Risk-Minimization for Life Insurance Liabilities with Basis Risk

Francesca Biagini* Thorsten Rheinländer† Irene Schreiber‡

Abstract

In this paper we study the pricing and hedging of typical life insurance payment processes for a homogeneous insurance portfolio by means of the well-known risk-minimization approach. We find the price and risk-minimizing strategy in a financial market where we allow for investments in a risky asset and a bank account, as well as a hedging instrument based on a longevity index, representing the systematic mortality risk. Main features of this work are that we allow for hedging of the risk inherent in the life insurance liabilities by investing not only in the risky asset and the money market account, but also in an instrument representing the systematic mortality risk. Thereby we take into account and model the basis risk that arises due to the fact that the insurance company cannot perfectly hedge its exposure by investing in a hedging instrument that is based on the longevity index, not on the insurance portfolio itself. The dependency between the index and the insurance portfolio is described by means of an affine mean-reverting diffusion process with stochastic drift.

Mathematics Subject Classification (2010): 62P05; 91G80, 91G20, 62P20.
JEL Classification: C02, G19, G10.
Key words: life insurance payment processes, affine mortality structure, risk-minimization, martingale representation, basis risk.

*Department of Mathematics, University of Munich, Theresienstraße 39, 80333 Munich, Germany. Email: francesca.biagini@math.lmu.de.

†Financial and Actuarial Mathematics Group, Vienna University of Technology, Wiedner Hauptstrasse 8/105-1, 1040 Vienna, Austria. Email: rheinlan@fam.tuwien.ac.at.

‡Department of Mathematics, University of Munich, Theresienstraße 39, 80333 Munich, Germany. Email: irene.schreiber@math.lmu.de.

The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no [228087].

1 Introduction

Mortality and longevity is a primary source of risk for many insurance and pension products. The traditional method of dealing with mortality risk is through suitable insurance or reinsurance contracts. However, reinsurers are often reluctant to take on the aggregated bulk risk typical of these transactions, thus leading to securitization as a new form of risk transfer and consequently to the creation of a new life market, see, e.g., Blake et al. [10]. In this context pricing and modeling of life insurance liabilities has been studied extensively in the literature, for an overview on the valuation and securitization of mortality risk we refer to Cairns et al. [12].

Mortality risk can essentially be split into systematic risk represented by the mortality intensity, i.e. the risk that the mortality rate of an age cohort differs from the one expected at inception, and idiosyncratic or unsystematic risk, i.e. the risk that the mortality rate of the individual is different from that of its age cohort. The first kind of risk may be hedged by investing in a longevity bond, see, e.g., Cairns et al. [12]. This bond pays out the conditional survival probability at maturity as a function of the hazard rate or mortality intensity, which is given by a so-called longevity or survivor index. Survivor indices, provided by various investment banks, consist of publicly available mortality data aggregated by population, hence providing a good proxy for the systematic component of the mortality risk. One of the important features of our approach is to allow for hedging of the risk inherent in the life insurance liabilities by investing not only in the stock and money market account, but also in the longevity bond, accounting for the systematic mortality risk.

Besides that we explicitly model the longevity basis risk that arises due to the fact, that the hedging instrument is based on an index representing the whole population, not on the insurance portfolio itself. Because of differences in socioeconomic profiles (with respect to e.g., health, income or lifestyle), the mortality rates of the population typically differ from those of the insurance portfolio. Hence the hedge will be imperfect, leaving a residual amount of risk, know as basis risk. There exist a number of empirical studies concerned with quantifying and modeling mortality basis risk, see, e.g., Cairns et al. [13], Coughlan et al. [14], Dowd et al. [19], Jarner and Kryger [25], Li and Hardy [27] and Li and Lee [28]. Li and Lee [28] are the first to study the mortality rate of closely related populations within a global modeling context. They extend the well-known Lee-Carter model by introducing the concept of a global improvement process together with mean-reverting idiosyncratic variations for each population. Cairns et al. [13], Dowd et al. [19] and Jarner and Kryger [25] model the mortality rates of a small population that is a subpopulation of a larger reference population, where the relationship between the large and small population's mortality rates is determined by a mean-reverting stochastic spread. In this paper, similarly as in Biffis [9], we model the mortality intensity of the insurance portfolio together with the intensity of the population by means of a multivariate affine square-root diffusion. The dependency between

