The formation of financial bubbles in defaultable markets

Francesca Biagini* Sorin Nedelcu*

29 April 2015

Abstract

In this paper we study the formation of financial bubbles in the valuation of defaultable claims in a reduced form setting. The birth of a bubble is caused by the impact of trading activity of investors, who consider the claim to be a safe investment under some circumstances. We also show how microeconomic interactions may at an aggregate level determine a shift in the martingale measure. In this way we establish a connection between our approach and the martingale theory of bubbles, see [2] and [27]. This is illustrated by a characterization of the space of equivalent local martingale measures by measure pasting. Furthermore our model is consistent with the no-arbitrage framework, as we show in a concrete example.

1 Introduction

The aim of this paper is to propose a mathematical model for bubble formation in the valuation of defaultable claims in a reduced form setting. In the economic literature, microeconomic theories of bubble formation refer to investor heterogeneity and limits to arbitrage as possible factors determining the formation of asset price bubbles. The latter factor can be the result of short-selling constraints (see e.g. Miller [30]) or shocks to funding liquidity (see e.g. Schleifer and Vishny [35]). Investor heterogeneity can arise when agents in the economy may disagree on the value of future dividends (see e.g.Harrison and Kreps [17]) or may overestimate the importance of certain signals, i.e. exhibit overconfidence -the tendency of exaggerating the precision of their knowledge, see Scheinkman and Xiong [34]). Moreover, as pointed out in Föllmer et al.[15], investors may use different predictors when forecasting the future prices and this creates in certain time periods heterogeneity among their views.

*Department of Mathematics, Ludwig-Maximilians Universität, Theresienstraße 39, 80333 Munich, Germany. Emails: biagini@math.lmu.de, nedelcu@math.lmu.de

From the economic point of view, the main challenge consists in explaining how such bubbles are generated at the microeconomic level by the interaction of market participants; see for instance Harrison and Kreps [17], DeLong, Shleifer, Summers and Waldmann [12], Föllmer, Horst and Kirman [15], Abreu and Brunnenmeyer [1], Scheinkman and Xiong [34], Tirole [37] and the references therein.

From the mathematical point of view asset price bubbles have been mainly studied by using the local martingale framework, see for instance Loewenstein and Willard [29], Cox and Hobson [9], Jarrow [21], Jarrow, Protter et al. [26], [27], [23], [22], [24] and Biagini, Föllmer and Nedelcu [2]. In these papers a bubble appears if for some reason there is a shift in the martingale measure: a non-trivial bubble will be generated if the (discounted) wealth process is no longer a uniformly integrable martingale with respect to the underlying pricing measure. For a comprehensive survey of the recent mathematical literature on financial bubbles, we refer to Protter [32].

A first attempt to explain in a mathematical model how dynamics at the microeconomic level of interacting market participants influence asset price formation is presented in Jarrow, Protter and Roch [25], where bubble generation is determined by the impact of trading volume on asset prices. In [25] the asset's fundamental price process is exogenously given and asset price bubbles are endogenously determined by the impact of liquidity risk and studied through a detailed analysis of the liquidity supply curve. In contrast, the martingale approach in [9] and [27] to modeling price bubbles assumes that the asset's market price process is exogenous and the fundamental price is given by the expected future cash flows computed under a martingale measure.

In this paper we propose a constructive model for bubble formation in defaultable markets and study its relation to the martingale theory of bubbles. In a reduced form setting, see Bielecki and Rutkowski [4], we consider a market model which includes the possibility of investing in defaultable claims, i.e. contingent agreements traded over-the-counter between default-prone parties. For the sake of simplicity, the money market account is supposed to be constantly equal to one.

Initially a given defaultable claim is evaluated by using the underlying pricing measure, as it is usual in the reduced form setting, see Definition 8.1.2 in [4]. After a certain time the claim starts to be considered safe enough, if the conditional probability of having a default in the remaining time interval becomes small enough. The trading activity of the investors determines a deviation from the initially estimated wealth via a factor f, which is a function of time and of the credibility process introduced in Definition 2.2. We define a bubble as the difference between the modified wealth process (called market wealth process) and the risk neutral valuation of the defaultable claim.

We then study the relation between our model and the martingale theory of bubbles of [27] and [2]. In particular, we establish a connection between our approach and the setting of [27] and [2] through the characterization of the set of equivalent martingale measures for the market wealth process W of a defaultable claim via measure pasting in Theorem 4.6. For further details on measure pasting, see Definition 4.2 and Section 6.4 of Föllmer and Schied [16]. If σ_1 denotes the starting moment of the influence of the credibility process on the contract value, in Theorem 4.6 we prove that all equivalent martingale measures for W are given by the pasting in σ_1 of an equivalent martingale measure for the initially estimated wealth up to σ_1 with an equivalent martingale measure for $(W - W_{\sigma_1})1_{\{\cdot \geq \sigma_1\}}$, on $\{0 < \sigma_1 < T\}$. This result describes rigorously that, since the wealth process changes its form at σ_1, the corresponding martingale measure has to readapt. In this way we directly connect a shift in the martingale measure to a change in the dynamics of the market wealth process which is caused by the resulting trading activity due to the influence of many micro-economic interactions.
An outline of the paper is the following. In Section 2 we describe the setting of a reduced-form credit risk model and define the credibility process. In Section 3 we introduce our definition of bubble and compare it with the classical martingale theory of bubbles introduced in [27]. We provide a set of conditions when an increase in the market wealth, due to the investors' trading activity, can lead to an increase in the asset's fundamental value. In Section 4 we provide a characterization of the equivalent martingale measures for W by measure pasting. Section 5 concludes our paper with an example that illustrates the results of the previous sections.

2 The Setting

For a fixed time horizon $T > 0$ we consider a market model defined on the filtered probability space $(\Omega, \mathcal{G}, \mathbb{F}, P)$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfy the usual conditions of right-continuity and completeness. The market model contains a defaultable asset with maturity date T and a money market account constantly equal to 1.
The random time of default is represented by a non-negative $\mathcal{G}$-measurable random variable $\tau : \Omega \to [0, +\infty]$, with $P(\tau = 0) = 0$ and $P(\tau > t) > 0$ for each $t \in [0, T]$. The last condition means that the default may not occur on the interval $[0, T]$. The random time τ is not an $\mathbb{F}$-stopping time. For the default time τ, we introduce the associated default process $H = (H_t)_{t\in[0,T]}$ given by $H_t = 1_{\{\tau \leq t\}}$, $t \in [0, T]$, and denote by $\mathbb{H} = (\mathcal{H}_t)_{0\leq t\leq T}$ the filtration generated by the process H, i.e. $\mathcal{H}_t = \sigma(H_u; u \leq t)$ for any $t \in [0, T]$.
Let $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ be the filtration obtained by progressively enlarging the filtration $\mathbb{F}$ with the random time τ, i.e. $\mathbb{G} = \mathbb{F}\vee\mathbb{H}$. For the sake of simplicity we assume $\mathcal{G}_0 = \mathcal{F}_0 = \mathcal{H}_0 = \{\emptyset, \Omega\}$ and $\mathcal{G} = \mathcal{G}_T = \mathcal{F}_T \vee \mathcal{H}_T$. Note that τ is

a stopping time with respect to the filtration $\mathbb{G}$.
Consider the (Azéma) $\mathbb{F}$-supermartingale $\mathcal{Z} = (\mathcal{Z}_t)_{t\in[0,T]}$ defined by

$$\mathcal{Z}_t = P(\tau > t|\mathcal{F}_t), \quad t \in [0, T],$$

and chosen to be càdlàg. We assume that $\mathcal{Z}_t > 0$ for all $t \in [0, T]$. Then the *hazard process* $\Gamma = (\Gamma_t)_{t\in[0,T]}$ of τ under P given by

$$\Gamma_t = -\ln \mathcal{Z}_t = -\ln P(\tau > t|\mathcal{F}_t)$$

is well defined for every $t \in [0, T]$. We make the following Assumptions that hold for the rest of the paper:

Assumption 2.1. *We consider that:*

i) The immersion property holds under the measure P, i.e. all $(\mathbb{F}, P)$-martingales are also $(\mathbb{G}, P)$-martingales.

ii) The hazard process Γ admits the representation

$$\Gamma_t = \int_0^t \mu_s ds, \quad t \in [0, T],$$

where $\mu = (\mu_t)_{t\in[0,T]}$ is an $\mathbb{F}$-adapted process such that $\int_0^t \mu_s ds < \infty$ a.s. for all $t \in [0, T]$.

If the immersion property holds, it follows from Corollary 3.9 of Coculescu et al.[7] that the Azéma supermartingale $\mathcal{Z}$ is a decreasing process. Hence Γ is increasing, which implies that $(\mu_t)_{t\in[0,T]}$ is a non-negative process. The process μ is called the *stochastic intensity* or *hazard rate* of τ. The existence of the intensity implies that τ is a totally inaccessible $\mathbb{G}$-stopping time. Furthermore, since the Azéma supermartingale $\mathcal{Z}$ is continuous and decreasing, it follows from Corollary 3.4 of Coculescu and Nikeghbali [8] that τ avoids all $\mathbb{F}$-stopping times.
We define the compensated process $\hat{M} = (\hat{M}_t)_{t\in[0,T]}$ by

$$\hat{M}_t := H_t - \int_0^{t\wedge\tau} \mu_s ds = H_t - \int_0^t \hat{\mu}_s ds, \quad t \in [0, T].$$

Notice that for the sake of brevity we put $\hat{\mu}_t := \mu_t 1_{\{\tau\geq t\}}$. It follows from Proposition 5.1.3 of Bielecki and Rutkowski [4] that $\hat{M}$ is a $\mathbb{G}$-martingale.

In this setting a defaultable claim is given by a triplet $H = (X, R, \tau)$, where:

1. the *promised contingent claim* $X \in L^1(\mathcal{F}_T)$[1] represents the non-negative payoff received by the owner of the claim at time T, if there was no default prior to or at time T.

[1]Note that the payoff X must not be bounded above, otherwise no bubbles are possible, see Protter [32]. We refer to Remark 3.3 for further details.

2. the *recovery process* R represents the recovery payoff at time τ of default if default occurs prior to or at the maturity date T, and it is assumed to be a strictly positive, continuous, $\mathbb{F}$-adapted process that satisfies

$$\mathbb{E}_P\Big[\sup_{t\in[0,T]} R_t\Big] < \infty. \tag{2.1}$$

We postulate that the underlying probability measure P is a martingale measure. Note that this assumption implies that there is no free-lunch with vanishing risk, see [10]. By following the reduced-form model approach, see Definition 8.1.2 of Bielecki and Rutkowski [4], the risk-neutral valuation of the defaultable claim $H = (X.R, \tau)$ introduced above is given by

$$W_t^e := \mathbb{E}_P[X1_{\{\tau>T\}} + R_\tau 1_{\{\tau\le T\}}|\mathcal{G}_t], \quad t\in[0,T].$$

If we put

$$\Lambda_t := \mathbb{E}_P[X1_{\{\tau>T\}} + R_\tau 1_{\{\tau\le T\}}|\mathcal{G}_t]1_{\{\tau>t\}}, \quad t\in[0,T]. \tag{2.2}$$

then

$$W_t^e = \Lambda_t + R_\tau 1_{\{\tau\le t\}}, \quad t\in[0,T].$$

In the sequel we model the impact of the trading activity of investors, who consider the defaultable claim as a safe investment if some circumstances are verified, as we explain more in detail below. To this purpose we first introduce the following notion of *credibility process*.

Definition 2.2. *For any $t \le T$ the credibility process $F = (F_t)_{t\in[0,T]}$ is defined as*

$$F_t = P(t < \tau \le T|\mathcal{G}_t),$$

for all $t\in[0,T]$.

Then we can deduce the following property of the credibility process:

Lemma 2.3. *The process $F = (F_t)_{t\in[0,T]}$ is a $(\mathbb{G}, P)$-supermartingale.*

Proof. It is easy to see that F can be written in the form

$$F_t = \mathbb{E}_P[1_{\{\tau\le T\}}|\mathcal{G}_t] - 1_{\{\tau\le t\}}.$$

□

This property is intuitively clear, since the probability that the asset defaults on the remaining time interval declines in expectation as we approach the maturity date T.

3 Bubbles in defaultable claim valuation

Let now

$$f : [0, T] \times (0, 1] \to [1, \infty) \tag{3.1}$$

be a deterministic function in $C^{1,2}([0, T] \times (0, 1])$. Fix $p \in (0, 1)$ with $p < P(0 < \tau \leq T)$. We assume that

i) For all $t \in [0, T]$, $f(t, x) = 1$ for all $x \geq p$ and $f(t, x) > 1$ for $x < p$,

ii) f is strictly decreasing in both arguments for $x < p$.

iii) $lim_{t \to T} f(t, x) = 1$ for all $x \in (0, 1]$.

Definition 3.1. *The* market wealth process $W = (W_t)_{t \in [0,T]}$ *of the defaultable asset is defined as*

$$W_t = f(t, F_t)\Lambda_t + R_\tau 1_{\{\tau \leq t\}}, \tag{3.2}$$

for all $t \in [0, T]$.

