Risk-Minimization for Life Insurance Liabilities

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Abstract

In this paper we study the pricing and hedging of a very general class of life insurance liabilities by means of the risk-minimization approach. We find the price and risk-minimizing strategy in two cases, first in the case when the financial market consists only of one risky asset, e.g. a stock, and a bank account, and second in an extended financial market, allowing for investments in two additional traded assets, representing the systematic and unsystematic mortality risk. We also provide an application in the case of a unit-linked term insurance contract in a jump-diffusion model for the stock price and affine stochastic mortality intensity. Main novelties of this work are that we allow for hedging of the risk inherent in the insurance liabilities by investing not only in the stock and money market account, but also in a longevity bond, representing the systematic mortality risk and a pure endowment contract, accounting for the unsystematic mortality risk. Besides that we consider a very general setting regarding the underlying asset price and the structure of the insurance payment process studied, i.e. we work outside the Brownian setting, in particular the asset price may have jumps. Finally we are able to relax certain technical assumptions such as the existence of the mortality intensity and we do not require the independence of the underlying processes.

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1 Introduction

In this paper we study the problem of pricing and hedging life insurance liabilities in a very general setting by means of the well-known risk-minimization approach. First we consider a financial market model with one risky primary asset, e.g. a stock, and the riskless money market account. In this setting we compute the price and hedging strategy for an insurance payment process whose value may depend on the primary assets as well as on the time of death of a single individual, such as a unit-linked life insurance contract. In a second step we extend the financial market by introducing two mortality-linked securities, a longevity bond, incorporating the systematic longevity risk, and a pure endowment contract, representing the unsystematic mortality risk. The main idea then is to hedge the financial and (systematic and unsystematic) mortality risk by investing in both the stock and the bank account, as well as in the two mortality-linked securities.

Mortality or longevity is a primary source of risk for many insurance and pension products. For example, annuity providers face the risk that the mortality rates of pensioners might fall at a faster rate than expected, whereas life insurers are exposed to the risk of unexpected increases in mortality. The traditional method of dealing with mortality risk is through suitable insurance or reinsurance contracts. However, reinsurers are often reluctant to take on the aggregated bulk risk typical of these transactions, thus leading to securitization as a new form of risk transfer and consequently to the creation of a new life market, see, e.g., Blake et al. [10]. In this context pricing and modeling of mortality-linked securities has been studied extensively in the literature, for an overview on the valuation and securitization of mortality risk we refer to Cairns et al. [12].

The mortality risk incorporated in life insurance liabilities can essentially be split into systematic risk, i.e. the risk that the mortality rate of an age cohort differs from the one expected at inception, and idiosyncratic or unsystematic risk, i.e. the risk that the mortality rate of the individual is different from that of its age cohort. The first kind of risk may be hedged by investing in a longevity bond representing the systematic mortality risk, see, e.g., Cairns et al. [12]. This bond pays out the conditional survival probability at maturity as a function of the hazard rate or mortality intensity, which is given by a so-called survivor index. Survivor indices, provided by various investment banks, consist of publicly available mortality data aggregated by population, hence providing a good proxy for the systematic component of the mortality risk. The unsystematic risk however, can only be eliminated by trading in products that depend directly on the time of death, such as a pure endowment contract, i.e. a financial contract that pays 1 at maturity if the individual survived. One of the novelties in our approach is to allow for hedging of the risk inherent in the life insurance liabilities by investing not only in the stock and money market account, but also in the longevity bond, accounting for the systematic mortality risk, and in the pure endowment contract, representing the idiosyncratic mortality risk. We would like to emphasize, that hedging with these two mortality-linked securities is intrinsic in our modeling context, in a sense that

it does not depend on the specific form of the insurance payment process. One may argue, that in the case where large portfolios with independent risks are pooled by insurance companies, the unsystematic risk might be eliminated by law of large number arguments. However, in many cases portfolios with a smaller number of insured lives are of interest. Furthermore in some situations, for instance in the case of catastrophic mortality events, it is not realistic to assume independence between members of the portfolio. Hence hedging of both the systematic and unsystematic mortality risk and thus completely eliminating the cost term may be of great value in many practical applications.
When modeling life insurance liabilities, it is feasible to make use of the similarities between mortality and credit risk, as the time of death can be treated in a very similar way as the default-time of a company. Here we follow the intensity-based or hazard rate approach of reduced-form modeling, see, e.g., Bielecki and Rutkowski [7]. Since the time of death, represented by a totally inaccessible stopping time τ, occurs as a surprise for the market participants, it is impossible to hedge it by using a portfolio consisting only of the primary assets. Hence the primary market extended with the insurance payment process is incomplete and it is thus necessary to select one of the techniques for pricing and hedging in incomplete markets. Here we make use of the popular risk-minimization method first introduced by Föllmer and Sondermann [20]. The idea of this technique is to find a replicating strategy for a given claim, that in general might not necessarily be self-financing, but instead may have a cost. The aim is then to find the replicating strategy with minimal cost in a sense that we discuss in Section 3. This hedging technique has been applied in various areas within financial modeling of incomplete markets, such as for pricing credit derivatives and insurance products that are influenced by an orthogonal source of randomness like mortality and catastrophic risks. There exist a number of studies that focus on applications of the risk-minimization approach in the context of mortality modeling, see, e.g., Biagini et al. [6], Dahl and Møller [16], Dahl et al. [17] and Møller [24, 25]. These authors study quadratic hedging for very specific insurance products in a Brownian setting, whereas we allow for more general assumptions regarding the given filtrations and the structure of the insurance liabilities. Also some authors such as Møller [24, 25] assume independence between the financial market and the insurance model. In the context of credit risk modeling Biagini and Cretarola [3, 4, 5] study local risk-minimization for defaultable claims, again in a Brownian setting. Here we allow for mutual dependence between the time of death and the asset prices behavior as in Biagini and Cretarola [4, 5] and Biagini et al. [6], however we extend their results since we allow for a more general structure of the insurance payment process and we do not require the existence of the mortality intensity. Besides that, similarly as in Barbarin [2], we work outside the Brownian setting, in particular we allow for jumps in the asset price. Hence in this paper we extend earlier work on risk-minimization for insurance products in several directions: we allow for a very general setting regarding the underlying asset price and the structure of the payment process studied, we are able to relax certain technical assumptions such

as the existence of the mortality intensity and we do not require the independence of the underlying processes. We also allow for hedging by investing in the primary assets as well as in two further mortality-linked securities.
The remainder of this paper is organized as follows: Section 2 introduces the general setup and Section 3 briefly reviews the risk-minimization approach in our context. In Section 4, we provide our main result by computing the Galtchouk-Kunita-Watanabe decomposition and finding the price and risk-minimizing strategy of the life insurance liabilities. The financial market is extended by two tradable mortality-linked securities representing the systematic and unsystematic mortality risk in Section 5, thus completing the market and eliminating the cost process. Section 6 then concludes this paper with a specific example where we consider a unit-linked term insurance contract in a jump-diffusion model for the asset price with affine stochastic mortality intensity.

2 The setting

For a fixed time horizon $T > 0$ we consider a simple financial market model defined on a given probability space $(\Omega, \mathcal{G}, \mathbb{P})$ consisting of one risky asset with discounted asset price X and discounted bank account X^0, i.e. $X^0_t \equiv 1$, $t \in [0, T]$. On this probability space we assume given a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$, such that X is a local $(\mathbb{P}, \mathbb{F})$-martingale, i.e. the financial market given by X is arbitrage-free.
We now introduce the *time of death* of an individual, given by a strictly positive random variable $\tau : \Omega \to [0, T] \cup \{\infty\}$, defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$ for each $t \in [0, T]$. Note that since the time horizon T is usually fixed as the maturity of the life insurance contract, in order to ensure that $\mathbb{P}(\tau > T) > 0$ (the remaining lifetime τ is not necessarily bounded by T) it is necessary to allow τ to take values larger than T, indicated here by the convention that τ can assume the value infinity. We define the *death process* $H_t = \mathbb{1}_{\{\tau\leq t\}}$ and denote by $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ the filtration generated by this process.
In this setting we consider the extended market $\mathbb{G} = \mathbb{F}\vee\mathbb{H}$, such that the information available to all agents in the market at time t is assumed to be $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ and we put $\mathcal{G} = \mathcal{G}_T$. It is clear that τ is an $\mathbb{H}$-stopping time, as well as a $\mathbb{G}$-stopping time, but not necessarily an $\mathbb{F}$-stopping time. In fact here we assume that the random time τ avoids every $\mathbb{F}$-stopping time $\tilde{\tau}$, i.e. $\mathbb{P}(\tau = \tilde{\tau}) = 0$, and under this hypothesis we have that τ is a totally inaccessible $\mathbb{G}$-stopping time and $\Delta U_\tau = 0$ for any $\mathbb{F}$-adapted càdlàg process U (see, e.g., Coculescu et al. [13] or Blanchet-Scalliet and Jeanblanc [11]). All filtrations are assumed to satisfy the usual hypotheses of completeness and right-continuity. We postulate that all $\mathbb{F}$-local martingales are also $\mathbb{G}$-local martingales, and in the sequel we refer to this hypothesis as Hypothesis (H). This hypothesis is well-known in the literature of reduced form approaches for valuating defaultable claims, for a discussion of this