the two populations is captured by the fact that the intensity of the insurance portfolio is fluctuating around a stochastic drift, which is given by the mortality intensity of the reference index. This model is intuitive in its interpretation, as well as analytically tractable through its affine structure. Affine models have recently become very popular in many areas of applied financial mathematics, such as exotic option pricing, or interest rate and credit risk modeling. An overview of the theory of affine processes can be found in Duffie et al. [21], as well as in Filipović and Mayerhofer [22] for the case of affine diffusions.
When modeling life insurance liabilities we make use of the similarities between mortality and credit risk and follow the intensity-based or hazard rate approach of reduced-form modeling, see, e.g., Bielecki and Rutkowski [8]. Since it is impossible to completely hedge the financial and mortality risk inherent in the liabilities of the insurance company, even in this setting where we allow for investments in a product representing the systematic mortality risk, the market is incomplete and it is thus necessary to select one of the techniques for pricing and hedging in incomplete markets. Here we make use of the popular risk-minimization method first introduced by Föllmer and Sondermann [23]. The idea of this technique is to allow for a wide class of admissible strategies that in general might not necessarily be self-financing, and to find an optimal hedging strategy with "minimal risk" within this class of strategies that perfectly replicates the given claim. For a survey on risk-minimization and other quadratic hedging methods we refer to Schweizer [33].
There exist a number of studies that focus on applications of the risk-minimization approach in the context of mortality modeling or in related areas such as credit risk, see, e.g., Barbarin [2], Biagini et al. [3, 4, 5, 6, 7], Møller et al. [15, 16, 29, 30] and Riesner [32]. However, some authors such as Møller [29, 30] and Riesner [32] assume independence between the financial market and the insurance model. Furthermore, most authors consider very specific payoff structures. Here we work in a more general setting, i.e. we allow for mutual dependence between the times of death and the financial market, as well as for more general payoff structures similarly as in Barbarin [2] and Biagini et al. [4, 5, 6]. Besides that, similarly as in Biagini et al. [6, 7] and Dahl et al. [16], we allow for hedging of the insurance liabilities by investing not only in the primary financial market, but also in an instrument representing the systematic mortality risk. Dahl et al. [16] also model the dependency between two death counting processes, the first one representing an insurance portfolio and the second one the whole population. They allow for dependency between the mortality intensities via correlated diffusion terms. Here we consider an affine mean-reverting diffusion model with stochastic drift and model the portfolio mortality intensity as depending on the evolution of the intensity of the population. This has the great advantage, of capturing the basis risk between the insurance portfolio and the longevity index in a very natural way, thereby offering an intuitive interpretation while remaining analytically tractable due to the affine structure. Also in this way it is not necessary to artificially introduce a second death counting process representing the population.

Hence in this paper we extend earlier work on risk-minimization for insurance products in several directions: we provide explicit computations of risk-minimizing strategies for a portfolio of life insurance liabilities in a complex setting. Thereby we explicitly take into account and model the basis risk between the insurance portfolio and the longevity index and allow for investments in hedging instruments representing the systematic mortality risk. Besides that we allow for a general structure of the insurance products studied and we do not require certain technical assumptions such as the independence of the financial market and the insurance model.
The remainder of this paper is organized as follows: Section 2 introduces the general setup, including the structure of the insurance portfolio and the financial market. In Section 3 we compute the price and the risk-minimizing strategy of the life insurance payment streams. We also provide specific applications in the context of unit-linked life insurance contracts.