In our model we then assume that the initial value estimation Λ is affected via the function f by the fluctuations of the credibility process F and by the length of the remaining time interval $[t, T]$ to maturity. Here the value $p \in (0, 1)$ acts as a threshold in the sense that, if the conditional probability of default F goes below the value p, then the asset is perceived as a safe investment by the traders (i.e. the asset becomes "credible" enough). Furthermore we also take into account the fact that the asset is perceived as safe at an earlier date impacts the price in a more significant way than at a later date, i.e. if $F_{t_1} = F_{t_2}$ for $t_1 < t_2$, then $f(t_1, F_{t_1}) > f(t_2, F_{t_2})$. Note that by Proposition 2.3 the conditional probability of default F decreases in expectation as we approach maturity.
A possible motivation for our model is that the credibility process F is capturing the views of a very big investor who will buy the claim when the credibility process goes below the threshold p. Everybody in the market will then follow the big investor, generating the bubble. Other explanations for this model are of course possible, see for example Brunnermeyer and Oehmke [6], Hugonnier [19], Scheinkman [33].

Remark 3.2. *The impact of the credibility affects the value of the defaultable asset only strictly prior to the default time* τ. *If* τ *occurs before or at* T, *the recovery payment* R_τ *will be paid, as established at the beginning in the contractual agreement underlying the claim. Hence* R *is not influenced by the credibility process. Analogously at* $t = T$ *we have*

$$W_T = X 1_{\{\tau > T\}} + R_\tau 1_{\{\tau \leq T\}} = W^e_T,$$

since the payment at time of maturity and at default is determined by the contractual agreement.

Remark 3.3. *Note that the payoff of the defaultable claim (and the corresponding wealth process associated to the claim) must not be not upper bounded, since the martingale theory of financial bubbles does not allow for bubbles in the price of bounded asset prices. A possible way of avoiding this limitation is by introducing the concept of a* relative asset price bubble*, see Bilina Falafala, Jarrow, and Protter [5].*

In the sequel we denote by σ_1 the starting moment of the influence of F on Λ, i.e.

$$\sigma_1 := \inf\{t \in [0,T]; F_t < p\}. \tag{3.3}$$

Note that $\sigma_1 \leq \tau$, see also Proposition 3.9.

Following the approach of [27] we denote by $\mathcal{M}_{loc}(W)$ the set of probability measures $Q \approx P$ defined on $(\Omega, \mathbb{G})$ under which the market wealth process W is a $(\mathbb{G}, Q)$-local martingale. We have that

$$\mathcal{M}_{loc}(W) = \mathcal{M}_{UI}(W) \cup \mathcal{M}_{NUI}(W),$$

where, in the notation of [27], $\mathcal{M}_{UI}(W)$ denotes the class of measures $Q \approx P$ such that W is a uniformly integrable martingale under Q, and $\mathcal{M}_{NUI}(W)$ represents the class of measures $Q \approx P$ such that W is a non-uniformly integrable martingale. Since we work on a finite time horizon, the two classes correspond to the class of true martingales and the class of strict local martingales, respectively. Typically, the classes $\mathcal{M}_{UI}(W)$ and $\mathcal{M}_{NUI}(W)$ will both be non-empty, see Delbaen and Schachermayer [11] and the examples in Section 4 and 5 of [2].

Remark 3.4. *Note that we have $\mathcal{M}_{loc}(W) \cap \mathcal{M}_{loc}(W^e) = \emptyset$ if $P(0 < \sigma_1 < T) > 0$. Suppose on the contrary that there exists $Q \in \mathcal{M}_{loc}(W) \cap \mathcal{M}_{loc}(W^e)$. There exists $\epsilon > 0$ such that $P(\sigma_1 + \epsilon < T) > 0$ and $F_{\sigma_1+\epsilon} < p$ on $\{\sigma_1 + \epsilon < T\}$. Therefore $f(\sigma_1 + \epsilon, F_{\sigma_1+\epsilon}) > 1$ on $\{\sigma_1 + \epsilon < T\}$, and we obtain the following contradiction*

$$0 < \mathbb{E}_Q[W_{(\sigma_1+\epsilon)\wedge T} - W^e_{(\sigma_1+\epsilon)\wedge T}] \leq \mathbb{E}_Q[W_0 - W^e_0] = 0,$$

since $W^e_0 = W_0$ and Q is equivalent to P.

Definition 3.5. *Let $Q \in \mathcal{M}_{loc}(W)$. The process W^Q defined by*

$$W^Q_t = \mathbb{E}_Q[X 1_{\{\tau > T\}} + R_\tau 1_{\{\tau \leq T\}} | \mathcal{G}_t], \quad t \in [0,T],$$

is called the fundamental wealth process *of the defaultable claim perceived under the measure Q.*

In particular, we have that

$$W_t \geq \mathbb{E}_Q[W_T | \mathcal{G}_t] = \mathbb{E}_Q[\mathbb{E}_Q[W_T | \mathcal{G}_T | \mathcal{G}_t] = \mathbb{E}_Q[W^Q_T | \mathcal{G}_t] = W^Q_t$$

for any $Q \in \mathcal{M}_{loc}(W)$, with strict inequality if W is a strict local martingale under Q.

We now consider an alternative way of defining a bubble for defaultable claims.

Definition 3.6. *For any $t \in [0, T]$, we define the* bubble $\beta^o = (\beta^o_t)_{t\in[0,T]}$ *by*

$$\beta^o_t = W_t - W^e_t = (f(t, F_t) - 1)\Lambda_t 1_{\{\tau>t\}}, \quad t \in [0, T]. \tag{3.4}$$

The bubble represents the difference between the market wealth W and the risk-neutral valuation of the claim W^e, which is generated by the impact of the credibility process.

Remark 3.7. *Note that the model could also be modified by choosing a different function f in order to include the appearance of negative bubbles. It may happen that for some reason a particular asset is seen as extremely dangerous by a consistent number of investors. In this case the asset could experience a decrease in the market value that may not be motivated by the underlying economic and financial conditions.*

3.1 Relation with the martingale theory of bubbles

Let $Q \in \mathcal{M}_{loc}(W)$. We now examine the relation between our Definition 3.6 of bubble and the concept of Q-bubble, as introduced in [26] and [27]. We start by recalling the definition of a Q-bubble as in the approach of [27].

Definition 3.8. *For any $Q \in \mathcal{M}_{loc}(W)$, the non-negative adapted process $\beta^Q = (\beta^Q_t)_{t\in[0,T]}$ defined by*

$$\beta^Q_t = W_t - W^Q_t \geq 0,$$

is called the bubble *perceived under the measure Q or Q-bubble.*

The existence and the size of the Q-bubble β^Q depends on the choice of the martingale measure. If $Q \in \mathcal{M}_{UI}(W)$, then the Q-bubble reduces to the trivial case $\beta^Q = 0$. For $Q \in \mathcal{M}_{NUI}(W)$ the Q-bubble is a non-negative local martingale with $\beta^Q_T = 0$. Furthermore it is also clear that there is no bubble at time T, since at time of maturity the asset X must be delivered according to contractual obligations. Analogously the market wealth process exhibits no bubbles after default, as stated by the following Proposition.

Proposition 3.9. *On the set $\{t \geq \tau\}$ we have*

$$\beta^Q_t = W_t - W^Q_t = 0,$$

for any $Q \in \mathcal{M}_{loc}(W)$.

Proof. Let $Q \in \mathcal{M}_{loc}(W)$. Then

$$\begin{aligned} W_t^Q 1_{\{\tau\le t\}} &= \mathbb{E}_Q[X 1_{\{\tau>T\}} + R_\tau 1_{\{\tau\le T\}}|\mathcal{G}_t]1_{\{\tau\le t\}} \\ &= \mathbb{E}_Q[X 1_{\{\tau>T\}}|\mathcal{G}_t]1_{\{\tau\le t\}} + \mathbb{E}_Q[R_\tau 1_{\{\tau\le T\}}|\mathcal{G}_t]1_{\{\tau\le t\}} \\ &= \mathbb{E}_Q[R_\tau 1_{\{\tau\le t\}}|\mathcal{G}_t] = R_\tau 1_{\{\tau\le t\}} = W_t 1_{\{\tau\le t\}}. \end{aligned} \tag{3.5}$$

□

By using Definition 3.8, we can rewrite (3.4) as the sum of two components

$$\begin{aligned} \beta_t^o &= W_t - W_t^e = (W_t - W_t^Q) + (W_t^Q - W_t^e) \\ &= \beta_t^Q + (W_t^Q - W_t^e) \ge 0. \end{aligned}$$

In particular if $Q \in \mathcal{M}_{UI}(W)$, then $\beta_t^Q = 0$. This in turn implies that the bubble β^o is equal to

$$\beta_t^o = W_t^Q - W_t^e = W_t - W_t^e \ge 0, \quad t \in [0,T].$$

Therefore an increase in the market wealth leads to the creation of bubble at a time t or to a difference between the initial estimation of the wealth W^e and its current value W. Nevertheless, this may not create a Q-bubble in the martingale sense, if the new pricing measure Q corresponding to the wealth process W belongs to the set $\mathcal{M}_{UI}(W)$.
We now investigate when the second component of the bubble is also non-negative. In the rest of the paper we will use the abbreviated notation:

$$Z|_{\mathcal{G}_t} := \mathbb{E}[Z|\mathcal{G}_t],$$

for a random variable Z.

Proposition 3.10. *Let $Q \in \mathcal{M}_{loc}(W)$ with Radon-Nikodym density process $Z = (Z_t)_{t\in[0,T]}$ i.e $Z_t = \frac{dQ}{dP}|_{\mathcal{G}_t}$, $t \in [0,T]$. If the process $W^e Z$ is a P-submartingale, then*

$$W_t^Q \ge W_t^e, \tag{3.6}$$

for all $t \in [0,T]$.

Proof. By applying Bayes' theorem we obtain

$$\begin{aligned} W_t^Q - W_t^e &= \mathbb{E}_Q[X 1_{\{\tau>T\}} + R_\tau 1_{\{\tau\le T\}}|\mathcal{G}_t] - \mathbb{E}_P[X 1_{\{\tau>T\}} + R_\tau 1_{\{\tau\le T\}}|\mathcal{G}_t] \\ &= \frac{1}{Z_t}\mathbb{E}_P[(X 1_{\{\tau>T\}} + R_\tau 1_{\{\tau\le T\}})Z_T|\mathcal{G}_t] \\ &- \mathbb{E}_P[X 1_{\{\tau>T\}} + R_\tau 1_{\{\tau\le T\}}|\mathcal{G}_t] \\ &= \frac{1}{Z_t}(\mathbb{E}_P[W_T^e Z_T|\mathcal{G}_t] - W_t^e Z_t). \end{aligned}$$

Therefore it is enough to have that $W^e Z$ is a $(\mathbb{G}, P)$-submartingale for (3.6) to hold. □

4 Characterization of $\mathcal{M}_{loc}(W)$ by measure pasting

In this section we characterise $\mathcal{M}_{loc}(W)$, by using the concept of *pasting of measures*, see Section 6.4 of Föllmer and Schied [16]. By this method we establish a deeper connection between the martingale theory of bubbles and Definition 3.4.

We rewrite the market wealth process W in the following form:

$$\begin{aligned} W_t &= W_t^{\sigma_1} + (W_t - W_{\sigma_1})1_{\{t \geq \sigma_1\}} \\ &= W_t^{(1)} + W_t^{(2)}, \end{aligned} \tag{4.1}$$

where $W_t^{(1)} = W_t^{\sigma_1}$ and $W_t^{(2)} = (W_t - W_{\sigma_1})1_{\{t \geq \sigma_1\}}$ for $t \in [0, T]$. Note also that $W_t^{(1)} = W_t^{\sigma_1} = W_t^e$ on $\{\sigma_1 > t\}$.

Assumption 4.1. *i) We have* $0 < \sigma_1 < T$.

ii) For $i = 1, 2$, $\mathcal{M}_{loc}(W^{(i)}) \neq \emptyset$.

Assumption 4.1 *i)* is done for the sake of simplicity and without loss of generality. In fact, if $0 < P(0 < \sigma_1 < T) < 1$, for $Q \in \mathcal{M}_{loc}(W)$ decomposition (4.7) will hold on the set $\{0 < \sigma_1 < T\}$. Our aim is now to find a characterization of $\mathcal{M}_{loc}(W)$ that reflects the following facts. The wealth process W coincides with W^e until the starting time σ_1 of the bubble. After σ_1 the impact of the credibility induces an alteration of the total wealth process W, that deviates from W^e. Hence an equivalent measure $Q \in \mathcal{M}_{loc}(W)$ must take account of this change after σ_1. We can interpret this as a shift of martingale measures caused by a change in the underlying microeconomic conditions. This explains in an endogenous way the dynamic in the space of equivalent martingale measures as in the approach of [2] and [27], where the bubble is generated by a change in the underlying pricing measure. In this way we connect a constructive approach, where the bubble originates because the asset price is distorted by an excessive market confidence, with the martingale theory of bubbles, where a bubble is generated by a switch in the chosen pricing measure. In particular we now prove that every $Q \in \mathcal{M}_{loc}(W)$ is obtained by the pasting in σ_1 of $Q_1 \in \mathcal{M}_{loc}(W^{(1)})$ and $Q_2 \in \mathcal{M}_{loc}(W^{(2)})$. To this purpose we first recall and prove some results on measure pasting. In the sequel let Q_1 and Q_2 be two equivalent measures on $(\Omega, \mathcal{G}_T)$ and η be a $\mathcal{G}$-stopping time with $0 \leq \eta \leq T$.