hypothesis we refer to Blanchet-Scalliet and Jeanblanc [11]. In this setting we follow the hazard rate or intensity-based approach, well-known from reduced-form modeling of credit derivatives (see, e.g., Bielecki and Rutkowski [7]), which means that as opposed to the structural approach the default time occurs as a surprise for the market participants, since the time of death τ is a totally inaccessible stopping time. Therefore it is not possible to predict τ, and an important role is then played by the conditional distribution function of τ, given by

$$F_t = \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t),$$

and we assume $F_t < 1$ for all $t \in [0, T]$. Then the *hazard process* Γ of τ under $\mathbb{P}$

$$\Gamma_t = -\ln(1 - F_t) = -\ln \mathbb{E}[\mathbb{1}_{\{\tau > t\}} \,|\, \mathcal{F}_t]$$

is well-defined for every $t \in [0, T]$. In particular under the above conditions the hazard process Γ is continuous and increasing (see, e.g., Coculescu et al. [13]) and we additionally assume that Γ_T is bounded. Note that this rather strong assumption is not always required in concrete examples, since it may be possible to directly check the necessary integrability conditions (see also Section 6). The process

$$e^{-\Gamma_t} = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t), \quad t \in [0, T],$$

is often called a *survivor index* and according to Cairns et al. [12] can be seen as the basic building block for many other mortality-linked securities. The need for standardization in the life markets has led to the creation of various such indices by investment banks comprising publicly available mortality data for various age cohorts across populations of many different countries. Therefore many market-traded securities have payments linked to a survivor index, e.g. they pay out the survivor index or a function of the survivor index at maturity T. Hence a fundamental role is played by the $\mathbb{F}$-martingale

$$\mathbb{E}[e^{-\Gamma_T} \,|\, \mathcal{G}_t] = \mathbb{E}[e^{-\Gamma_T} \,|\, \mathcal{F}_t] = \mathbb{E}[\mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{F}_t], \quad t \in [0, T], \tag{2.1}$$

since it represents the information on the mortality risk contained in the filtration $\mathbb{F}$, i.e. it describes the systematic mortality risk as we will see in Section 4 and 5. Note that in the first equation of (2.1) we have used that Hypothesis (H) is equivalent to the fact that conditioning on $\mathcal{G}_t$ can be replaced by conditioning on $\mathcal{F}_t$ for $\mathcal{F}_T$-measurable random variables (see, e.g., Bielecki and Rutkowski [7]).

Commonly the hazard rate process Γ is represented as an integral over the mortality intensity, which itself is given by a diffusion. Here we work in a more general setting, since we do not require the existence of the mortality intensity. Instead we describe the systematic mortality risk component $\mathbb{E}[\mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{F}_t]$, $t \in [0, T]$, as driven by a local $\mathbb{F}$-martingale Y strongly orthogonal to X, see (5.1) in Section 5

We recall that two local martingales X, Y are said to be *strongly orthogonal* if the product $(X_t Y_t)_{t \in [0,T]}$ is a local martingale.

and also (6.4) in Section 6, where Y is given by a Brownian motion. In general, financial markets may be affected by consistent or sudden variations of the mortality rate, hence we a priori do not consider X and Y to be independent. However, we suppose that they are strongly orthogonal, since mortality is external to the financial markets, and not hedgeable by investing only in the primary assets.
By Proposition 5.1.3 of Bielecki and Rutkowski [7] we obtain that the compensated process M given by

$$M_t = H_t - \Gamma_{t \wedge \tau}, \quad t \in [0, T], \tag{2.2}$$

follows a $\mathbb{G}$-martingale. Since M is a finite variation process and X has no jump in τ, by Proposition 4.52 of Jacod and Shiryaev [21, Chapter I] for the square bracket process we have

$$\begin{aligned}[X, M]_t &= \langle X^C, M^C \rangle_t + \sum_{0 \le s \le t} \Delta X_s \Delta M_s \\ &= \sum_{0 \le s \le t} \Delta X_s \Delta M_s = X_0 M_0 = 0,\end{aligned}$$

$t \in [0, T]$, where X^C and M^C denote the continuous martingale parts of X and M. Hence by Proposition 4.50 of Jacod and Shiryaev [21, Chapter I], XM is a local martingale, i.e. X and M are strongly orthogonal. Note that by the same arguments M is in fact strongly orthogonal to any $\mathbb{F}$-adapted local martingale.
In this setting we now introduce a square integrable (discounted) *life insurance payment process* A:

$$A_t = \mathbb{1}_{\{\tau \le t\}} \bar{A}_\tau + \mathbb{1}_{\{t = T\}} \mathbb{1}_{\{\tau > T\}} \tilde{A}, \tag{2.3}$$

where $\bar{A} = (\bar{A}_t)_{t \in [0,T]}$ is an $\mathbb{F}$-predictable process, such that $\mathbb{E}\left[\sup_{t \in [0,T]} \bar{A}_t^2\right] < \infty$ and $\tilde{A}$ is a $\mathcal{G}_T$-measurable random variable, such that $\mathbb{E}[\tilde{A}^2] < \infty$.

Remark 2.1. *We would like to comment on the structure of A as defined in* (2.3)*. The first part*

$$\mathbb{1}_{\{\tau \le T\}} \bar{A}_\tau$$

consists of a so-called term insurance contract*, i.e. the contract pays out $\bar{A}_\tau$ at the random time τ in case of death before T. The second part*

$$\mathbb{1}_{\{\tau > T\}} \tilde{A}$$

is a pure endowment contract*, i.e. the contract pays out $\tilde{A}$ in case of survival until T. It is now widely acknowledged (see, e.g. Barbarin [2], Biffis [8] and Møller [24]) that most mortality linked securities of practical relevance are of the form* (2.3)*. For example, consider an* annuity contract *with accumulated payments up to the time of death given by $C = (C_t)_{t \in [0,T]}$, where C is an $\mathbb{F}$-adapted, non-negative continuous increasing process such that $C_0 = 0$. Then the accumulated payoff can be decomposed as*

$$\int_0^T \mathbb{1}_{\{\tau > s\}} \mathrm{d}C_s = C_\tau \mathbb{1}_{\{\tau \le T\}} + C_T \mathbb{1}_{\{\tau > T\}},$$

i.e. the payoff is given by (2.3) *with* $\bar{A}_t = C_t$ *and* $\tilde{A} = C_T$*. Also note that the form of* A *is in fact very general as a consequence of Lemma 4.4 in Chapter IV.2 of Jeulin [22], where the general form of a* $\mathbb{G}$*-predictable process in terms of* $\mathbb{F}$*-predictable processes is given.*

Recall that the primary financial market is arbitrage-free (but not necessarily complete), and it is a well-known fact that Hypothesis (H) is a sufficient condition for the market given by the larger filtration $\mathbb{G}$ to be arbitrage-free, see, e.g., Blanchet-Scalliet and Jeanblanc [11]. Nevertheless, since it is impossible to hedge a short position in A by investing in a portfolio consisting only of the primary assets our extended market model $\mathbb{G}$ is incomplete even in the case where the reduced market generated by X is complete. In order to find a price and hedge for the life insurance liabilities we therefore make use of a well-known quadratic hedging method for pricing and hedging in incomplete markets, the (local) risk-minimization approach that will be briefly discussed in the following section.

Remark 2.2. *In this work our focus is on risk-minimization with an application to life markets in a general setting, i.e. our modeling framework regarding the two primary assets is generic in a sense that we do not specify the dynamics of the bank account, but instead directly consider everything in a discounted world. In fact various choices for the discounting factor are feasible in this context, such as the so-called* $\mathbb{P}$*-numéraire portfolio, under which according to Platen and Heath [26] the discounted asset prices are local martingales if they are described by continuous processes or in a wide class of jump-diffusion models.*

3 Risk-minimization

The (local) risk-minimization method is a quadratic hedging approach that was first introduced by Föllmer and Sondermann [20] in the case of European type contingent claims and later extended to the case of payment processes by Møller [25] and later Schweizer [29] and Barbarin [2, Chapter 4]. It has since become a popular method for pricing derivative products in incomplete markets that are influenced by orthogonal sources of randomness. In this section for the readers convenience we briefly review all aspects of the theoretical background that are relevant for our purposes. Note that this borrows extensively from Møller [25] and Schweizer [28].

