Bootstrap percolation in directed and inhomogeneous random graphs

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Abstract

Bootstrap percolation is a process that is used to model the spread of an infection on a given graph. In the model considered here each vertex is equipped with an individual threshold. As soon as the number of infected neighbors exceeds that threshold, the vertex gets infected as well and remains so forever. In this paper we perform a thorough analysis of bootstrap percolation on a novel model of directed and inhomogeneous random graphs, where the distribution of the edges is specified by assigning two distinct weights to each vertex, describing the tendency of it to receive edges from or to send edges to other vertices. Under the assumption that the limiting degree distribution of the graph is integrable we determine the typical fraction of infected vertices. Our model allows us to study settings that were outside the reach of current methods, in particular the prominent case in which the degree distribution has an unbounded variance. Among other results, we quantify the notion of "systemic risk", that is, to what extent local adverse shocks can propagate to large parts of the graph through a cascade, and discover novel features that make graphs prone/resilient to initially small infections.

1 Introduction

In this paper we study *bootstrap percolation*, which is a classical mathematical model that is used to describe how a certain activity disperses on a given finite graph. In the "bare bones" variant of the model, one starts with a non-empty subset of the vertices, the so-called *initially infected set*. The process continues in distinct rounds, and further vertices become active as soon as they have at least a certain fixed number $c \in \mathbb{N}$ of infected neighbors. After a finite number of steps the set of infected vertices will eventually stabilize, and the important question is to quantify its shape as a function of the underlying graph and the initially infected set.

The study of bootstrap percolation has a rather long history, beginning with its invention in 1979 in [9], where it was used to investigate the demagnetisation properties of certain crystals. Since then, many important statistics of it have been studied in a broad variety of different settings, including for example the breakthrough result [7] in the case where the underlying

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graph is the d-dimensional finite grid $[n]^d$, and the extensive study in the case of Erdos-Rényi random graphs [8]. From today's perspective, however, the underlying graphs that we wish to study are more complex and heterogeneous, and the details of the infection process are more intricate. Let us mention briefly two characteristic examples that will motivate the definition of our model:

- *Financial networks.* The vertices are financial institutions (like banks or insurance companies) and the edges describe dependencies between them, for example a loan from one bank to another. If some institutions go bankrupt, then this may result in a cascade of credit defaults, depending on how much each remaining institution can withstand.
- *Social networks.* The vertices are individuals, who exchange information through announcing messages; in *Twitter* for example, the users may broadcast a message to all of their followers. They, in turn, can broadcast it further, resulting again in a cascade of message transmitions.

The graphs in the two previous examples, as well as many others that appear in a variety of similar contexts, have three relevant characteristics. First, they are *heterogeneous*, in the sense that the degree distribution (i.e., the probability that a uniformly random vertex has a given number of neighbors) is far from uniform – it typically has a heavy tail. This has been verified empirically in a vast number of studies [1, 11, 2]. Second, the graphs are *directed*, meaning that the induced relation among the vertices is not necessarily symmetric. Finally, each vertex has an *individual threshold level* according to which it becomes infected; some vertices are more sensitive to activity in their neighborhood than others.

The model that we propose and study in this paper encompasses these characteristics. It contains two basic ingredients: a model for random directed graphs, where the degree distribution regarding both incoming and outgoing edges can be prescribed, and a model for bootstrap percolation, where each vertex has its own individual infection threshold. The main results of this paper include an extensive study with respect to all parameters, and we will sketch here the contribution, while leaving the details to the following sections.

Let $n \in \mathbb{N}$. In our random graph model each vertex $i \in [n] = \{1, \dots, n\}$ is associated with three parameters $w_i^-, w_i^+ \in \mathbb{R}_+$ and $c_i \in \mathbb{N}_0$. The first parameter w_i^- quantifies the tendency of i to receive edges from other vertices, and similarly w_i^+ quantifies its tendency to connect to other vertices. In particular, the probability that the *directed* edge (i, j), where $i \neq j$, is in the graph, is given by

$$p_{i,j} = \min\left\{1, w_i^+ w_j^- / n\right\}.$$

Moreover, all these events are assumed to be independent. This model is a generalization of the popular Chung-Lu model [10], see also [21] and the extensive study [8], to the setting of directed graphs. We show that if the joint empirical distribution of the weight sequences $(w_i^-, w_i^+)_{i\in[n]}$ converges to the distribution of an integrable random variable (W^-, W^+), then the resulting in-degree and out-degree sequences are close to a bivariate mixed Poisson distribution with mixing variable (W^-, W^+). Random directed graphs that were proposed prior to our work are based on the configuration model, where the *actual* in- and out degrees are specified for each vertex, see e.g. [4, 12]. Despite being quite powerful and sufficient in many situations, these models suffer from the disadvantage that the resulting graphs are simple, meaning that they have no self-loops and multiple edges, only if the degree sequence fulfills a second moment condition [18]. This condition is invalid in many settings that are considered here and that appear in several intended applications, where the degree sequence is so heavy tailed that it has a variance that grows with the number n of vertices – the most prominent case is a power-law distribution with exponent $2 < \beta < 3$, a situation that arises frequently in observed real-world networks [2, 11].

The third set of parameters in our model, i.e., the quantities $(c_i)_{i\in[n]}$, describe the sensitiveness of the vertices with respect to activity in their neighborhoods. Our bootstrap percolation process on a given graph G with vertex set $[n]$ is a deterministic procedure that works as follows. There is an initially infected set of vertices given by

$$\mathcal{A}_0 = \{i \in [n] \mid c_i = 0\}.$$

For a vertex i let $N_G^-(i) = \{j \in [n] \mid (j, i) \text{ is an edge of } G\}$ be the in-neighborhood of i in G. In the k-th generation, where $k \in \mathbb{N}$, the infection spreads to the set $\mathcal{A}_k$ given by

$$\mathcal{A}_k = \big\{i \in [n] \mid |N_G^-(i) \cap \mathcal{A}_{k-1}| \geq c_i\big\}.$$

That is, as soon as there are c_i infected in-neighbors of i, that vertex gets infected as well and remains so forever. A straightforward consequence of this definition is that the sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ stabilizes after at most $n-1$ generations. We say that $\mathcal{A}_n$ is the set of *finally infected vertices*. The main result of this paper, Theorem 4.1, establishes in several relevant cases the typical value of $|\mathcal{A}_n|$ if the underlying graph is a directed inhomogeneous random graph in the setting described previously, and if the *joint* empirical distribution of $(w_i^-, w_i^+, c_i)_{i\in[n]}$ converges to the distribution of a random vector (W^-, W^+, C), where, as before, we assume that (W^-, W^+) is integrable. Our setting is thus as general as possible, and it allows in particular for intricate correlation structures among the in-/out-degrees of the vertices and the infection threshold, which are expected to exist in many natural models; for example, in specific settings it is certainly expected that vertices with high degrees have also a higher threshold, as might be the case in a financial network. Related settings with varying infection thresholds have been considered before, in particular in the notable works [20, 3, 4]; however, in this papers the underlying model for the graph is the configuration model, in which it is not possible to study the prominent case in which the degrees have a variance that is not bounded. Here we overcome this shortcoming by using the more flexible model of random directed inhomogeneous graphs.

Applications Our results enable us to study a variety of explicit scenarios, for example the case $\mathbb{P}(C = 0) \to 0$, i.e., where the initially infected vertices constitute only an insignificant fraction of all vertices in the graph. This is a particularly important situation, as in typical applications cascades are triggered by only a small number of vertices in the network – in a financial crisis, for example, a small number of market participants defaults initially, but this may have a severe effect on a huge part of the network. In particular, we determine under which conditions on (W^-, W^+, C) the finally infected set contains a (large) *positive* fraction of all vertices that is *independent* of the probability of being initially infected, as long as this is > 0. Our results thus put us in a position to quantify the notion of "systemic risk" by determining to what extent local adverse shocks can propagate to large parts of the system through a cascade.

Our formal description of "systemic risk" allows us also to discover novel features in networks that make them prone to initially small infections. In particular, in previous works [15, 4] the same bootstrap percolation process was studied on random graphs that are generated according to the configuration model, where the sequence of degrees has a bounded second moment. There it was shown that an initially small infection can propagate to a big part of the graph *if and only if* the subgraph induced by the so-called *contagious* edges is large; an edge (i, j) is contagious if j has an infection threshold equal to one. Moreover, especially in the context of financial mathematics the key concept in studying the effect of initial defaults is precisely that of contagious edges [14, 16]. Our results are in stark contrast to that; that is, our analysis reveals that in general, even if $c_i \neq 1$ for all $i \in [n]$ and $\mathbb{P}(C = 0) > 0$ arbitrarily small, the initial shock can propagate to large parts of the system. The ultimate reason for this dramatic effect is the rapidly expanding structure of the graph if the second moment of the degree distribution is not bounded. We present several natural examples demonstrating this effect.

As further applications, in a forthcoming paper [13] we will elaborate more on the implications of our results for the financial system. A similar phenomenon in the restricted setting of undirected graphs with power-law degree sequence and with constant activation threshold was observed in [5, 6].

Paper & Proof Outline The paper is structured as follows. In Section 2 we propose our random graph model, state the required regularity properties and derive the asymptotic shape of the degree sequence. In a second step we associate with each vertex $i \in [n]$ its sensitivity c_i.

In Section 3 we consider vertex sequences that approximate random vectors (W^-, W^+, C) supporting only *finitely* many values. We reformulate the infection process in a sequential form such that at each time-step only the infection from one vertex is considered. This reformulation allows us to approximate the dynamics of the system with differential equations, using the method in [23], to derive a law of large number for the process in the sequential description. This yields a lower bound for the size of $\mathcal{A}_n$. Further, if the fixpoint of a certain functional of (W^-, W^+, C) is stable, we can determine the exact size of $\mathcal{A}_n$. The use of this method is quite common for such problems, see e.g. [6], but in our context the application is both conceptually and technically complex due to the three-dimensional nature of our parameter space. Moreover, since the functions defining the differential equations are only Lipchitz continuous on a domain smaller than the one of interest, we use a novel probabilistic argument to show that the activations outside the considered domain are negligible, given the stability of the fixpoint.

In Section 4, we extend our results to the general setting by developing several couplings of the original vertex sequence to tailor-made finitary sequences. Difficulties arise here due to the multi-dimensionality of the vector (W^-, W^+, C) and the fact that it is eventually unbounded. Further, based on these generalizations we can study the case $\mathbb{P}(C = 0) \to 0$ in full generality. Section 5.2 provides examples of possible specifications for (W^-, W^+, C). We investigate the stability of the fixpoint and the case $\mathbb{P}(C = 0) \to 0$ and/or $\mathbb{P}(C = 1) = 0$.

2 Directed Inhomogeneous Random Graphs

In this section we introduce the directed inhomogeneous random graph and study some basic properties, as the in- and out degrees of single vertex and the degree sequence. Further, in Section 2.2, we equip each vertex in the graph with a percolation threshold, which determines the number of neighbors of a vertex $i \in [n]$ that need to be infected such that i is infected itself.

2.1 The Model

For each $n \in \mathbb{N}$ we consider the vertex set $[n] := \{1, \ldots, n\}$ and the set of directed edges $E := \{(i, j) \mid i, j \in [n], i \neq j\}$. Let $\Omega := \{0, 1\}^{|E|}$ and $\mathcal{F} := 2^{\Omega}$. We define a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ in the following way. To each vertex $i \in [n]$ we assign two deterministic weights w_i^- and $w_i^+ \in \mathbb{R}_+$ and define the probability $p_{i,j} = p_{i,j}(n)$ for $i \neq j$ that there is a directed edge from vertex i to vertex j by

$$p_{i,j} = \min\{1, w_i^+ w_j^- / n\}. \tag{2.1}$$

Furthermore, we assume that the events that an edge is present happens independent of the presence of all other edges. The role of w_i^- respectively w_i^+ is to determine the tendency of vertex $i \in [n]$ to have incoming respectively outgoing edges. Let further $(\mathbf{w}^-)(n) = (w_1^-(n), \ldots, w_n^-(n))$ and $(\mathbf{w}^+)(n) = (w_1^+(n), \ldots, w_n^+(n))$ be the in- and out weight sequences. Observe that all the quantities including $\mathbb{P}, \Omega$ and $\mathcal{F}$ depend on n. However, to simplify notation we often neglect this dependency in the notation.

We denote the resulting random graph by $G_n(\mathbf{w}^-, \mathbf{w}^+)$ and by $X_{i,j} = X_{i,j}(n)$ the indicator function that there is a directed edge from vertex i to vertex j. Furthermore, define the in-degree D_i^- and out degree D_i^+ of vertex $i \in [n]$ by

$$D_i^- = \sum_{j \neq i} X_{j,i} \quad \text{and} \quad D_i^+ = \sum_{j \neq i} X_{i,j}. \tag{2.2}$$

For a pair of in- and out weight sequences $(\mathbf{w}^-, \mathbf{w}^+)$ we define their empirical distribution by

$$F_n(x,y) = n^{-1} \sum_{i \in [n]} \mathbf{1}\{w_i^-(n) \leq x, w_i^+(n) \leq y\}, \quad \forall x, y \in [0, \infty). \tag{2.3}$$

Let in the following (W_n^-, W_n^+) be a random vector with distribution function $F_n(x,y)$. We shall pose some mild assumptions on the weight sequences.

Definition 2.1 (**Regular Weight Sequence).** We say that the pair of weight sequences $(\mathbf{w}^-, \mathbf{w}^+)$ is *regular*, if it satisfies the following conditions:

1. **Convergence of weights**: There exists a distribution function $F : [0,\infty) \times [0,\infty) \to [0,1]$ such that for all (x,y) where F is continuous, $\lim_{n\to\infty} F_n(x,y) = F(x,y)$.

2. **Convergence of average weights:** Let (W_F^-, W_F^+) be a random variable with distribution F. Then $\lim_{n\to\infty} \mathbb{E}[(W_n^-, W_n^+)] = \mathbb{E}[(W_F^-, W_F^+)] = (\lambda^-, \lambda^+)$ for some $0 < \lambda^-, \lambda^+ < \infty$.

It can easily be seen that for a regular pair of weight sequences $\max_{i\in[n]} w_i^+ = o(n)$ and $\max_{i\in[n]} w_i^- = o(n)$. We shall use this observation frequently. We assume in the following that we are given a regular pair of weight sequences $(\mathbf{w}^-, \mathbf{w}^+)$ with limiting distribution function F. The regularity property allows to prove the following theorem about the total number of edges in the graph, Theorem 2.5, a result about the in- and out degree of a single vertex in $[n]$, and Theorem 2.4 about the sequence of in- and out degrees in the graph.