Definition 4.2. *The probability measure* Q

$$Q(A) := \mathbb{E}_{Q_1}[Q_2(A|\mathcal{G}_\eta)], \quad A \in \mathcal{G}_T,$$

is called the pasting of Q_1 *and* Q_2 *in* η.

We remind the reader of the following results.

Lemma 4.3. *If Q is the pasting of Q_1 and Q_2 in η, then for all stopping times ξ and all $\mathcal{G}_T$-measurable random variables $Y \geq 0$ it holds that*

$$\mathbb{E}_Q[Y|\mathcal{G}_\xi] = \mathbb{E}_{Q_1}[\mathbb{E}_{Q_2}[Y|\mathcal{G}_{\eta\vee\xi}]|\mathcal{G}_\xi].$$

Proof. See Lemma 6.40 in [16]. □

For $i = 1, 2$ let $Z^{(i)} := (Z_t^{(i)})_{t\in[0,T]}$ be the corresponding Radon-Nikodym density process

$$Z_t^{(i)} = \frac{dQ_i}{dP}|_{\mathcal{G}_t}, \quad t \in [0, T]. \tag{4.2}$$

We put

$$U_t = \frac{dQ_2}{dQ_1}|_{\mathcal{G}_t}, \ t \in [0, T]. \tag{4.3}$$

Lemma 4.4. *The pasting Q of Q_1 and Q_2 in η is equivalent to Q_1 and satisfies*

$$\frac{dQ}{dQ_1} = \frac{U_T}{U_\eta}, \tag{4.4}$$

where U is introduced in (4.3).

Proof. See Lemma 6.39 in [16]. □

It follows from Lemma 4.4 that

$$\frac{dQ}{dQ_1}|_{\mathcal{G}_t} = \frac{U_t}{U_{t\wedge\eta}}, \quad t \in [0, T],$$

since

$$\begin{aligned}
\mathbb{E}_{Q_1}[\frac{dQ}{dQ_1}|\mathcal{G}_t] &= \mathbb{E}_{Q_1}[\frac{U_T}{U_\eta}1_{\{\eta\leq t\}}|\mathcal{G}_t] + \mathbb{E}_{Q_1}[\frac{U_T}{U_\eta}1_{\{t<\eta\}}|\mathcal{G}_t] \\
&= \frac{1}{U_\eta}U_t 1_{\{\eta\leq t\}} + \mathbb{E}_{Q_1}[\mathbb{E}_{Q_1}[\frac{U_T}{U_\eta}1_{\{t<\eta\}}|\mathcal{G}_\eta]|\mathcal{G}_t] \\
&= \frac{1}{U_\eta}U_t 1_{\{\eta\leq t\}} + \mathbb{E}_{Q_1}[1_{\{t<\eta\}}|\mathcal{G}_t] \\
&= \frac{U_t}{U_{t\wedge\eta}}1_{\{\eta\leq t\}} + \frac{U_t}{U_{t\wedge\eta}}1_{\{t<\eta\}} \\
&= \frac{U_t}{U_{t\wedge\eta}}.
\end{aligned} \tag{4.5}$$

This allows us to obtain the Radon-Nykodim density process $Z = (Z_t)_{t\in[0,T]}$ of the measure Q obtained through pasting, i.e. $Z_t = \frac{dQ}{dP}|_{\mathcal{G}_t}$, $t \in [0, T]$.

Corollary 4.5. *The process $Z = (Z_t)_{t\in[0,T]}$ given by*

$$Z_t = Z_t^{(1)} \frac{U_t}{U_{t\wedge\eta}}, \quad t \in [0,T], \tag{4.6}$$

is a P-martingale with respect to the filtration $\mathbb{G}$ and $\frac{dQ}{dP}|_{\mathcal{G}_t} = Z_t$, $t \in [0,T]$. Furthermore

$$Z_t = Z_{t\wedge\eta}^{(1)} \frac{Z_t^{(2)}}{Z_{t\wedge\eta}^{(2)}}, \quad t \in [0,T],$$

where $Z^{(1)}$, $Z^{(2)}$ are given in (4.2).

Proof. The result is a simple consequence of Lemma 4.4. By applying Bayes formula we obtain

$$\begin{aligned} Z_t = Z_t^{(1)} \frac{U_t}{U_{t\wedge\eta}} &= Z_t^{(1)} \frac{\mathbb{E}_P[\frac{dQ_2}{dP}|\mathcal{G}_t]}{\mathbb{E}_P[\frac{dQ_1}{dP}|\mathcal{G}_t]} \frac{\mathbb{E}_P[\frac{dQ_1}{dP}|\mathcal{G}_{t\wedge\eta}]}{\mathbb{E}_P[\frac{dQ_2}{dP}|\mathcal{G}_{t\wedge\eta}]} \\ &= Z_t^{(1)} \frac{Z_t^{(2)}}{Z_t^{(1)}} \frac{Z_{t\wedge\eta}^{(1)}}{Z_{t\wedge\eta}^{(2)}} = Z_{t\wedge\eta}^{(1)} \frac{Z_t^{(2)}}{Z_{t\wedge\eta}^{(2)}}, \quad t \in [0,T]. \end{aligned}$$

□

We now apply these general results on measure pasting to characterize the set $\mathcal{M}_{loc}(W)$. The following result represents the central theorem of this section.

Theorem 4.6. *We assume that Assumption 4.1 holds.*

i) Let $Q_i \in \mathcal{M}_{loc}(W^i)$, $i = 1,2$, and let $Q \approx P$ be the measure obtained by the pasting of Q_1 and Q_2 in σ_1, with Radon-Nikodym density process $Z_t = \frac{dQ}{dP}|_{\mathcal{G}_t}$, $t \in [0,T]$. Then $Q \in \mathcal{M}_{loc}(W)$.
In addition, if $Q_1 \in \mathcal{M}_{NUI}(W^{(1)})$ or $Q_2 \in \mathcal{M}_{NUI}(W^{(2)})$, then $Q \in \mathcal{M}_{NUI}(W)$.

ii) On the other hand, let $Q \in \mathcal{M}_{loc}(W)$ with Radon-Nikodym density $Z = (Z_t)_{t\in[0,T]}$, i.e. $Z_t = \frac{dQ}{dP}|_{\mathcal{G}_t}$, $t \in [0,T]$. There exist $Q_i \in \mathcal{M}_{loc}(W^{(i)})$, $i = 1,2$, with corresponding Radon-Nikodym density processes $Z_t^{(i)} = \frac{dQ_i}{dP}|_{\mathcal{G}_t}$ given by $Z_t^{(1)} = Z_{t\wedge\sigma_1}$ and $Z_t^{(2)} = \frac{Z_t}{Z_{t\wedge\sigma_1}}$ for all $t \in [0,T]$, such that Z can be written in the form

$$Z_t = Z_t^{(1)} Z_t^{(2)}, \quad t \in [0,T], \tag{4.7}$$

and Q is the pasting of Q_1 and Q_2 in σ_1.

Proof. i) Let $(\tau_n^i)_{n\geq 0}$ be a localizing sequence such that $W^{(i),\tau_n^i}$ is a Q_i-martingale on $[0,T]$ for $i=1,2$. We define the sequence of stopping times $(\tau_n)_{n\geq 0}$ by $\tau_n := \tau_n^1 \wedge \tau_n^2$, $n \geq 0$. We show that W^{τ_n} is a Q-martingale on $[0,T]$, where Q is the pasting of Q_1 and Q_2 in σ_1. For any $s \leq t$, it follows from Lemma 4.3 that

$$
\begin{aligned}
\mathbb{E}_Q[W_{t\wedge\tau_n}|\mathcal{G}_s] &= \mathbb{E}_{Q_1}[\mathbb{E}_{Q_2}[W_{t\wedge\tau_n}|\mathcal{G}_{s\vee\sigma_1}]|\mathcal{G}_s] \\
&= \mathbb{E}_{Q_1}[\mathbb{E}_{Q_2}[W_{t\wedge\tau_n\wedge\sigma_1} + W^{(2)}_{t\wedge\tau_n}|\mathcal{G}_{s\vee\sigma_1}]|\mathcal{G}_s] \\
&= \mathbb{E}_{Q_1}[W_{t\wedge\tau_n\wedge\sigma_1} + W^{(2)}_{(t\wedge\tau_n)\wedge(s\vee\sigma_1)}|\mathcal{G}_s] \\
&= W_{s\wedge\tau_n\wedge\sigma_1} + \mathbb{E}_{Q_1}[W^{(2)}_{(t\wedge\tau_n)\wedge(s\vee\sigma_1)}|\mathcal{G}_s] \\
&= W_{s\wedge\tau_n\wedge\sigma_1} + \mathbb{E}_{Q_1}[1_{\{s\wedge\tau_n<\sigma_1\}}W^{(2)}_{(t\wedge\tau_n\wedge\sigma_1)\vee(s\wedge\tau_n)}|\mathcal{G}_s] \\
&\quad + \mathbb{E}_{Q_1}[1_{\{s\wedge\tau_n\geq\sigma_1\}}W^{(2)}_{(t\wedge\tau_n\wedge\sigma_1)\vee(s\wedge\tau_n)}|\mathcal{G}_s] \\
&= W_{s\wedge\tau_n\wedge\sigma_1} + \mathbb{E}_{Q_1}[1_{\{s\wedge\tau_n<\sigma_1\}}W^{(2)}_{t\wedge\tau_n\wedge\sigma_1}|\mathcal{G}_s] \\
&\quad + \mathbb{E}_{Q_1}[1_{\{s\wedge\tau_n\geq\sigma_1\}}W^{(2)}_{s\wedge\tau_n}|\mathcal{G}_s] \\
&= W_{s\wedge\tau_n\wedge\sigma_1} + W^{(2)}_{s\wedge\tau_n}1_{\{s\wedge\tau_n\geq\sigma_1\}} = W_{s\wedge\tau_n},
\end{aligned}
$$

since $W^{(2)}_{t\wedge\tau_n\wedge\sigma_1} = (W_{t\wedge\tau_n\wedge\sigma_1} - W_{\sigma_1})1_{\{t\wedge\tau_n\wedge\sigma_1\geq\sigma_1\}} = 0$.
Hence $Q \in \mathcal{M}_{loc}(W)$. For the second part of i) we use the fact that $Q \in \mathcal{M}_{NUI}(W)$ is equivalent to

$$W_t > \mathbb{E}_Q[W_T|\mathcal{G}_t],$$

for some $t \in [0,T]$. By applying Lemma 4.3 we obtain

$$
\begin{aligned}
\mathbb{E}_Q[W_T|\mathcal{G}_t] &= \mathbb{E}_{Q_1}[\mathbb{E}_{Q_2}[W_T|\mathcal{G}_{\sigma_1\vee t}]|\mathcal{G}_t] \\
&\leq \mathbb{E}_{Q_1}[W_{\sigma_1\vee t}|\mathcal{G}_t] \leq W_t,
\end{aligned}
\tag{4.8}
$$

where one of the inequalities is strict for some t if Q_1 or Q_2 belong to the set $\mathcal{M}_{NUI}(W)$.
ii) We introduce the set

$$\mathcal{Z}(W) := \{Z; ZW \text{ is a } P-\text{local martingale}\}.$$

Then the Radon-Nikodym density process Z belongs to $\mathcal{Z}(W)$. As shown in Lemma 2.3. of Stricker and Yan [36], since $\sigma_1 < T$, we have that $Z \in \mathcal{Z}(W^{\sigma_1})$ i.e. ZW^{σ_1} is a P-local martingale on $[0,T]$. We define the measure $Q_1 \approx P$ by

$$\frac{dQ_1}{dP} = Z_{\sigma_1}.$$

Furthermore $\frac{Z_t}{Z_{t\wedge\sigma_1}}$ is a Radon-Nikodym density process for a measure $Q_2 \approx P$ such that $W^{(2)}$ is a Q_2-local martingale on $[0,T]$ as we now prove. Let