A strictly positive, finite, self-financing portfolio V^* with initial capital 1 is called $\mathbb{P}$-*numéraire portfolio*, if every nonnegative, finite, self-financing portfolio V with initial capital 1, when denominated in units of V^*, forms a supermartingale, that is for $0 \leq s \leq t \leq T$:

$$\mathbb{E}\left[\frac{V_t}{V_t^*}\,\middle|\,\mathcal{F}_s\right] \leq \frac{V_s}{V_s^*}.$$

It is our aim to find a price and hedge of a square integrable American type (discounted) life insurance payment process $A = (A_t)_{t\in[0,T]}$ in the financial market defined in Section 2 by means of the risk-minimization method. Recall that since the discounted asset price X is a local $\mathbb{P}$-martingale the market is arbitrage-free, in particular the measure $\mathbb{P}$ itself belongs to the set of equivalent local martingale measures. However, the market given by X might not be complete, and in particular the extended financial market defined by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ is not complete, since the time of death τ occurs as a surprise to the market and hence represents a kind of "orthogonal" risk. Therefore any structured product relying on information of the time of death cannot be completely hedged by investing in X. Since the market is incomplete, it is in general not possible to find a self-financing hedging strategy that perfectly replicates the insurance payment process. The idea of risk-minimization is to relax the self-financing assumption, allowing for a wider class of admissible strategies (that may not necessarily be self-financing), and to find an optimal hedging strategy with "minimal risk" within this class of strategies that perfectly replicates the life insurance payment process. In the following we now explain how to find the risk-minimizing strategy and explain in what sense this strategy is optimal. We begin with some definitions.

Definition 3.1. *An L^2-*strategy *is a pair $\varphi = (\xi, \xi^0)$, such that ξ is a $\mathbb{G}$-predictable process belonging to $L^2(X)$, with*

$$L^2(X) := \left\{ \xi \,\middle|\, \xi \ \mathbb{G}\text{-predictable},\ \mathbb{E}\left[\int_0^T \xi_s^2 \,\mathrm{d}[X]_s\right]^{1/2} < \infty \right\},$$

and ξ^0 is a $\mathbb{G}$-adapted process such that the discounted value process

$$V_t(\varphi) = \xi_t X_t + \xi_t^0, \quad t \in [0,T],$$

is right-continuous and square integrable.

Note that both the investment in the risky asset ξ as well as the investment in the bank account ξ^0 are both allowed to be $\mathbb{G}$-adapted, i.e. we assume that all agents invest according to information incorporating both the asset price *and* the time of death. For an L^2-strategy the *(cumulative) cost process* $C(\varphi)$ is defined by

$$C_t(\varphi) = V_t(\varphi) - \int_{]0,t]} \xi_s \,\mathrm{d}X_s + A_t, \quad t \in [0,T],$$

describing the accumulated costs of the trading strategy φ during $[0,t]$ including the payments A_t. Note that $V_t(\varphi)$ should therefore be interpreted as the discounted value of the portfolio φ_t held at time t *after* the payments A_t have been made. In particular, $V_T(\varphi)$ is the value of the portfolio upon settlement of all liabilities, and a natural condition is then to restrict to 0-*admissible strategies* satisfying

$$V_T(\varphi) = 0 \quad \mathbb{P}\text{-a.s.}$$

The *risk process* of φ is given by the conditional expected value of the squared future costs

$$R_t(\varphi) = \mathbb{E}[(C_T(\varphi) - C_t(\varphi))^2 \,|\, \mathcal{G}_t], \quad t \in [0,T], \tag{3.1}$$

and is taken as a measure of the hedger's remaining risk. We would like to find a trading strategy that minimizes the risk in a sense we define now.

Definition 3.2. *An L^2-strategy $\varphi = (\xi, \xi^0)$ is called* risk-minimizing*, if for any L^2-strategy $\tilde{\varphi} = (\tilde{\xi}, \tilde{\xi}^0)$ such that $V_T(\tilde{\varphi}) = V_T(\varphi) = 0$ $\mathbb{P}$-a.s., we have*

$$R_t(\varphi) \leq R_t(\tilde{\varphi}), \quad t \in [0,T],$$

i.e. φ pointwise minimizes the risk process introduced in (3.1)*.*

The key to finding the strategy with minimal risk is the well-known Galtchouk-Kunita-Watanabe (GKW) decomposition, see Ansel and Stricker [1]. Since A is square integrable, the expected accumulated total payments may be decomposed by use of the GKW decomposition as

$$\mathbb{E}[A_T \,|\, \mathcal{G}_t] = \mathbb{E}[A_T \,|\, \mathcal{G}_0] + \int_{]0,t]} \xi^A_s \,\mathrm{d}X_s + L^A_t, \quad t \in [0,T], \tag{3.2}$$

where $\xi^A \in L^2(X)$ and L^A is a square integrable martingale null at 0 that is strongly orthogonal to the space of stochastic integrals with respect to X

$$\mathcal{I}^2(X) := \left\{ \int \psi \,\mathrm{d}X \,\middle|\, \psi \in L^2(X) \right\},$$

i.e. for $\psi \in L^2(X)$, $L^A_t \int_0^t \psi \,\mathrm{d}X$, $t \in [0,T]$, is a (uniformly integrable) martingale.

Theorem 3.3. *There exists a unique 0-admissible risk-minimizing L^2-strategy $\varphi = (\xi, \xi^0)$, given by*

$$\begin{aligned} \xi_t &:= \xi^A_t, \\ \xi^0_t &:= V_t(\varphi) - \xi_t X_t, \end{aligned}$$

with discounted value process

$$V_t(\varphi) = \mathbb{E}[A_T \,|\, \mathcal{G}_t] - A_t = \mathbb{E}[A_T \,|\, \mathcal{G}_0] + \int_{]0,t]} \xi_s \,\mathrm{d}X_s + L^A_t - A_t,$$

discounted optimal cost process

$$C_t(\varphi) = \mathbb{E}[A_T \,|\, \mathcal{G}_0] + L^A_t = C_0(\varphi) + L^A_t,$$

and minimal risk process

$$R_t(\varphi) = \mathbb{E}[(L^A_T - L^A_t)^2 \,|\, \mathcal{G}_t],$$

$t \in [0,T]$, where ξ^A and L^A are given by (3.2)*.*

Proof. See Schweizer [28] for the single payoff case or Møller [25] and Schweizer [29] for the extension to the case of payment streams. □

Note that the preceding approach relies heavily on the fact that the discounted asset prices are local martingales under the original measure $\mathbb{P}$. In a more general setting, when the discounted asset price is merely required to be a semimartingale under $\mathbb{P}$, one finds the price by following the *local* risk-minimization technique, see Schweizer [29] or Barbarin [2, Chapter 4]. For more information on (local) risk-minimization and other quadratic hedging approaches we would like to refer the interested reader to the survey paper of Schweizer [28].

4 Risk-minimization for life insurance liabilities

Under the hypotheses of Section 2 we now compute the price and hedging strategy for the life insurance payment process A as introduced in (2.3) by applying the results of Section 3. In order to find a hedging strategy with optimal cost, we compute the GKW decomposition of

$$\mathbb{E}[A_T \mid \mathcal{G}_t] = \underbrace{\mathbb{E}[\mathbb{1}_{\{\tau\leq T\}}\bar{A}_\tau \mid \mathcal{G}_t]}_{a)} + \underbrace{\mathbb{E}[\mathbb{1}_{\{\tau>T\}}\tilde{A} \mid \mathcal{G}_t]}_{b)}, \quad t \in [0,T]. \tag{4.1}$$

We now separately compute the terms $a)$ and $b)$ in (4.1). We start with $a)$.