Theorem 2.2. *Denote by $e(G_n(\mathbf{w}^-, \mathbf{w}^+))$ the number of edges in $G_n(\mathbf{w}^-, \mathbf{w}^+)$. Then*

$$n^{-1} e(G_n(\mathbf{w}^-, \mathbf{w}^+)) \xrightarrow{p} \lambda^- \lambda^+. \tag{2.4}$$

Proof. We first calculate $\lim_{n\to\infty} \mathbb{E}[e(G_n(\mathbf{w}^-, \mathbf{w}^+))]$. Since $\mathbb{E}[X_{i,j}] = p_{i,j}$

$$\mathbb{E}[e(G_n(\mathbf{w}^-, \mathbf{w}^+))] \leq \sum_{i,j\in[n], i\neq j} \frac{w_i^+ w_j^-}{n} \leq \sum_{i\in[n]} w_i^+ \sum_{j\in[n]} \frac{w_j^-}{n}. \tag{2.5}$$

By Definition 2.1, $n^{-1}\sum_{j\in[n]} w_j^- = \lambda^- + o(1)$ and $n^{-1}\sum_{j\in[n]} w_j^+ = \lambda^+ + o(1)$. This implies that the right hand side of (2.5) equals $n(\lambda^+\lambda^- + o(1))$. To derive a lower bound for $\mathbb{E}[e(G_n(\mathbf{w}^-, \mathbf{w}^+))]$, note that in order to have $w_i^+ w_j^- > n$ and the minimization to 1 in (2.1) to be relevant, at least one of the two factors w_i^+ and w_j^- has to be greater than $\sqrt{n}$. So,

$$\mathbb{E}[e(G_n(\mathbf{w}^-, \mathbf{w}^+))] \geq n^{-1} \sum_{i\in[n], w_i^+ \leq \sqrt{n}} w_i^+ \sum_{j\in[n]\setminus\{i\}, w_i^- \leq \sqrt{n}} w_j^-. \tag{2.6}$$

By $\lim_{n\to\infty}\mathbb{E}[W_n^-]=\lambda^-$ it follows that

$$\lim_{n\to\infty} n^{-1}\sum_{w_j^->\sqrt{n}} w_j^- = \lim_{n\to\infty}\mathbb{E}[W_n^-\mathbf{1}_{W_n^->\sqrt{n}}]=0.$$

This, together with the same argument for the sum involving the w_i^+ and the fact that $\max_{i\in[n]} w_i^- = o(n)$ shows that the right hand side of (2.6) equals $n(\lambda^+\lambda^-+o(1))$ and therefore

$$\lim_{n\to\infty} n^{-1}\mathbb{E}[e(G_n(\mathbf{w}^-,\mathbf{w}^+))]=\lambda^+\lambda^-.$$

Since $e(G_n(\mathbf{w}^-,\mathbf{w}^+))$ is the sum of independent indicator functions, it follows that

$$\mathrm{Var}(e(G_n(\mathbf{w}^-,\mathbf{w}^+)))\le\mathbb{E}[e(G_n(\mathbf{w}^-,\mathbf{w}^+))]$$

and applying the second moment method establishes the claim. □

Before stating the next theorem we need the following definition (see [17] for a treatment of uni-variate mixed Poisson distributions)

Definition 2.3. Multivariate mixed Poisson distribution: A random vector $X=(X_1,\dots,X_n)$ has a mixed Poisson distribution with mixing distribution F_Y, if for every $k=(k_1,...,k_n)\in\mathbb{N}_0^n$,

$$\mathbb{P}(X=k)=\mathbb{E}\left[\prod_{1\le i\le n} e^{-Y_i}\frac{Y_i^{k_i}}{k_i!}\right], \tag{2.7}$$

where $Y=(Y_1,\dots,Y_n)$ is a random vector with distribution function F_Y.

In the following we denote by $\mathrm{Poi}(Y)$ a random vector having a mixed Poisson distribution with mixing vector Y. It can be easily seen that $\mathbb{E}[\mathrm{Poi}(Y_i)]=\mathbb{E}[Y_i]$ and $\mathrm{Cov}(\mathrm{Poi}(Y))=\mathrm{Cov}(Y)+\mathrm{Diag}(\mathbb{E}[Y_1],\dots,\mathbb{E}[Y_n])$, where $\mathrm{Diag}(a_1,\dots,a_n)$ denotes the matrix with entries $a_1,\dots,a_n$ on the diagonal and zero elsewhere. The following Theorems 2.4 and 2.5 are directed versions of known results for undirected inhomogeneous random graphs (see [22, Thm. 6.7., Cor. 6.9] or [8, Thm. 3.13]).

Let $P_n(k,j)$ be the random distribution function defined by

$$P_n(k,j) \quad = \quad n^{-1}\sum_{i\in[n]}\mathbf{1}_{\{D_i^-=k,D_i^+=j\}},\quad \forall k,j\in\mathbb{N}_0. \tag{2.8}$$

Theorem 2.4. *Let $p_n(k,j)$ be the probability mass function of the mixed Poisson random variable $(Poi(W_F^-\lambda^+),Poi(W_F^+\lambda^-))$ given by*

$$p(k,j)=\mathbb{E}\left[e^{\left(-(W_F^-\lambda^++W_F^+\lambda^-)\right)}\frac{(W_F^-\lambda^+)^k(W_F^+\lambda^-)^j}{k!j!}\right]. \tag{2.9}$$

Then for all $\epsilon>0$

$$\mathbb{P}\left(\sum_{k,j}|p(k,j)-P_n(k,j)|>\epsilon\right)\to 0 \tag{2.10}$$

as $n\to\infty$.

Proof. Using Theorem 2.5 below, the proof in [8, Thm. 3.13] or [22, Thm. 6.10.] can be applied with some minor changes reflecting the in- and out weights and the minimization with respect to 1 in (2.1). ☐

Theorem 2.5. *Let $k \in [n]$. There exists a coupling (D_k^-, Z_k^-) and (D_k^+, Z_k^+) of D_k^- and D_k^+ respectively, where Z_k^- and Z_k^+ are Poisson random variables with parameters $w_k^-\lambda^+$ and $w_k^+\lambda^-$ such that*

$$\begin{aligned} \mathbb{P}(D_k^- \neq Z_k^-) &\leq \left((w_k^-)^2 + w_k^-\right) o(1) &(2.11)\\ \mathbb{P}(D_k^+ \neq Z_k^+) &\leq \left((w_k^+)^2 + w_k^+\right) o(1). &(2.12)\end{aligned}$$

Proof. We provide the proof for D_k^- only, as the argument for D_k^+ is similar. By definition, and similar considerations as in the proof of Theorem 2.2 we can define random variables $\overline{D}_k^-$ and $\underline{D}_k^-$ by

$$\underline{D}_k^- := \sum_{j\in[n]\setminus k, w_k^- w_j^+ \leq n} Be\left(\frac{w_k^- w_j^+}{n}\right) \quad \text{and} \quad \overline{D}_k^- := \sum_{j\in[n]} Be\left(\frac{w_k^- w_j^+}{n}\right). \tag{2.13}$$

such that $\underline{D}_k^- \preceq D_k^- \preceq \overline{D}_k^-$, where $\preceq$ denotes stochastic ordering. Consider Poisson random variables $\underline{V}_k^-$ and $\overline{V}_k^-$ with parameters

$$n^{-1} \sum_{j\in[n]\setminus k, w_k^- w_j^+ \leq n} w_k^- w_j^+ \quad \text{and} \quad n^{-1}\sum_{j\in[n]} w_k^- w_j^+ \tag{2.14}$$

respectively. It follows that

$$\mathbb{P}(\underline{D}_k^- \neq \underline{V}_k^-) \leq \sum_{j\in[n]} \frac{(w_j^+)^2(w_k^-)^2}{n^2} \leq (w_k^-)^2 \max_{i\in[n]} w_i^+ \sum_{j\in[n]} \frac{(w_j^+)}{n^2} = (w_k^-)^2 o(1), \tag{2.15}$$

since $\max_{i\in[n]} w_i^+ = o(n)$ and by Definition 2.1. The same estimate holds with $\underline{D}_k^-$ and $\underline{V}_k^-$ replaced by $\overline{D}_k^-$ and $\overline{V}_k^-$. To complete the proof we will couple $\overline{V}_k^-$ and $\underline{V}_k^-$ to a Poisson random variable Z_k^- with parameter $w_k^-\lambda^+$.

Define $\overline{\eta}_k^-$ and $\underline{\eta}_k^-$ by

$$\overline{\eta}_k^- := w_k^-\lambda^+ - \sum_{j\in[n]} \frac{w_k^- w_j^+}{n} \text{ and } \underline{\eta}_k^- := \sum_{w_k^- w_j^+ > n} \frac{w_k^- w_j^+}{n}. \tag{2.16}$$

Then by Definition 2.1 $\overline{\eta}_k^- = w_k^- o(1)$ and $\underline{\eta}_k^- = w_k^- o(1)$. If $\overline{\eta}_k^- > 0$, let $\overline{Y}_k^-$ be a Poisson distributed random variable with parameter $\overline{\eta}_k^-$ and we define $Z_k^- := \overline{Y}_k^- + \overline{V}_k^-$. If $\overline{\eta}_k^- < 0$, we may assume that $\overline{V}_k^-$ is the sum of two independent Poisson distributed random variables Z_k^- and $\overline{Y}_k^-$ with parameters $w_k^-\lambda^+$ and $-\overline{\eta}_k^-$. In any case Z_k^- is Poisson distributed with parameter $w_k^-\lambda^+$ and

$$\mathbb{P}(Z_k^- \neq \overline{V}_k^-) = \mathbb{P}(\mathrm{Poi}(|\overline{\eta}_k^-|) \geq 1) \leq \mathbb{E}[\mathrm{Poi}(|\overline{\eta}_k^-|)] = |\overline{\eta}_k^-|, \tag{2.17}$$

due to Markovs inequality. By a similar observation we find that $\mathbb{P}(\underline{V}_k^- \neq \overline{V}_k^-) \leq \underline{\eta}_k^-$. Observing that,

$$\begin{aligned}\mathbb{P}(D_k^- \neq Z_k^-) &\leq \mathbb{P}(\underline{D}_k^- \neq Z_k^-) + \mathbb{P}(\overline{D}_k^- \neq Z_k^-) \\ &\leq \mathbb{P}(\underline{D}_k^- \neq \underline{V}_k^-) + \mathbb{P}(\underline{V}_k^- \neq \overline{V}_k^-) + \mathbb{P}(Z_k^- \neq \overline{V}_k^-) + \mathbb{P}(\overline{D}_k^- \neq \overline{V}_k^-)\end{aligned}$$

the claim follows by combining the considerations above. □

2.2 Bootstrap Percolation with Infection Thresholds

In addition to the weights, we assume that each vertex $i \in [n]$ is associated with an infection threshold c_i. The vertex i becomes infected after c_i of the vertices that have a directed edge to it are infected. We allow for vertices that can never be infected and assume that they have threshold ∞, a choice done for convenience. Set $\mathbb{N}^\infty := \mathbb{N} \cup \{\infty\}$ and $\mathbb{N}_0^\infty := \mathbb{N}_0 \cup \{\infty\}$. As in the case of the weights, we assume that we are given a threshold sequence $\mathbf{c}(n) = (c_1(n), \dots, c_n(n)) \in (\mathbb{N}_0^\infty)^n$.

Definition 2.6 (Regular Vertex Sequence). Let $(\mathbf{w}^-, \mathbf{w}^+)$ be a regular pair of weight sequences and $\mathbf{c}$ a percolation threshold. We call $(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$ a *regular vertex sequence* if there exists a distribution function $G : \mathbb{R} \times \mathbb{R} \times \mathbb{N}_0^\infty \to [0, 1]$ such that for all points $(x, y, l) \in \mathbb{R} \times \mathbb{R} \times \mathbb{N}_0^\infty$ for which $G_l(x, y) := G(x, y, l)$ is continuous in (x, y) we have $\lim_{n\to\infty} G_n(x, y, l) = G(x, y, l)$, where $G_n(x, y, l)$ is the empirical distribution function defined by

$$G_n(x, y, l) = n^{-1} \sum_{i\in[n]} \mathbf{1}\{w_i^-(n) \leq x, w_i^+(n) \leq y, c_i(n) \leq l\}, \quad \forall (x, y, l) \in \mathbb{R} \times \mathbb{R} \times \mathbb{N}_0^\infty. \tag{2.18}$$

Note that in contrast to Definition 2.1 of a regular weight sequence we do not pose any integrability assumptions on the threshold value.

Given a directed graph and the threshold sequence, a bootstrap percolation process is triggered by the initial set infected vertices $\mathcal{A}_0 := \{i \in [n] \mid c_i = 0\}$, and in the $k-$th generation, the infection is spread to the set

$$\mathcal{A}_k := \Big\{ i \in [n] \mid c_i \leq \sum_{j\in\mathcal{A}_{k-1}} X_{j,i} \Big\}. \tag{2.19}$$

One can easily see that after at most $n - 1$ rounds, the chain $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \dots$ stabilizes and $\mathcal{A}_{n-1} = \mathcal{A}_n$. We call $\mathcal{A}_n$ the final set of infected vertices.

3 Bootstrap percolation for finitary vertex type sequences

In this section we study bootstrap percolation in directed inhomogeneous random graphs with so-called finitary vertex sequences that are defined below. We extend the results later in Section 4 by approximating the general weight sequences by a finitary ones.

Definition 3.1. (Regular finitary vertex sequence) We call a regular vertex sequence $(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$ *finitary* if there exist positive integers $l_1, l_2, c_{\max} \in \mathbb{N}$ such that the following conditions are satisfied.

1. There exist *weight levels* $0 < \tilde{w}_1^- < \tilde{w}_2^- < \cdots < \tilde{w}_{l_1}^-$ and $0 < \tilde{w}_1^+ < \tilde{w}_2^+ < \cdots < \tilde{w}_{l_2}^+$ such that $\forall i \in [n]$, $w_i^- \in \cup_{j=1}^{l_1} \{\tilde{w}_j^-\}$ and $w_i^+ \in \cup_{j=1}^{l_2} \{\tilde{w}_j^+\}$, that is, the weights take only finitely many values.
2. $\forall i \in [n]$, either $c_i \leq c_{\max}$ or $c_i = \infty$.

Observe that for a finitary vertex sequence there exists a partition of $[n]$ given by

$$[n] = \bigcup_{\substack{0\leq j\leq l_1, 0\leq k\leq l_2\\ m\in[c_{\max}]\cup\{\infty\}}} I_{j,k;\,m}\ , \tag{3.1}$$

into subsets with constant threshold and in and out weights, i.e. $I_{j,k;m} := \{i \in [n] \mid (w_i^-, w_i^+) = (\tilde{w}_j^-, \tilde{w}_k^+), c_i = m\}$. Furthermore due to the regularity, there exist $\gamma_{j,k;m}$ with $1 = \sum \gamma_{j,k;m}$ such that $|I_{j,k;m}| = \gamma_{j,k;m} n(1 + o(1))$.