$(\tau_n)_{n\in\mathbb{N}}$ be a localizing sequence for the Q-local martingale W. Then

$$\begin{aligned}
\mathbb{E}_P\Big[\frac{Z_t}{Z_{t\wedge\sigma_1}}W^{(2)}_{t\wedge\tau_n}|\mathcal{G}_s\Big] &= \mathbb{E}_P\Big[\frac{Z_t}{Z_{t\wedge\sigma_1}}(W_{t\wedge\tau_n}-W_{\sigma_1})1_{\{t\wedge\tau_n\geq\sigma_1\}}|\mathcal{G}_s\Big]\\
&= \mathbb{E}_P\Big[\frac{Z_t}{Z_{t\wedge\sigma_1}}(W_{t\wedge\tau_n}-W_{\sigma_1})1_{\{t\wedge\tau_n\geq\sigma_1\}}(1_{\{s\wedge\tau_n\geq\sigma_1\}}\\
&\quad + 1_{\{s\wedge\tau_n<\sigma_1\leq t\wedge\tau_n\}} + 1_{\{t\wedge\tau_n<\sigma_1\}})|\mathcal{G}_s\Big]\\
&= \mathbb{E}_P\Big[\frac{Z_t}{Z_{t\wedge\sigma_1}}(W_{t\wedge\tau_n}-W_{\sigma_1})1_{\{s\wedge\tau_n\geq\sigma_1\}}|\mathcal{G}_s\Big]\\
&\quad + \mathbb{E}_P\Big[\frac{Z_t}{Z_{t\wedge\sigma_1}}(W_{t\wedge\tau_n}-W_{\sigma_1})1_{\{s\wedge\tau_n<\sigma_1\leq t\wedge\tau_n\}}|\mathcal{G}_s\Big]\\
&= 1_{\{s\wedge\tau_n\geq\sigma_1\}}\frac{1}{Z_{\sigma_1}}(\mathbb{E}_P[Z_tW_{t\wedge\tau_n}|\mathcal{G}_s]-\mathbb{E}_P[Z_tW_{\sigma_1}|\mathcal{G}_s])\\
&\quad + \mathbb{E}_P\Big[\mathbb{E}_P\Big[\frac{Z_t}{Z_{t\wedge\sigma_1}}(W_{t\wedge\tau_n}-W_{\sigma_1})1_{\{s\wedge\tau_n<\sigma_1\leq t\wedge\tau_n\}}|\mathcal{G}_{s\vee\sigma_1}\Big]|\mathcal{G}_s\Big]\\
&= \frac{Z_s}{Z_{\sigma_1\wedge s}}(W_{s\wedge\tau_n}-W_{\sigma_1})1_{\{s\wedge\tau_n\geq\sigma_1\}}\\
&\quad + \mathbb{E}_P\Big[1_{\{s\wedge\tau_n<\sigma_1\leq t\wedge\tau_n\}}\frac{1}{Z_{\sigma_1}}(\mathbb{E}_P[Z_tW_{t\wedge\tau_n}|\mathcal{G}_{s\vee\sigma_1}]\\
&\quad - \mathbb{E}_P[W_{\sigma_1}Z_t|\mathcal{G}_{s\vee\sigma_1}])|\mathcal{G}_s\Big]\\
&= \frac{Z_s}{Z_{\sigma_1\wedge s}}W^{(2)}_{s\wedge\tau_n} + \mathbb{E}_P\Big[1_{\{s\wedge\tau_n<\sigma_1\leq t\wedge\tau_n\}}\frac{1}{Z_{\sigma_1}}(Z_{t\wedge(\sigma_1\vee s)}W_{(t\wedge\tau_n)\wedge(\sigma_1\vee s)}\\
&\quad - W_{\sigma_1}Z_{t\wedge(\sigma_1\vee s)})|\mathcal{G}_s\Big]\\
&= \frac{Z_s}{Z_{\sigma_1\wedge s}}W^{(2)}_{s\wedge\tau_n} + \mathbb{E}_P\Big[1_{\{s\wedge\tau_n<\sigma_1\leq t\wedge\tau_n\}}\frac{Z_{t\wedge(\sigma_1\vee s)}}{Z_{\sigma_1}}(W_{t\wedge\tau_n\wedge\sigma_1}-W_{\sigma_1})|\mathcal{G}_s\Big]\\
&= \frac{Z_s}{Z_{\sigma_1\wedge s}}W^{(2)}_{s\wedge\tau_n}.
\end{aligned}$$

Hence $(\frac{1}{Z_{\sigma_1\wedge t}}Z_tW^{(2)}_t)_{t\in[0,T]}$ is a P-local martingale. Thus we can define

$$\frac{dQ_2}{dP}|_{\mathcal{G}_t} := \frac{Z_t}{Z_{\sigma_1\wedge t}}, \quad t\in[0,T],$$

and this concludes the proof. □

Theorem 4.6 shows that a change in the dynamics of the market wealth process can possibly lead to a switch to a martingale measure belonging to $\mathcal{M}_{NUI}(W)$ with consequent formation of a bubble in the sense of Definition 3.8. Here this shift is directly generated by the impact of the resulting trading activity due to the influence of many micro-economic interactions.

5 Example

We now consider a specific setting to illustrate the concepts presented in Section 2. Let $B = (B^1, B^2)$ be a 2-dimensional Brownian motion and set $\mathbb{F} = \mathbb{F}^1 \vee \mathbb{F}^2$, where $\mathbb{F}^1$ and $\mathbb{F}^2$ are the natural filtrations associated to B^1 and B^2 respectively. Let $X = (X_t)_{t\in[0,T]}$ be a process satisfying the following dynamics

$$dX_t = \sigma X_t dB_t^2, \quad X_0 = x_0,$$

with $x_0 \in \mathbb{R}_+$ and consider a defaultable claim such that $X = X_T$ and $R_t = cX_t$ for all $t \in [0,T]$ and some $c \in (0,1)$. It follows from the Doob's maximal inequality applied to the martingale X that our chosen R satisfies (2.1). We assume that the stochastic intensity μ is given by a Cox-Ingersoll-Ross model

$$\begin{aligned} d\mu_t &= (a + b\mu_t)dt + \theta\sqrt{\mu_t}dB_t^1, \\ \mu_0 &= \tilde{\mu}, \end{aligned} \tag{5.1}$$

where $a, \theta, \tilde{\mu} > 0$ and $b \in \mathbb{R}$.

Proposition 5.1. *The credibility process F satisfies under P the following equation*

$$dF_t = -\psi_t L_t \sqrt{\mu_t} dB_t^1 - \tilde{F}_t L_{t-} d\hat{M}_t - e^{-\Gamma_t} L_t \mu_t dt, \tag{5.2}$$

where $L_t = (1 - H_t)e^{\Gamma_t}$ and $\tilde{F}_t = P(t < \tau \leq T|\mathcal{F}_t)$ for all $t \in [0,T]$, and

$$\psi_t = \theta\beta(t)e^{\alpha(t)+\beta(t)\mu_t - \Gamma_t}, \tag{5.3}$$

with

$$\alpha(t) = \frac{2a}{\theta^2}\ln\Big(\frac{2\lambda e^{\frac{(\lambda-b)(T-t)}{2}}}{(\lambda-b)(e^{\lambda(T-t)}-1)+2\lambda}\Big), \tag{5.4}$$

and

$$\beta(t) = -\frac{2(e^{\lambda(T-t)}-1)}{(\lambda-b)(e^{\lambda(T-t)}-1)+2\lambda}, \tag{5.5}$$

for all $t \in [0,T]$ with $\lambda := \sqrt{b^2 + 2\theta^2}$.

Proof. It follows from Corollary 5.1.1 of Bielecki and Rutkowski [4] that

$$\begin{aligned} F_t = P(t < \tau \leq T|\mathcal{G}_t) &= 1_{\{\tau > t\}}\mathbb{E}_P[1_{\{t<\tau\leq T\}}e^{\Gamma_t}|\mathcal{F}_t] \\ &= L_t P(t < \tau \leq T|\mathcal{F}_t) = L_t\tilde{F}_t, \end{aligned}$$

where we have denoted $L_t = (1-H_t)e^{\Gamma_t}$, $t \in [0,T]$ and $\tilde{F}_t = P(t < \tau \leq T|\mathcal{F}_t)$ for all $t \in [0,T]$. Using the definition of the $\mathbb{F}$-hazard process, F can be written in the form

$$\tilde{F}_t = -\mathbb{E}_P[e^{-\Gamma_T}|\mathcal{F}_t] + e^{-\Gamma_t}.$$

Since μ is an affine process, e.g. by Filipovic [14], we have that

$$\tilde{F}_t = e^{-\Gamma_t}(1 - \mathbb{E}_P[e^{-(\Gamma_T - \Gamma_t)}|\mathcal{F}_t^1]) = -e^{-\Gamma_t}e^{\alpha(t)+\beta(t)\mu_t} + e^{-\Gamma_t}, \tag{5.6}$$

where $\alpha(t)$ and $\beta(t)$ are given by (5.4) and (5.5) respectively. By applying Itô's formula and using (5.1) we obtain that $\tilde{F}$ satisfies the equation

$$\begin{aligned} d\tilde{F}_t &= -e^{-\Gamma_t}e^{\alpha(t)+\beta(t)\mu_t}\theta\sqrt{\mu_t}\beta(t)dB_t^1 - e^{-\Gamma_t}\mu_t dt \\ &= -\psi_t\sqrt{\mu_t}dB_t^1 - e^{-\Gamma_t}\mu_t dt \end{aligned} \tag{5.7}$$

for all $t \in [0,T]$, where ψ is given by (5.3). By the integration by parts formula we have that

$$\begin{aligned} dF_t &= L_t d\tilde{F}_t + \tilde{F}_t dL_t + d[F,L]_t \\ &= -\psi_t L_t\sqrt{\mu_t}dB_t^1 - \tilde{F}_t L_{t-}d\hat{M}_t - e^{-\Gamma_t}L_t\mu_t dt. \end{aligned}$$

□

We examine the dynamics of the process Λ in this setting.

Theorem 5.2. *The process Λ satisfies the following equation*

$$d\Lambda_t = \Lambda_{t-}(\xi_t^1 dB_t^1 + \sigma dB_t^2 - d\hat{M}_t + \xi_t^2 dt), \tag{5.8}$$

for all $t \in [0,T]$, where ξ^1 and ξ^2 are given by

$$\xi_t^1 = \frac{(1-c)\psi_t\sqrt{\mu_t}e^{\Gamma_t}}{(1-c)e^{\alpha(t)+\beta(t)\mu_t} + c} \quad \textit{and} \quad \xi_t^2 = -\frac{c\mu_t}{(1-c)e^{\alpha(t)+\beta(t)\mu_t} + c} \tag{5.9}$$

for all $t \in [0,T]$.

Proof. By (2.2) we have

$$\Lambda_t = \mathbb{E}_P[X_T 1_{\{\tau>T\}}|\mathcal{G}_t] + \mathbb{E}_P[cX_\tau 1_{\{\tau\le T\}}|\mathcal{G}_t]1_{\{\tau>t\}}. \tag{5.10}$$

It follows from Corollary 5.1.1. of [4] that

$$\mathbb{E}_P[X_T 1_{\{\tau>T\}}|\mathcal{G}_T] = L_t\mathbb{E}_P[X_T 1_{\{\tau>T\}}|\mathcal{F}_t], \tag{5.11}$$

where $L_t = (1-H_t)e^{\Gamma_t}$, $t \in [0,T]$. Hence

$$\begin{aligned} \mathbb{E}_P[X_T 1_{\{\tau>T\}}|\mathcal{G}_t] &= L_t\mathbb{E}_P[1_{\{\tau>T\}}X_T|\mathcal{F}_t] = L_t\mathbb{E}_P[\mathbb{E}_P[X_T 1_{\{\tau>T\}}|\mathcal{F}_T]|\mathcal{F}_t] \\ &= L_t\mathbb{E}_P[X_T P(\tau>T|\mathcal{F}_T)|\mathcal{F}_t] = L_t\mathbb{E}_P[X_T e^{-\Gamma_T}|\mathcal{F}_t] \\ &= L_t\mathbb{E}_P[X_T|\mathcal{F}_t^2]\mathbb{E}_P[e^{-\Gamma_T}|\mathcal{F}_t^1] \\ &= L_t X_t e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t}. \end{aligned}$$

By Lemma 4.1. of [3] we then have

$$\begin{aligned}
\mathbb{E}_P[cX_\tau 1_{\{\tau\le T\}}|\mathcal{G}_t]1_{\{\tau>t\}} &= cL_t\mathbb{E}_P[\int_t^T X_s e^{-\Gamma_s}d\Gamma_s|\mathcal{F}_t] = cL_t\int_t^T \mathbb{E}_P[X_s e^{-\Gamma_s}\mu_s|\mathcal{F}_t]ds \\
&= cL_t\int_t^T \mathbb{E}_P[X_s|\mathcal{F}_t^2]\mathbb{E}_P[e^{-\Gamma_s}\mu_s|\mathcal{F}_t^1]ds \\
&= cL_tX_t\int_t^T \mathbb{E}_P[e^{-\Gamma_s}\mu_s|\mathcal{F}_t^1]ds \\
&= cL_tX_t(\mathbb{E}_P[\int_0^T e^{-\Gamma_s}\mu_s ds|\mathcal{F}_t^1] - \int_0^t e^{-\Gamma_s}\mu_s ds).
\end{aligned}$$

Since

$$\begin{aligned}
\mathbb{E}_P[\int_0^T e^{-\Gamma_s}\mu_s ds|\mathcal{F}_t] &= \mathbb{E}_P[1-e^{-\Gamma_T}|\mathcal{F}_t] = 1-\mathbb{E}_P[e^{-\Gamma_T}|\mathcal{F}_t] \\
&= 1-e^{-\Gamma_t}\mathbb{E}_P[e^{-\int_t^T\mu_s ds}|\mathcal{F}_t] = 1-e^{-\Gamma_t}e^{\alpha(t)+\beta(t)\mu_t},
\end{aligned}$$

we obtain

$$\begin{aligned}
\Lambda_t &= L_tX_t(c+(1-c)e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t} - c\int_0^t e^{-\Gamma_s}\mu_s ds) \\
&= L_tX_tD_t,
\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}
D_t &:= c+(1-c)e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t} - c\int_0^t e^{-\Gamma_s}\mu_s ds \\
&= (1-c)e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t} + ce^{-\Gamma_t} > 0,
\end{aligned} \tag{5.13}$$