2 The model

Let $T > 0$ be a fixed finite time horizon and $(\Omega, \mathcal{G}, \mathbb{P})$ a probability space equipped with a filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ which contains all available information. We define $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, and put $\mathcal{G} = \mathcal{G}_T$, where $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ is generated by the death counting processes of the insurance portfolio (see Subsection 2.1). The *background filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ contains all information available except the information regarding the individual survival times. Here we define $\mathcal{F}_t = \sigma\{(W_s, W^\mu_s, W^{\bar{\mu}}_s) : 0 \le s \le t\}$, $t \in [0,T]$, where W, W^μ and $W^{\bar{\mu}}$ are independent Brownian motions driving the financial market and the mortality intensities (see Subsections 2.1 and 2.2). In the following we introduce the three components of the model: the insurance portfolio, the financial market and the combined model.

2.1 Insurance portfolio and mortality intensities

We consider a portfolio of n lives all aged x at time 0, with *death counting process*

$$N_t = \sum_{i=1}^{n} \mathbb{1}_{\{\tau^{x,i} \le t\}}, \quad t \in [0,T],$$

where $\tau^{x,i} : \Omega \to [0,T] \cup \{\infty\}$, and for convenience in the following we omit the dependency on x. We assume that $\mathbb{P}(\tau^i = 0) = 0$ and $\mathbb{P}(\tau^i > t) > 0$ for $i = 1,\dots,n$ and $t \in [0,T]$. Note that since the time horizon T is usually fixed as the maturity of the life insurance liabilities, in order to ensure that $\mathbb{P}(\tau^i > T) > 0$ for $i = 1,\dots,n$ (the remaining lifetimes are not necessarily bounded by T), it is necessary to allow τ^i to take values larger than T, indicated here by the convention that τ^i can assume the value infinity. We define $\mathcal{H}_t = \mathcal{H}^1_t \vee \dots \vee \mathcal{H}^n_t$, with $\mathcal{H}^i_t = \sigma\{H^i_s : 0 \le s \le t\}$ and $H^i_t = \mathbb{1}_{\{\tau^i \le t\}}$. We assume that the times of death τ^i

are totally inaccessible $\mathbb{G}$-stopping times, and an important role is then played by the conditional distribution function of τ^i, given by

$$F_t^i = \mathbb{P}(\tau^i \leq t \,|\, \mathcal{F}_t), \quad i = 1, \dots, n,$$

and we assume $F_t^i < 1$ for all $t \in [0, T]$. Then the *hazard process* Γ^i of τ^i under $\mathbb{P}$

$$\Gamma_t^i = -\ln(1 - F_t^i) = -\ln \mathbb{E}[\mathbb{1}_{\{\tau^i > t\}} \,|\, \mathcal{F}_t], \tag{2.1}$$

is well-defined for every $t \in [0, T]$. Since the insurance portfolio is homogenous in the sense that all individuals belong to the same age cohort, we set $\Gamma^i = \Gamma$, where

$$\Gamma_t = \int_0^t \mu_s \,\mathrm{d}s, \quad t \in [0, T].$$

Similarly as in Biffis [9], we assume that the *mortality intensity* μ is given as the solution of the following set of stochastic differential equations:

$$\mathrm{d}\mu_t = \gamma_1(\bar{\mu}_t - \mu_t)\mathrm{d}t + \sigma_1\sqrt{\mu_t}\,\mathrm{d}W_t^{\mu}, \tag{2.2}$$

$$\mathrm{d}\bar{\mu}_t = \gamma_2(m(t) - \bar{\mu}_t)\mathrm{d}t + \sigma_2\sqrt{\bar{\mu}_t}\,\mathrm{d}W_t^{\bar{\mu}}, \tag{2.3}$$