For $r \in \mathbb{N}_0^\infty$ let $\psi_r(x)$ denote the probability that a Poisson distributed random variable with parameter $x \geq 0$ is at least r, i.e.,

$$\psi_r(x) := \begin{cases} \mathbb{P}(\mathrm{Poi}(x) \geq r) = \sum_{j=r}^\infty \frac{x^j}{j!} e^{-x}, & r \geq 0 \\ 0 & r = \infty. \end{cases} \tag{3.2}$$

Before we state the main theorem of this section we define some functions that will play a crucial role. Let $(X, Y, Z) : \Omega \to \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}_0^\infty$ be a random variable defined on some probability space Ω such that $\mathbb{E}[Y] < \infty$. Define the function $f : \mathbb{R}_+ \to \mathbb{R}$ by

$$f(z; (X, Y, Z)) := \mathbb{E}[Y \psi_Z(Xz)] - z. \tag{3.3}$$

Further, define the function $g : \mathbb{R}_+ \to \mathbb{R}$ by

$$g(z; (X, Z)) := \mathbb{E}[\psi_Z(Xz)]. \tag{3.4}$$

In the proof of Theorem 3.3 we shall make use of the following simple property.

Lemma 3.2. *The functions $f(z; (X, Y, Z))$ and $g(z; (X, Z))$ are continuous. Furthermore, the equation $f(z; (X, Y, Z)) = 0$ has a solution $\hat{z} \in [0, \mathbb{E}[Y]]$.*

Proof. For continuity, first observe that $Y\psi_Z(Xz)$ is continuous point-wise, which implies

$$\lim_{h\to 0} |Y(\psi_Z(X(z + h)) - \psi_Z(X(z)))| = 0.$$

Furthermore, $|Y(\psi_Z(X(z + h)) - \psi_Z(X(z)))|$ is bounded by Y, and by assumption $\mathbb{E}[Y] < \infty$. The Dominated Convergence Theorem yields that

$$\lim_{h\to 0} |\mathbb{E}[Y\psi_Z(X(z + h))] - \mathbb{E}[Y\psi_Z(X(z))]| = 0,$$

from which continuity of f follows. By a similar argument, the continuity of g follows.

For the solution $f(z; (X, Y, Z)) = 0$, first observe that $f(0; (X, Y, Z)) \geq 0$ and

$$\mathbb{E}[Y\psi_Z(Xz)] \leq \mathbb{E}[Y],$$

for all $z \in \mathbb{R}$ and therefore $f(\mathbb{E}[Y]; (X, Y, Z)) \leq 0$, which implies that a smallest solution exists by the continuity of $f(z; (X, Y, Z))$. □

Denote by $G_n(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$ the directed random graph with vertex sequence $(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$. The following theorem about the size of the final set of infected vertices for finitary vertex sequences is the main result of this section. We shall extend it to non-finitary vertex sequences in Section 4 by approximating more general vertex sequences by finitary ones.

Theorem 3.3. *Let $(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$ be a finitary regular vertex sequence with limiting distribution (W^-, W^+, C) and $\mathbb{P}(C = 0) > 0$. Let further $\hat{z}$ be the smallest positive solution of*

$$f(z; (W^-, W^+, C) \stackrel{!}{=} 0.$$

Let $\mathcal{A}_n$ be the set of infected vertices at the end of the infection process in $G_n(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$. Then the following holds:

1. *For all $\epsilon > 0$ with high probability:*

$$n^{-1} |\mathcal{A}_n| \geq \mathbb{E}[\psi_C(W^- \hat{z})] - \epsilon. \tag{3.5}$$

2. *If $f'(\hat{z}; (W^-, W^+, C) < 0$, then*

$$n^{-1} |\mathcal{A}_n| \xrightarrow{p} \mathbb{E}[\psi_C(W^- \hat{z})], \text{ as } n \to \infty. \tag{3.6}$$

Proof. We first show 1. We partition $[n]$ as in Definition 3.1. We shall determine the final size of the set of infected vertices by sequentially exposing the neighbors of all vertices that are either infected initially or become infected during the process. At each step only a single infected vertex $i \in [n]$ is considered and its neighbors are exposed. If a neighbors of i becomes infected due to the new edge that is send from i it is added to the set of unexposed infected vertices and will be exposed at a later step. Otherwise, its threshold value is reduced by 1.

During this process we need to keep track of the size of the set of unexposed, infected vertices, its total outgoing weight and the sizes of the sets of vertices with a given percolation threshold and number of infected neighbors. In step t, let

$$I^l_{j,k;m}(t) := \{i \in I_{j,k;m} | \; i \text{ has } l \text{ exposed, infected neighbors at step } t\}$$

for $0 \leq j \leq l_1, 0 \leq k \leq l_2, 0 \leq l < c_{\max}$ and let $U(t)$ be the set of infected, unexposed vertices. The set $U(0)$ contains the initially infected vertices. Furthermore, let $\overline{U}(t)$ be the set of newly infected vertices at time step t, a set we only need temporary in each step to define the procedure outlined below. The sets $I^l_{j,k;m}(0)$ and the set $\overline{U}(0)$ are empty. At step $t \geq 1$ we update the sets in the following way:

1. A vertex $v \in U(t-1)$ is chosen uniformly at random.
2. Expose all neighbors of v in the sets $I^l_{j,k;m}(t-1)$ to determine the sets $I^l_{j,k;m}(t)$ and $\overline{U}(t)$.
3. Set $U(t) := (U(t-1) \setminus \{v\}) \cup \overline{U}(t)$.

The above steps are repeated until step $\hat{t}$, the first time the set $U(t)$ is empty. Note that $\hat{t}$ is the final number of infected vertices and further, that during step t a vertex $w \in I^l_{j,k;m}(t-1)$ is either placed in $I^{l+1}_{j,k;m}(t)$ or in $I^l_{j,k;m}(t)$ for $l \leq m-2$, and in $\overline{U}(t)$ or $I^l_{j,k;m}(t)$ for $l = m-1$, depending on whether there is a directed edge from v to w or not. Denote by $c^l_{j,k;m}(t)$ the size of $I^l_{j,k;m}(t)$ and by $u(t)$ the size of $U(t)$. Additionally we need to keep track of the total out going weights in the set $U(t)$, which we shall denote by $w(t)$. Further we use $h(t)$ to describe the state of the entire system, that is $h(t) = (u(t), w(t), c^l_{j,k;m}(t))$. First observe that for n sufficiently large, we can ignore the minimization in (2.1), since $w^+_i w^-_i$ is bounded for finitary weight sequences and the denominator in (2.1) is n. Conditioning on the weight of the selected vertex and using the law of total expectation one obtains that the expected evolution of the system is governed by the following equations:

$$
\begin{aligned}
\mathbb{E}[u(t)-u(t-1)|h(t-1)] &= -1+n^{-1}\sum_j\left(\sum_{k,m}c^{m-1}_{j,k;m}(t-1)\right)\frac{\tilde{w}^-_j w(t-1)}{u(t-1)},\\
\mathbb{E}[w(t)-w(t-1)|h(t-1)] &= -\frac{w(t-1)}{u(t-1)}\\
&+ n^{-1}\sum_k \tilde{w}^+_k\left(\sum_{j,m}c^{m-1}_{j,k;m}(t-1)\frac{\tilde{w}^-_j w(t-1)}{u(t-1)}\right),
\end{aligned}
$$

and for the index set $0\le j\le l_1, 0\le k\le l_2, 1\le m< c_{\max}$ and $0\le l<m$:

$$
\mathbb{E}[c^l_{j,k;m}(t)-c^l_{j,k;m}(t-1)|h(t-1)] = n^{-1}\left(\mathbf{1}_{\{l\neq 0\}}c^{l-1}_{j,k;m}(t-1)-c^l_{j,k;m}(t-1)\right)\left(\frac{\tilde{w}^-_j w(t-1)}{u(t-1)}\right).
$$

We will approximate the quantities $h(t)/n$ using the method proposed in [23] by functions $(\nu(\tau),\mu(\tau),\gamma^l_{j,k;m}(\tau))$ solving the following system of ordinary differential equations:

$$
\frac{\mathrm{d}\gamma^l_{j,k;m}(\tau)}{\mathrm{d}\tau} = \left(\mathbf{1}_{\{l\neq 0\}}\gamma^{l-1}_{j,k;m}(\tau)-\gamma^l_{j,k;m}(\tau)\right)\left(\frac{\tilde{w}^-_j\mu(\tau)}{\nu(\tau)}\right), \tag{3.7}
$$

$$
\frac{\mathrm{d}\nu(\tau)}{\mathrm{d}\tau} = -1+\sum_j\left(\sum_{k,m}\gamma^{m-1}_{j,k;m}(\tau)\right)\frac{\tilde{w}^-_j\mu(\tau)}{\nu(\tau)}, \tag{3.8}
$$

$$
\frac{\mathrm{d}\mu(\tau)}{\mathrm{d}\tau} = -\frac{\mu(\tau)}{\nu(\tau)}+\sum_k\tilde{w}^+_k\left(\sum_{j,m}\gamma^{m-1}_{j,k;m}(\tau)\frac{\tilde{w}^-_j\mu(\tau)}{\nu(\tau)}\right), \tag{3.9}
$$

with initial conditions

$$
\nu(0) = \mathbb{P}(C=0), \tag{3.10}
$$

$$
\mu(0) = \sum_k\tilde{w}^+_k\mathbb{P}(W^+=\tilde{w}^+_k, C=0), \tag{3.11}
$$

$$
\gamma^0_{j,k;m}(0) = \mathbb{P}(W^-=\tilde{w}^-_j, W^+=\tilde{w}^+_k, C=m), \tag{3.12}
$$

$$
\gamma^l_{j,k;m}(0) = 0,\ \text{for } 0<l<m. \tag{3.13}
$$

For $\delta_1,\delta_2>0$ we consider the domain

$$
\begin{aligned}
D_{\delta_1,\delta_2}=\Big\{(\tau,\nu,\mu,\gamma^l_{j,k;m})\in\mathbb{R}^{b+1}\ \ |\ \ &-\delta_1<\tau<1, -\delta_1<\frac{\mu}{\nu}<2\tilde{w}^+_{l_2}, -\delta_1<\gamma^l_{j,k;m}<\gamma_{j,k;m},\\
&\delta_2<\nu<1, 0<\mu<2\tilde{w}^+_{l_2}\Big\}.
\end{aligned} \tag{3.14}
$$

The system (3.7)-(3.8) fulfills a Lipschitz condition on D_{δ_1,δ_2} for $\delta_2>0$ as can be easily seen by calculating the partial derivatives. Further, in order to apply [23, Thm. 1] we have to show that there are functions $\lambda(n)=\lambda$ with $\lambda\to\infty$ and $\omega(n)=\omega$ such that $\lambda^4\log n<\omega<n^{2/3}/\lambda$ and

$$
\mathbb{P}\left(\|h(t+1)-h(t)\|_{\max}>\frac{\sqrt{\omega}}{\lambda^2\sqrt{\log n}}|h(t)\right)=o(n^{-3}). \tag{3.15}
$$

If one chooses $\lambda(n) = n^{1/8}$ and $\omega(n) = B^2 n^{1/96}$ where $B > 0$ is a constant, then

$$\frac{\sqrt{\omega}}{\lambda^2\sqrt{\log n}} = Bn^{19/24}/\sqrt{\log n}, \tag{3.16}$$

and it remains to show that the maximal degree is bounded by $n^{1/96}/\sqrt{\log n}$, since due to the bounded weights, B can be chosen such that (3.15) holds. This is done in Lemma 3.4. According to [23, Thm. 1] we get

$$\begin{aligned} c^l_{j,k;m}(t)/n &= \gamma^l_{j,k;m}(t/n) + o_p(1) &(3.17)\\ u(t)/n &= \nu(t/n) + o_p(1) &(3.18)\\ w(t)/n &= \mu(t/n) + o_p(1) &(3.19)\end{aligned}$$

where $\gamma^l_{j,k;m}(\tau), \nu(\tau)$ and $\mu(\tau)$ solve (3.7)-(3.9) and where (3.17)-(3.19) holds until the solution leaves D_{δ_1,δ_2}. Since δ_2 can be chosen arbitrarily close to 0, it is clear that the solution can be extended to the region $D_{\delta_1,0}$.

An easy but tedious calculation shows that the solutions ν and μ are given by

$$\nu(\tau) = \nu(0) - \tau + \sum_{j,k}\left(\sum_m \gamma^0_{j,k;m}(0)\mathbb{P}\left[\mathrm{Poi}\left(\tilde{w}^-_j \int_0^\tau \frac{\mu(s)}{\nu(s)}\mathrm{d}s\right) \geq m\right]\right) \tag{3.20}$$

and

$$\mu(\tau) = \mu(0) - \int_0^\tau \frac{\mu(s)}{\nu(s)}\mathrm{d}s + \sum_{j,k,m} \tilde{w}^+_k \gamma^0_{j,k;m}(0)\mathbb{P}\left[\mathrm{Poi}\left(\tilde{w}^-_j \int_0^\tau \frac{\mu(s)}{\nu(s)}\mathrm{d}s\right) \geq m\right]. \tag{3.21}$$

Define

$$\tau_{D_{\delta_1,\delta_2}} = \min\{\tau|(\tau, \nu(\tau), \mu(\tau), \gamma^l_{j,k;m}(\tau)) \notin D_{\delta_1,\delta_2})\} \tag{3.22}$$

Observe that $f(z(\tau)) = \mu(\tau)$ with $z(\tau) = \left(\int_0^\tau \frac{\mu(s)}{\nu(s)}\mathrm{d}s\right)$ for $\tau < \tau_{D_{\delta_1,0}}$. Since $\left(\int_0^\tau \frac{\mu(s)}{\nu(s)}\mathrm{d}s\right)$ is strictly increasing in τ as long as $(\tau, \nu(\tau), \mu(\tau), \gamma^l_{j,k;m}(\tau)) \in D_{\delta_1,0}$ the function $z(\tau)$ is injective. We need to ensure that we can chose δ_2 small enough such that the process can be approximated arbitrarily close to $\hat{\tau}$, which is such that $z(\hat{\tau}) = \left(\int_0^{\hat{\tau}} \frac{\mu(s)}{\nu(s)}\mathrm{d}s\right)$ equals $\hat{z}$, the first zero of f.

Therefore we need to show that for any given ϵ we can chose δ_2 small enough such that there exists $\tau_\epsilon < \tau_{D_{\delta_1,\delta_2}}$ with

$$\hat{z} - \epsilon < \int_0^{\tau_\epsilon} \frac{\mu(s)}{\nu(s)}\mathrm{d}s. \tag{3.23}$$

Since $\hat{z}$ is assumed to be the first zero of f and since f is continuous on the compact set $[0, \hat{z}-\epsilon]$ it attains its minimum at some point $z_{\min} \in [0, \hat{z} - \epsilon]$. Further, observe that we must have $\nu(\tau) \geq \mu(\tau)/\tilde{w}^+_{l_2}$ such that choosing

$$0 < \delta_2(\epsilon) < f(z_{\min})/\tilde{w}^+_{l_2} \tag{3.24}$$

ensures that there exists $\tau_\epsilon < \tau_{D_{\delta_1,\delta_2(\epsilon)}}$ such that the inequality in (3.23) holds and the convergence in (3.17) holds at least until τ_ϵ. Since ϵ can be chosen arbitrarily close to 0, we can conclude that $\mu(\tau)$ converges to 0 as $z(\tau)$ approaches $\hat{z}$ by continuity of f. Because $\tilde{w}^+_1\nu(\tau) \leq \mu(\tau) \leq \tilde{w}^+_{l_2}\nu(\tau)$ on $D_{\delta_1,0}$ we know that also $\nu(\tau)$ converges to 0. For any given ϵ we get that

$$u(\lfloor \tau_\epsilon n \rfloor)/n = \nu(\tau_\epsilon) + o_p(1). \tag{3.25}$$

From $\nu(\hat{\tau}) = 0$ it follows that

$$\begin{aligned}
\hat{\tau} &= \nu(0) + \sum_{j,k} \left(\sum_m \gamma^0_{j,k;m}(0) \mathbb{P}\left[\mathrm{Poi}\left(\tilde{w}^-_j \int_0^{\hat{\tau}} \frac{\mu(s)}{\nu(s)} \mathrm{d}s \right) > m \right] \right) \\
&= \nu(0) + \sum_{j,k} \left(\sum_m \gamma^0_{j,k;m}(0) \mathbb{P}\left[\mathrm{Poi}\left(\tilde{w}^-_j \hat{z} \right) > m \right] \right) \\
&= \mathbb{E}[\psi_C(W^- \hat{z})]. \qquad (3.26)
\end{aligned}$$

Since $|\mathcal{A}_n|/n \geq \hat{\tau} + o_p(1)$, the claim follows.