$a.s.$ for all $t\in[0,T]$. Since

$$d\Big(e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t}\Big) = \theta\beta(t)\sqrt{\mu_t}e^{\alpha(t)+\beta(t)\mu_t-\Gamma_t}dB_t^1 = \psi_t\sqrt{\mu_t}dB_t^1,$$

where ψ is defined in (5.3). We have

$$dD_t = D_t\Big(\frac{(1-c)\psi_t\sqrt{\mu_t}}{D_t}dB_t^1 - \frac{ce^{-\Gamma_t}\mu_t}{D_t}dt\Big) = D_t(\xi_t^1 dB_t^1 + \xi_t^2 dt),$$

where

$$\xi_t^1 = \frac{(1-c)\psi_t\sqrt{\mu_t}}{D_t} = \frac{(1-c)\psi_t\sqrt{\mu_t}e^{\Gamma_t}}{(1-c)e^{\alpha(t)+\beta(t)\mu_t}+c} \le 0.$$

and

$$\xi_t^2 = -\frac{ce^{-\Gamma_t}\mu_t}{D_t} = -\frac{c\mu_t}{(1-c)e^{\alpha(t)+\beta(t)\mu_t}+c} \le 0.$$

An application of Itô's product formula yields

$$d\Lambda_t = \Lambda_{t-}(\xi_t^1 dB_t^1 + \sigma dB_t^2 - d\hat{M}_t + \xi_t^2 dt)$$

and this concludes the proof. □

Let us now examine the structure of the market wealth process W. We remind that in our setting the immersion property holds under the measure P.

Theorem 5.3. *The market wealth process W is a $(\mathbb{G}, P)$-semimartingale that admits the canonical decomposition*

$$W_t = M_t + A_t,$$

for all $t \in [0,T]$, where the local martingale part M is given by

$$\begin{aligned} dM_t = &(f(t,F_t)\xi_t^1 - f_x(t,F_t)L_t\psi_t\sqrt{\mu_t})\Lambda_t dB_t^1 + \sigma f(t,F_t)\Lambda_t dB_t^2 \\ &+ (cX_t - f(t,F_t)\Lambda_{t-})d\hat{M}_t, \end{aligned} \tag{5.14}$$

and the finite variation part A is given by

$$\begin{aligned} dA_t = \Big\{ &[f(t,F_t)\xi_t^2 + f_t(t,F_t) - f_x(t,F_t)L_t e^{-\Gamma_t}\mu_t + \frac{1}{2} f_{xx}(t,F_t)L_t^2\psi_t^2\mu_t \\ &+ f_x(t,F_t)L_t\tilde{F}_t\hat{\mu}_t - f_x(t,F_t)L_t\psi_t\sqrt{\mu_t}\xi_t^1]\Lambda_t + cX_t\hat{\mu}_t\Big\} dt. \end{aligned} \tag{5.15}$$

Proof. By applying the integration by parts formula we obtain

$$dW_t = f(t,F_t)d\Lambda_t + \Lambda_t df(t,F_t) + d[\Lambda, f(\cdot,F)]_t + cX_t dH_t. \tag{5.16}$$

We start by determining the dynamics of $f(t,F_t)$. Using Itô's formula (see Theorem II.32 in Protter [31]) and (5.2) we have

$$\begin{aligned} f(t,F_t) &= f(0,F_0) + \int_0^t f_s(s,F_s)ds + \int_0^t f_x(s,F_{s-})dF_s + \frac{1}{2}\int_0^t f_{xx}(s,F_{s-})d[F,F]_s^c \\ &\quad + \sum_{0<s\le t} \{f(s,F_s) - f(s,F_{s-}) - f_x(s,F_{s-})\Delta F_s)\} \\ &= f(0,F_0) + \int_0^t f_s(s,F_s)ds - \int_0^t f_x(s,F_s)L_s\psi_s\sqrt{\mu_s}dB_s^1 \\ &\quad - \int_0^t f_x(s,F_{s-})\tilde{F}_{s-}L_{s-}d\hat{M}_s - \int_0^t f_x(s,F_s)L_s e^{-\Gamma_s}\mu_s ds \\ &\quad + \frac{1}{2}\int_0^t f_{xx}(s,F_s)L_s^2\psi_s^2\mu_s ds \\ &\quad + \int_0^t \{f(s,F_s) - f(s,F_{s-}) + f_x(s,F_{s-})\tilde{F}_{s-}L_{s-}\}d\hat{M}_s \\ &\quad + \int_0^t f_x(s,F_s)\tilde{F}_s L_s\hat{\mu}_s ds, \end{aligned}$$

where we wrote the sum of jumps as a stochastic integral as shown below

$$
\begin{aligned}
&\sum_{0<s\le t}\{f(s,F_s)-f(s,F_{s-})-f_x(s,F_{s-})\Delta F_s\}\\
&\quad=\sum_{0<s\le t}\{f(s,F_s)-f(s,F_{s-})+f_x(s,F_{s-})\tilde{F}_sL_{s-}\Delta H_s\}\\
&\quad=\sum_{0<s\le t}\{f(s,F_s)-f(s,F_{s-})\}\Delta H_s+\sum_{0<s\le t}f_x(s,F_{s-})\tilde{F}_sL_{s-}\Delta H_s\\
&\quad=\int_0^t(f(s,F_s)-f(s,F_{s-}))dH_s+\int_0^t f_x(s,F_{s-})\tilde{F}_sL_{s-}dH_s\\
&\quad=\int_0^t\{f(s,F_s)-f(s,F_{s-})+f_x(s,F_{s-})\tilde{F}_sL_{s-}\}d\hat{M}_s+\int_0^t f_x(s,F_{s-})\tilde{F}_sL_{s-}\hat{\mu}_sds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f(t,F_t)=f(0,F_0)&-\int_0^t f_x(s,F_s)L_s\psi_s\sqrt{\mu_s}dB^1_s+\int_0^t\Big(f(s,F_s)-f(s,F_{s-})\Big)d\hat{M}_s\\
&+\int_0^t\Big(f_s(s,F_s)-f_x(s,F_s)L_se^{-\Gamma_s}\mu_s+\frac{1}{2}f_{xx}(s,F_s)L_s^2\psi_s^2\mu_s\\
&+f_x(s,F_s)\tilde{F}_sL_s\hat{\mu}_s\Big)ds.
\end{aligned}
$$

It follows from Theorem 5.2 and the expression of $f(t,F_t)$ that the quadratic covariation $[\Lambda,f(\cdot,F)]$ is equal to

$$
\begin{aligned}
d[\Lambda,f(\cdot,F)]_t&=-(f(t,F_t)-f(t,F_{t-}))\Lambda_{t-}dH_t-f_x(t,F_t)L_t\psi_t\xi^1_t\sqrt{\mu_t}\Lambda_tdt\\
&=-(f(t,F_t)-f(t,F_{t-}))\Lambda_{t-}d\hat{M}_t-f_x(t,F_t)L_t\psi_t\xi^1_t\sqrt{\mu_t}\Lambda_tdt.
\end{aligned}
$$

By replacing the expressions of $[\Lambda,f(\cdot,F)]$ and $f(t,F_t)$ in (5.16) and by using

(5.8) we obtain

$$
\begin{aligned}
dW_t &= f(t,F_t)\xi_t^1\Lambda_t dB_t^1 + \sigma f(t,F_t)\Lambda_t dB_t^2 - f(t,F_{t-})\Lambda_{t-}d\hat{M}_t \\
&\quad + f(t,F_t)\xi_t^2\Lambda_t dt - f_x(t,F_t)L_t\psi_t\sqrt{\mu_t}\Lambda_t dB_t^1 \\
&\quad + (f(t,F_t) - f(t,F_{t-}))\Lambda_{t-}d\hat{M}_t + \Big(f_t(t,F_t) - f_x(t,F_t)L_t e^{-\Gamma_t}\mu_t \\
&\quad + \frac{1}{2}f_{xx}(t,F_t)L_t^2\psi_t^2\mu_t + f_x(t,F_t)\tilde{F}_t L_t\hat{\mu}_t\Big)\Lambda_t dt \\
&\quad - f_x(t,F_t)L_t\psi_t\sqrt{\mu_t}\xi_t^1\Lambda_t dt - (f(t,F_t) - f(t,F_{t-}))\Lambda_{t-}d\hat{M}_t \\
&\quad + cX_t d\hat{M}_t + cX_t\hat{\mu}_t dt \\
&= \Big(f(t,F_t)\xi_t^1 - f_x(t,F_t)L_t\psi_t\sqrt{\mu_t}\Big)\Lambda_t dB_t^1 + \sigma f(t,F_t)\Lambda_t dB_t^2 \\
&\quad + (cX_t - f(t,F_{t-})\Lambda_{t-})d\hat{M}_t \\
&\quad + \Big\{\Big(f(t,F_t)\xi_t^2 + f_t(t,F_t) - f_x(t,F_t)L_t e^{-\Gamma_t}\mu_t \\
&\quad + \frac{1}{2}f_{xx}(t,F_t)L_t^2\psi_t^2\mu_t + f_x(t,F_t)L_t\tilde{F}_t\hat{\mu}_t \\
&\quad - f_x(t,F_t)L_t\psi_t\xi_t^1\sqrt{\mu_t}\Big)\Lambda_t + cX_t\hat{\mu}_t\Big\}dt.
\end{aligned}
$$

□

We now assume that $\mathcal{M}_{loc}(W) \neq \emptyset$ and derive first a general form for the Radon-Nikodym density process associated to a measure $Q \in \mathcal{M}_{loc}(W)$.

Proposition 5.4. *Let $Q \in \mathcal{M}_{loc}(W)$ with Radon-Nikodym density process $Z = (Z_t)_{t\in[0,T]}$ i.e. $Z_t = \frac{dQ}{dP}|_{\mathcal{G}_t}$, $t \in [0,T]$. Furthermore, we assume that the quadratic covariation $[Z,M]$ is locally integrable. Then Z admits the representation*

$$
dZ_t = Z_{t-}(b_t^{(1)}dB_t^1 + b_t^{(2)}dB_t^2 + b_{t-}^{(3)}d\hat{M}_t), \quad t \in [0,T], \tag{5.17}
$$

where $(b_t^{(i)})_{t\in[0,T]}$ are $\mathbb{G}$-predictable processes for all $i = 1,2,3$, satisfying

$$
\begin{aligned}
0 = dA_t + \Big[\Big(&b_t^{(1)}(\xi_t^1 f(t,F_t) - f_x(t,F_t)L_t\psi_t\sqrt{\mu_t}) \\
&+ \sigma b_t^{(2)}f(t,F_t)\Big)\Lambda_t + b_t^{(3)}(cX_t - f(t,F_t)\Lambda_{t-})\hat{\mu}_t\Big]dt
\end{aligned} \tag{5.18}
$$

$dt \otimes dP$-almost surely on $[0,T]\times\Omega$.

Proof. Since the process Z is strictly positive, representation (5.17) follows by the martingale representation theorem with respect to $(\mathbb{G},P)$, see [4]. From the predictable version of Girsanov's Theorem we obtain that W admits under Q the following decomposition

$$
W_t = M_t - \int_0^t \frac{1}{Z_{s-}}d\langle Z,M\rangle_s + A_t + \int_0^t \frac{1}{Z_{s-}}d\langle Z,M\rangle_s, \quad t \in [0,T].
$$

Since $Q \in \mathcal{M}_{loc}(W)$, this implies that

$$A_t + \int_0^t \frac{1}{Z_{s-}} d\langle Z, M\rangle_s = 0$$

$dt \otimes dP$-a.s. which is equivalent to (5.18). $\square$

We now state an auxiliary result, that will be used later in the proofs.

