1}} Y_j A_{T_2 - \tau_j}^j \right\} \middle| \mathcal{F}_t \right], \quad u \in \mathbb{R}, \quad (7)$$

for $t \in [0, T_2]$. We define the conditional characteristic function of the reestimation martingale A^j as

$$\psi_t^{A^j}(s, u) := E \left[e^{iuA_s^j} \middle| \mathcal{F}_t^{A^j} \right], \quad 0 \leq t \leq s \leq T_2, \quad (8)$$

where $\mathcal{F}_t^{A^j} := \sigma(A_s^j, 0 \leq s \leq t)$ is the filtration generated by the j -th reestimation factor. Then our main result is

Theorem 3.3. *The conditional characteristic function (7) of the loss index L_{T_2} is given*

1. for $t < T_1$ by

$$c_t(u) = \exp \left\{ -\lambda(T_1 - t) \left(1 - E \left[\psi_0^A(T_2 - U, uY) \right] \right) \right\} \\ \cdot \prod_{j=1}^{N_t} \psi_{t-s_j}^{A^j}(T_2 - s_j, uy_j) \Big|_{s_j=\tau_j, y_j=Y_j}, \quad u \in \mathbb{R};$$

2. for $t \in [T_1, T_2]$ by

$$c_t(u) = \prod_{j=1}^{N_{T_1}} \psi_{t-s_j}^{A^j}(T_2 - s_j, uy_j) \Big|_{s_j=\tau_j, y_j=Y_j}, \quad u \in \mathbb{R}.$$

Here U is a uniformly distributed random variable on $[t, T_1]$, and Y is a random variable with distribution function F^Y and independent of U .

Note that in Theorem 3.3, the times of catastrophe occurrence τ_j and the initial loss estimates Y_j up to time t is known data.

Proof. We distinguish the computations over the two periods.

1) For $t \in [0, T_1]$ we get by Assumption (H4) and by the independent increments of N_t that

$$c_t(u) = E \left[\exp \left\{ iu \left(\sum_{j=1}^{N_t} Y_j A_{T_2-\tau_j}^j + \sum_{j=N_t+1}^{N_{T_1}} Y_j A_{T_2-\tau_j}^j \right) \right\} \middle| \mathcal{F}_t \right] \\ = E \left[\underbrace{\exp \left\{ iu \sum_{j=1}^{N_t} Y_j A_{T_2-\tau_j}^j \right\}}_{:=a_t(u)} \middle| \mathcal{F}_t \right] \cdot E \left[\underbrace{\exp \left\{ iu \sum_{j=N_t+1}^{N_{T_1}} Y_j A_{T_2-\tau_j}^j \right\}}_{:=b_t(u)} \middle| \mathcal{F}_t \right]. \quad (9)$$

We compute separately the terms a_t and b_t in (9). By assumption (H4) we have for $a_t(u)$, $u \in \mathbb{R}$,

$$\begin{aligned}
a_t(u) &= E \left[\exp \left\{ iu \sum_{j=1}^{N_t} Y_j A_{T_2 - \tau_j}^j \right\} \middle| \mathcal{F}_t \right] \\
&= E \left[\exp \left\{ iu \sum_{j=1}^n y_j A_{T_2 - s_j}^j \right\} \middle| \mathcal{F}_t \right]_{n=N_t, s_j=\tau_j, y_j=Y_j} \\
&= \prod_{j=1}^{N_t} E \left[\exp \left\{ iu y_j A_{T_2 - s_j}^j \right\} \middle| \mathcal{F}_t \right]_{s_j=\tau_j, y_j=Y_j} \\
&= \prod_{j=1}^{N_t} E \left[\exp \left\{ iu y_j A_{T_2 - s_j}^j \right\} \middle| \mathcal{F}_{t-s_j}^{A_j} \right]_{s_j=\tau_j, y_j=Y_j} \\
&= \prod_{j=1}^{N_t} \psi_{t-s_j}^{A_j}(T_2 - s_j, u y_j) \Big|_{s_j=\tau_j, y_j=Y_j}.
\end{aligned}$$

Note that for this first term the Y_j 's, τ_j 's and N_t is known data, because the corresponding catastrophes have happened before t .