In order to prove 2. we need to show that the process $u(t)$ becomes zero soon after the $\tau_\epsilon n$ steps or equivalently, that the remaining infections triggered by vertices in $U(\lfloor \tau_\epsilon n \rfloor)$ are negligible. We shall expose all remaining vertices in U at once and bound the number of infections triggered by $U(\lfloor \tau_\epsilon n \rfloor)$. Denote by $\mathcal{W} := \bigcup_{j,k,m} I^{m-1}_{j,k;m}$ the set of (weak) vertices that need only one more infected neighbor to become infected and by $\mathcal{S} := \bigcup_{j,k,m \geq l+2} I^l_{j,k;m}$ the set of (strong) vertices that need at least two more infected vertices to become infected. Further, denote $\mathcal{N}_l \subset \mathcal{W} \cup \mathcal{S}$ the set of vertices that become infected in the lth round after exposing $U(\tau_\epsilon n)$ and define

$$\begin{aligned}
\mathcal{W}_l &:= \mathcal{W} \cap \mathcal{N}_l \\
\mathcal{S}_l &:= \mathcal{S} \cap \mathcal{N}_l.
\end{aligned}$$

First observe that, since (W^-, W^+, C) takes only finitely many values, differentiation under the integral sign can be justified and one easily observes that f is continuously differentiable in $(0, \infty)$. By Equation (3.9) and the fact that $f'(\hat{z}) = \kappa$ with $\kappa < 0$ by assumption and by continuity of f' one easily observes that we can chose ϵ_κ such that

$$\left(\sum_{j,k,m} \gamma^{m-1}_{j,k;m}(\tau_\epsilon) \tilde{w}^-_j \tilde{w}^+_k \right) \leq 1 + \kappa/2 < 1$$

for $\epsilon < \epsilon_\kappa$. Further observe that $\sum_{j,k,m} \gamma^{m-1}_{j,k;m}(\tau_\epsilon)$ is bounded by 1 for $\epsilon < \epsilon_\kappa$. Set

$$c_1 := \max\{1 + \kappa/2, \sum_{j,k,m} \gamma^{m-1}_{j,k;m}(\tau_{\epsilon_\kappa})\} < 1$$

and chose $0 < c_2, c < 1$ such that $0 \leq c_1 + c_2 \leq c < 1$. Further define

$$\begin{aligned}
C_1 &:= \max\{\tilde{w}^-_{l_1}, 1\} \\
C_2 &:= \max\{\left(\tilde{w}^-_{l_1}\right)^2, (\tilde{w}^-_{l_1} \tilde{w}^+_{l_2})^2\} \frac{C_1^2}{1-c} \\
C_3 &:= \max\{\tilde{w}^-_{l_1} \tilde{w}^+_{l_2}, 1\}
\end{aligned}$$

and chose x_0 such that

$$C_2 x^2 \leq ((c_1 c_2 c)/C_3) x \qquad (3.27)$$

for $0 < x \leq x_0$. We shall prove by induction that for $\epsilon < \epsilon_0$

$$\begin{aligned}
\mathbb{E}[|\mathcal{W}_l|]/n &\leq c_1 c^{l-1} \mu(\tau_\epsilon) C_1 \qquad (3.28) \\
\mathbb{E}[|\mathcal{S}_l|]/n &\leq C_2 \mu(\tau_\epsilon)^2 c^{l-2} \leq (c_2/C_3) c^{l-1} \mu(\tau_\epsilon). \qquad (3.29)
\end{aligned}$$

The estimates (3.28) and (3.29) especially imply that

$$\mathbb{E}[|\mathcal{W}_l| + |\mathcal{S}_l|]/n \leq c^l C_1 \mu(\tau_\epsilon)$$

and

$$\left(\sum_l \mathbb{E}[|\mathcal{W}_l| + |\mathcal{S}_l|]\right)/n \leq \frac{C_1}{1-c}\mu(\tau_\epsilon). \tag{3.30}$$

For $l = 1$ observe that for a vertex $x \in \mathcal{I}^1$ the probability to become infected by vertices in $U(\tau_\epsilon n)$ is bounded by

$$\tilde{w}_{l_1}^- \left(\mu(\tau_\epsilon) + o_p(1)\right)$$

such that

$$\mathbb{E}[|\mathcal{W}_1|]/n \leq \sum_{j,k,m} \gamma_{j,k;m}^{m-1}(\tau_\epsilon)\tilde{w}_{l_1}^- \left(\mu(\tau_\epsilon) + o_p(1)\right) \leq c_1\mu(\tau_\epsilon)C_1. \tag{3.31}$$

Further, the probability for a vertex to be in $\mathcal{S}_1$ is bounded by

$$\left(\tilde{w}_{l_1}^- \left(\mu(\tau_\epsilon) + o_p(1)\right)\right)^2$$

and thus choosing $\epsilon < \epsilon_0$ such that $\mu(\tau_\epsilon) < x_0$ yields $\mathbb{E}[|\mathcal{S}_1|]/n \leq C_2\mu(\tau_\epsilon)^2 \leq c_2/C_3\mu(\tau_\epsilon)$ by definition of C_2 and x_0 for $n \geq n_0$ fulfilling $\mu(\tau_\epsilon) + o_p(1) < x_0$.

Assume now that (3.28) and (3.29) hold for $1 \leq k \leq l$. For a vertex $x \in \mathcal{W}$ to be in $\mathcal{W}_{l+1}$ it needs to have a least one neighbor in $\mathcal{N}_{l-1}$. We shall show the slightly stricter recursion for $\mathcal{W}_l$, namely

$$\frac{1}{n}\left(\sum_{x\in\mathcal{W}} w_x^-\right)\sum_{y\in\mathcal{W}\cup\mathcal{S}} \frac{w_y^+\mathbb{P}(y\in\mathcal{N}_l)}{n} \leq c_1c^l\mu(\tau_\epsilon)C_1$$

from which clearly (3.28) follows (note that for $l = 1$ this was captured in (3.31) already) since

$$\mathbb{E}[|\mathcal{W}_{l+1}|] \leq \sum_{x\in\mathcal{W},y\in\mathcal{W}\cup\mathcal{S}} \mathbb{P}(\{xy\})\mathbb{P}(y\in\mathcal{N}_l).$$

First observe that

$$\begin{aligned}
& \frac{1}{n}\left(\sum_{x\in\mathcal{W}} w_x^-\right)\sum_{y\in\mathcal{W}\cup\mathcal{S}} \frac{w_y^+\mathbb{P}(y\in\mathcal{N}_l)}{n} \\
= & \frac{1}{n}\sum_{x\in\mathcal{W},y\in\mathcal{W}} \mathbb{P}(\{xy\})\mathbb{P}(y\in\mathcal{W}_l) + \frac{1}{n}\sum_{x\in\mathcal{W},y\in\mathcal{S}} \mathbb{P}(\{xy\})\mathbb{P}(y\in\mathcal{S}_l) \\
\leq & \frac{1}{n}\left(\sum_{x\in\mathcal{W}} w_x^-\right)\sum_{y\in\mathcal{W}} \frac{w_y^+\mathbb{P}(y\in\mathcal{W}_l)}{n} + \frac{1}{n}\tilde{w}_{l_1}^-\tilde{w}_{l_2}^+\mathbb{E}[|\mathcal{S}_l|] \\
\leq & \frac{1}{n}\left(\sum_{x\in\mathcal{W}} w_x^-\right)\left(\sum_{z\in\mathcal{W}} \frac{w_z^+w_z^-}{n}\right)\sum_{y\in\mathcal{W}\cup\mathcal{S}} \frac{w_y^+\mathbb{P}(y\in\mathcal{N}_{l-1})}{n} + \frac{1}{n}\tilde{w}_{l_1}^-\tilde{w}_{l_2}^+\mathbb{E}[|\mathcal{S}_l|].
\end{aligned} \tag{3.32}$$

The middle factor in the first summand is bounded by c_1 by definition of c_1. The induction step then implies that (3.32) is bounded by

$$c_1c_1c^{l-1}\mu(\tau_\epsilon)C_1 + C_3C_2\mu(\tau_\epsilon)^2c^{l-2} \leq c_1c^l\mu(\tau_\epsilon)C_1.$$

To calculate $\mathbb{E}[|\mathcal{S}_{l+1}|]$, we first observe that for a vertex to be in $\mathcal{S}_{l+1}$ it needs to have at least one neighbor in $\mathcal{N}_l$ and one in $\cup_{k\le l}\mathcal{N}_k$. Using $\mathbb{E}[|\mathcal{W}_k|] = \sum_{x\in\mathcal{W}} \mathbb{P}(x \in \mathcal{W}_k)$ and $\mathbb{E}[|\mathcal{S}_{l+1}|] = \sum_{x\in\mathcal{S}} \mathbb{P}(x \in \mathcal{S}_{l+1})$ for $k \le l$ we find

$$\mathbb{E}[|\mathcal{S}_{l+1}|] \le n(\tilde{w}^-_{l_1}\tilde{w}^+_{l_2}/n)^2 \sum_{x\in\mathcal{W}\cup\mathcal{S}} \mathbb{P}(x \in \mathcal{N}_l) \sum_{x\in\mathcal{W}\cup\mathcal{S}} \mathbb{P}(x \in \cup_{k\le l}\mathcal{N}_k).$$

However, by the induction step we know that

$$\sum_{x\in\mathcal{W}\cup\mathcal{S}} \mathbb{P}(x \in \mathcal{N}_l) \le nc^l \mu(\tau_\epsilon) C_1$$

and

$$\sum_{x\in\mathcal{W}\cup\mathcal{S}} \mathbb{P}(x \in \cup_{k\le l}\mathcal{N}_k) \le \sum_{k\le l} nc^k C_1 \mu(\tau_\epsilon) \le n\frac{C_1}{1-c}\mu(\tau_\epsilon),$$

yielding

$$n^{-1}\mathbb{E}[|\mathcal{S}_{l+1}|] \le C_2 c^{l-1}\mu(\tau_\epsilon)\mu(\tau_\epsilon) \le (c_2/C_3)c^l\mu(\tau_\epsilon),$$

proving (3.29). By (3.30) and Markov's inequality we get

$$\mathbb{P}\left(n^{-1}\sum_l(|\mathcal{W}_l| + |\mathcal{S}_l|) \ge \sqrt{\frac{C_1}{1-c}\mu(\tau_\epsilon)}\right) \le \sqrt{\frac{C_1}{1-c}\mu(\tau_\epsilon)}. \tag{3.33}$$

Since $|\mathcal{A}_n|/n \le \hat{\tau} + \left(\sum_l |\mathcal{W}_l| + |\mathcal{S}_l|\right)/n$, the claim in (3.6) follows from (3.33) together with (3.26). □

In the proof of the last theorem we needed the following simple result about the maximum degree in the random graph.

Lemma 3.4. *Let $D^- := \max_{i\in[n]} D^-_i$ and $D^+ := \max_{i\in[n]} D^+_i$ the maximal degrees in $G_n(\mathbf{w}^-, \mathbf{w}^+)$, where $(\mathbf{w}^-, \mathbf{w}^+)$ is finitary. Then $\mathbb{P}(D^+, D^- \ge n^{1/100}) = o(n^{-3})$.*

Proof. We shall prove the bound for D^-, the bound for D^+ is analogue. Let $w := \max\{\tilde{w}^-_{l_1}, \tilde{w}^+_{l_2}\}$ and observe that

$$\mathbb{P}(D^- \ge k) \le n\binom{n-1}{k}\left(\frac{w^2}{n}\right)^k \le n\frac{w^{2k}}{k!}.$$

The proof is completed by noting that for $k \ge n^{1/100}$ and large n we have $k! \ge n^5 w^{2k}$. □

4 General Vertex Sequences

In this section we show that Theorem 3.3 extends to non-finitary vertex sequences.

Theorem 4.1. *Let $(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$ be a regular vertex sequence with limiting distribution $G : \mathbb{R}\times\mathbb{R}\times\mathbb{N}_0^\infty \to [0,1]$ such that $G_l(x, y) := G(x, y, l)$ is continuous for each $l \in \mathbb{N}$ and there exists some $x_0 > 0$ such that $G(x, y, l) = 0$ for $\min\{x, y\} < x_0$, that is, the in and out weights W^- and W^+ are bounded from below away from zero. Further, let $\mathbb{P}(C = 0) > 0$. Denote by $\hat{z}$ the smallest positive solution of*

$$f(z; (W^-, W^+, C)) \stackrel{!}{=} 0, \tag{4.1}$$

with f as defined in (3.3). Then the following holds:

1. *For all $\epsilon > 0$ with high probability:*

$$n^{-1}|\mathcal{A}_n| \geq \mathbb{E}[\psi_C(W^-\hat{z})] - \epsilon. \tag{4.2}$$

2. *If $f'(\hat{z}; (W^-, W^+, C) < 0$, then*

$$n^{-1}|\mathcal{A}_n| \xrightarrow{p} g(\hat{z}; (W^-, C)) = \mathbb{E}[\psi_C(W^-\hat{z})], \text{ as } n \to \infty. \tag{4.3}$$

We use the results of the last section to approximate general vertex sequences by two finitary ones in a tailor-made way. The approximation is such that one sequence ultimately generates a graph that gives a lower bound for the final fraction of infected vertices, while the second sequence generates a graph that provides an upper bound. A sandwich type argument then allows us to determine $|\mathcal{A}_n|$ in Section 4.4. For this argument we first have to show that the functions defined in (3.4) and (3.3) depend continuously on the random variables involved. For bounded domains, the necessary results are provided by Helly's theorem, which states that if a sequence of distribution functions G^i converges point-wise to G and the function h is continuous and bounded on some bounded domain D, then $\int_D h\mathrm{d}G^i \to \int_D h\mathrm{d}G$. However, to investigate convergence of the functions defined in (3.4) and (3.3) we are faced with an unbounded domain and unbounded support. Therefore Helly's theorem cannot be applied and we shall use a tailor-made approximation of G that ensures that the integrals over several functions relevant in the following analysis are convergent.