Lemma 5.5. *On the set $\{\tau > t\}$ we have*

$$R_t < \Lambda_t. \tag{5.19}$$

Proof. Since $R_t = cX_t$ for all $t \geq 0$, we obtain

$$\begin{aligned}
R_t 1_{\{\tau>t\}} &= cX_t 1_{\{\tau>t\}} = c\mathbb{E}_P[X_T|\mathcal{G}_t]1_{\{\tau>t\}} \\
&= c1_{\{\tau>t\}}\mathbb{E}_P[X_T 1_{\{\tau>T\}} + X_T 1_{\{\tau\leq T\}}|\mathcal{G}_t] \\
&= c(1_{\{\tau>t\}}\mathbb{E}_P[X_T 1_{\{\tau>T\}}|\mathcal{G}_t] + 1_{\{\tau>t\}}\mathbb{E}_P[\mathbb{E}_P[X_T 1_{\{t<\tau\leq T\}}|\mathcal{G}_\tau]|\mathcal{G}_t]) \\
&= c1_{\{\tau>t\}}\mathbb{E}_P[X_T 1_{\{\tau>T\}} + X_\tau 1_{\{t<\tau\leq T\}}|\mathcal{G}_t] < \Lambda_t 1_{\{\tau>t\}},
\end{aligned}$$

since $c \in (0,1)$. $\square$

The next result provides a concrete example (by specifying a function $f(t,x)$) when the set $\mathcal{M}_{loc}(W)$ is non-empty. We also compute a specific form of the density process. Let

$$f(t,x) := \begin{cases} 1 + k(T-t)(1-\frac{x}{p})^3 & \text{if } x \leq p,\ t \in [0,T], \\ 1 & \text{if } x > p,\ t \in [0,T]. \end{cases} \tag{5.20}$$

where $k > 0$ is a positive constant. The partial derivatives of $f(t,x)$ will be equal to

$$f_t(t,x) := \begin{cases} -k(1-\frac{x}{p})^3 & \text{if } x \leq p,\ t \in [0,T], \\ 0 & \text{if } x > p,\ t \in [0,T]. \end{cases}$$

and

$$f_x(t,x) := \begin{cases} -\frac{3k}{p}(T-t)(1-\frac{x}{p})^2 & \text{if } x \leq p,\ t \in [0,T], \\ 0 & \text{if } x > p,\ t \in [0,T] \end{cases}$$

and the second derivative of $f(t,x)$ with respect to x is given by

$$f_{xx}(t,x) := \begin{cases} \frac{6k}{p^2}(T-t)(1-\frac{x}{p}) & \text{if } x \leq p,\ t \in [0,T], \\ 0 & \text{if } x > p,\ t \in [0,T] \end{cases}$$

Hence $f(t,x)$ has bounded first and second order partial derivatives. Note that, in the setting of this example, the impact of the credibility process F on the wealth process W is bounded. However the wealth process is not bounded.

Theorem 5.6. *If the function $f(t,x)$ is defined by* (5.20)*, then $\mathcal{M}_{loc}(W) \neq \emptyset$.*

Proof. We now rewrite the expression of the finite variation part A of W as given in (5.15) in a simpler form. .By (5.3), (5.6) and (5.9), we have

$$\begin{aligned} dA_t = &\Big\{f(t,F_t)\xi_t^2 + f_t(t,F_t)\Big\}\Lambda_t dt + \Big\{ - f_x(t,F_t)e^{\alpha(t)+\beta(t)\mu_t} \\ &\cdot \Big(1 + \frac{1-c}{(1-c)e^{\alpha(t)+\beta(t)\mu_t}+c}\theta^2\beta^2(t)e^{\alpha(t)+\beta(t)\mu_t}\Big)\Lambda_t \\ &+ \frac{1}{2}f_{xx}(t,F_t)\theta^2\beta^2(t)e^{2\alpha(t)+2\beta(t)\mu_t}\Lambda_t + cX_t\Big\}\hat{\mu}_t dt. \end{aligned}$$

We denote

$$\begin{aligned} \delta_t := &-f_x(t,F_t)e^{\alpha(t)+\beta(t)\mu_t}\Big(1 + \frac{(1-c)e^{\alpha(t)+\beta(t)\mu_t}}{(1-c)e^{\alpha(t)+\beta(t)\mu_t}+c}\theta^2\beta^2(t)\Big) \\ &+ \frac{1}{2}f_{xx}(t,F_t)\theta^2\beta^2(t)e^{2\alpha(t)+2\beta(t)\mu_t}, \end{aligned} \tag{5.21}$$

for all $t \in [0,T]$. It is easy to see that $0 \leq \delta_t < C_\delta$ for some constant $C_\delta > 0$ a.s. for all $t \in [0,T]$. Therefore

$$dA_t = \Big\{f(t,F_t)\xi_t^2 + f_t(t,F_t)\Big\}\Lambda_t dt + \Big(\delta_t\Lambda_t + cX_t\Big)\hat{\mu}_t dt.$$

We define the process $Z^{(1)} = (Z_t^{(1)})_{t\in[0,T]}$ by

$$dZ_t^{(1)} = b_t^{(1)}Z_t^{(1)}dB_t^2, \tag{5.22}$$

and

$$\begin{aligned} b_t^{(1)} &= -\frac{f(t,F_t)\xi_t^2 + f_t(t,F_t) + \frac{cX_t}{2\Lambda_t}\mu_t 1_{\{\tau>t\}}}{\sigma f(t,F_t)} \\ &= -\frac{1}{\sigma}\xi_t^2 - \frac{f_t(t,F_t)}{\sigma f(t,F_t)} - \frac{cX_t}{2\sigma f(t,F_t)\Lambda_t}\mu_t 1_{\{\tau>t\}} \\ &= \frac{1}{\sigma}\frac{c}{(1-c)e^{\alpha(t)+\beta(t)\mu_t}+c}\mu_t + k(1-\frac{F_t}{p})\frac{1}{\sigma f(t,F_t)}1_{\{\sigma_1\leq t\}} \\ &\quad - \frac{1}{2\sigma f(t,F_t)}\frac{c\mu_t}{(1-c)e^{\alpha(t)+\beta(t)\mu_t}+c}1_{\{\tau>t\}} \\ &= \frac{1}{\sigma}\frac{c\mu_t}{(1-c)e^{\alpha(t)+\beta(t)\mu_t}+c}\Big(1 - \frac{1}{2f(t,F_t)}1_{\{\tau>t\}}\Big) + k(1-\frac{F_t}{p})\frac{1}{\sigma f(t,F_t)}1_{\{\sigma_1\leq t\}}, \end{aligned}$$

where we have used the fact that $\Lambda_t = X_tL_tD_t$ with D_t defined in (5.13). By Proposition 5.7 we have that $Z^{(1)}$ is a P-martingale. We define the measure Q^1 by

$$\frac{dQ^1}{dP}|_{\mathcal{G}_T} = Z_T^{(1)}, \tag{5.23}$$

Under the measure Q^1, the process

$$\tilde{B}_t^2 = B_t^2 - \int_0^t b_s^{(1)} ds, \quad t \in [0, T],$$

is a Brownian motion with respect to the filtration $\mathbb{G}$. By applying Girsanov's theorem we obtain that W admits the following decomposition under Q^1:

$$W_t = M_t^1 + A_t^1, \quad t \in [0, T],$$

with the local martingale part $(M_t^1)_{t\in[0,T]}$ equal to

$$\begin{aligned} dM_t^1 = &\Big(f(t, F_t)\xi_t^1 - f_x(t, F_t) L_t \psi_t \sqrt{\mu_t} \Big) \Lambda_t dB_t^1 + \sigma f(t, F_t) \Lambda_t d\tilde{B}_t^2 \\ &+ (cX_t - f(t, F_t)\Lambda_t) d\hat{M}_t, \end{aligned}$$

and the finite variation part $(A_t^1)_{t\in[0,T]}$ is given by

$$dA_t^1 = (\frac{c}{2} X_t + \delta_t \Lambda_t) \hat{\mu}_t dt.$$

We define the process $Z^{(2)} = (Z^{(2)})_{t\in[0,T]}$ as the solution of

$$dZ_t^{(2)} = b_{t-} Z_{t-}^{(2)} d\hat{M}_t, \quad t \in [0, T], \tag{5.24}$$

with

$$b_t = \frac{\frac{c}{2} X_t + \delta_t \Lambda_t}{f(t, F_t)\Lambda_t - cX_t}, \quad t \in [0, T]. \tag{5.25}$$

We prove that $Z^{(2)}$ is a Q^1-martingale. We start by showing that $(b_t)_{t\in[0,T]}$ defined in (5.25) satisfies $b_t > -1$ for all $t \in [0, T]$. We have

$$\begin{aligned} b_t &= \frac{\frac{c}{2} X_t + \delta_t \Lambda_t}{f(t, F_t)\Lambda_t - cX_t} 1_{\{\tau \le t\}} + \frac{\frac{c}{2} X_t + \delta_t \Lambda_t}{f(t, F_t)\Lambda_t - cX_t} 1_{\{\tau > t\}} \\ &= -\frac{1}{2} 1_{\{\tau \le t\}} + \frac{\frac{c}{2} X_t + \delta_t \Lambda_t}{f(t, F_t)\Lambda_t - cX_t} 1_{\{\tau > t\}}. \end{aligned} \tag{5.26}$$

On the set $\{\tau > t\}$ we have by Lemma 5.5, (3.1) and (5.21) that

$$cX_t < \Lambda_t \le f(t, F_t)\Lambda_t, \tag{5.27}$$

so

$$b_t \ge 0 \quad \text{on} \quad \{\tau > t\}. \tag{5.28}$$

Moreover $b_t > -1$ for all $t \in [0,T]$. It follows from Theorem II.37 in Protter [31] that the unique solution of equation (5.24) is given by

$$
\begin{aligned}
Z_t^{(2)} &= \exp\Big(\int_0^t b_s d\hat{M}_s\Big)\Pi_{s\le t}(1+b_s\Delta H_s)\exp(-b_s\Delta H_s)\\
&= \exp\Big(\int_0^t b_s dH_s - \int_0^{t\wedge\tau} b_s\mu_s ds\Big)(1+b_\tau 1_{\{\tau\le t\}})\exp(-b_\tau 1_{\{\tau\le t\}})\\
&= 1_{\{\tau>t\}}\exp\Big(-\int_0^{t\wedge\tau} b_s\mu_s ds\Big) + \frac{1}{2}1_{\{\tau\le t\}}\exp\Big(-\int_0^{t\wedge\tau} b_s\mu_s ds\Big)\\
&\le 1_{\{\tau>t\}} + \frac{1}{2}1_{\{\tau\le t\}} < \frac{3}{2},
\end{aligned}
\tag{5.29}
$$

where we have used (5.26) and (5.28). Therefore $Z^{(2)}$ is a positive Q^1-martingale since it is a bounded local martingale. Analogously we obtain that

$$|\Delta Z_t^{(2)}| = |Z_t^{(2)} - Z_{t-}^{(2)}| \le \mathcal{K}_\Delta,$$

for some $\mathcal{K}_\Delta > 0$. Hence $Z^{(2)}$ has bounded jumps. It follows from Lemma 3.14 in [20] that $[Z^{(2)}, M^1]$ has locally integrable variation. Therefore its Q^1-compensator $\langle Z^{(2)}, M^1\rangle$ exists and is well defined.
We now define the measure Q^2 by

$$\frac{dQ^2}{dQ^1}|_{\mathcal{G}_T} = Z_T^{(2)}. \tag{5.30}$$

Since $\langle Z^{(2)}, M^1\rangle$ exists under Q^1, the predictable version of Girsanov's theorem (see Theorem III.40 in [31]) yields the following canonical decomposition of W under Q^2

$$W_t = M_t^2 + A_t^2, \quad t\in[0,T],$$

where the local martingale part M^2 is given by

$$M_t^2 = M_t^1 - \int_0^t \frac{1}{Z_{s-}^{(2)}} d\langle Z^{(2)}, M^1\rangle_s$$

and the finite variation part A^2 by

$$
\begin{aligned}
A_t^2 &= A_t^1 + \int_0^t \frac{1}{Z_{s-}^{(2)}} d\langle Z^{(2)}, M^1\rangle_s\\
&= \int_0^t (\frac{c}{2}X_s + \delta_s\Lambda_s)\hat{\mu}_s ds + \int_0^t b_s d\langle \hat{M}, M^1\rangle_s\\
&= \int_0^t (\frac{c}{2}X_s + \delta_s\Lambda_s)\hat{\mu}_s ds + \int_0^t b_s(cX_s - f(s,F_s)\Lambda_s)\hat{\mu}_s ds\\
&= 0,
\end{aligned}
$$

where we have used (5.25). Hence W is a local martingale under Q^2. Then the equivalent probability measure $Q \approx P$ defined by

$$\frac{dQ}{dP} = \frac{dQ^1}{dP}\frac{dQ^2}{dQ^1} \tag{5.31}$$

where Q^1 and Q^2 are defined in (5.23) and (5.30) respectively, belongs to $\mathcal{M}_{loc}(W)$, i.e $\mathcal{M}_{loc}(W) \neq \emptyset$. The following proposition concludes the proof. $\square$

Proposition 5.7. *The process $Z^{(1)} = (Z_t^{(1)})_{t\in[0,T]}$ defined by*

$$dZ_t^{(1)} = b_t^{(1)} Z_t^{(1)} dB_t^2,$$

with $(b_t^{(1)})_{t\in[0,T]}$ given by (5.22) is a P-martingale.