For the second term $b_t(u)$, $u \in \mathbb{R}$, we get again by (H4) and the independent increments of the Poisson process N

$$\begin{aligned}
b_t(u) &= E \left[\exp \left\{ iu \sum_{j=N_t+1}^{N_{T_1}} Y_j A_{T_2 - \tau_j}^j \right\} \middle| \mathcal{F}_t \right] = E \left[\exp \left\{ iu \sum_{j=N_t+1}^{N_{T_1}} Y_j A_{T_2 - \tau_j}^j \right\} \right] \\
&= E \left[E \left[e^{iu \sum_{j=1}^n y_j A_{T_2 - s_j}^j} \middle| N_{T_1} - N_t, Y_1, \tau_1, \dots, \tau_{N_{T_1} - N_t}, \dots, Y_{N_{T_1} - N_t} \right]_{n=N_{T_1} - N_t, y_j=Y_j, s_j=\tau_j} \right] \\
&= E \left[E \left[\prod_{j=1}^n \psi_0^{A_j}(T_2 - s_j, u y_j) \middle| N_{T_1} - N_t, Y_1, \tau_1, \dots, \tau_{N_{T_1} - N_t}, \dots, Y_{N_{T_1} - N_t} \right]_{n=N_{T_1} - N_t, y_j=Y_j, s_j=\tau_j} \right] \\
&= E \left[\prod_{j=N_t+1}^{N_{T_1}} \psi_0^A(T_2 - \tau_j, u Y_j) \right]. \tag{10}
\end{aligned}$$

By Theorem 5.2.1 of [15] we obtain the following result:

Lemma 3.4. *Let N_t be a Poisson process with jump times τ_j , $j = 1, 2, \dots$. Then for all $0 \leq t \leq T$,*

$$(\tau_{N_t+1}, \dots, \tau_{N_T} | N_T - N_t = n)$$

has the same distribution as the order statistics $(U_{(1)}, \dots, U_{(n)})$, where U_j , $j = 1, \dots, n$ are i.i.d. uniformly distributed on the interval $[t, T]$.

Using Lemma 3.4 and again assumption (H4), we can replace the τ_j 's in (10) with the order statistics $U_{(j)}$ of i.i.d. uniformly distributed random variables on the interval $[t, T_1]$ and get

$$b_t(u) = E \left[\prod_{j=N_t+1}^{N_{T_1}} \psi_0^A(T_2 - U_{(j)}, uY_j) \right], \quad u \in \mathbb{R}. \quad (11)$$

Next, we note the following simple help lemma

Lemma 3.5. *Consider the order statistics $U_{(1)}, \dots, U_{(n)}$ of n i.i.d. random variables U_1, \dots, U_n and a bounded measurable function $f(x_1, \dots, x_n)$ symmetric in its arguments. Then*

$$E [f(U_{(1)}, \dots, U_{(n)})] = E [f(U_1, \dots, U_n)].$$

Proof. We denote by Σ_n the set of all permutations of $\{1, \dots, n\}$

$$\begin{aligned} E [f(U_{(1)}, \dots, U_{(n)})] &= E \left[\sum_{\sigma \in \Sigma_n} f(U_{\sigma(1)}, \dots, U_{\sigma(n)}) I_{\{U_{\sigma(1)} < \dots < U_{\sigma(n)}\}} \right] \\ &= E \left[f(U_1, \dots, U_n) \underbrace{\sum_{\sigma \in \Sigma_n} I_{\{U_{\sigma(1)} < \dots < U_{\sigma(n)}\}}}_1 \right] \\ &= E [f(U_1, \dots, U_n)]. \end{aligned}$$

□

By the i.i.d. assumption of the Y_j 's and A^j 's, we see that the function

$$f_u^n(s_1, \dots, s_n) := E \left[\prod_{j=1}^n \psi_0^A(T_2 - s_j, uY_j) \right], \quad u \in \mathbb{R},$$

is symmetric in s_1, \dots, s_n . It is then not difficult to see, using Lemma 3.5, that