Let (W^-, W^+, C) be a random vector with distribution function G fulfilling the properties of Theorem 4.1. To avoid confusion we use the expectation operator $\mathbb{E}$ only with respect to the measure defined by G on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}_0^\infty$. For the approximating measures we use the integral notation. For the approximation we can restrict to the set $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}_0 \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}_0^\infty$, since all involved functions are based on $\psi_r(x)$ and are zero for $r = \infty$. Define the sets

$$D_i := \{(x, y, l) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}_0 \mid x, y, l \leq i\}, D_\infty := \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}_0 \quad \text{and} \quad D_i^c := \mathbb{R}_+^3 \backslash D_i. \tag{4.4}$$

Definition 4.2. Let H be a set of functions such that each $h \in H$ maps from $\mathbb{R} \times \mathbb{R} \times \mathbb{N}_0$ to $\mathbb{R}$. A sequence $\{G^i\}_{i \in \mathbb{N}}$ of distribution functions defined on $\mathbb{R} \times \mathbb{R} \times \mathbb{N}_0$ is called G-convergent with respect to H if

1. $\forall\, (x, y, l) \in \mathbb{R} \times \mathbb{R} \times \mathbb{N}_0, G^i(x, y, l) \to G(x, y, l)$, as $i \to \infty$.
2. For each $h \in H$, $\int_{D_\infty} h(x, y, l)\mathrm{d}G(x, y, l) < \infty$ and uniformly over H:
   (a) $\lim_{i\to\infty} \int_{D_i^c} h(x, y, l)\mathrm{d}G^i(x, y, l) = 0$ and $\lim_{i\to\infty} \int_{D_i^c} h(x, y, l)\mathrm{d}G(x, y, l) = 0$,
   (b) $\lim_{i\to\infty} \left| \int_{D_i} h(x, y, l)\mathrm{d}G^i(x, y, l) - \int_{D_i} h(x, y, l)\mathrm{d}G(x, y, l) \right| = 0$.

Property 1 is the usual convergence in distribution. To understand Property 2a, note that from $\int_{D_\infty} h(x, y, l)\mathrm{d}G(x, y, l) < \infty$ it follows trivially that $\lim_{i\to\infty} \int_{D_i^c} h(x, y, l)\mathrm{d}G(x, y, l) = 0$. Property 2a ensures that as $i \to \infty$ the tail probabilities of the measure implied by G^i are decreasing fast enough such that integrals over functions $h \in H$ still have this convergence property. Property 2b is concerned with the approximation in the center. Its crucial point is that the integration domain is becoming larger, since as mentioned above for a fixed domain D, the convergence property holds automatically, at least for continuous functions. The assumption that the convergence in Property 2a and 2b is uniform will simplify the following analysis.

For our purpose the functions G^i will be the limiting distribution functions of finitary vertex sequences $(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i$ for which Theorem 3.3 holds. With increasing integer i the granularity is increasing, that is, there are more weight levels in the sequence and at the same time the approximated range will become larger. The construction of the sequence is done in Section 4.2 and incorporates the set H, which will contain functions with unbounded support and therefore Helly's theorem does not allow to conclude convergence of the integral. We shall first show some simple implications of Definition 4.2, which we need later on.

4.1 Convergence of some relevant functions

We saw already in Theorem 3.3 that the functions g and f defined in (3.3) and (3.4) play a crucial role in determining the final fraction of defaults. We shall show in Proposition 4.4 that the convergence stated in Definition 4.2 ensures a certain convergence of these quantities. To prove Proposition 4.4 we need the following lemma.

Lemma 4.3. *Let $\{G^i\}_{i\in\mathbb{N}}$ be a G-convergent sequence of distribution functions with respect to H. Let the function $h : [z_{min}, z_{max}] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}_0 \to \mathbb{R}$ be such that the functions $h_l(z, x, y) := h(z, x, y, l)$ are continuous for all l and $h(z, x, y, l) \in H$ for each $z \in [z_{min}, z_{max}]$.*

Let $\bar{z} \in [z_{min}, z_{max}]$. Then, there exist two sequences $\{\varepsilon_i\}_{i\in\mathbb{N}}$ with $\varepsilon_i > 0$ and $\lim_{i\to\infty} \varepsilon_i = 0$ and $\{\delta_i\}_{i\in\mathbb{N}}$ with $\delta_i > 0$ respectively, independent of $\bar{z}$, such that for any $z_i \in [z_{min}, z_{max}]$ with $|\bar{z} - z_i| \leq \delta_i$,

$$\left|\int_{D_\infty} h_l(z_i, x, y)\mathrm{d}G^i(x, y, l) - \mathbb{E}[h_C(\bar{z}, W^-, W^+)]\right| \quad < \quad \varepsilon_i. \tag{4.5}$$

Proof. Using the triangle inequality we find that

$$\left|\int_{D_\infty} h_l(z_i, x, y)\mathrm{d}G^i(x, y, l) - \mathbb{E}[h_C(\bar{z}, W^-, W^+)]\right| \tag{4.6}$$

$$\leq \left|\int_{D_\infty} h_l(z_i, x, y)\mathrm{d}G^i(x, y, l) - \int_{D_\infty} h_l(\bar{z}, x, y)\mathrm{d}G^i(x, y, l)\right| \tag{4.7}$$

$$+ \left|\int_{D_\infty} h_l(\bar{z}, x, y)\mathrm{d}G^i(x, y, l) - \int_{D_\infty} h_l(\bar{z}, x, y)\mathrm{d}G(x, y, l)\right|. \tag{4.8}$$

We first bound the term in (4.8). Let the sequence $\{a_i\}_{i\in\mathbb{N}}$ be defined by

$$a_i = \sup_{\bar{z}\in[z_{min},z_{max}]} \max\left\{\left|\int_{D_i^c} h_l(\bar{z}, x, y)\mathrm{d}G^i(x, y, l)\right|, \left|\int_{D_i^c} h_l(\bar{z}, x, y)\mathrm{d}G(x, y, l)\right|\right\}$$

and observe that $\lim_{i\to\infty} a_i = 0$ by Property 2. of Definition 4.2. Using that $h(\bar{z}, x, y, l) \in H$ it follows that $\lim_{i\to\infty} b_i = 0$ with b_i defined by

$$b_i := \sup_{\bar{z}\in[z_{min},z_{max}]} \left|\int_{D_i} h_l(\bar{z}, x, y)\mathrm{d}G^i(x, y, l) - \int_{D_i} h_l(\bar{z}, x, y)\mathrm{d}G(x, y, l)\right|.$$

Using the triangular inequality once more, and observing that $D_\infty = D_i \cup D_i^c$ shows that the term in (4.8) is bounded by $2a_i + b_i$.

Consider now the term in (4.7) and observe that by linearity of the integral and the definition of a_i

$$\left|\int_{D_i} |h_l(z_i, x, y) - h_l(\bar{z}, x, y)|\, \mathrm{d}G^i(x, y, l)\right| + 2a_i. \tag{4.9}$$

Define $\varepsilon_i = 4a_i + b_i + 1/i$. We bound the integral in (4.9) by $1/i$. Chose now δ_i such that $|h_l(z_i,x,y) - h_l(\bar{z},x,y)| \leq 1/i$ for $|\bar{z} - z_i| \leq \delta_i$ and each $(x,y,l) \in D_i$, which is possible since h_l is continuous and therefore uniformly continuous on $[z_{min}, z_{max}] \times [0,i] \times [0,i]$ for each $l \leq i$ by the Heine-Cantor theorem. Then (4.7) is bounded by

$$\int_{D_i} |h_l(z_i,x,y) - h_l(\bar{z},x,y)| \, \mathrm{d}G^i(x,y,l) \leq 1/i$$

and since $D_\infty = D_i \cup D_i^c$ it follows that (4.7) is bounded by $2a_i + 1/i$, and altogether we find that (4.6) is bounded by ε_i. $\square$

To shorten notation we shall define

$$f^i(z) \quad := \quad f(z; W^{-i}, W^{+i}, C^i) \quad \text{and} \quad f(z) := f(z; W^-, W^+, C)$$

and $g^i(y)$ and $g(y)$ accordingly, where (W^{-i}, W^{+i}, C^i) is a random vector with distribution function G^i and (W^-, W^+, C) a random vector with distribution function G. Moreover, define the functions $h_{1,z}(x,y,l) := y\psi_l(xz)$ and $h_{2,z}(x,y,l) := \psi_l(xz)$ for $z \in [0,1]$. Define $H := \big(\cup_{z\in[0,1]} h_{1,z}(x,y,l)\big) \cup \big(\cup_{z\in[0,1]} h_{2,z}(x,y,l)\big)$. If f is continuously differentiable for $z \in (\hat{z}-\delta, \hat{z}+\delta)$ for some $\delta > 0$ and $f^{'}(\hat{z}) < 0$, define further $h_{3,z}(x,y,l) := xy\mathbb{P}(\text{Poi}(\hat{z}x) = l-1)$ for $z \in (\hat{z}-\delta_2, \hat{z}+\delta_2)$, where δ_2 is chosen such that $f^{'}(z) < 0$ for $z \in (\hat{z}-\delta_2, \hat{z}+\delta_2)$. In that case define

$$H := \big(\cup_{z\in[0,1]} h_{1,z}(x,y,l)\big) \cup \big(\cup_{z\in[0,1]} h_{2,z}(x,y,l)\big) \cup \big(\cup_{z\in[0,1]} h_{3,z}(x,y,l)\big)$$

The previous lemma allows us to prove the following proposition about the convergence properties of the functions f and g for a $G-$convergent sequence with respect to the set H. The construction of such a sequence is outlined in Section 4.2.

Proposition 4.4. *Let $\{G^i\}_{i\in\mathbb{N}}$ be a $G-$convergent sequence with respect to the set H as defined above. Let $\hat{z}$ be the first positive zero of f. Further, define $\hat{z}^i$ to be the first zero of $f^i(z)$. Then there exists a sub-sequence $\{l_i\}_{i\in\mathbb{N}}$ such that*

$$\liminf_{i\to\infty} \hat{z}^{l_i} \quad \geq \quad \hat{z} \tag{4.10}$$

$$\liminf_{i\to\infty} g^{l_i}(\hat{z}^{l_i}) \quad \geq \quad g(\hat{z}) \tag{4.11}$$

Furthermore, if f is continuously differentiable for $z \in (\hat{z}-\delta, \hat{z}+\delta)$ for some $\delta > 0$ and $f^{'}(\hat{z}) < 0$, then $f'^{l_i}(z_i) < 0$ and

$$\lim_{i\to\infty} \hat{z}^{l_i} \quad = \quad \hat{z} \quad \textit{and} \quad \lim_{i\to\infty} g^{l_i}(\hat{z}^{l_i}) = g(\hat{z}).$$

Proof. We prove the claim only when f is continuously differentiable and $f^{'}(\hat{z}) < 0$. The lower bounds in (4.10) and (4.11) without this assumptions can be shown by similar means, observing that the functions g^i are monotonically increasing in z. First observe that since $f^{'}(\hat{z}) < 0$ we can chose δ_0 as small as we like such that

$$f(\hat{z}+\delta_0) < 0 \text{ and } f(\hat{z}-\delta_0) > 0,$$

We apply Lemma 4.3 to the functions $h_{1,z} \subset H$. Let $\{\varepsilon_{1,i}\}_{i\in\mathbb{N}}$ and $\{\delta_{1,i}\}_{i\in\mathbb{N}}$ be the resulting sequences. We can conclude that

$$\big|(f^i(z_i) + z_i) - (f(z) + z)\big| \leq \varepsilon_{1,i},$$

for $|z_i - z| \leq \delta_{1,i}$. Further there exists $\{\hat{\varepsilon}_{1,i}\}_{i \in \mathbb{N}}$ with $\lim_{i \to \infty} \hat{\varepsilon}_{1,i} = 0$ such that also

$$\left|(f^i(z_i) - (f(z))\right| \leq \hat{\varepsilon}_{1,i}. \tag{4.12}$$

(and especially $\left|(f^i(z) - (f(z))\right| \leq \hat{\varepsilon}_{1,i}$) for $|z - z_i| \leq \delta_{1,i}$.

This observation implies that

$$f^i(\hat{z} + \delta_0) < 0 \text{ and } f^i(\hat{z} - \delta_0) > 0,$$

for $i \geq i_0$ and that $f^i(z)$ has a root in $[\hat{z} - \delta_0, \hat{z} + \delta_0]$ for $i \geq i_0$. Denote by $\hat{z}^i$ the first zero of $f^i(z)$. We need to show that $\hat{z}$ is a limit point of $\{\hat{z}^i\}_{i \in \mathbb{N}}$. Assume for the sake of contradiction that there exists a sub-sequence $\{k_i\}$ with $\hat{z}^{k_i} \in [0, \hat{z} - \delta_0]$. Since $[0, \hat{z} - \delta_0]$ is compact there exists a limit point $\bar{z}$ approached by some subsub-sequence $\{l_{k_i}\}$ such that $\lim_{i \to \infty} \hat{z}^{l_{k_i}} = \bar{z} \leq \hat{z} - \delta_0$. By continuity, the function f attains its minimum M on $[0, \hat{z} - \delta_0]$. However, then by the observation (4.12), there exists i_2 such that $\left|f^{l_{k_i}}(\hat{z}^{l_{k_i}}) - f(\bar{z})\right| = f(\bar{z}) \leq M/2$ for $k_i \geq i_2$, providing the contradiction. Since δ_0 can be chosen arbitrarily small it follows that

$$\liminf_{i \to \infty} \hat{z}^i = \hat{z}.$$

Applying Lemma 4.3 twice, once with the functions $h_{2,z} \subset H$, and once with the functions $h_{3,z} \subset H$ provides the sequences $\{\varepsilon_{2,i}\}_{i \in \mathbb{N}}$ and $\{\delta_{2,i}\}_{i \in \mathbb{N}}$ and $\{\varepsilon_{3,i}\}_{i \in \mathbb{N}}$ and $\{\delta_{3,i}\}_{i \in \mathbb{N}}$, respectively, such that

$$\left|g^i(z_i) - g(z)\right| \leq \varepsilon_{2,i} \tag{4.13}$$

for $|z - z_i| \leq \delta_{2,i}$ and

$$\left|\mathbb{E}[W^+ W^- \mathbb{P}(\mathrm{Poi}(\hat{z} W^-) = C - 1)] - \int_{D_\infty} xy \mathbb{P}(\mathrm{Poi}(z_i x) = l - 1) \mathrm{d}G^i(x, y, l)\right| \leq \varepsilon_{3,i}$$

for $|z - z_i| \leq \delta_{3,i}$. Using Lemma 4.5 and differentiating $f^i(z)$, where differentiation under the integral sign is justified by the fact that G^i assigns measure only to finitely many values, this implies that $f'^i(z_i) < 0$ for $i \geq i_1$ and $|z - z_i| \leq \delta_{3,i}$. The claim follows by choosing a sub-sequence l_i such that $\left|\hat{z} - \hat{z}^{l_i}\right| \leq \min\{\delta_{1,i}, \delta_{2,i}, \delta_{3,i}\}$. □