for $t \in [0, T]$ and $\mu_0 = \bar{\mu}_0 = 0$, where γ_1, γ_2, σ_1, $\sigma_2 > 0$, and $m : [0, T] \to \mathbb{R}_+$ is a continuous deterministic function. The existence and uniqueness of a strong solution $(\mu, \bar{\mu})$ to the set of stochastic differential equations (2.2) - (2.3) is proved in Appendix E of Biffis [9] by using Proposition 2.13 and 2.18 in Chapter 5 of Karatzas and Shreve [26], as well as results of Deelstra and Delbaen [18]. The process $\bar{\mu}$ represents the mortality intensity of the equivalent age cohort of the population, and can be derived by means of publicly available data of the *survivor index*

$$S_t^{\bar{\mu}} = \exp\left(-\int_0^t \bar{\mu}_s \,\mathrm{d}s\right), \quad t \in [0, T]. \tag{2.4}$$

According to Cairns et al. [12] survivor indices can be seen as basic building blocks for many mortality-linked securities. The need for standardization in the life markets has led to the creation of various such indices by investment banks and many market traded securities have payments linked to survivor indices. The dynamics of $\bar{\mu}$ in (2.3) are given by a non-negative affine square root diffusion, mean-reverting towards the deterministic drift m, which can be seen as best estimate for $\bar{\mu}$, such as an available mortality table. Hence μ as defined in (2.2) is a non-negative affine process, fluctuating around a stochastic drift given by the mortality intensity $\bar{\mu}$ of the respective age cohort of the population. Note that many empirical studies have shown that the mortality of life insurance portfolios is often lower than that of the equivalent age cohort of the population due to socioeconomic factors such as lifestyle, income, etc. This characteristic feature can easily be incorporated in our model e.g., by replacing the stochastic drift $\bar{\mu}$ by $\bar{\mu} - \varepsilon$ and m by $m - \varepsilon$ for a constant $\varepsilon > 0$ in (2.2) - (2.3).
We also assume that for $i \neq j$, τ^i, τ^j are conditionally independent given $\mathcal{F}_T$, i.e.

$$\mathbb{E}[\mathbb{1}_{\{\tau^i > t\}}\mathbb{1}_{\{\tau^j > s\}} \,|\, \mathcal{F}_T] = \mathbb{E}[\mathbb{1}_{\{\tau^i > t\}} \,|\, \mathcal{F}_T]\,\mathbb{E}[\mathbb{1}_{\{\tau^j > s\}} \,|\, \mathcal{F}_T], \quad 0 \leq s, t \leq T. \tag{2.5}$$

This assumption is well-known in the literature of credit risk modeling, see, e.g., Chapter 9 of Bielecki and Rutkowski [8]. All individuals within the insurance portfolio are subject to idiosyncratic risk factors, as well as common risk factors, given by the information represented by the background filtration $\mathbb{F}$. Intuitively, the assumption of conditional independence means that given all common risk factors are known, the idiosyncratic risk factors become independent of each other.

2.2 The financial market

Since our focus is on modeling the basis risk between the insurance portfolio and the longevity index, for simplicity we consider a rather simple model of a financial market defined on $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ consisting of a bank account or numéraire B with constant short rate $r > 0$, i.e.

$$B_t = \exp\{rt\}, \quad t \in [0, T],$$

as well as two risky assets with asset prices S and P. We assume that S follows the $\mathbb{P}$-dynamics

$$\mathrm{d}S_t = S_t \left(r\mathrm{d}t + \sigma(t, S_t)\, \mathrm{d}W_t \right), \quad t \in [0, T], \tag{2.6}$$

with $S_0 = s$ and we assume that σ satisfies certain regularity conditions that ensure the existence and uniqueness of a solution to (2.6). We denote by $X = S/B$ the discounted asset price, i.e. the dynamics of X are given by

$$\mathrm{d}X_t = \mathrm{d}\left(\frac{S_t}{B_t} \right) = \sigma(t, S_t) X_t \, \mathrm{d}W_t, \quad t \in [0, T]. \tag{2.7}$$