$$\begin{aligned}
b_t(u) &= E \left[E \left[\prod_{j=1}^n \psi_0^A(T_2 - s_j, uY_j) \middle| N_{T_1} - N_t, U_{(1)}, \dots, U_{(N_{T_1} - N_t)} \right]_{n=N_{T_1} - N_t, s_j=U_{(j)}} \right] \\
&= E \left[f_u^n(s_1, \dots, s_n) \middle|_{n=N_{T_1} - N_t, s_j=U_{(j)}} \right] \\
&= E \left[f_u^n(U_{(1)}, \dots, U_{(n)}) \middle|_{n=N_{T_1} - N_t} \right] \\
&= E \left[f_u^n(U_1, \dots, U_n) \middle|_{n=N_{T_1} - N_t} \right] \\
&= E \left[\prod_{j=N_t+1}^{N_{T_1}} \psi_0^A(T_2 - U_j, uY_j) \right] \\
&= E \left[\exp \left\{ iu \sum_{j=N_t+1}^{N_{T_1}} Y_j A_{T_2 - U_j}^j \right\} \right], \tag{12}
\end{aligned}$$

where we have now substituted the order statistics $U_{(j)}$ with the i.i.d. uniform variables U_j .

Note that (12) coincides with the characteristic function of a compound Poisson process of the form

$$\sum_{j=N_t+1}^{N_{T_1}} Z^j,$$

where $Z^j = Y_j A_{T_2 - U_j}^j$, $j = 1, 2, \dots$, are i.i.d. The form of the characteristic function is in this case well-known. Thus we can rewrite (12) as

$$\begin{aligned}
E \left[\exp \left\{ iu \sum_{j=N_t+1}^{N_{T_1}} Y_j A_{T_2 - U_j}^j \right\} \right] &= \exp \left\{ -\lambda(T_1 - t) \left(1 - E \left[e^{iuZ^1} \right] \right) \right\} \\
&= \exp \left\{ -\lambda(T_1 - t) \left(1 - E \left[\psi_0^A(T_2 - U, uY) \right] \right) \right\}.
\end{aligned}$$

This completes the proof for the case $t \leq T_1$.

2) For the case when $t > T_1$, we get

$$c_t(u) = \prod_{j=1}^{N_{T_1}} \psi_{t-s_j}^{A_j}(T_2 - s_j, uy_j) \Big|_{s_j=\tau_j, y_j=Y_j}; \quad u \in \mathbb{R},$$

as for the term a_t in the case $0 \leq t \leq T_1$.

This completes the proof of Theorem 3.3. \square

Remark 3.6. In [17] a special case of our model is presented where the reestimation factor A is geometric Brownian motion. In this case, the conditional characteristic function of the reestimation factor can be computed by numeric integration by

$$\begin{aligned} \psi_t^A(s, u) &= E \left[e^{iu \exp(B_s - \frac{1}{2}s)} \middle| \mathcal{F}_t \right] = E \left[e^{iu \exp(B_t - \frac{1}{2}t) \exp\{B_s - B_t - \frac{1}{2}(s-t)\}} \middle| \mathcal{F}_t \right] \\ &= E \left[e^{iuw_t \exp(B_{s-t} - \frac{1}{2}(s-t))} \middle| w_t = e^{B_t - \frac{1}{2}t} \right] \\ &= \int e^{iuw_t e^y} e^{-\frac{(y - \frac{1}{2}(s-t))^2}{2(s-t)}} dy \bigg|_{w_t = e^{B_t - \frac{1}{2}t}}. \end{aligned}$$

For further results about the characteristic function of lognormal random variables, we refer also to [10].

In the next section we turn the attention to a class of reestimation processes where the conditional characteristic function is numerically tractable and in some cases analytically obtainable: affine processes. For further information on affine processes and their applications to mathematical finance, we refer to [5], [6] and [7].

4 Reestimation with positive affine processes

We suppose that the reestimation factors are given by positive affine processes. Affine processes constitute a reach class of processes suitable to model a wide range of phenomena. At the same time the advantage is that the conditional characteristic function can be obtained explicitly up to the solution of two Riccati equations.

Definition 4.1. A Markov process $A = (A_t, \mathbb{P}_x)$ on $[0, \infty]$ is called an affine process if there exist \mathbb{C} -valued functions $\phi(t, u)$ and $\psi(t, u)$, defined on $\mathbb{R}_+ \times \mathbb{R}$, such that

$$E \left[e^{iuA_{T_2}} \middle| \mathcal{F}_t \right] = e^{\phi(T_2-t, u) + \psi(T_2-t, u)A_t}. \quad (13)$$

for $t \geq 0$.

We assume that

(A1) A is conservative, i.e. for every $t > 0$ and $x \geq 0$

$$\mathbb{P}_x[A_t < \infty] = 1.$$

(A2) A is stochastically continuous for every \mathbb{P}_x .