Lemma 4.5. *Let the random vector (W^-, W^+, C) be such that $f'(\bar{z}; W^-, W^+, C) < 0$ for some $\bar{z} > 0$. Then*

$$\mathbb{E}[W^+ W^- \mathbb{P}(\mathrm{Poi}(\bar{z} W^-) = C - 1) \mathbf{1}_{C \geq 1}] < 1. \tag{4.14}$$

Proof. Since $f'(\bar{z}; W^-, W^+, C) < 0$, it follows by the definition of f that

$$\lim_{h \to 0} \left(\mathbb{E}[W^+ \psi_C((\bar{z} + h) W^-)] - \mathbb{E}[W^+ \psi_C((\bar{z}) W^-)]\right)/h < 1.$$

Furthermore

$$\frac{\partial}{\partial z} \psi_l(x\bar{z}) = \frac{\partial}{\partial z} \left(\sum_{r=l}^{\infty} e^{-x\bar{z}} \frac{(x\bar{z})^r}{r!}\right) = x e^{-x\bar{z}} \frac{(x\bar{z})^{l-1}}{(l-1)!} = x \mathbb{P}(\mathrm{Poi}(\bar{z} x) = l - 1),$$

for $l \geq 1$ and $\frac{\partial}{\partial z} \psi_0(x\bar{z}) = 0$, such that the left hand side of (4.14) can be written as

$$\mathbb{E}\left[\lim_{h \to \infty} W^+ \left(\psi_C((\bar{z} + h) W^-) - \psi_C(\bar{z} W^-)\right) \mathbf{1}_{C \geq 1}/h\right],$$

where $W^+\left(\psi_C((\bar z+h)W^-)-\psi_C(\bar z W^-)\right)\mathbf{1}_{C\geq 1}/h>0$ for every $h>0$. By Fatou's lemma, this observation allows us to conclude that for any sequence $\{h_i\}_{i\in\mathbb{N}}$, with $\lim_{i\to\infty}h_i=0$

$$\begin{aligned}
&\mathbb{E}[\liminf_{i\to\infty} W^+\left(\psi_C((\bar z+h_i)W^-)-\psi_C(\bar z W^-)\right)\mathbf{1}_{C\geq 1}/h_i]\\
\leq\ &\liminf_{i\to\infty}\mathbb{E}[W^+\left(\psi_C((\bar z+h_i)W^-)-\psi_C(\bar z W^-)\right)\mathbf{1}_{C\geq 1}/h_i]\\
=\ &\lim_{i\to\infty}\{\left(\mathbb{E}[W^+\psi_C((\bar z+h_i)W^-)]-\mathbb{E}[W^+\psi_C(\bar z W^-)]\right)/h_i\}<1,
\end{aligned}$$

from which (4.14) follows by the differentiability of $\psi_l(zx)$ with respect to x. $\square$

4.2 Constructing a $G-$convergent sequence

The following lemma is crucial in our construction of G-convergent sequences.

Lemma 4.6. *Let h be a function defined on $[z_{min},z_{max}]\times\mathbb{R}\times\mathbb{R}\times\mathbb{N}_0$, such that for each $l\in\mathbb{N}_0$, the function $h_l(z,x,y):=h(z,x,y,l)$ is continuous on $[z_{min},z_{max}]\times\mathbb{R}\times\mathbb{R}$. Let $D\subset\mathbb{R}\times\mathbb{R}\times\mathbb{N}_0$ be a closed, bounded rectangular domain and $\{G_i\}_{i\in\mathbb{N}}$ a sequence of distribution functions on $\mathbb{R}\times\mathbb{R}\times\mathbb{N}_0$ such that for each $(x,y,l)\in D$, $\lim_{i\to\infty}G_i(x,y,l)=G(x,y,l)$. Then for every $\varepsilon>0$, there exists $i_0\in\mathbb{N}$ such that for each $i\geq i_0$*

$$\left|\int_D h(z,x,y,l)\mathrm{d}G^i(x,y,l)-\int_D h(z,x,y,l)\mathrm{d}G(x,y,l)\right|\leq\varepsilon, \tag{4.15}$$

for every $z\in[z_{min},z_{max}]$.

Proof. Choose a step function $\hat h(z,x,y,l)$ with lm steps $h_{k,j}, 0\leq j\leq l, 0\leq k\leq m$ such that

$$\hat h(z,x,y,l)=h_{k,j},\ \text{ for } (z,x,y,l)\in J_k\times I_j$$

where I_j are equally sized cubes with $D=\cup_j I_j$ and J_k equally sized intervals with $[z_{min},z_{max}]=\cup_j J_k$ such that $\left|\hat h(z,x,y,l)-h(z,x,y,l)\right|\leq\varepsilon/3$ on D for every $z\in[z_{min},z_{max}]$. This choice is possible due to the fact that both, D and $[z_{min},z_{max}]$ bounded and closed and $h_l(z,x,y)$ continuous for each l. Then $\forall i$

$$\left|\int_D \hat h(z,x,y,l)-h(z,x,y,l)\mathrm{d}G^i(x,y,l)\right|\leq\varepsilon/3,$$

and

$$\left|\int_D \hat h(z,x,y,l)-h(z,x,y,l)\mathrm{d}G(x,y,l)\right|\leq\varepsilon/3.$$

Observe that for $z\in J_k$

$$\int_D \hat h(z,x,y,l)\mathrm{d}G^i(x,y,l)-\int_D \hat h(z,x,y,l)\mathrm{d}G(x,y,l)=\sum_j h_{k,j}(\mu^i(I_j)-\mu(I_j)),$$

where μ^i and μ are the measures implied by G^i and G respectively. Since $\lim_{i\to\infty}\mu^i(I_j)=\mu(I_j)$, by the fact that the I_j are intervals (i.e. the measures $\mu^i(I_j)$ and $\mu(I_j)$ are determined by the values of G^i and G at the endpoints of I_j), there exists i_0 such that for $i\geq i_0$

$$\sum_j h_{k,j}(\mu^i(I_j)-\mu(I_j))\leq\varepsilon/3.$$

By the triangular inequality it can be easily seen that (4.15) holds for $i \geq i_0$. $\square$

Clearly one can chose k_i such that the last Lemma holds for a finite set of functions H, where each $h \in H$ fulfills the assumptions of the proposition, an observation we shall use in our construction below.

In a first step we construct two G-convergent sequences of finitary distribution functions $\{G^i_A\}_{i\in\mathbb{N}}$ and $\{G^i_B\}_{i\in\mathbb{N}}$ with respect to a given set H of functions. One sequence is such that it generates random graph with asymptotically less infections than in a random graph with limiting distribution G and the second with asymptotically more infections. In a second step we couple a vertex sequence $(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$ with limiting distribution G to sequences with limiting distribution G^i_A and G^i_B for each $i \in \mathbb{N}$. This coupling allows for a sandwich type argument in the proof of Theorem 4.1. For simplicity we pose the following restriction on the limiting distribution of the vertex sequence. The restriction is not mandatory but simplifies the exposition.

Assumption 4.7. *The limiting distribution $G : \mathbb{R} \times \mathbb{R} \times \mathbb{N}_0^\infty \to [0,1]$ of the regular vertex sequence $(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$ is such that for each l the function $G_l(x,y) := G(x,y,l)$ is continuous.*

Defining a G-convergent family of finitary distribution functions is rather straightforward under Assumption 4.7. For this let H be the set of functions defined in Section 4.1 and observe that each $h \in H$ is such that $h_l(x,y) := h(x,y,l)$ is continuous for each l. For a given $i \in \mathbb{N}$ we determine two distribution functions $G^i_A(x,y,l)$ and $G^i_B(x,y,l)$ for random vectors $(W_A^{-i}, W_A^{+i}, C^i_A)$ and $(W_B^{-i}, W_B^{+i}, C^i_B)$, respectively. The random vector $(W_A^{-i}, W_A^{+i}, C^i_A)$ is constructed in a way such that it can be coupled to (W^-, W^+, C) with the probabilities $\mathbb{P}((W^-, W^+) \leq (W_A^{-i}, W_A^{+i}))$ and $\mathbb{P}(C^i_A \leq C)$ very large, where the first inequality is meant component-wise. This ensures that random graphs approximating $(W_A^{-i}, W_A^{+i}, C^i_A)$ have more edges and lower threshold values, and in turn more infections. For $(W_B^{-i}, W_B^{+i}, C^i_B)$, the inequalities are reversed and a random graph with limiting distribution $(W_B^{-i}, W_B^{+i}, C^i_B)$ will have less infections than a random graph approximating (W^-, W^+, C).

For each $i \in \mathbb{N}$ we partition $[x_0, i] \times [x_0, i]$ into $L(i)L(i)$ equally spaced half open squares $D^i_{k,j} := [p_k, p_{k+1}) \times [p_j, p_{j+1}), 1 \leq j,k \leq L(i)$, where $L(i)$ is chosen such that $\forall h \in H$

$$\left| \int_{D_i} h(x,y,l) \mathrm{d}\bar{G}(x,y,l) - \int_{D_i} h(x,y,l) \mathrm{d}G(x,y,l) \right| \leq 1/i, \text{ for } \bar{G} \in \{\bar{G}^i_B, \bar{G}^i_A\}, \tag{4.16}$$

where $\bar{G}^i_A$ and $\bar{G}^i_B$ are defined on D_i by

$$\bar{G}^i_A(x,y,l) := G(p_k, p_j, l) \text{ if } (x,y) \in D^i_{k,j} \tag{4.17}$$

$$\bar{G}^i_B(x,y,l) := G(p_{k+1}, p_{j+1}, l) \text{ if } (x,y) \in D^i_{k,j}. \tag{4.18}$$

To see that this choice is possible one can start with sequences $\{\bar{G}^{i,m}_A\}_{m\in\mathbb{N}}$ and $\{\bar{G}^{i,m}_B\}_{m\in\mathbb{N}}$, which are defined by partitioning into mm half open cubes for each l, and chose $L(i)$ to be the smallest natural number such that (4.16) is fulfilled. To guarantee the existence of such an $L(i)$ one can use Lemma 4.6 for each of the families $\{h_{i,q}\}, q \in \{1,2,3\})$ and chose the maximum in an obvious way.

Define the function $\gamma : \mathbb{N} \to \mathbb{R}_+$ by

$$\gamma(i) = \mathbb{P}\left(\{W^+ > i\} \cup \{W^- > i\} \cup \{\infty > C > i\}\right). \tag{4.19}$$

Observe that $\gamma(i) \leq \mathbb{P}(W^+ > i) + \mathbb{P}(W^- > i) + \mathbb{P}(\infty > C > i)$ and $\lim_{i\to\infty} \gamma(i) = 0$.

Now define distribution functions $\{G_B^i\}_{i\in\mathbb{N}}$ and $\{G_A^i\}_{i\in\mathbb{N}}$ by

$$G_B^i(x,y,l) := \begin{cases} 0 & \text{if } (x < x_0) \vee (y < x_0) \\ \bar{G}_B^i(\min\{x,i\},\min\{y,i\},\min\{l,i\}) & \text{if } (l < \infty) \wedge ((x \geq x_0) \wedge (y \geq x_0)) \\ 1 & \text{if } (l = \infty) \wedge ((x \geq x_0) \wedge (y \geq x_0)), \end{cases} \tag{4.20}$$

where $\wedge$ denotes logical *and* and $\vee$ logical *or*.

To specify G_A^i, we determine for each i a value $\bar{w}_i^+$ such that

$$\bar{w}_i^+\gamma(i) = 2\int_{D_i^c} y\mathrm{d}G(x,y,l), \tag{4.21}$$

and observe that $\lim_{i\to\infty} \bar{w}_i^+\gamma(i) = 0$. Define

$$G_A^i(x,y,l) := \begin{cases} 0 & \text{if } (x < x_0) \vee (y < x_0) \\ \bar{G}_A^i(\min\{x,i\},\min\{y,i\},\min\{l,i\}) & \text{if } (l < \infty) \wedge ((x \geq x_0) \wedge (\bar{w}_i^+ \geq y \geq x_0)) \\ \gamma(i) + \bar{G}_A^i(x_0,i,0) & \text{if } (l < \infty) \wedge (x \geq x_0) \wedge (y \geq \bar{w}_i^+) \\ 1 & \text{if } (l = \infty) \wedge (x \geq x_0) \wedge (y \geq x_0). \end{cases} \tag{4.22}$$

Proposition 4.8. *The two sequences of distributions $\{G_A^i\}_{i\in\mathbb{N}}$ and $\{G_B^i\}_{i\in\mathbb{N}}$ are $G-$convergent with respect to the set H.*

Proof. Property 1 and 2b of Definition 4.2 are obvious from the construction of the two sequences and Lemma 4.6. We show only Property 2a. For G_B^i by construction $\int_{D_i^c} \mathrm{d}G_B^i(x,y,l) = 0$, and therefore for any function h it follows that $\int_{D_i^c} h(x,y,l)\mathrm{d}G_B^i(x,y,l) = 0$. For G_A^i we find

$$\int_{D_i^c} 1\mathrm{d}G_A^i(x,y,l) = \gamma(i) \tag{4.23}$$

$$\int_{D_i^c} y\mathrm{d}G_A^i(x,y,l) = \bar{w}^+\gamma(i) = 2\int_{D_i} y\mathrm{d}G(x,y,l) \tag{4.24}$$

$$\int_{D_i^c} xy\mathbb{P}(\mathrm{Poi}(zx) = l-1)\mathrm{d}G_A^i(x,y,l) \leq \int_{D_i^c} xy\mathrm{d}G_A^i(x,y,l) = \bar{w}_i^+x_0\gamma(i), \tag{4.25}$$

and all the quantities on the right hand side converge to 0. Since they are only finitely many, they converge uniformly. Checking the definition of $h_{1,z}, h_{2,z}$ and $h_{3,z}$ shows that the integral with respect to these functions is bounded by one of the above quantities, such that the uniformity holds over H. $\square$

4.3 Coupling to the original vertex sequence

To allow for a sandwich type argument that squeezes our original sequence with limiting distribution (W^-, W^+, C) between two finitary ones, we will develop a specific coupling. Let as above $(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$ be a regular vertex sequence with limiting distribution $G : \mathbb{R}\times\mathbb{R}\times\mathbb{N}_0^\infty \to [0,1]$ and further (W^-, W^+, C) a random vector with distribution function G. Recall the definition of D_i in (4.4). For any given $i \in \mathbb{N}$ we construct two finitary regular vertex sequences $(\mathbf{w}_A^{-i}, \mathbf{w}_A^{+i}, \mathbf{c}_A^i)$ and $(\mathbf{w}_B^{-i}, \mathbf{w}_B^{+i}, \mathbf{c}_B^i)$ on the same index set $[n]$ with limiting distributions $G_A^i, G_B^i : \mathbb{R}\times\mathbb{R}\times\mathbb{N}_0 \to [0,1]$ as follows:

1. Define $\mathbf{D}_i^c := \{m \in [n] \mid (w_m^-(n), w_m^+(n), c_m(n)) \in D_i^c\}$. We consider a partition of $[n]\backslash\mathbf{D}_i^c$ into $i\cdot L(i)\cdot L(i)$ parts $\mathbf{D}_{k,j,l} := \{m \in [n] \mid (w_m^-(n), w_m^+(n)) \in D_{k,j}, c_m(n) = l\}$, where $L(i)$

is the number of half open intervals chosen in the definition of G^i_A and G^i_B (see (4.16)).