Lemma 4.1. *Let $\bar{A} = (\bar{A}_t)_{t\in[0,T]}$ be given as in* (2.3)*. Then for $a)$ in* (4.1) *we have the following decomposition:*

$$\mathbb{E}[\mathbb{1}_{\{\tau\leq T\}}\bar{A}_\tau \mid \mathcal{G}_t] = \bar{m}_0 + \int_{]0,t]} \mathbb{1}_{\{\tau\geq s\}} e^{\Gamma_s}\,\mathrm{d}\bar{m}_s + \int_{]0,t]} (\bar{A}_s - e^{\Gamma_s}U_s)\,\mathrm{d}M_s, \quad t \in [0,T],$$

where

$$U_t = \bar{m}_t - \int_0^t \bar{A}_s e^{-\Gamma_s}\,\mathrm{d}\Gamma_s \tag{4.2}$$

and

$$\bar{m}_t = \mathbb{E}\left[\int_0^T \bar{A}_s e^{-\Gamma_s}\,\mathrm{d}\Gamma_s \,\Big|\, \mathcal{F}_t\right]. \tag{4.3}$$

Proof. First note that since Γ is continuous and increasing and M is stopped in τ, by Proposition 5.1.3 of Bielecki and Rutkowski [7] we have that

$$L_t := \mathbb{1}_{\{\tau>t\}} e^{\Gamma_t} = 1 - \int_{]0,t]} L_{s-}\,\mathrm{d}M_s = 1 - \int_{]0,t]} e^{\Gamma_s}\,\mathrm{d}M_s, \quad t \in [0,T]. \tag{4.4}$$

Then by Corollary 5.1.3 of Bielecki and Rutkowski [7] for $t \in [0,T]$ we have

$$\mathbb{E}[\mathbb{1}_{\{\tau\leq T\}}\bar{A}_\tau \mid \mathcal{G}_t] = \mathbb{1}_{\{\tau\leq t\}}\bar{A}_\tau + L_t U_t,$$

where U is given by (4.2) and $\bar{m}$ is given by (4.3). By (4.4) and an application of Itô's formula we get

$$\begin{aligned}L_t U_t &= \bar{m}_0 + \int_{]0,t]} L_{s-}\,\mathrm{d}U_s + \int_{]0,t]} U_{s-}\,\mathrm{d}L_s \\ &= \bar{m}_0 + \int_{]0,t]} \mathbb{1}_{\{\tau\geq s\}} e^{\Gamma_s}\,\mathrm{d}\bar{m}_s - \int_0^t \mathbb{1}_{\{\tau\geq s\}} \bar{A}_s\,\mathrm{d}\Gamma_s - \int_{]0,t]} e^{\Gamma_s} U_s\,\mathrm{d}M_s,\end{aligned}$$

since U has no jumps in τ. Hence for $t \in [0,T]$ we obtain that

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{\{\tau\leq T\}} \bar{A}_\tau \,|\, \mathcal{G}_t] = H_t \bar{A}_\tau + \bar{m}_0 &+ \int_{]0,t]} \mathbb{1}_{\{\tau\geq s\}} e^{\Gamma_s}\,\mathrm{d}\bar{m}_s - \int_0^t \mathbb{1}_{\{\tau\geq s\}} \bar{A}_s\,\mathrm{d}\Gamma_s \\ &- \int_{]0,t]} e^{\Gamma_s} U_s\,\mathrm{d}M_s \\ = \bar{m}_0 &+ \int_{]0,t]} \mathbb{1}_{\{\tau\geq s\}} e^{\Gamma_s}\,\mathrm{d}\bar{m}_s + \int_{]0,t]} (\bar{A}_s - e^{\Gamma_s} U_s)\,\mathrm{d}M_s,\end{aligned}$$

and the result follows. □

Note that Corollary 5.1.3 of Bielecki and Rutkowski [7] requires $\bar{A}$ to be bounded. However, it can be easily seen that this result also holds if $\mathbb{E}[\sup_{t\in[0,T]} \bar{A}_t^2] < \infty$ and we may therefore apply it in our setting. For the second term *b*) of (4.1) we have the following result.

Lemma 4.2. *Let* $\tilde{A} \in L^2(\mathcal{G}_T, \mathbb{P})$. *Then for b*) *in* (4.1) *we have the following decomposition:*

$$\mathbb{E}[\mathbb{1}_{\{\tau>T\}} \tilde{A} \,|\, \mathcal{G}_t] = \tilde{m}_0 + \int_{]0,t]} \mathbb{1}_{\{\tau\geq s\}} e^{\Gamma_s}\,\mathrm{d}\tilde{m}_s - \int_{]0,t]} e^{\Gamma_s} \tilde{m}_s\,\mathrm{d}M_s, \quad t \in [0,T],$$

where

$$\tilde{m}_t = \mathbb{E}[\mathbb{1}_{\{\tau>T\}} \tilde{A} \,|\, \mathcal{F}_t]. \tag{4.5}$$

Proof. By Corollary 5.1.1 of Bielecki and Rutkowski [7] we have

$$\mathbb{E}[\mathbb{1}_{\{\tau>T\}} \tilde{A} \,|\, \mathcal{G}_t] = L_t \tilde{m}_t, \quad t \in [0,T],$$

where $\tilde{m}$ is given by (4.5). By the same arguments as in the proof of Lemma 4.1 we get

$$\begin{aligned}L_t \tilde{m}_t &= \tilde{m}_0 + \int_{]0,t]} L_{s-}\,\mathrm{d}\tilde{m}_s + \int_{]0,t]} \tilde{m}_{s-}\,\mathrm{d}L_s \\ &= \tilde{m}_0 + \int_{]0,t]} \mathbb{1}_{\{\tau\geq s\}} e^{\Gamma_s}\,\mathrm{d}\tilde{m}_s - \int_{]0,t]} e^{\Gamma_s} \tilde{m}_s\,\mathrm{d}M_s,\end{aligned}$$

hence the result follows. □

The next theorem states the most important result of this work, providing the risk-minimizing strategy of the insurance payment process as defined in (2.3).

Theorem 4.3. *In the market model outlined in Section 2, every insurance payment process admits a risk-minimizing strategy* $\varphi = (\xi, \xi^0)$ *given by*

$$\begin{aligned}\xi_t &= \mathbb{1}_{\{\tau \geq t\}} e^{\Gamma_t} \xi_t^m, \\ \xi_t^0 &= V_t - \xi_t X_t = V_t - \mathbb{1}_{\{\tau \geq t\}} e^{\Gamma_t} \xi_t^m X_t,\end{aligned}$$

with discounted value process

$$\begin{aligned}V_t(\varphi) &= \mathbb{E}[A_T \,|\, \mathcal{G}_t] - A_t \\ &= m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \xi_s^m \,\mathrm{d}X_s + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \eta_s^m \,\mathrm{d}Y_s \\ &\quad + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \,\mathrm{d}C_s^m + \int_{]0,t]} \psi_s^M \,\mathrm{d}M_s - A_t \end{aligned} \tag{4.6}$$

and optimal cost process

$$C_t(\varphi) = m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \eta_s^m \,\mathrm{d}Y_s + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \,\mathrm{d}C_s^m + \int_{]0,t]} \psi_s^M \,\mathrm{d}M_s, \tag{4.7}$$

for $t \in [0,T]$*, where the processes* M*,* m*,* ψ^M*,* ξ^m*,* η^m *and* C^m *are introduced respectively in* (2.2) *and* (4.9) - (4.11).

Proof. By Lemma 4.1 and Lemma 4.2 for $t \in [0,T]$ we have that

$$V_t^A := \mathbb{E}[A_T \,|\, \mathcal{G}_t] = m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \,\mathrm{d}m_s + \int_{]0,t]} \psi_s^M \,\mathrm{d}M_s, \tag{4.8}$$

with

$$m_t = \bar{m}_t + \tilde{m}_t = \mathbb{E}\left[\int_0^T \bar{A}_s e^{-\Gamma_s} \,\mathrm{d}\Gamma_s \,\Big|\, \mathcal{F}_t\right] + \mathbb{E}[\mathbb{1}_{\{\tau > T\}} \tilde{A} \,|\, \mathcal{F}_t] \tag{4.9}$$

and

$$\psi_t^M = \bar{A}_t - e^{\Gamma_t}(U_t + \tilde{m}_t), \tag{4.10}$$

where $\bar{m}$, $\tilde{m}$, and U are defined in (4.2), (4.3) and (4.5).