Following Cairns et al. [12] we assume that P is the price process of a *longevity bond* with maturity T representing the systematic mortality risk, i.e. P is defined as a zero-coupon bond that pays out the value of the survivor or longevity index as defined in (2.4) at T. This means the discounted value process $Y = P/B$ is given by

$$Y_t = \mathbb{E}\left[\frac{S_T^{\bar{\mu}}}{B_T} \,\middle|\, \mathcal{G}_t \right] = \mathbb{E}\left[\frac{\exp(-\int_0^T \bar{\mu}_s \, \mathrm{d}s)}{B_T} \,\middle|\, \mathcal{G}_t \right], \quad t \in [0, T]. \tag{2.8}$$

Thus the discounted asset prices X, Y are continuous (local) $(\mathbb{P}, \mathbb{F})$-martingales, i.e. the financial market is arbitrage-free, since the physical measure $\mathbb{P}$ belongs to the set of equivalent local martingale measures. Note that the asset prices are $\mathbb{F}$-adapted, however the trading strategies are allowed to be $\mathbb{G}$-adapted, i.e. in the following sections we consider (discounted) hedging portfolios

$$V_t(\varphi) = \xi_t^X X_t + \xi_t^Y Y_t + \xi_t^0, \quad t \in [0, T],$$

where $\varphi = (\xi^X, \xi^Y, \xi^0)$ is a $\mathbb{G}$-adapted process (see also Definition A.1 in Appendix A). This implies that all agents invest according to information incorporating the common risk factors such as the financial market and the mortality intensities, as well as the individual times of death.

2.3 The combined model

We consider the extended market $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, such that the information available in the market at time $t \in [0, T]$ is assumed to be $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. All filtrations are assumed to satisfy the usual hypotheses of completeness and right-continuity. We postulate that all $\mathbb{F}$-local martingales are also $\mathbb{G}$-local martingales, and in the sequel we refer to this hypothesis as Hypothesis (H). This hypothesis is well-known in the literature on enlargements of filtrations, for a discussion of this hypothesis we refer to Blanchet-Scalliet and Jeanblanc [11] and Bielecki and Rutkowski [8, Chapter 6]. In this setting we study life insurance liabilities in form of insurance payment streams as introduced by Møller [30]. It is now widely acknowledged (see, e.g. Barbarin [2], Biffis [9] and Møller [29]) that most payment streams of practical relevance are covered by the three building blocks pure endowment-, term insurance-, and annuity contracts. Following Barbarin [2], the *pure endowment contract* consists of a payoff

$$C^{pe}\left(n - N_T\right) \tag{2.9}$$

at T, where C^{pe} is a non-negative $\mathcal{F}_T$-measurable random variable such that $\mathbb{E}[(C^{pe})^2] < \infty$, i.e. the insurer pays the amount C^{pe} at the term T of the contract to every policyholder of the portfolio who has survived until T. The *term insurance contract* is defined as

$$\int_0^T C_s^{ti}\,\mathrm{d}N_s = \sum_{i=1}^n \int_0^T C_s^{ti}\,\mathrm{d}H_s^i = \sum_{i=1}^n \mathbb{1}_{\{\tau^i \le T\}} C_{\tau^i}^{ti}, \tag{2.10}$$

where C^{ti} is assumed to be a non-negative $\mathbb{F}$-predictable process such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} (C_t^{ti})^2\right] < \infty,$$

i.e. the amount $C_{\tau^i}^{ti}$ is payed at the time of death τ^i to every policyholder i, $i = 1, \dots, n$. The *annuity contract* consists of multiple payoffs the insurer has to pay as long as the policyholders are alive. We model these payoffs through their cumulative value C_t^a up to time t, where C^a is assumed to be a right-continuous, non-negative increasing $\mathbb{F}$-adapted process such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} (C_t^{a})^2\right] < \infty.$$

The cumulative payment up to time T is then given by

$$\int_0^T (n - N_s)\,\mathrm{d}C_s^a = \sum_{i=1}^n \int_0^T \left(1 - H_s^i\right)\mathrm{d}C_s^a. \tag{2.11}$$