2. Construct a vertex sequence $(\mathbf{w}^{-i}_B, \mathbf{w}^{+i}_B, \mathbf{c}^i_B)$ on $[n]$ by $(w^{-i}_{m,B}, w^{+i}_{m,B}, c^i_{m,B}) = (p_k, p_j, l)$ for $m \in \mathbf{D}_{k,j,l}$ and $(w^{-i}_{m,B}, w^{+i}_{m,B}, c^i_{m,B}) = (x_0, x_0, \infty)$ for $m \in [n]\backslash \mathbf{D}^c_i$.

3. Construct a sequence $(\mathbf{w}^{-i}_{m,A}, \mathbf{w}^{+i}_{m,A}, \mathbf{c}^i_{m,A})$ on $[n]$ by $(w^{-i}_{m,A}, w^{+i}_{m,A}, c^i_{m,A}) = (p_{k+1}, p_{j+1}, l)$ for $m \in \mathbf{D}_{k,j,l}$ and $(\mathbf{w}^{-i}_{m,A}, \mathbf{w}^{+i}_{m,A}, \mathbf{c}^i_{m,A}) = (x_0, \bar{w}^+_i, 0)$.

It can easily be seen that the resulting sequences have the required convergence properties, i.e. are regular vertex sequences. In 2., the choice of the in- and outweights is irrelevant for vertices with percolation threshold infinity since they cannot spread the contagion process. In 3., for vertices with percolation threshold equal to 0, the in-weight is irrelevant and was chosen in order to have a bound on the integral in (4.25) and the resulting uniform convergence property for the functions $h_{3,z}$.

4.4 Proof of Theorem 4.1

Denote by $\mathcal{A}_n(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$ the set of infected vertices for $G(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$, and by $\mathcal{A}_{f_{A,i}}((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_A)$ and $\mathcal{A}_{f_{B,i}}((\mathbf{w}^-, w^+, \mathbf{c})^i_B)$ those for $G^{A,i}((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_A)$ and $G^{B,i}((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_B)$ respectively.

Proof of Theorem 4.1. Again we consider only the slightly more complicated result when f is differentiable and $f'(\hat{z}) < 0$. The vertex sequence $(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_B$ has been constructed such that for each vertex $m \in \cup \mathbf{D}_{k,j,l}$ the threshold value agrees with its counterpart in $(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$ and its in- and outweights are lower. Furthermore, the vertices in $[n] \backslash \cup \mathbf{D}_{k,j,l}$ are uninfectable in $G^{B,i}((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_B)$. These considerations imply that $G((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c}))$ and $G^{B,i}((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_B)$ can be coupled such that

$$\mathcal{A}_{f_{B,i}}((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_B) \preceq \mathcal{A}_f((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})), \tag{4.26}$$

where $\preceq$ denotes stochastic ordering.

To compare $\mathcal{A}_{f_{A,i}}((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_A)$ with $\mathcal{A}_f((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c}))$, first note that for each $m \in \cup \mathbf{D}_{k,j,l}$ the threshold values are the same for $G((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c}))$ and $G^{A,i}((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_A)$ but in and out weight are larger in $G^{A,i}((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_A)$. Further any $m \in [n] \setminus (\cup \mathbf{D}_{k,j,l})$ is a defaulted knot in $G^{A,i}((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_A)$. The total out-weight of the vertex set $[n] \setminus (\cup \mathbf{D}_{k,j,l})$ in $G^{A,i}((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_A)$ is $2n \int_{D^c_i} y \mathrm{d}G(x, y, l)$ while in $G((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c}))$ it is $n \int_{D^c_i} y \mathrm{d}G(x, y, l)$. This implies that for each vertex $v \in \cup \mathbf{D}_{k,j,l}$ with in-weight w^-, the in-degree D^-_v and $D^-_{v,A}$ of v in $\mathbf{D}^c_i$ can be coupled (similar as in the proof of Theorem 2.5) to Poisson random variables Z_v and $Z_{v,A}$ (depending on i) with parameter $w^- 2 \int_{D^c_i} y \mathrm{d}G(x, y, l)$ and $w^- \int_{D^c_i} y \mathrm{d}G(x, y, l)$, respectively, such that

$$\mathbb{P}(D^-_v \neq Z_v) \leq ((w^-)^2 + w^-) o(1) = o(1) \tag{4.27}$$

$$\mathbb{P}(D^-_{v,A} \neq Z_{v,A}) \leq ((w^-)^2 + w^-) o(1) = o(1). \tag{4.28}$$

Since $\mathbb{P}(Z_v \geq l) < \mathbb{P}(Z_{v,A} \geq l)$ for all $l \in \mathbb{N}$, and w^-_v and c_v are bounded by i, it follows that $\mathbb{P}(D^-_v \geq l) < \mathbb{P}(D^-_{v,A} \geq l)$ for $l \leq i$. Since all vertices in $\mathbf{D}^c_i$ are infected in the random graph $G^{A,i}((w^-, w^+, c)^i_A)$, the probability that the vertex v has egdes to at least l infected vertices in $\mathbf{D}^c_i$ for $l \leq i$ is larger in $G^{A,i}((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_A)$ than in $G((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c}))$. These considerations imply that

$$\mathcal{A}_{f_{B,i}}((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_B) \preceq \mathcal{A}_f((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})) \preceq \mathcal{A}_{f_{A,i}}((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})^i_A)). \tag{4.29}$$

Chose two sub-sequences $\{l_{A,i}\}_{i\in\mathbb{N}}$ and $\{l_{B,i}\}_{i\in\mathbb{N}}$ for the G-convergent sequences $(W_A^{-i}, W_A^{+i}, C_A^i)$ and $(W_B^{-i}, W_B^{+i}, C_B^i)$ as provided by Proposition 4.4 such that for $\hat{z}_A^{l_{A,i}}$ and $\hat{z}_B^{l_{B,i}}$, the first zeros of $f(z,(W_A^{-l_{A,i}}, W_A^{+l_{A,i}}, C_A^{l_{A,i}})$ and $f(z,(W_B^{-l_{B,i}}, W_B^{+l_{B,i}}, C_B^{l_{B,i}})$ respectively, we have $\lim_{i\to\infty} \hat{z}_A^{l_{A,i}} = \lim_{i\to\infty} \hat{z}_B^{l_{B,i}} = \hat{z}$. Further $\lim_{i\to\infty} g^{l_{A,i}}(\hat{z}^{l_{A,i}}) = \lim_{i\to\infty} = g^{l_{B,i}}(\hat{z}^{l_{B,i}}) = g(\hat{z})$ holds. By Theorem 3.3 together with (4.29), it follows that for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}\left(g^{l_{B,i}}(\hat{z}^{l_{B,i}}) - \varepsilon \le n^{-1}\left|\mathcal{A}_f((\mathbf{w}^-, \mathbf{w}^+, \mathbf{c}))\right| \le g^{l_{A,i}}(\hat{z}^{l_{A,i}}) + \varepsilon\right) = 0,$$

proving Theorem 4.1. □

5 Applications

5.1 Quantifying Systemic Risk

In this section we investigate under which conditions even a very small set of infected vertices can cause a large fraction of infected vertices at the end of the process. Let as before $(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$ be a regular vertex sequence with limiting distribution G and (W^-, W^+, C) a random variable with distribution G.

Our model for studying the effect of very small initially infected sets is as follows. We assume that $\mathbb{P}(C = 0) = 0$, that is, (asymptotically) there are no initial infections. Moreover, we assume that some vertices $i \in [n]$ are being infected *ex post*. In this process all vertices $i \in [n]$ receive a binary mark m_i, which is either 1 or 0, where 1 means that the vertex keeps its initial infection threshold and 0 that it becomes infected. Let $\mathbf{m}$ be the sequence of marks. We define, similar as in Definition 3.1, the function $\bar{G}_n(x, y, l, m) : \mathbb{R} \times \mathbb{R} \times \mathbb{N}_0^\infty \times \{0, 1\} \to [0, 1]$ by

$$\bar{G}_n(x, y, l, k) \quad = \quad n^{-1} \sum_{i\in[n]} \mathbf{1}\{w_i^-(n) \le x, w_i^+(n) \le y, c_i(n) \le l, m_i(n) \le k\}$$

and assume that $\lim_{n\to\infty} \bar{G}_n(x, y, l, m) = \bar{G}(x, y, l, m)$ for each (x, y, l, m) and some distribution function $\bar{G}$. Let (W^-, W^+, C, M) be a random vector distributed according to $\bar{G}$. Let $\mathbb{P}(M = 0) > 0$ and define the random variable $C_M := CM$.

The following proposition investigates under which condition the fraction of infected vertices at the end of the process can be bounded away from 0 *independent* of M for random graphs parametrized by (W^-, W^+, C_M).

Proposition 5.1. *Assume that (W^-, W^+, C) is such that there exists $z_0 > 0$ and $\delta > 0$ such that for any $0 < z < z_0$*

$$\mathbb{E}[W^+\psi_C(zW^-)] > z \tag{5.1}$$

Then, for all M with $\mathbb{P}(M = 0) > 0$ with high probability

$$n^{-1}|\mathcal{A}_n| \ge \mathbb{E}[\psi_C(W^- z_0)] > 0. \tag{5.2}$$

Networks that fulfill the assumption of the last proposition are very prone to small initial infections. In particular, in Example 6 below we will show that (5.1) often holds even if $\mathbb{P}(C = 1) = 0$, that is, when there are no *weak* vertices with infection threshold equal to 1; the crucial property driving $\mathbb{E}[W^+\psi_C(zW^-)]$ up will be the non-existence of the second moment of the distribution of W^-. As mentioned in the introduction, this result is in contrast to the general assumption (particularly in the financial mathematics literature) that networks with $\mathbb{P}(C = 1) = 0$ are resilient (compare Thm. 3.10 in [4]), and provides a new global feature that enable us to study the vulnerability of networks.

Proof. Since $\psi_0(x) = 1$ for $x > 0$ and especially $\psi_0(x) > \psi_r(x)$ for $r \in \mathbb{N}^\infty$ and further W^+ strictly positive, it follows that

$$\mathbb{E}[W^+\psi_{C_M}(zW^-)] > \mathbb{E}[W^+\psi_C(zW^-)] \geq z, \tag{5.3}$$

such that $\mathbb{E}[W^+\psi_{C_M}(zW^-)] > 0$ for $z \in (0, z_0]$. To see that the left inequality in (5.3) is really strict, chose $\bar{w}$ with $\mathbb{P}((W^- \leq \bar{w}) \wedge (M = 0)) \geq \mathbb{P}(C_M = 0)/2$, then it follows that

$$\mathbb{E}[W^+\psi_{C_M}(zW^-)] - \mathbb{E}[W^+\psi_C(zW^-)] \geq x_0(1/2)\mathbb{P}(C_M = 0)(1 - \psi_1(z\bar{w})) > 0.$$

By continuity of $\mathbb{E}[W^+\psi_C(zW^-)]$ (compare with the proof of Lemma 3.2) it is easy to see that $\mathbb{E}[W^+\psi_C(zW^-)] > 0$ for $z \in (0, z_0 + \delta]$ for some $\delta > 0$. Furthermore, since $\mathbb{P}(C_M = 0) > 0$ and $W^+ \geq x_0$, it follows that $\mathbb{E}[W^+\psi_{C_M}(0W^-)] > 0$. Let $\hat{z}$ be the smallest positive solution of

$$f(z; (W^-, W^+, C_M)) = 0.$$

The above considerations imply that $\hat{z} > z_0$. Similarly $\mathbb{E}[\psi_{C_M}(W^-\hat{z})] > \mathbb{E}[\psi_C(W^- z_0)]$. By Theorem 4.1 for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}(n^{-1}|\mathcal{A}_n| < \mathbb{E}[\psi_{C_M}(W^-\hat{z})] - \varepsilon) = 0.$$

Chose $\varepsilon = \mathbb{E}[\psi_{C_M}(W^-\hat{z})] - \mathbb{E}[\psi_C(W^- z_0)] > 0$ and (5.2) follows. □

A sufficient condition for the assumptions of the last proposition to hold is that f is right differentiable in 0 with positive derivative. To see that the bound in (5.2) can in general not be improved consider a vertex sequence with limiting distribution (W^-, W^+, C) fulfilling (5.1) with $f(z; (W^-, W^+, C))$ continuously differentiable and such that $f'(\hat{z}; (W^-, W^+, C)) < 0$ where $\hat{z} > 0$ is the smallest strictly positive solution of

$$f(z; (W^-, W^+, C)) = 0.$$

Now infect ex post all vertices i.i.d. with probability $p > 0$. The resulting vertex sequence is close to (W^-, W^+, C_{M_p}) with $C_{M_p} := CM_p$, where M_p is a Bernoulli random variable independent of all others with success probability $1 - p$. Conditioning on M_p shows that

$$f(z; (W^-, W^+, C_{M_p})) = (1 - p)\mathbb{E}[W^+\psi_C(zW^-)] + p\mathbb{E}[W^+] - z.$$

Choose $\delta_0 > 0$ such that $f'(z; (W^-, W^+, C)) < 0$ for all $z \in (\hat{z} - \delta_0, \hat{z} + \delta_0)$. Since we have that $\frac{\partial}{\partial z}\mathbb{E}[W^+\psi_C(zW^-)] \geq 0$, it follows that

$$f'(z; (W^-, W^+, C_{M_p})) \leq f'(z; (W^-, W^+, C)) < 0, \tag{5.4}$$

for $z \in (\hat{z}-\delta_0, \hat{z}+\delta_0)$. Let $\hat{z}_p \geq \hat{z}$ be the first positive solution of $f(z; (W^-, W^+, C_{M_p})) = 0$. Then one easily observes that $\lim_{p\to 0} \hat{z}_p = \hat{z}$ and $\lim_{p\to 0} \mathbb{E}[W^+\psi_{C_{M_p}}(\hat{z}_p W^-)] = \mathbb{E}[W^+\psi_C(\hat{z}W^-)]$. This implies that there exists $p_0 > 0$ such that $\hat{z}_p \in (\hat{z} - \delta_0, \hat{z} + \delta_0)$ for $p < p_0$ and therefore $f'(\hat{z}_p; (W^-, W^+, C_{M_p})) < 0$ by (5.4). By Theorem 4.1, for any $\delta > 0$ there exists p such that with high probability

$$n^{-1}|\mathcal{A}_n| \leq \mathbb{E}[\psi_C(W^-\hat{z})] + \delta,$$

which shows that the bound in (5.2) is best possible. The following corollary shows that in a network satisfying the requirements of Proposition 5.1 a sublinear set of initially infected vertics is sufficient for the infection to spread to a linear set.