We now compute the martingale representation for the process m as defined in (4.9) in terms of the underlying driving process X and Y, as introduced in Section 2. By Lemma 2.1 of Schweizer [28] for all ξ, η $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T \xi_s^2 \,\mathrm{d}[X]_s\right], \; \mathbb{E}\left[\int_0^T \eta_s^2 \,\mathrm{d}[Y]_s\right] < \infty,$$

we have that the integral processes $\int \xi_s \,\mathrm{d}X_s$, $\int \eta_s \,\mathrm{d}Y_s$ are square integrable $\mathbb{F}$-martingales. Furthermore, since X and Y are strongly orthogonal, by Proposition 4.50 of Jacod and Shiryaev [21, Chapter I] the bracket process $[X,Y]$ is a local martingale, hence

$$\left[\int \xi_s \,\mathrm{d}X_s, \int \eta_s \,\mathrm{d}Y_s\right]_t = \int_0^t \xi_s \eta_s \,\mathrm{d}[X,Y], \quad t \in [0,T],$$

is a local martingale, e.g. by Jacod and Shiryaev [21, Chapter I, 3.23], and since by the Kunita-Watanabe inequality we have

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[0,T]}\Big|\int_0^t \xi_s\eta_s\,\mathrm{d}[X,Y]_s\Big|\right]\\&\leq \mathbb{E}\left[\int_0^T \xi_s^2\,\mathrm{d}[X]_s\right]^{1/2}\mathbb{E}\left[\int_0^T \eta_s^2\,\mathrm{d}[Y]_s\right]^{1/2}<\infty,\end{aligned}$$

it is in fact a (uniformly integrable) martingale, and therefore (again by Proposition 4.50 of Jacod and Shiryaev [21, Chapter I]) the product

$$\int_0^t \xi_s\,\mathrm{d}X_s\cdot\int_0^t \eta_s\,\mathrm{d}Y_s,\quad t\in[0,T],$$

is a (uniformly integrable) martingale, i.e. the two processes are strongly orthogonal. Since by (2.3) and Jensen's inequality for any $t\in[0,T]$ we have

$$\mathbb{E}[\bar{m}_t^2]\leq \mathbb{E}\left[\left(\int_0^T \bar{A}_s\,\mathrm{d}(e^{-\Gamma_s})\right)^2\right]\leq \mathbb{E}\left[\sup_{t\in[0,T]}\bar{A}_t^2\right]<\infty,$$

as well as

$$\mathbb{E}[\tilde{m}_t^2]\leq \mathbb{E}[\tilde{A}^2]<\infty,$$

the process m as given in (4.9) is a square integrable $\mathbb{F}$-martingale as a sum of square integrable martingales. Hence, e.g. by Protter [27, Chapter IV.3], m admits a decomposition

$$m_t = m_0+\int_{]0,t]}\xi_s^m\,\mathrm{d}X_s+\int_{]0,t]}\eta_s^m\,\mathrm{d}Y_s+C_t^m,\quad t\in[0,T],\tag{4.11}$$

where ξ^m, η^m are $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T (\xi_s^m)^2\,\mathrm{d}[X]_s\right],\ \mathbb{E}\left[\int_0^T (\eta_s^m)^2\,\mathrm{d}[Y]_s\right]<\infty,$$

and C^m is a square integrable martingale strongly orthogonal to $\int\xi_s^m\,\mathrm{d}X_s$ and $\int\eta_s^m\,\mathrm{d}Y_s$, i.e. $\int\xi_s^m\,\mathrm{d}X_s\cdot C^m$, $\int\eta_s^m\,\mathrm{d}Y_s\cdot C^m$ are (uniformly integrable) martingales. Therefore we have that

$$\begin{aligned}V_t^A=\mathbb{E}[A_T\,|\,\mathcal{G}_t]=m_0&+\int_{]0,t]}\mathbb{1}_{\{\tau\geq s\}}e^{\Gamma_s}\xi_s^m\,\mathrm{d}X_s+\int_{]0,t]}\mathbb{1}_{\{\tau\geq s\}}e^{\Gamma_s}\eta_s^m\,\mathrm{d}Y_s\\&+\int_{]0,t]}\mathbb{1}_{\{\tau\geq s\}}e^{\Gamma_s}\,\mathrm{d}C_s^m+\int_{]0,t]}\psi_s^M\,\mathrm{d}M_s,\quad t\in[0,T].\end{aligned}\tag{4.12}$$

We now prove that (4.12) is indeed the GKW decomposition of $\mathbb{E}[A_T\,|\,\mathcal{G}_t]$. To this end we need to show that the integral with respect to X in (4.12) is square integrable, and that the process

$$L_t^A:=\int_{]0,t]}\mathbb{1}_{\{\tau\geq s\}}e^{\Gamma_s}\eta_s^m\,\mathrm{d}Y_s+\int_{]0,t]}\mathbb{1}_{\{\tau\geq s\}}e^{\Gamma_s}\,\mathrm{d}C_s^m+\int_{]0,t]}\psi_s^M\,\mathrm{d}M_s,$$

$t \in [0,T]$, is square integrable and strongly orthogonal to the space $\mathcal{I}^2(X)$ of stochastic integrals with respect to X. First note that by (2.3) we have

$$\mathbb{E}[(V_t^A)^2] \le \mathbb{E}[A_T^2] < \infty, \quad t \in [0,T],$$

hence V^A is a square integrable martingale and $\mathbb{E}[[V^A]_T] < \infty$ (see, e.g. Corollary 3 of Theorem 27 in Chapter II of Protter [27]). Since

$$\mathbb{E}[[V^A]_T] = \mathbb{E}\left[\int_0^T (\mathbb{1}_{\{\tau\ge s\}} e^{\Gamma_s})^2 \,\mathrm{d}[m]_s\right] + \mathbb{E}\left[\int_0^T (\psi_s^M)^2 \,\mathrm{d}[M]_s\right],$$

where we have used (4.8) and the fact that m has no jumps in τ, i.e.

$$[m,M]_t = \sum_{0\le s\le t} \Delta m_s \Delta M_s = 0, \quad t \in [0,T],$$

we have that

$$\mathbb{E}\left[\int_0^T (\psi_s^M)^2 \,\mathrm{d}[M]_s\right] < \infty.$$

Besides that since Γ is increasing and Γ_T is bounded, we have that

$$\mathbb{E}\left[\int_0^T (\mathbb{1}_{\{\tau\ge s\}} e^{\Gamma_s} \xi_s^m)^2 \,\mathrm{d}[X]_s\right] < \infty, \tag{4.13}$$

and analogously for the second and third term in (4.12) we have that

$$\mathbb{E}\left[\int_0^T (\mathbb{1}_{\{\tau\ge s\}} e^{\Gamma_s} \eta_s^m)^2 \,\mathrm{d}[Y]_s\right] < \infty, \tag{4.14}$$

as well as

$$\mathbb{E}\left[\int_0^T (\mathbb{1}_{\{\tau\ge s\}} e^{\Gamma_s})^2 \,\mathrm{d}[C^m]_s\right] < \infty.$$

Hence all integrals in (4.12) are square integrable martingales by Lemma 2.1 of Schweizer [28], and L^A is square integrable as the sum of square integrable martingales. Furthermore for a $\mathbb{G}$-predictable process $\psi \in L^2(X)$, i.e.

$$\mathbb{E}\left[\int_0^T \psi_s^2 \,\mathrm{d}[X]_s\right] < \infty, \tag{4.15}$$

we have that

$$\left[\int \psi_s \,\mathrm{d}X_s, \int \mathbb{1}_{\{\tau\ge s\}} e^{\Gamma_s} \eta_s^m \,\mathrm{d}Y_s\right]_t = \int_0^t \mathbb{1}_{\{\tau\ge s\}} e^{\Gamma_s} \psi_s \eta_s^m \,\mathrm{d}[X,Y], \quad t \in [0,T],$$

is a local martingale, e.g. by Jacod and Shiryaev [21, Chapter I, 3.23], and in view of (4.14) and (4.15) and again by the Kunita-Watanabe inequality we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \mathbb{1}_{\{\tau\ge s\}} e^{\Gamma_s} \psi_s \eta_s^m \,\mathrm{d}[X,Y]_s\right|\right] < \infty,$$

i.e. the bracket process is in fact a (uniformly integrable) martingale, and therefore by Proposition 4.50 of Jacod and Shiryaev [21, Chapter I] the product

$$\int_0^t \psi_s \,\mathrm{d}X_s \cdot \int_0^t \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \eta_s^m \,\mathrm{d}Y_s, \quad t \in [0,T],$$

is a (uniformly integrable) martingale. With the same arguments it can easily be seen that

$$\int_0^t \psi_s \,\mathrm{d}X_s \cdot \int_0^t \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \,\mathrm{d}C_s^m, \quad t \in [0,T],$$

is a (uniformly integrable) martingale. Finally

$$\begin{aligned}&\left[\int \psi_s \,\mathrm{d}X_s, \int \psi_s^M \,\mathrm{d}M_s\right]_t \\ &= \int_0^t \psi_s \psi_s^M \,\mathrm{d}[X,M]_s = 0,\end{aligned}$$

for $t \in [0,T]$ since X and M are strongly orthogonal and X has no jumps in τ, i.e. the product

$$\int_0^t \psi_s \,\mathrm{d}X_s \cdot \int_0^t \psi_s^M \,\mathrm{d}M_s, \quad t \in [0,T],$$

is also a (uniformly integrable) martingale. Putting everything together we have obtained that

$$\int_0^t \psi_s \,\mathrm{d}X_s \cdot L_t^A, \quad t \in [0,T],$$

is a (uniformly integrable) martingale for $\psi \in L^2(X)$, i.e. L^A is strongly orthogonal to $\mathcal{I}^2(X)$ and thus (4.12) is the GKW decomposition of $\mathbb{E}[A_T \,|\, \mathcal{G}_t]$. By the results of Section 3 it then follows that the risk-minimizing strategy $\varphi = (\xi, \xi^0)$ is given by

$$\begin{aligned}\xi_t &= \mathbb{1}_{\{\tau \geq t\}} e^{\Gamma_t} \xi_t^m, \\ \xi_t^0 &= V_t - \xi_t X_t,\end{aligned}$$

for $t \in [0,T]$, with discounted value process $V_t(\varphi) = \mathbb{E}[A_T \,|\, \mathcal{G}_t] - A_t$ and optimal cost process $C_t(\varphi) = m_0 + L_t^A$. □