Proof. The process $b^{(1)}$ can be written under the form

$$b_t^{(1)} = k_t^1 \mu_t + k_t^2, \quad t \in [0,T],$$

where $(k_t^1)_{t\in[0,T]}$ and $(k_t^2)_{t\in[0,T]}$ are càdlàg adapted bounded processes, given by

$$k_t^1 = \frac{1}{\sigma}\frac{c}{(1-c)e^{\alpha(t)+\beta(t)\mu_t}+c}\Big(1 - \frac{1}{2f(t,F_t)}1_{\{\tau>t\}}\Big)$$

and

$$k_t^2 = k(1-\frac{F_t}{p})\frac{1}{\sigma f(t,F_t)}1_{\{\sigma_1\le t\}}. \tag{5.32}$$

Let $K>0$ be such that $0 \le k_t^i \le K$ for all $t\in[0,T]$ and $i=1,2$.
Since the default intensity process μ given by (5.1) is a Cox-Ingersoll-Ross process, it can be written as the finite sum of squared Ornstein-Uhlenbeck processes, see Dufresne [13]. For simplicity, we assume that μ can be written under the form

$$\mu_t = r_t^2, \quad t\in[0,T], \tag{5.33}$$

where $(r_t)_{t\in[0,T]}$ is an Ornstein-Uhlenbeck process satisfying the equation

$$dr_t = -mr_t dt + \zeta dB_t^1, \quad r_0 = \bar{r} > 0, \tag{5.34}$$

where $m\in\mathbb{R}$ and $\zeta>0$. Equation (5.34) admits the solution

$$r_t = \bar{r}e^{-mt} + \zeta e^{-mt}\int_0^t e^{ms} dB_s^1, \quad t\in[0,T].$$

However, the proof can be easily extended to the general case when μ is a finite sum of squared Ornstein-Uhlenbeck processes. Let now $\sigma_n = \inf\{t \ge 0; b_t^{(1)} \ge n\}$ and denote $Z_t^n := Z_{t\wedge\sigma_n}^{(1)}$, for all $t\in[0,T]$. For each $n\in\mathbb{N}$, the process $(Z_t^n)_{t\in[0,T]}$ is a P-martingale, since the Novikov condition is trivially satisfied:

$$\mathbb{E}_P\Big[\exp\Big(\frac{1}{2}\int_0^{T\wedge\sigma_n}(b_s^{(1)})^2ds\Big)\Big]\le\mathbb{E}_P[\exp(\frac{1}{2}n^2T)]<\infty.$$

In order to show that the positive local martingale $Z^{(1)}$ is a P-martingale, we now prove that

$$\mathbb{E}_P[Z_T^{(1)}]=1.$$

To this purpose, it is sufficient to show that the family $(Z_T^n)_{n\in\mathbb{N}}$ is uniformly integrable, since an application of Lebesgue's dominated convergence theorem yields

$$1=\lim_{n\to\infty}\mathbb{E}_P[Z_T^n]=\mathbb{E}_P[\lim_{n\to\infty}Z_T^n]=\mathbb{E}_P[Z_T^{(1)}].$$

As in Theorem 2 of Hitsuda [18] and Theorem 2.1 of Klebaner and Liptser [28], the uniform integrability of the family $(Z_T^n)_{n\in\mathbb{N}}$ follows by applying the de la Vallée-Poussin Theorem with $g(x)=x\log x$, $x\ge 0$, and showing that

$$\sup_n\mathbb{E}_P[g(Z_T^n)]<\infty.$$

We have

$$\begin{aligned}\mathbb{E}_P[g(Z_T^n)]&=\mathbb{E}_P[Z_T^n\log Z_T^n]\\&=\mathbb{E}_P\Big[Z_T^n\Big(\int_0^T 1_{\{\sigma_n\ge s\}}b_s^{(1)}dB_s^2-\frac{1}{2}\int_0^T 1_{\{\sigma_n\ge s\}}(b_s^{(1)})^2ds\Big)\Big]\\&\le\mathbb{E}_P\Big[Z_T^n(\int_0^T 1_{\{\sigma_n\ge s\}}b_s^{(1)}dB_s^2)\Big]=\mathbb{E}_{P_n}\Big[\int_0^T 1_{\{\sigma_n\ge s\}}b_s^{(1)}dB_s^2\Big],\end{aligned}$$

where the probability measure $P_n\approx P$ is defined by

$$\frac{dP_n}{dP}=Z_T^n.$$

Under P_n, the process $(B_t^n)_{t\ge0}$ given by

$$B_t^n=B_{t\wedge\sigma_n}^2-\int_0^{\sigma_n\wedge t}b_s^{(1)}ds,\quad t\in[0,T],$$

is a Brownian motion. Hence

$$\begin{aligned}\mathbb{E}_{P_n}\Big[\int_0^T 1_{\{\sigma_n\ge s\}}b_s^{(1)}dB_s^2\Big]&=\mathbb{E}_{P_n}\Big[\int_0^T 1_{\{\sigma_n\ge s\}}b_s^{(1)}dB_s^n\Big]+\mathbb{E}_{P_n}\Big[\int_0^T 1_{\{\sigma_n\ge s\}}(b_s^{(1)})^2ds\Big]\\&\le\mathbb{E}_{P_n}\Big[\int_0^T(b_s^{(1)})^2ds\Big]\le 2K^2\mathbb{E}_{P_n}\Big[\int_0^T(\mu_s^2+1)ds\Big]\\&=2K^2\Big(\int_0^T\mathbb{E}_{P_n}[r_s^4]ds+T\Big).\end{aligned}$$

Since r does not change its dynamics under the measure P_n and B^1 remains a Brownian Motion under P_n, we have that

$$\begin{aligned}\mathbb{E}_{P_n}[r_s^4] &= \mathbb{E}_{P_n}\Big[\Big(\bar{r}e^{-ms} + \zeta e^{-ms}\int_0^s e^{mu}dB_u^1\Big)^4\Big] \\ &= \bar{r}^4 e^{-4ms} + \frac{3\zeta^2\bar{r}^2}{m}(e^{-2ms} - e^{-4ms}) + \frac{3\zeta^4}{4m^2}(1-e^{-2ms})^2 := \psi(s),\end{aligned}$$

and therefore

$$\begin{aligned}\mathbb{E}_{P_n}\Big[\int_0^T 1_{\{\sigma_n \geq s\}} b_s^{(1)} dB_s^2\Big] &\leq 2K^2\Big(\int_0^T \mathbb{E}_{P_n}[r_s^4]ds + T\Big) \\ &= 2K^2\Big(\int_0^T \psi(s)ds + T\Big) < \infty.\end{aligned}$$

Hence

$$\sup_n \mathbb{E}_P[Z_T^n] < \infty,$$

and this implies that the family $(Z_T^n)_{n\in\mathbb{N}}$ is uniformly integrable.

□

The following proposition provides us with a general criterion for checking when an element of $\mathcal{M}_{loc}(W)$ is an element of $\mathcal{M}_{NUI}(W)$.

Proposition 5.8. *Let $Q \in \mathcal{M}_{loc}(W)$ and $P(0 < \sigma_1 < T) > 0$. If the process $(W_t^e)_{t\in[0,T]}$ is a Q-supermartingale with respect to the filtration $\mathbb{G}$, then $Q \in \mathcal{M}_{NUI}(W)$.*

Proof. Let $Q \in \mathcal{M}_{loc}(W)$ such that $(W_t^e)_{t\in[0,T]}$ is a $(\mathbb{G}, Q)$-supermartingale. Let $\epsilon > 0$ such that $P(\sigma_1 + \epsilon < T) > 0$ and $F_{\sigma_1+\epsilon} < p$ on $\{\sigma_1 + \epsilon > T\}$. We have

$$\begin{aligned}\mathbb{E}_Q[W_T] = \mathbb{E}_Q[W_T^e] = \mathbb{E}_Q[\mathbb{E}_Q[W_T^e|\mathcal{G}_{\sigma_1+\epsilon}]] &\leq \mathbb{E}_Q[W^e_{(\sigma_1+\epsilon)\wedge T}] \\ < \mathbb{E}_Q[W_{(\sigma_1+\epsilon)\wedge T}] &\leq \mathbb{E}_Q[W_0].\end{aligned}$$

Therefore W is a Q-strict local martingale on $[0, T]$. □

The following result shows that the assumptions of Proposition 3.10 hold in this context.

Proposition 5.9. *Let $Q \in \mathcal{M}_{loc}(W)$ be defined in (5.31), with Radon-Nikodym density*

$$Z_t = \frac{dQ}{dP}|_{\mathcal{F}_t}, \quad t \in [0,T].$$

Then $W^e Z$ is a P-submartingale.

Proof. By using Theorem 5.2 we have that W^e satisfies under P the equation

$$\begin{aligned} dW_t^e &= d\Lambda_t + R_t dH_t \\ &= \Lambda_{t-}(\xi_t^1 dB_t^1 + \sigma dB_t^2 - d\hat{M}_t + \xi_t^2 dt) + R_{t-}d\hat{M}_t + R_t\hat{\mu}_t dt \\ &= \Lambda_t(\xi_t^1 dB_t^1 + \sigma dB_t^2) + (cX_t - \Lambda_{t-})d\hat{M}_t + (\xi_t^2\Lambda_t + cX_t\hat{\mu}_t)dt \\ &= \Lambda_t(\xi_t^1 dB_t^1 + \sigma dB_t^2) + (cX_t - \Lambda_{t-})d\hat{M}_t, \end{aligned}$$

since by (5.9) we have

$$\xi_t^2 = -\frac{ce^{-\Gamma_t}\mu_t}{D_t}, \quad t \in [0,T],$$

and therefore by (5.12), we have P-a.s.

$$cX_t\hat{\mu}_t + \xi_t^2\Lambda_t = cX_t\mu_t 1_{\{\tau \geq t\}} - \frac{ce^{-\Gamma_t}\mu_t}{D_t}L_t X_t D_t = cX_t\mu_t 1_{\{\tau\geq t\}} - ce^{-\Gamma_t}\mu_t X_t L_t = 0.$$

By applying the integration by parts formula we obtain the canonical semimartingale decomposition of W^eZ

$$(W^eZ)_t = m_t + a_t, \quad t \in [0,T],$$

where the local martingale part $(m_t)_{t\in[0,T]}$ is given by

$$dm_t = Z_{t-}dW_t^e + W_{t-}^e dZ_t,$$

and the finite variation part $(a_t)_{t\in[0,T]}$ is equal to

$$\begin{aligned} da_t = d[W^e, Z]_t &= Z_t^{(2)} d[W^e, Z^{(1)}]_t + Z_{t-}^{(1)} d[W^e, Z^{(2)}]_t \\ &= \sigma b_t^{(1)} Z_t \Lambda_t dt + b_{t-}Z_{t-}(cX_t - \Lambda_{t-})dH_t. \end{aligned}$$

Hence, it follows from (5.26) that

$$a_t = a_0 + \int_0^t \sigma b_s^{(1)} Z_s \Lambda_s ds + \frac{1}{2} Z_\tau(\Lambda_{\tau-} - cX_\tau)1_{\{\tau\leq t\}},$$

Therefore $(a_t)_{t\in[0,T]}$ is an increasing process since $b_t^{(1)} > 0$ for all $t \in [0,T]$ and this implies that W^eZ is a P-local submartingale. To conclude the proof it is sufficient to show that the local martingale part m of W^eZ is a P-martingale. The process W^e satisfies the inequalities

$$0 < W_t^e = \mathbb{E}_P[X_T 1_{\{\tau>T\}} + cX_\tau 1_{\{\tau\leq T\}}|\mathcal{G}_t] \leq X_{t\wedge\tau}, \tag{5.35}$$

for all $t \in [0,T]$, since $c < 1$. Hence, it follows from (5.29) and (5.35) that

$$0 \leq W_t^e Z_t \leq \frac{3}{2} X_t^\tau Z_t^{(1)}.$$

Let $Y_t := Z_t^{(1)} X_t^\tau$. We have

$$\begin{aligned}
dY_t &= Z_t^{(1)} dX_t^\tau + X_t^\tau dZ_t^{(1)} + d[Z^{(1)}, X^\tau]_t \\
&= \sigma Z_t^{(1)} X_t^\tau dB_t^2 + b_t^{(1)} Z_t^{(1)} X_t^\tau dB_t^2 + \sigma b_t^{(1)} Z_t^{(1)} X_t^\tau dt \\
&= (\sigma + b_t^{(1)}) Y_t dB_t^2 + \sigma b_t^{(1)} Y_t dt.
\end{aligned}$$

Therefore Y is a P-local submartingale. However Y can be written under the following multiplicative decomposition

$$Y_t = M_t^Y A_t^Y,$$

where M^Y is a local martingale given by

$$M_t^Y = \exp\Big(\int_0^t (\sigma + b_s^{(1)}) dB_s^2 - \frac{1}{2} \int_0^t (\sigma + b_s^{(1)})^2 ds \Big), \quad t \in [0, T],$$

and A^Y is a continuous finite variation process given by

$$A_t^Y = \exp\Big(\int_0^t \sigma b_s^{(1)} ds \Big), \quad t \in [0, T].$$

With the same argument as in the proof of Proposition 5.7 with $\sigma + b^{(1)}$ instead of $b^{(1)}$, we obtain that $(M_t^Y)_{t \in [0,T]}$ is a P-martingale by the de la Vallée-Poussin Theorem. Therefore by $\sigma b_s^{(1)} > 0$ a.s. for all $s \geq 0$, we have

$$\mathbb{E}_P[Y_t | \mathcal{G}_s] = \mathbb{E}_P[M_t^Y A_t^Y | \mathcal{G}_s] \geq \mathbb{E}_P[M_t^Y | \mathcal{G}_s] A_s^Y = M_s^Y A_s^Y = Y_s,$$

which implies that $(Y_t)_{t \in [0,T]}$ is a P-submartingale, closed by Y_T, i.e

$$Y_t \leq \mathbb{E}_P[Y_T | \mathcal{G}_t], \quad t \in [0, T].$$

Since $0 \leq W^e Z_t \leq \frac{3}{2} Y_t$ for all $t \in [0, T]$, we have

$$\sup_{t \in [0,T]} \mathbb{E}_P[W_t^e Z_t] \leq \frac{3}{2} \sup_{t \in [0,T]} \mathbb{E}_P[Y_t] \leq \frac{3}{2} \mathbb{E}_P[Y_T] < \infty.$$