Corollary 5.2. *Let* $(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$ *be a regular vertex sequence and* (W^-, W^+, C) *a random vector with distribution as the limiting distribution of* $(\mathbf{w}^-, \mathbf{w}^+, \mathbf{c})$*. Assume that* (W^-, W^+, C) *satisfies* (5.1) *for* $z < z_0$*. Then, there exists a sequence* $\varepsilon(n)$ *with* $\lim_{n\to\infty} \varepsilon(n) = 0$ *such that if we infect each vertex* $i \in [n]$ *independently with probability* $\varepsilon(n)$*, then with high probability*

$$n^{-1} |\mathcal{A}_n| \geq \mathbb{E}[\psi_C(W^- z_0)],$$

where $\mathcal{A}_n$ *is the final set of infected vertices at the end of the infection process.*

Proof. Let $\varepsilon_i := 1/i$ and as before M_{ε_i} a Bernoulli random variable independent of all other variables with success probability $1 - \varepsilon_i$. By Proposition 5.1, we can define n_i such that

$$\mathbb{P}\left(n^{-1} |\mathcal{A}_n| < \mathbb{E}[\psi_C(W^- z_0)]\right) \leq 1/i$$

for $n \geq n_i$ and $n_i > n_{i-1}$ (to ensure that the sequence $\{n_i\}_{i\in\mathbb{N}}$ is strictly increasing) in the random graph parametrized by $(W^-, W^+, C_{M_{\varepsilon_i}})$. Define $\varepsilon(n) = \varepsilon_i$ for $n_i \leq n < n_{i+1}$. $\square$

We finish this section with a reverse answer to Proposition 5.1, describing the situation when the network is resilient.

Proposition 5.3. *Assume that* (W^-, W^+, C) *is such that there exists* $z_0 > 0$ *with* f *continuously differentiable on* $[0, z_0)$ *and let the right derivative* $f'(0, (W^-, W^+, C)) < 0$*. Then, for any sequences of ex post infections* $\{M_i\}_{i\in\mathbb{N}}$ *with* $\lim_{i\to\infty} \mathbb{P}(M_i = 0) = 0$*, it follows that for any* $\varepsilon > 0$*, there exists* i_ε *such that for* $i \geq i_\varepsilon$ *with high probability*

$$n^{-1} |\mathcal{A}_n| \leq \varepsilon. \tag{5.5}$$

A network for which the assumption of the proposition holds can be considered as being resilient to small infections, since the final fraction of infected vertices will still be small. In Example 7 we describe a family of graphs that are resilient.

Proof. Let $\varepsilon > 0$, chose δ such that $g(z; (W^-, W^+, C)) \leq \varepsilon/2$ for $z \in [0, \delta]$. This choice is possible since $g(z; (W^-, W^+, C)$ is continuous by Lemma 3.2 and $g(0; (W^-, W^+, C) = 0$. In order to prove the claim, we first show that there exists i_0 such that $f(z, (W^-, W^+, C_{M_i})$ has a stable fixpoint before δ for $i \geq i_0$. Let $\kappa = f'(0, (W^-, W^+, C)) < 0$ and $\delta > \delta_2 > 0$ chosen such that $f'(z, (W^-, W^+, C) < \kappa/2$ for $z \in [0, \delta_2]$. It follows that $f(z, (W^-, W^+, C) \leq (\kappa/2)z$ for $z \in [0, \delta_2]$. Further,

$$\begin{aligned}
& f(\delta_2, (W^-, W^+, C_{M_i}) \\
= \;& f(\delta_2, (W^-, W^+, C) + \mathbb{E}[W^+(\psi_{C_{M_i}}(\delta_2 W^-) - \psi_C(\delta_2 W^-))] \\
\leq \;& (\kappa/2)\delta_2 + \mathbb{E}[W^+ \mathbf{1}_{\{M_i=0\}}]
\end{aligned} \tag{5.6}$$

Since $\lim_{i\to\infty} \mathbb{P}(M_i = 0) = 0$ it follows that $\lim_{i\to\infty} \mathbb{E}[W^+ \mathbf{1}_{\{M_i=0\}}] = 0$ by $\mathbb{E}[W^+] < \infty$. This observation allows us to chose i_0 such that the right hand side of (5.6) is strictly smaller than 0. This ensures that $\hat{z}_{M_i} \leq \delta_2$ for $i \geq i_0$, where $\hat{z}_{M_i}$ is the first positive zero of $f(z, (W^-, W^+, C_{M_i})$. Further, since $f'(\hat{z}_{M_i}, (W^-, W^+, C)) < 0$, Lemma 4.5 yields that

$$\mathbb{E}[W^+ W^- \mathbb{P}(\mathrm{Poi}(\hat{z}_{M_i} W^-) = C - 1)\mathbf{1}_{\{C\geq 1\}}] < 1.$$

As a consequence, since $P((C_{M_i} = C) \cup (C_{M_i} = 0)) = 1$,

$$\mathbb{E}[W^+ W^- \mathbb{P}(\mathrm{Poi}(\hat{z}_{M_i} W^-) = C_{M_i} - 1) \mathbf{1}_{\{C_{M_i} \geq 1\}}] < 1.$$

Differentiation under the integral sign shows that $f^{'}(\hat{z}_{M_i}, (W^-, W^+, C_{M_i})) < 0$. Furthermore, the monotony of g and $\hat{z}_{M_i} < \delta$ implies $g(\hat{z}_{M_i}; (W^-, W^+, C) < g(\delta; (W^-, W^+, C) \leq \varepsilon/2$. Further

$$g(\hat{z}_{M_i}; (W^-, W^+, C_{M_i}) \leq g(\hat{z}_{M_i}; (W^-, W^+, C)) + \mathbb{P}(M_i = 0), \tag{5.7}$$

and choosing $i_\varepsilon \geq i_0$ such that $\mathbb{P}(M_i = 0) < \varepsilon/2$ the claim follows again by Theorem 4.1. $\square$

In [5] undirected random graphs which exhibit a power law with exponent $\beta \in (2, 3)$ and where each vertex has the same threshold value $r \in \mathbb{N}$ are investigated. Under some assumptions on the weight sequence, a threshold function $a_c(n) = o(n)$ for the percolation process is derived. If $a(n)$ is the number of randomly initially infected vertices, then for $a(n) \ll a_c(n)$ the infection does not spread at all, while for $a(n) \gg a_c(n)$ there exists $\epsilon > 0$ such that the infection spreads to at least ϵn vertices. Under very mild assumptions on the vertex sequence, Proposition 5.1 and Proposition 5.3 answer the question when a sublinear set of infected vertices is enough to spread to a linear fraction. To derive a threshold function in this general setting is beyond the scope of this paper and is left as an open problem.

5.2 Examples

In this section we give some example distributions of random vectors (W^-, W^+, C) and check the stability of the fixpoint of $f(z, (W^-, W^+, C)$ and the conditions of Proposition5.1. The first example shows that in the absence of a second moment of (W^-, W^+), condition (5.1) can be satisfied even when not a single vertex with threshold value 1 exists.

Example 6. *Assume that $W = W^- = W^+$ and $\mathbb{P}(C = r) = 1$ for some $r \in \mathbb{N}, r \geq 2$. Furthermore, let W be power-law distributed with exponent $\beta \in (2, 3)$ and density function $h(w)$ given by*

$$h(w) := \begin{cases} Cw^{-\beta}, & \text{if } w \geq 1 \\ 0, & \text{else} \end{cases},$$

where C is a normalizing constant. Since $\psi_r(zW) > e^{-zW} \frac{(zW)^r}{r!}$, it follows that

$$\begin{aligned} \mathbb{E}[W^+ \psi_C(zW^-)] &\geq \frac{z^r}{r!} C \int_1^\infty w^{(r+1)-\beta} e^{-zw} \mathrm{d}w \\ &\geq z^r C_1 \int_1^{1/z} w^{(r+1)-\beta} e^{-zw} \mathrm{d}w \\ &\geq z^r C_1 \int_1^{1/z} w^{(r+1)-\beta} \mathrm{d}w \geq C_1(z^{\beta-2} - z^r), \end{aligned}$$

where it was used that $e^{-zw} \geq e^{-1}$ for $w \leq 1/z$ and C_1 is a constant changing from line to line but independent of z. Now choose z_0 such that $C_1(z^{\beta-2} - z^r) > z$ for $z \in (0, z_0)$.

The last example can be easily generalized to the situation where $\mathbb{P}(C = r) > 0$ for some $r \geq 2, r \in \mathbb{N}$ and $W|(C = r)$ is power-law distributed with parameter $\beta \in (2, 3)$. In contrast, note that if we choose the exponent of the power-law to be > 3, then a simple calculation shows that the assumptions of Propostion 5.3 are satisfied, and thus the network will be always resilient (see also [4]).

In our next example we show a natural specification of (W^-, W^+, C) such that f has a stable fixpoint.

Example 7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with positive random variable $W \in \mathbb{R}$ and random vector $(W^-, W^+, C) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}$ defined. Let further W^+ be independent of $\{W^-, C\}$ given W $(W^+ \perp\!\!\!\perp_{\sigma(W)} (W^-, C))$ and let $C|W^-$ be uniformly distributed on $\{0, 1, 2, \dots, \lceil W^- \rceil\}$.*

Let $\hat{z}$ be the first positive zero of $f(z, (W^-, W^+, C))$. We show that $f^{'}(\hat{z}, (W^-, W^+, C)) < 0$. For this we differentiate $f(z, (W^-, W^+, C)$ below the integral sign and obtain that

$$\mathbb{E}\left[\frac{\partial W^+ \psi_C(zW^-)}{\partial z}\right] = \mathbb{E}\left[W^+ W^- \mathbb{P}\left(\mathrm{Poi}(zW^-) = C - 1\right) \mathbf{1}_{C>0}\right]. \tag{7.1}$$

Conditioning on W and using that $W^+ \perp\!\!\!\perp_{\sigma(W)} (W^-, C))$ yields

$$\begin{aligned} & \mathbb{E}[\mathbb{E}[W^+|W]\mathbb{E}[W^- \mathbb{P}\left(\mathrm{Poi}(zW^-) = C - 1\right) \mathbf{1}_{\{C>0\}}|W]] \\ = \; & \mathbb{E}[\mathbb{E}[W^+|W]\mathbb{E}[\mathbb{E}[W^- \mathbb{P}\left(\mathrm{Poi}(zW^-) = C - 1\right) \mathbf{1}_{\{C>0\}}|W, W^-]|W]]. \end{aligned} \tag{7.2}$$

Since the distribution of $C|W, W^-$ is uniform on $\{0, 1, 2, 3, \dots, \lceil W^- \rceil\}$ using the disintegration theorem ([19, Thm. 6.4.]), it follows that

$$\begin{aligned} \mathbb{E}[W^- \mathbb{P}\left(\mathrm{Poi}(zW^-) = C - 1\right) \mathbf{1}_{C>0}|W, W^-] &= W^- \mathbb{E}[\mathbb{P}\left(\mathrm{Poi}(zW^-) = C - 1\right) \mathbf{1}_{C>0}|W, W^-] \\ &= W^- \sum_{c=1}^{\lceil W^- \rceil} \frac{1}{\lceil W^- \rceil + 1} \mathbb{P}\left(\mathrm{Poi}(zW^-) = c - 1\right) \\ &= \frac{W^-}{\lceil W^- \rceil + 1} \mathbb{P}\left(\mathrm{Poi}(zW^-) \le \lceil W^- \rceil - 1\right) \le 1 \end{aligned}$$

These considerations imply that

$$\mathbb{E}[W^+ W^- \mathbb{P}\left(\mathrm{Poi}(zW^-) = C - 1\right) \mathbf{1}_{C>0}] \le \mathbb{E}[W^+] \tag{7.3}$$

for all z, which justifies differentiation under the integral sign and shows that

$$f^{'}(z, (W^+, W^-, C)) = \mathbb{E}[\mathbb{E}[W^+|W]]\mathbb{E}[\frac{W^-}{\lceil W^- + 1 \rceil} \mathbb{P}\left(\mathrm{Poi}(zW^-) \le \lceil W^- \rceil - 1\right)]] - 1. \tag{7.4}$$

This observation implies that the derivative is strictly decreasing in z and that the fixpoint is stable. The example can be adjusted slightly by choosing $C|W, W^-$ uniform on $\{1, 2, 3, \dots, \lceil W^- \rceil\}$, then $\mathbb{P}(C = 0) = 0$ and analog observations as above show that for $\mathbb{E}[W^+] < 1$ the derivative $f^{'}(z, (W^+, W^-, C))$ is negative for all z and the network is resilient according to Proposition 5.3.

A particularly interesting case in the previous example is given by choosing W power law with some parameter α and $W^-|W$ and $W^+|W$ both Poisson distributed with parameter W. Then both W^- and W^+ are power law distributed with parameter α (see e.g. [22] for a related situation).

In our final example (5.1) is satisfied and therefore any small infection spreads to a positive fraction of the random graph. In contrast to Example 6, the infection is spread mainly by vertices with threshold function 1 and the distribution of (W^-, W^+) can have all moments.

Example 8. *We adjust Example 7 slightly such that $\mathbb{P}(C = 0) = 0$ and $\mathbb{P}(C = 1|W^-) \ge (1 + \delta)/W^-$ for some $\delta > 0$ (this can be done for example by choosing $\mathbb{P}(W^- \ge 1 + \delta) = 1$ and the conditional distribution of $C|(W, W^-)$ such that $\mathbb{P}(C = 1|(W, W^-)) = (1 + \delta)/W^-$ and*

$\mathbb{P}(C = j|(W, W^-)) = (1-(1+\delta)/W^-)/(\lceil W^- \rceil - 1))$ *for* $j \in \{2, 3, \ldots, \lceil W^- \rceil\}$. *Furthermore, assume that* $\mathbb{E}[W^+] > 1$. *To see that Condition 5.1 holds, observe that*

$$\begin{aligned} \mathbb{E}[\mathbb{E}[W^+\mathbb{P}(\mathrm{Poi}(\varepsilon W^-) \geq C)|(W, W^-)]] &\geq \mathbb{E}[\mathbb{E}[W^+|(W, W^-)]\mathbb{E}[\mathbb{P}(\mathrm{Poi}(\varepsilon W^-) \geq 1)\frac{1+\delta}{W^-}] \\ &= \mathbb{E}[\mathbb{E}[W^+|W]e^{-\varepsilon W^-}\varepsilon W^- \frac{1+\delta}{W^-}] \\ &\geq \varepsilon\mathbb{E}[\mathbb{E}[W^+|W]e^{-\varepsilon\bar{W}}(1+\delta)\mathbf{1}_{\{W^- \leq \bar{W}\}}]. \end{aligned} \tag{8.1}$$

Choose $\bar{W}$ *such that* $\mathbb{E}[\mathbb{E}[W^+|W]\mathbf{1}_{\{W^- \leq \bar{W}\}}] \geq (1-\delta_1)\mathbb{E}[W^+]$ *for some* δ_1 *with* $(1-\delta_1)(1+\delta) > 1$. *Then there exists* ε_0 *such that for* $\varepsilon \leq \varepsilon_0$ *this quantity is larger than* $\varepsilon(1+\delta_2)$ *for some* $\delta_2 > 0$ *by* $\mathbb{E}[W^+] > 1$. *Condition 5.1 follows.*

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