By Proposition 1.1 in [8] Assumption (A2) is equivalent to the assumption that $\phi(t, u)$ and $\psi(t, u)$ are continuous in t for each u .

In the framework of our model, the computation of the conditional characteristic function reduces to the computation of ϕ and ψ . In some cases these are explicitly known, otherwise they can be obtained numerically. In the particular case when the reestimation factors remain positive affine martingales under the pricing measure, we are able to prove the following characterization, which provides a useful simplification of the conditional characteristic function.

Theorem 4.2. *Let A be an affine process, satisfying Assumptions (A1) and (A2). A is a positive local martingale if and only if A admits the following semimartingale characteristics (B, C, ν) :*

$$\begin{aligned} B_t &= \beta \int_0^t A_s ds, \\ C_t &= \alpha \int_0^t A_s ds, \quad \text{and} \\ \nu(dt, dy) &= A_t \mu(dy) dt, \end{aligned}$$

where

$$\beta = \mu[1, \infty) - \int_1^\infty y \mu(dy),$$

$\alpha \geq 0$ and μ is a Lévy measure on $(0, \infty)$.

Proof. Since A_t satisfies Assumptions (A1) and (A2), by Theorem 1.1 in [8] and by Theorem 2.12 in [6] A_t is a positive affine semimartingale if and only if A_t admits the following characteristics (B, C, ν) :

$$\begin{aligned} B_t &= \int_0^t (\tilde{b} + \beta A_s) ds, \\ C_t &= \alpha \int_0^t A_s ds, \quad \text{and} \\ \nu(dt, dy) &= (m(dy) + A_t \mu(dy)) dt, \end{aligned}$$

for every \mathbb{P}_x , where

$$\tilde{b} = b + \int_{(0,\infty)} (1 \wedge y)m(dy),$$

$\alpha, b \geq 0, \beta \in \mathbb{R}$, m and μ are Lévy measures on $(0, \infty)$, such that

$$\int_{(0,\infty)} (y \wedge 1)m(dy) < \infty.$$

See also [8]. By (A2) and Theorem 7.16 in [3] the following operator L

$$\begin{aligned} Lf(x) &= \frac{1}{2}\alpha x f''(x) + (b + \beta x)f'(x) + \int_{(0,\infty)} (f(x+y) - f(x))m(dy) \\ &\quad + x \int_{(0,\infty)} (f(x+y) - f(x) - f'(x)(1 \wedge y))\mu(dy) \end{aligned} \quad (14)$$

on $C^2(\mathbb{R}_+)$ is a version of the restriction of the extended infinitesimal generator¹ of A to $C^2(\mathbb{R}_+)$. Then A is a local martingale, if and only if

$$Lf(x) \equiv 0 \quad \text{for } f(x) = x.$$

Substituting $f(x) = x$ in (14), we get

$$\begin{aligned} Lx &= b + \beta x + \int_{(0,\infty)} ym(dy) + x \int_{(0,\infty)} (y - (1 \wedge y))\mu(dy) \\ &= (\beta + \int_1^\infty (y-1)\mu(dy))x + b + \int_{(0,\infty)} ym(dy). \end{aligned}$$

Hence, A is a local martingale if and only if

$$(\beta + \int_1^\infty (y-1)\mu(dy))x + b + \int_{(0,\infty)} ym(dy) = 0. \quad (15)$$

for any $x \in \mathbb{R}_+$. Since $b \geq 0$ and m is a non-negative measure, condition (15) means that

$$b = 0, \quad m \equiv 0, \quad \text{and} \quad \beta = \mu[1, \infty) - \int_1^\infty y\mu(dy). \quad (16)$$

¹An operator L with domain \mathcal{D}_L is said to be an extended generator for A if \mathcal{D}_L consists of those Borel functions f for which there exists a Borel function Lf such that the process

$$L_t^f = f(A_t) - f(A_0) - \int_0^t Lf(X_s)ds$$

is a local martingale.