Furthermore, for every $\epsilon > 0$, there exists $\delta > 0$ such that if $A \in \mathcal{G}$ and $P(A) < \delta$ we have $\mathbb{E}_P[Y_t 1_A] < \epsilon$ for all $t \in [0, T]$. Hence

$$\mathbb{E}_P[W_t^e Z_t 1_A] \leq \frac{3}{2} \mathbb{E}_P[Y_t 1_A] < \epsilon.$$

Therefore $W^e Z$ is also a uniformly integrable local submartingale. Let $(\sigma_n)_{n \geq 0}$ be a localizing sequence for $W^e Z$. An application of Lebesgue's dominated convergence theorem yields

$$\begin{aligned}
\mathbb{E}_P[W_t^e Z_t | \mathcal{G}_s] &= \mathbb{E}_P[\lim_{n \to \infty} W_{t \wedge \sigma_n}^e Z_{t \wedge \sigma_n} | \mathcal{G}_s] = \lim_{n \to \infty} \mathbb{E}_P[W_{t \wedge \sigma_n}^e Z_{t \wedge \sigma_n} | \mathcal{G}_s] \\
&\geq \lim_{n \to \infty} W_{s \wedge \sigma_n}^e Z_{s \wedge \sigma_n} = W_s^e Z_s.
\end{aligned}$$

Hence $W^e Z$ is a P-submartingale.

$\square$

Corollary 5.10. *If $Q \in \mathcal{M}_{loc}(W)$ satisfies the assumptions of Proposition 5.9, then*

$$W_t^Q - W_t^e \geq 0,$$

for all $t \in [0,T]$

Proof. It follows by Propositions 3.10 and 5.9. □

The following proposition provides a sufficient condition to guarantee that the process W^e exhibits a R-supermartingale behavior on the interval $[0,T]$, under some measure $R \in \mathcal{M}_{loc}(W)$.

Proposition 5.11. *Let $R \in \mathcal{M}_{loc}(W)$ with Radon-Nikodym density process $Z = (Z_t)_{t\in[0,T]}$, where $Z_t = \frac{dR}{dP}|_{\mathcal{G}_t}$, for $t \in [0,T]$. We assume that the quadratic covariation process $[Z,M]$ is locally integrable.*
If the processes $b^{(i)}$, $i = 1,2,3$, in the representation (5.17) *of Z satisfy the inequality*

$$(\xi_t^1 b_t^{(1)} + \sigma b_t^{(2)})\Big(1 + \frac{c}{1-c}e^{-\alpha(t)-\beta(t)\mu_t}\Big) \leq b_t^{(3)}\mu_t, \tag{5.36}$$

on the set $\{\tau > t\}$, then W^e is a R-supermartingale on $[0,T]$. In particular $R \in \mathcal{M}_{NUI}(W)$.

Proof. It follows from the predictable version of the Girsanov theorem that W^e admits the following semimartingale decomposition under R

$$W_t^e = \mathcal{N}_t + \mathcal{A}_t, \quad t \in [0,T]$$

where the local martingale part $\mathcal{N} = (\mathcal{N}_t)_{t\in[0,T]}$ is given by

$$\mathcal{N}_t = W_t^e - \int_0^t \frac{1}{Z_s} d\langle Z, W^e\rangle, \quad t \in [0,T],$$

and the finite variation part $\mathcal{A} = (\mathcal{A}_t)_{t\in[0,T]}$ is equal to

$$\mathcal{A}_t = \int_0^t \frac{1}{Z_s} d\langle Z, W^e\rangle, \quad t \in [0,T].$$

We have

$$\mathcal{A}_t = \int_0^t \Big((\xi_s^1 b_s^{(1)} + \sigma b_s^{(2)})\Lambda_s + (cX_s - \Lambda_s) b_s^{(3)}\hat{\mu}_s\Big) ds.$$

Therefore $\mathcal{A}$ is decreasing if

$$(\xi_t^1 b_t^{(1)} + \sigma b_t^{(2)})\Lambda_t + (cX_t - \Lambda_t) b_t^{(3)}\hat{\mu}_t \leq 0, \quad t \in [0,T],$$

or equivalently

$$(\xi_t^1 b_t^{(1)} + \sigma b_t^{(2)})\Lambda_t \leq (\Lambda_t - cX_t) b_t^{(3)}\mu_t \tag{5.37}$$

on $\{\tau > t\}$. By replacing $\Lambda_t = L_t D_t X_t$, where $(D_t)_{t\in[0,T]}$ is defined in (5.13), (5.37) becomes

$$(\xi^1_t b^{(1)}_t + \sigma b^{(2)}_t) L_t D_t \leq (L_t D_t - c) b^{(3)}_t \mu_t$$

on $\{\tau > t\}$, or

$$(\xi^1_t b^{(1)}_t + \sigma b^{(2)}_t)\Big(1 + \frac{c}{1-c} e^{-\alpha(t)-\beta(t)\mu_t}\Big) \leq b^{(3)}_t \mu_t,$$

on the set $\{\tau > t\}$. Hence W^e is a local R-supermartingale. Since $W^e_t = \mathcal{N}_t + \mathcal{A}_t \geq 0$, this implies $\mathcal{N}_t \geq -\mathcal{A}_t$, for all $t \in [0,T]$. Therefore $\mathcal{N}$ is a positive local R-martingale, and via an application of Fatou's lemma, a R-supermartingale. This implies

$$\mathbb{E}_R[W^e_t|\mathcal{G}_s] = \mathbb{E}_R[\mathcal{N}_t|\mathcal{G}_s] + \mathbb{E}_R[\mathcal{A}_t|\mathcal{G}_s] \leq \mathcal{N}_s + \mathcal{A}_s,$$

for any $s,t \in [0,T]$, with $s \leq t$. Hence W^e is an R-supermartingale. It follows from Proposition 5.8 that $R \in \mathcal{M}_{NUI}(W)$. □

6 Conclusion

In this paper we present a mathematical model for the formation of bubbles in the valuation of defaultable claims in reduced-form credit risk framework. We propose a constructive definition of bubble, which is triggered by the impact of the credibility process on the defaultable claim's risk neutral valuation. This setting is very flexible and can be readapted to include the influence of other (macro-economic) factors, which may induce bubble birth. Moreover it is consistent with the no-arbitrage (NFLVR) framework, as we show in a specific example, where the default intensity is given by a Cox-Ingersoll-Ross model. We also study the connection of our approach with the martingale theory of bubbles and provide a characterization of $\mathcal{M}_{loc}(W)$, which shows how shifts in the martingale measure may be determined by changes in the dynamics of the market wealth in our setting.

References

[1] D. Abreu and M.K. Brunnermeier. Bubbles and crashes. *Econometrica*, 71(1):173–204, 2003.

[2] F. Biagini, H. Föllmer, and S. Nedelcu. Shifting martingale measures and the birth of a bubble as a submartingale. *Finance and Stochastics*, 18(2):297–326, 2014.

[3] F. Biagini and I. Schreiber. Risk-minimization for life insurance liabilities. *SIAM Journal on Financial Mathematics*, 4(1):243–264, 2013.

[4] T. Bielecki and M. Rutkowski. *Credit Risk: Modeling, Valuation and Hedging*. Springer, 2002.

[5] R. Bilina Falafala, R. Jarrow, and P. Protter. Change of numeraires and relative asset price bubbles. Preprint, Available at SSRN: http://ssrn.com/abstract=2265465 or http://dx.doi.org/10.2139/ssrn.2265465, 2014.

[6] M.K. Brunnermeier and M. Oehmke. Bubbles, financial crashes and systemic risk. In E.N. White, editor, *Handbook Of The Economics Of Finance*. Elsevier, 2013.

[7] D. Coculescu, M. Jeanblanc, and A. Nikeghbali. Default times, no-arbitrage conditions and changes of probability measures. *Finance and Stochastics*, 16(3):513–535, 2012.

[8] D. Coculescu and A. Nikeghbali. Hazard processes and martingale hazard processes. *Mathematical Finance*, 22(3):519–537, 2012.

[9] A.M.G. Cox and D.G. Hobson. Local martingales, bubbles and option prices. *Finance and Stochastics*, 9(4):477–492, 2005.

[10] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300:463–520, 1994.

[11] F. Delbaen and W. Schachermayer. A simple counter-example to several problems in the theory of asset pricing. *Mathematical Finance*, 8:1–12, 1998.

[12] J.B. DeLong, A. Shleifer, L. Summers, and R. Waldmann. Noise trader risk in financial markets. *Journal of Political Economy*, 98(4):703–738, 1990.

[13] D. Dufresne. The integrated square-root process. Working paper, University of Montreal, 2001.

[14] D. Filipovic. *Term Structure Models: A Graduate Course*. Springer Finance, 2009.

[15] H. Föllmer, U. Horst, and A. Kirman. Equilibria in financial markets with heterogeneous agents: A probabilistic perspective. *Journal of Mathematical Economics*, 41(1-2):123–155, 2005.

[16] H. Föllmer and A. Schied. *Stochastic Finance: An introduction in discrete time*. De Gruyter Graduate. Walter De Gruyter, 3rd edition, 2010.

[17] J.M. Harrison and D.M. Kreps. Speculative investor behavior in a stock market with heterogeneous expectations. *The Quarterly Journal of Economics*, 92(2):323–336, 1978.

[18] M. Hitsuda. Representation of gaussian processes equivalent to wiener process. *Osaka Journal of Mathematics*, 5:299–312, 1968.

[19] J Hugonnier. Rational asset pricing bubbles and portfolio constraints. *Journal of Economic Theory*, 147:2260–2302, 2011.

[20] J. Jacod and A. Shiryaev. *Limit Theorems for Stochastic Processes.* A Series of Comprehensive Studies in Mathematics. Springer, 2nd edition, 2003.

[21] R. Jarrow. Market manipulation, bubbles, corners and short-squeezes. *Journal of Financial and Quantitative Analysis*, 27(3):311–338, 1992.

[22] R. Jarrow, Y. Kchia, and P. Protter. How to detect an asset bubble. *SIAM Journal on Financial Mathematics*, 2:839–865, 2011.

[23] R. Jarrow and P. Protter. Forward and futures prices with bubbles. *International Journal of Theoretical and Applied Finance*, 12(7):901–924, 2009.

[24] R. Jarrow and P. Protter. Foreign currency bubbles. *Review of Derivatives Research*, 14(1):67–83, 2011.

[25] R. Jarrow, P. Protter, and A Roch. A liquidity based model for asset price bubbles. *Quantitative Finance*, 12(9):1339–1349, 2012.

[26] R. Jarrow, P. Protter, and K. Shimbo. Asset price bubbles in complete markets. *Advances in Mathematical Finance*, Applied and Numerical Harmonic Analysis(Part II), 2007.

[27] R. Jarrow, P. Protter, and K. Shimbo. Asset price bubbles in incomplete markets. *Mathematical Finance*, 20(2):145–185, 2010.

[28] F. Klebaner and R. Liptser. Whena a stochastic exponential is a true martingale. extension of the benes method. *Theory of Probability and its Applications*, 58(1):38–62, 2014.

[29] M. Loewenstein and G.A. Willard. Rational equilibrium asset-pricing bubbles in continuous trading models. *Journal of Economic Theory*, 91(1):17–58, 2000.

[30] E.M. Miller. Risk, uncertainty, and divergence of opinion. *The Journal of Finance*, 32(4):1151–1168, 1977.

[31] P. Protter. *Stochastic Integration and Differential Equations (2nd edition).* Volume 21 of Applications of Mathematics. Springer-Verlag Berlin Heidelberg, 2004.

[32] P. Protter. A mathematical theory of financial bubbles. In V. Henderson and R. Sincar, editors, *Paris-Princeton Lectures on Mathematical Finance*, volume 2081 of *Lecture Notes in Mathematics*. Springer, 2013.

[33] J. Scheinkman. *Speculation, Trading and Bubbles*. Kenneth J. Arrow Lecture Series. Columbia University Press, 2014.

[34] J. Scheinkman and W. Xiong. Overconfidence and speculative bubbles. *Journal of Polical Economy*, 111(6):1183–1219, 2003.

[35] A. Schleifer and R.W. Vischny. The limits of arbitrage. *The Journal of Finance*, 52(1):35–55, 1997.

[36] C. Stricker and J.A. Yan. Some remarks on the optional decomposition theorem. *Seminaire de probabilites(Strasbourg)*, 32:56–66., 1998.

[37] J. Tirole. On the possibility of speculation under rational expectations. *Econometrica*, 53(6):1163–1182, 1982.