Note that in (4.6) every term is stopped in τ, i.e. the value process is constant after τ. Besides that every integral with respect to the local $\mathbb{F}$-martingales X, Y and C^m contains the ratio

$$\frac{\mathbb{1}_{\{\tau \geq t\}}}{\mathbb{P}(\tau > t \,|\, \mathcal{F}_t)},$$

i.e. the actual survival event divided by the conditional survival probability on $\mathbb{F}$. Also note that the cost process in (4.7) is essentially made up of three components, given in terms of orthogonal integrals with respect to the processes Y, C^m and M. While the component associated to C^m in general cannot be eliminated unless the processes X and Y have the predictable representation property (see Corollary 4.4), the other two integrals with respect to Y and M represent the systematic and unsystematic component of the mortality risk. As we will see in Section 5, these risks may be eliminated by introducing suitable mortality-linked products related to Y and M on the financial market.

Corollary 4.4. *Assume X and Y have the predictable representation property with respect to the filtration $\mathbb{F}$ (see, e.g., Protter [27, Chapter IV.3]). Then $C^m \equiv 0$ in decomposition* (4.11) *and the square integrable martingale m defined in* (4.9) *admits a decomposition*

$$m_t = m_0 + \int_{]0,t]} \tilde{\xi}_s^m \,\mathrm{d}X_s + \int_{]0,t]} \tilde{\eta}_s^m \,\mathrm{d}Y_s, \quad t \in [0,T], \tag{4.16}$$

where $\tilde{\xi}^m$ and $\tilde{\eta}^m$ are $\mathbb{F}$-predictable process satisfying

$$\mathbb{E}\left[\int_0^T (\tilde{\xi}_s^m)^2 \,\mathrm{d}[X]_s\right], \ \mathbb{E}\left[\int_0^T (\tilde{\eta}_s^m)^2 \,\mathrm{d}[Y]_s\right] < \infty.$$

In this case the insurance payment process A as defined in Section 2 admits a risk-minimizing strategy $\tilde{\varphi} = (\tilde{\xi}, \tilde{\xi}^0)$ given by

$$\begin{aligned} \tilde{\xi}_t &= \mathbb{1}_{\{\tau \geq t\}} e^{\Gamma_t} \tilde{\xi}_t^m, \\ \tilde{\xi}_t^0 &= V_t - \tilde{\xi}_t X_t = V_t - \mathbb{1}_{\{\tau \geq t\}} e^{\Gamma_t} \tilde{\xi}_t^m X_t, \end{aligned}$$

with discounted value process

$$\begin{aligned} V_t(\tilde{\varphi}) &= \mathbb{E}[A_T \mid \mathcal{G}_t] - A_t \\ &= m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \tilde{\xi}_s^m \,\mathrm{d}X_s \\ &\qquad + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \tilde{\eta}_s^m \,\mathrm{d}Y_s + \int_{]0,t]} \psi_s^M \,\mathrm{d}M_s - A_t \end{aligned} \tag{4.17}$$

and optimal cost process

$$C_t(\tilde{\varphi}) = m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \tilde{\eta}_s^m \,\mathrm{d}Y_s + \int_{]0,t]} \psi_s^M \,\mathrm{d}M_s,$$

for $t \in [0,T]$, where the processes M, m, ψ^M, $\tilde{\xi}^m$ and $\tilde{\eta}^m$ are introduced respectively in (2.2), (4.9), (4.10) *and* (4.16).

The predictable representation property is often associated with the completeness of the underlying financial market. However, assuming that the predictable representation property holds does not necessarily imply that the financial market is complete and vice versa, since these properties depend largely on the specific characteristics of the underlying driving price processes as well as the structure of the filtration (see, e.g., Cont and Tankov [14, Remark 9.1]).

Note that in Corollary 4.4, (4.16) means that the GKW decomposition of the square integrable $\mathbb{F}$-martingale m has a special structure, where the orthogonal part consists only of the integral with respect to Y. In particular we have that $C^m \equiv 0$ in (4.11) if (X, Y) have the predictable representation property with respect to the filtration $\mathbb{F}$. This is the case for example if $\mathbb{F}$ is generated by two independent Brownian motions driving X and Y. However in more general settings it often may not be possible to decompose m in this way, in fact this is the case in many jump diffusion models. However we will see in Section 6, that if the life insurance payment process has a special structure then it might be possible to find a decomposition of m as in (4.16), even if X and Y do *not* have the predictable representation property with respect to $\mathbb{F}$ (see (6.7) in Section 6).

5 Extending the financial market

We now turn to a more detailed analysis of the cost process in (4.7). If we consider the GKW decomposition as computed in (4.6) for a given payment process, we can see that the cost is generated by the following orthogonal components:

- Y, the driving process of the conditional survival probability,
- M, the compensated jump process of the time of death, and
- C^m, the orthogonal part due to the predictable decomposition of the $\mathbb{F}$-martingale m in (4.9).

Then a natural question is: Can we introduce mortality-linked products into the financial market, that can be used to hedge the cost parts due to Y and M? For illustration purposes in the following we set $C^m \equiv 0$ and for the short rate we assume $r_t \equiv 0$, $t \in [0, T]$, i.e. the bank account is constant. Following Cairns et al. [12] we now assume there exists another risky asset traded on the market, a zero-coupon *longevity bond* with maturity T, $(P_t^T)_{t\in[0,T]}$, with discounted asset price given by

$$P_t^T = \mathbb{E}[\, e^{-\Gamma_T} \mid \mathcal{G}_t] = \mathbb{E}[\, e^{-\Gamma_T} \mid \mathcal{F}_t] = \mathbb{E}[\mathbb{1}_{\{\tau > T\}} | \, \mathcal{F}_t], \quad t \in [0, T],$$

i.e. a zero-coupon bond that pays out the survivor index at maturity. As discussed in Section 2, since it is given by the conditional survival probability, it may be seen as incorporating the systematic mortality risk. Recall that we have defined Y as driving the martingale P^T, i.e.

$$P_t^T = P_0^T + \int_{]0,t]} \zeta_s P_{s-}^T \, \mathrm{d}Y_s, \quad t \in [0, T], \tag{5.1}$$

for an $\mathbb{F}$-predictable process ζ. If $\zeta_t \neq 0$ a.s. for all $t \in [0,T]$, inserting this in (4.17) immediately leads to

$$\begin{aligned} V_t(\tilde{\varphi}) &= \mathbb{E}[A_T \,|\, \mathcal{G}_t] - A_t \\ &= m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \tilde{\xi}^m_s \,\mathrm{d}X_s + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} \frac{e^{\Gamma_s} \tilde{\eta}^m_s}{\zeta_s P^T_{s-}} \,\mathrm{d}P^T_s \\ &\qquad + \int_{]0,t]} \psi^M_s \,\mathrm{d}M_s - A_t, \end{aligned} \tag{5.2}$$

i.e. we obtain the price of the life insurance payment process in terms of the investments in X and P^T, thereby eliminating the cost part associated to the systematic mortality risk. We now assume that there exists a third risky asset actively traded on the market, directly related to the time of death τ. This will finally enable us to eliminate the cost part due to M. We introduce a *pure endowment* contract $E = (E_t)_{t\in[0,T]}$, i.e. a life insurance contract that pays 1 at maturity T if the individual survived, with present value

$$E_t = \mathbb{E}[\mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t], \quad t \in [0,T].$$