□

Let A be an affine process, satisfying Assumptions (A1) and (A2). By Theorem 4.3 in [7] the conditional characteristic function of A satisfies (13), where $\phi(t, u)$ and $\psi(t, u)$ solve the equations

$$\phi(t, u) = \int_0^t F(\psi(s, u)) ds, \quad \text{and} \quad (17)$$

$$\partial_t \psi(t, u) = R(\psi(t, u)), \quad \psi(0, u) = iu, \quad (18)$$

where, for $z \in \{\mathbb{C} \mid \Re z \leq 0\}$,

$$R(z) = \frac{1}{2}\alpha z^2 + \beta z + \int_{(0, \infty)} (e^{zy} - 1 - z(y \wedge 1)) \mu(dy), \quad (19)$$

$$F(z) = bz + \int_{(0, \infty)} (e^{zy} - 1) m(dy), \quad (20)$$

and α, β, b, m, μ are parameters of infinitesimal generator (14) of A . If A is a local martingale, then by (16) we can simplify (20) and (19) as follows:

$$R(z) = \frac{1}{2}\alpha z^2 + \int_{(0, \infty)} (e^{zy} - zy - 1) \mu(dy), \quad \text{and} \quad (21)$$

$$F(z) \equiv 0. \quad (22)$$

From (22) and (17) we immediately obtain for positive affine local martingales that

$$\phi(t, u) \equiv 0.$$

In order to determine ψ , one has in general to solve (17) numerically. For some special cases, however, one can compute ψ analytically. We give two examples.

Example 4.3. If A has no jump part then A is called *Feller diffusion* (see e.g. [6]). In that case the positive affine martingale dynamics is given by

$$dA_t = \sqrt{\alpha A_t} dW_t,$$

where W_t is a standard Brownian motion. Consequently, we have $\mu = 0$ in (21) and we can rewrite (18) as

$$\psi'_t = \frac{1}{2}\alpha \psi_t^2. \quad (23)$$

Solving the differential equation (23) we get

$$\psi(t, u) \equiv 0 \quad \text{or} \quad \psi(t, u) = -\frac{1}{\frac{1}{2}\alpha t + C(u)}, \quad u \in \mathbb{R},$$

where $C(u)$ can be found from the boundary condition $\psi(0, u) = iu$. Substituting $C(u)$ in ψ , we get

$$\psi(t, u) \equiv 0 \quad \text{or} \quad \psi(t, u) = -\frac{1}{\frac{1}{2}\alpha t + \frac{i}{u}}, \quad u \in \mathbb{R}.$$

Note that if we have no jump part, then A has positive probability to be absorbed in 0. However, it may still be of interest to consider also the case of positive probability of absorption at zero, if we wish to include the possibility of fraud or falsified reporting of claims into the model. In this case, reestimation might discover the fraud and the previous fake evaluation will be set to zero.

Example 4.4. In order to give an example of a positive affine martingale including jumps where we can solve for ψ explicitly, we specify the jump density $\mu(dy)$ in the semimartingale characteristics in Theorem 4.2 as

$$\mu(dy) = \frac{3}{4\sqrt{\pi}} \frac{dy}{y^{5/2}}.$$

Then some calculations give $R(z)$ in (19)

$$R(z) = \frac{1}{2}\alpha z^2 + \frac{3}{4\sqrt{\pi}} \int_{(0, \infty)} (e^{zy} - zy - 1) \frac{dy}{y^{5/2}} \quad (24)$$

$$= \frac{1}{2}\alpha z^2 + (-z)^{3/2} \quad (25)$$

for $z \in \{\mathbb{C} \mid \Re z \leq 0\}$. Consider $\eta(t, u) := -\psi(t, u)$. By (18) we then have

$$-\eta'_t = \frac{1}{2}\alpha\eta^2 + \eta^{3/2}. \quad (26)$$

The solutions to (26) are $\eta(t, u) \equiv 0$ and

$$\eta(t, u) = \frac{4}{\alpha^2} (1 + W(-C(u)e^{-\frac{t}{\alpha}}))^{-2}, \quad (27)$$

where $W(\cdot)$ is the Lambert W function². The boundary condition $\eta(0, u) = -\psi(0, u) = iu$ yields

$$C(u) = - \left(-1 + \frac{2}{\alpha} \sqrt{\frac{i}{u}} \right) \exp \left(-1 + \frac{2}{\alpha} \sqrt{\frac{i}{u}} \right)$$

Substituting $C(u)$ in (27), we get for $\psi(t, u) = -\eta(t, u)$:

$$\psi(t, u) \equiv 0 \quad \text{or} \quad \psi(t, u) = -\frac{4}{\alpha^2} \left(1 + W \left(\left(-1 + \frac{2}{\alpha} \sqrt{\frac{i}{u}} \right) e^{-\frac{t}{\alpha} - 1 + \frac{2}{\alpha} \sqrt{\frac{i}{u}}} \right) \right)^{-2}.$$

5 Pricing of call and put spreads

We conclude this paper by showing that we can apply the developed pricing method to spread options, which are the typical catastrophe options traded in the market. A call spread option with strike prices $0 < K_1 < K_2$ is a European derivative with the payoff function at maturity given by

$$h(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq K_1; \\ x - K_1, & \text{if } K_1 < x \leq K_2; \\ K_2 - K_1, & \text{if } x > K_2. \end{cases}$$

The integrability condition $h(\cdot) - k \in L^2(\mathbb{R}_+)$ is satisfied for $k := K_2 - K_1$. In particular, $h(\cdot) - k \in L^1(\mathbb{R}_+)$.

To satisfy (C1) and (C3) we continuously extend h from \mathbb{R}_+ to \mathbb{R} by

$$\bar{h}(x) := \begin{cases} h(-x), & \text{if } x < 0; \\ h(x), & \text{if } x \geq 0. \end{cases}$$

Note that the price processes of the two corresponding options with payoffs $h(L_{T_2})$ respectively $\bar{h}(L_{T_2})$ remain the same, because $L_{T_2} \geq 0$.

Let

$$\hat{h}(u) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuz} (\bar{h}(z) - k) dz, \quad \forall u \in \mathbb{R},$$

²The *Lambert W function* $W(z)$ is defined to be the function satisfying $W(z)e^{W(z)} = z$, $z \in \mathbb{C}$. See also [4] for more details on the Lambert function.

be the Fourier transform of $\bar{h} - k$. Then

$$\begin{aligned}
\hat{h}(u) &= \frac{1}{2\pi} \left[\int_{-K_2}^{-K_1} e^{-iux} (-x - K_2) dx \right. \\
&\quad \left. + \int_{-K_1}^{K_1} e^{-iux} (K_1 - K_2) dx + \int_{K_1}^{K_2} e^{-iux} (x - K_2) dx \right] \\
&= \frac{1}{2\pi} \frac{1}{u^2} [e^{-iuK_2} + e^{iuK_2} - e^{-iuK_1} - e^{iuK_1}] \\
&= \frac{1}{\pi} \frac{1}{u^2} (\Re e^{iuK_2} - \Re e^{iuK_1}) = \frac{1}{\pi} \frac{1}{u^2} (\cos uK_2 - \cos uK_1) \in L^1(\mathbb{R}),
\end{aligned}$$

and by applying the inversion formula (4) to $\bar{h}(x)$ for $x \geq 0$, we obtain that (4) holds also for h , since $h(x) = \bar{h}(x)$ for $x \geq 0$.

In particular since $L_{T_2} \geq 0$ a.s., for the price of the call spread we can write

$$\begin{aligned}
\pi_t^{CS} &= E[h(L_{T_2}) - k | \mathcal{F}_t] + k = E[\bar{h}(L_{T_2}) - k | \mathcal{F}_t] + k \\
&= E \left[\int_{-\infty}^{\infty} e^{iuL_{T_2}} \hat{h}(u) du | \mathcal{F}_t \right] + k \\
&= \int_{-\infty}^{\infty} E[e^{iuL_{T_2}} | \mathcal{F}_t] \hat{h}(u) du + k \\
&= \int_{-\infty}^{\infty} c_t(u) \hat{h}(u) du + k \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{c_t(u)}{u^2} (\cos uK_2 - \cos uK_1) du + K_2 - K_1, \quad (28)
\end{aligned}$$

where $c_t(u)$ is defined in (6). Note that the integral in (28) is real, since $\Im c_t(-u) = -\Im c_t(u)$ by definition of c_t .

Analogously, for the put spread catastrophe option with payoff at the maturity given by

$$g(x) = \begin{cases} K_2 - K_1, & \text{if } 0 \leq x \leq K_1; \\ K_2 - x, & \text{if } K_1 < x \leq K_2; \\ 0, & \text{if } x > K_2, \end{cases}$$

we obtain

$$\pi_t^{PS} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{c_t(u)}{u^2} (\cos uK_1 - \cos uK_2) du. \quad (29)$$

Note that the call-put parity is satisfied:

$$\pi_t^{PS} = K_2 - K_1 - \pi_t^{CS}.$$

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