As the following computations show, E, P^T and M are closely related to each other and we may find a representation of M in terms of P^T and E. By Lemma 5.1.2 of Bielecki and Rutkowski [7] we have that

$$E_t = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}[\mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{F}_t]}{\mathbb{P}(\tau > t \,|\, \mathcal{F}_t)} = L_t P^T_t, \quad t \in [0,T],$$

where $L_t = (1 - H_t) e^{\Gamma_t}$. By (4.4) and since P^T has no jumps in τ, it follows that

$$\begin{aligned} L_t P^T_t &= L_0 P^T_0 + \int_{]0,t]} L_{s-} \,\mathrm{d}P^T_s + \int_{]0,t]} P^T_{s-} \,\mathrm{d}L_s \\ &= L_0 P^T_0 + \int_{]0,t]} L_{s-} \,\mathrm{d}P^T_s - \int_{]0,t]} P^T_{s-} e^{\Gamma_s} \,\mathrm{d}M_s, \end{aligned}$$

i.e. for $t \in [0,T]$ we have

$$\mathrm{d}E_t = L_{t-} \,\mathrm{d}P^T_t - P^T_{t-} e^{\Gamma_t} \,\mathrm{d}M_t$$

and

$$\mathrm{d}M_t = \frac{\mathbb{1}_{\{\tau \geq t\}}}{P^T_{t-}} \,\mathrm{d}P^T_t - \frac{1}{P^T_{t-} e^{\Gamma_t}} \,\mathrm{d}E_t.$$

Note that $P^T_{t-}, e^{\Gamma_t} \neq 0$ for all $t \in [0,T]$. Hence by inserting this in (5.2) we obtain

$$\begin{aligned} V_t(\tilde{\varphi}) &= \mathbb{E}[A_T \,|\, \mathcal{G}_t] - A_t \\ &= m_0 + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} \tilde{\xi}^m_s \,\mathrm{d}X_s \\ &\qquad + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} \left(\frac{e^{\Gamma_s} \tilde{\eta}^m_s}{\zeta_s P^T_{s-}} + \frac{\psi^M_s}{P^T_{s-}} \right) \mathrm{d}P^T_s - \int_{]0,t]} \frac{\psi^M_s}{P^T_{s-} e^{\Gamma_s}} \,\mathrm{d}E_s - A_t. \end{aligned}$$

In practice insurance companies often trade mortality-linked contracts similar to P^T, where the payoff at maturity is directly linked to a survivor index. Examples of such products include e.g. the EIB/BNP and Swiss Re bonds in 2004 or futures and options on survivor indices (see, e.g., Blake et al. [9]). However, as shown very clearly by the above computations, by themselves these products are not able to offer complete protection against mortality risk, since a remaining source of randomness is directly related to the knowledge of τ, i.e. the unsystematic, individual mortality risk, and requires an additional asset in order to be hedged.

6 Example: unit-linked life insurance

In this section we assume given two independent Brownian motions $W = (W_t)_{t\in[0,T]}$, $W^\mu = (W_t^\mu)_{t\in[0,T]}$ and a compound Poisson process $Q = (Q_t)_{t\in[0,T]}$,

$$Q_t = \sum_{i=1}^{N_t} Y_i, \quad t \in [0,T],$$

where $N = (N_t)_{t\in[0,T]}$ is a Poisson process with intensity $\lambda > 0$ and Y_i are i.i.d. random variables independent of N with $Y_i > -1$ a.s., $i \geq 1$, such that $\mathbb{E}[Y_1] = \beta < \infty$ and $\mathbb{E}[Y_1^2] < \infty$. We then assume that the filtration $\mathbb{F}$ is generated by these three processes, i.e. $\mathbb{F} = \mathbb{F}^W \vee \mathbb{F}^{W^\mu} \vee \mathbb{F}^Q$, where $\mathbb{F}^W$, $\mathbb{F}^{W^\mu}$ and $\mathbb{F}^Q$ are the natural filtrations of W, W^μ and Q. For the discounted asset price process we assume a jump diffusion model

$$\mathrm{d}X_t = \sigma_t X_t \,\mathrm{d}W_t + X_{t-}\,\mathrm{d}\widetilde{Q}_t, \quad X_0 = x, \tag{6.1}$$

for $t \in [0,T]$, with $\widetilde{Q}_t = Q_t - \beta\lambda t$ and $\sigma = (\sigma_t)_{t\in[0,T]}$ is a bounded, $\mathbb{F}$-adapted process. Then X is a local martingale and the solution to (6.1) is given by

$$X_t = x \exp\left\{\int_0^t \sigma_s \,\mathrm{d}W_s - \left(\beta\lambda t + \frac{1}{2}\int_0^t \sigma_s^2 \,\mathrm{d}s\right)\right\} \prod_{i=1}^{N_t}(Y_i + 1), \quad t \in [0,T].$$

Since σ is bounded and by Doob's maximal inequalities we have

$$\mathbb{E}\left[\sup_{t\in[0,T]} X_t^2\right] \leq 4\mathbb{E}[X_T^2] \leq c_1 \exp\{(c-1)\lambda T\} < \infty, \tag{6.2}$$

where $c_1 \in \mathbb{R}_+$ and $c = \mathbb{E}[(Y_1+1)^2]$, hence X is in fact a (uniformly integrable) martingale. We assume that the hazard process admits the following representation:

$$\Gamma_t = \int_0^t \mu_s \,\mathrm{d}s, \quad t \in [0,T], \tag{6.3}$$

where the *default intensity* or *mortality rate* μ is a non-negative $\mathbb{F}$-progressively measurable process. Stochastic mortality modeling has been studied extensively in the literature, see, e.g., Biffis [8], Dahl [15] and Milevsky and Promislow [23] for different modeling approaches for the spot force of mortality in continuous time and Cairns et al. [12] for an overview of existing modeling frameworks for stochastic mortality. Here we follow the affine approach of Dahl [15] and assume that μ is given by the Cox-Ingersoll-Ross model

$$\mathrm{d}\mu_t = (a + b\mu_t)\,\mathrm{d}t + c\sqrt{\mu_t}\,\mathrm{d}W_t^{\mu}, \quad \mu_0 = 0,$$

for $t \in [0,T]$, $b \in \mathbb{R}$ and $a, c \in \mathbb{R}_+$. Note that the process Γ as introduced in (6.3) is not bounded, however we will show later, that the results of Section 4 remain valid in this setting even without this assumption, in particular equations (4.13) and (4.14) still hold. Since μ is an affine process, e.g. by Filipović [19] we have that

$$\mathbb{E}[e^{-\Gamma_T} \mid \mathcal{G}_t] = e^{-\Gamma_t}\mathbb{E}[e^{-\int_t^T \mu_s\,\mathrm{d}s} \mid \mathcal{F}_t^{W^\mu}] = e^{-\Gamma_t}e^{\alpha(t)+\beta(t)\mu_t}, \quad t \in [0,T],$$

where the functions $\alpha(t)$ and $\beta(t)$ satisfy the following equations:

$$\begin{aligned}
\partial_t\alpha(t) &= -a\beta(t), \quad \alpha(T) = 0,\\
\partial_t\beta(t) &= 1 - b\beta(t) - \frac{1}{2}c^2\beta^2(t), \quad \beta(T) = 0,
\end{aligned}$$

$t \in [0,T]$. It is well-known that the explicit solutions are given by

$$\begin{aligned}
\alpha(t) &= \frac{2a}{c^2}\ln\left(\frac{2\gamma e^{(\gamma-b)(T-t)/2}}{(\gamma-b)(e^{\gamma(T-t)}-1)+2\gamma}\right),\\
\beta(t) &= -\frac{2(e^{\gamma(T-t)}-1)}{(\gamma-b)(e^{\gamma(T-t)}-1)+2\gamma},
\end{aligned}$$

$t \in [0,T]$, where $\gamma := \sqrt{b^2+2c^2}$. Then by Itô's formula, since the process $e^{-\Gamma_t+\alpha(t)+\beta(t)\mu_t}$, $t \in [0,T]$, is continuous we get

$$\mathrm{d}(e^{-\Gamma_t}e^{\alpha(t)+\beta(t)\mu_t}) = e^{-\Gamma_t+\alpha(t)+\beta(t)\mu_t}c\sqrt{\mu_t}\beta(t)\,\mathrm{d}W_t^{\mu}, \tag{6.4}$$

hence in this setting the local martingale Y, introduced in Section 2 and (5.1) in Section 5 as the driving process of the martingale $\mathbb{E}[\exp\{-\Gamma_T\} \mid \mathcal{F}_t]$, is given by the Brownian motion W^μ. Note that Q and W, W^μ are independent (see, e.g., Chapter 11 of Shreve [30]) and by simple calculations it is easy to see that W, W^μ are independent if and only if they are strongly orthogonal. Similarly one can also show that $\widetilde{Q}W$ and $\widetilde{Q}W^\mu$ are martingales. Hence in this context we may apply the results of Section 4, since the underlying driving processes are strongly orthogonal martingales. We now study the case where the insurance payment process as defined in Section 2 is given by a (discounted) unit-linked term insurance contract:

$$A_T = \mathbb{1}_{\{\tau \le T\}}X_\tau,$$

i.e. a life insurance contract that pays out the discounted asset price in the case of death prior to maturity. Since X has no jumps in τ, we have that

$$\mathbb{1}_{\{\tau\le T\}}X_\tau = \mathbb{1}_{\{\tau\le T\}}X_{\tau-},$$

i.e. $\bar{A}_t = X_{t-}$ for $t\in[0,T]$ and $\tilde{A}=0$ in (2.3) and consequently for m as defined in (4.9) it follows that

$$\begin{aligned} m_t &= \mathbb{E}\left[\int_0^T X_{s-}e^{-\Gamma_s}\,\mathrm{d}\Gamma_s \,\middle|\, \mathcal{F}_t\right] \\ &= \int_0^t X_s e^{-\Gamma_s}\,\mathrm{d}\Gamma_s + \mathbb{E}\left[\int_t^T X_s e^{-\Gamma_s}\,\mathrm{d}\Gamma_s \,\middle|\, \mathcal{F}_t\right], \quad t\in[0,T]. \end{aligned} \tag{6.5}$$

By the independence of W^μ and W, Q we get

$$\begin{aligned} \mathbb{E}\left[\int_t^T X_s e^{-\Gamma_s}\,\mathrm{d}\Gamma_s \,\middle|\, \mathcal{F}_t\right] &= \int_t^T \mathbb{E}\left[X_s e^{-\Gamma_s}\mu_s \,\middle|\, \mathcal{F}_t\right]\mathrm{d}s \\ &= \int_t^T \mathbb{E}\left[X_s \,\middle|\, \mathcal{F}_t^W \vee \mathcal{F}_t^Q\right]\mathbb{E}\left[e^{-\Gamma_s}\mu_s \,\middle|\, \mathcal{F}_t^{W^\mu}\right]\mathrm{d}s \\ &= X_t\mathbb{E}\left[\int_t^T e^{-\Gamma_s}\mu_s\,\mathrm{d}s \,\middle|\, \mathcal{F}_t^{W^\mu}\right] \\ &= X_t\left(e^{-\Gamma_t} - \mathbb{E}\left[e^{-\Gamma_T} \,\middle|\, \mathcal{F}_t^{W^\mu}\right]\right), \quad t\in[0,T]. \end{aligned} \tag{6.6}$$

Then by (6.5) and (6.6) we have that

$$\begin{aligned} m_t &= \int_0^t X_s e^{-\Gamma_s}\,\mathrm{d}\Gamma_s + X_t e^{-\Gamma_t} - X_t\mathbb{E}\left[e^{-\Gamma_T} \,\middle|\, \mathcal{F}_t^{W^\mu}\right] \\ &= \int_0^t e^{-\Gamma_s}\,\mathrm{d}X_s - X_t\mathbb{E}\left[e^{-\Gamma_T} \,\middle|\, \mathcal{F}_t^{W^\mu}\right], \quad t\in[0,T], \end{aligned}$$

and by inserting (6.4) and again by the independence of W and W^μ we have

$$\mathrm{d}(X_t e^{-\Gamma_t+\alpha(t)+\beta(t)\mu_t}) = e^{-\Gamma_t+\alpha(t)+\beta(t)\mu_t}[\mathrm{d}X_t + X_t c\sqrt{\mu_t}\beta(t)\mathrm{d}W_t^\mu]$$

for $t\in[0,T]$. Therefore we obtain

$$\begin{aligned} m_t &= x(1-e^{\alpha(0)}) + \int_{]0,t]} e^{-\Gamma_s}(1-e^{\alpha(s)+\beta(s)\mu_s})\,\mathrm{d}X_s \\ &\quad - \int_0^t c\sqrt{\mu_s}\beta(s)X_s e^{-\Gamma_s+\alpha(s)+\beta(s)\mu_s}\,\mathrm{d}W_s^\mu, \quad t\in[0,T]. \end{aligned} \tag{6.7}$$

We would now like to comment on the integrability conditions as imposed in Section 4 and how they apply in this context. Note that (6.7) implies that the processes ξ^m and η^m as introduced in (4.11) are well-defined and in this setting given by

$$\xi_t^m = e^{-\Gamma_t}(1-e^{\alpha(t)+\beta(t)\mu_t}), \quad t\in[0,T],$$

and

$$\eta_t^m = -c\sqrt{\mu_t}\beta(t)X_t e^{-\Gamma_t+\alpha(t)+\beta(t)\mu_t}, \quad t \in [0,T].$$

Hence in (4.13) we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T (\mathbb{1}_{\{\tau\geq s\}} e^{\Gamma_s}\xi_s^m)^2 \,\mathrm{d}[X]_s\right] \\
&\leq \mathbb{E}\left[\int_0^T (1-e^{\alpha(s)+\beta(s)\mu_s})^2 \,\mathrm{d}[X]_s\right] \\
&\leq 4\mathbb{E}[[X]_T] \leq c_2\mathbb{E}\left[\sup_{t\in[0,T]} X_t^2\right] < \infty,
\end{aligned}$$

$c_2 \in \mathbb{R}_+$, where we have used (6.2) and the Burkholder-Davis-Gundy inequalities. Furthermore, in (4.14) we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T (\mathbb{1}_{\{\tau\geq s\}} e^{\Gamma_s}\eta_s^m)^2 \,\mathrm{d}[Y]_s\right] \\
&\leq \mathbb{E}\left[\int_0^T (c\sqrt{\mu_s}\beta(s)X_s e^{\alpha(s)+\beta(s)\mu_s})^2 \,\mathrm{d}[Y]_s\right] \\
&\leq c_3\mathbb{E}\left[\sup_{t\in[0,T]} X_t^2\right]\mathbb{E}\left[\int_0^T \mu_s \,\mathrm{d}s\right] < \infty,
\end{aligned}$$

$c_3 \in \mathbb{R}_+$, since $\beta(\,\cdot\,)$ is bounded on $[0,T]$ and the integral over the square root process μ has finite first moments, see, e.g., Dufresne [18]. Hence the results of Section 4 remain valid in the context of this model, even though Γ is not bounded. By (4.2), (6.5) and (6.6) for U_t we obtain

$$U_t = m_t - \int_0^t X_s e^{-\Gamma_s}\,\mathrm{d}\Gamma_s = X_t\left(e^{-\Gamma_t} - \mathbb{E}\left[e^{-\Gamma_T}\,\middle|\,\mathcal{F}_t^{W^\mu}\right]\right), \quad t \in [0,T],$$

hence for ψ_t^M as defined in (4.10) we have

$$\psi_t^M = \bar{A}_t - e^{\Gamma_t}U_t = X_{t-} - X_t(1-e^{\alpha(t)+\beta(t)\mu_t}), \quad t \in [0,T],$$

and again, since X has no jump in τ, in (4.6) we obtain:

$$\begin{aligned}
V_t &= \mathbb{E}[\mathbb{1}_{\{\tau\leq T\}}X_\tau \,|\, \mathcal{G}_t] - A_t \\
&= x(1-e^{\alpha(0)}) + \int_{]0,t]} \mathbb{1}_{\{\tau\geq s\}}(1-e^{\alpha(s)+\beta(s)\mu_s})\,\mathrm{d}X_s \\
&\quad - \int_0^t \mathbb{1}_{\{\tau\geq s\}} c\sqrt{\mu_s}\beta(s)X_s e^{\alpha(s)+\beta(s)\mu_s}\,\mathrm{d}W_s^\mu \\
&\quad + \int_{]0,t]} X_s e^{\alpha(s)+\beta(s)\mu_s}\mathrm{d}M_s - A_t, \quad t \in [0,T],
\end{aligned}$$

or, in the setting of Section 5 with two additional risky assets P^T and E,

$$\begin{aligned}
V_t &= x(1 - e^{\alpha(0)}) + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} (1 - e^{\alpha(s)+\beta(s)\mu_s}) \, \mathrm{d}X_s \\
&\quad - \int_0^t \mathbb{1}_{\{\tau \geq s\}} e^{\Gamma_s} X_s \, \mathrm{d}P_s^T + \int_0^t \frac{1}{P_s^T} \left(\mathbb{1}_{\{\tau \geq s\}} X_s e^{\alpha(s)+\beta(s)\mu_s} \right) \mathrm{d}P_s^T \\
&\quad - \int_{]0,t]} \frac{1}{P_s^T e^{\Gamma_s}} \left(X_s e^{\alpha(s)+\beta(s)\mu_s} \right) \mathrm{d}E_s - A_t \\
&= x(1 - e^{\alpha(0)}) + \int_{]0,t]} \mathbb{1}_{\{\tau \geq s\}} (1 - e^{\alpha(s)+\beta(s)\mu_s}) \, \mathrm{d}X_s \\
&\quad - \int_{]0,t]} X_s \, \mathrm{d}E_s - A_t, \quad t \in [0, T].
\end{aligned}$$

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