Similarly as in Møller [29] or Riesner [32] we also provide specific examples (see Corollary 3.6, Corollary 3.8 and Corollary 3.10) where in the context of unit-linked life insurance products we set $C^{pe} = f(S_T)$, $C^{ti}_t = f(S_t)$ and $C^a_t = \int_0^t f(S_u)\,\mathrm{d}u$ in (2.9) - (2.11) for a function f that satisfies sufficient regularity conditions. Recall that (X, Y) is a $(\mathbb{P}, \mathbb{F})$-local martingale, i.e. the market given by $(\mathbb{P}, \mathbb{F})$ is arbitrage-free, and Hypothesis (H) implies that the extended financial market defined by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ is also arbitrage-free. However, the market is not complete since the times of death occur as a surprise to the market and hence represent a kind of "orthogonal" risk. In particular any derivative relying on information of the individual times of death such as the insurance liabilities introduced in (2.9) - (2.11) cannot be perfectly hedged by investing in (X, Y). Therefore in the following section in order to find a price and hedging strategy for the insurance payment processes, we make use of a well-known quadratic hedging method for pricing and hedging in incomplete markets, the risk-minimization approach, a brief review of which is given in Appendix A.

Remark 2.1. *We would like to briefly comment on the fact that the insurance payment streams introduced in* (2.9) - (2.11) *can actually also be interpreted as T-claims, i.e. non-negative $\mathcal{G}_T$-measurable random variables, hence the risk-minimizing strategies may equivalently be found by means of the original method by Föllmer and Sondermann [23]. To this end note that the pure endowment contract consists of a single payoff at time T, hence it is a European type contingent claim by definition. Furthermore, the discounted term insurance contract has the same payoff as the discounted T-claim*

$$H = B_T^{-1} \sum_{i=1}^{n} \mathbb{1}_{\{\tau^i \le T\}} C^{ti}_{\tau^i} B^{-1}_{\tau^i} B_T = \sum_{i=1}^{n} \int_0^T C^{ti}_s B^{-1}_s \,\mathrm{d}H^i_s = \int_0^T C^{ti}_s B^{-1}_s \,\mathrm{d}N_s,$$

where the insurer's liabilities C^{ti} are deferred and accumulated using the riskless asset B. By the same arguments the annuity contract can also be interpreted as discounted T-claim. In Remark A.3 of Møller [30] it is shown how in this case the approaches of Föllmer and Sondermann [23] and Møller [30] coincide in the sense that they deliver equivalent risk-minimizing strategies. In particular the investment in the risky assets is equal in both settings. The portfolio value process and investment in the riskless asset differ only in the sense that the portfolio value in the payment stream setting is seen as after *the insurance payments have been settled, whereas the value process in the setting of Föllmer and Sondermann [23] accounts for the insurance liabilities by accumulating them on the bank account as deferred payments.*

3 Risk-minimization for life insurance liabilities

In the setting of Section 2 we now compute the price and hedging strategy for the life insurance liabilities by applying the results of Appendix A. We start with some preliminary results.

3.1 Preliminary results

For $i = 1, \dots, n$ we consider the finite variation process

$$L_t^i = (1 - H_t^i)e^{\Gamma_t}, \quad t \in [0,T],$$

then by Lemma 5.1.7 of Bielecki and Rutkowski [8] we have that L^i is a local $\mathbb{G}^i$-martingale, where $\mathbb{G}^i := (\mathcal{G}_t^i)_{t\in[0,T]}$ and $\mathcal{G}_t^i = \mathcal{F}_t \vee \mathcal{H}_t^i$, $t \in [0,T]$, $i = 1, \dots, n$. Since the hazard process Γ_t, $t \in [0,T]$, of τ^i exists and is continuous and increasing, by Proposition 5.1.3 of Bielecki and Rutkowski [8] we have that the compensated process M^i given by

$$M_t^i = H_t^i - \Gamma_{t\wedge\tau^i}, \quad t \in [0,T], \tag{3.1}$$

follows a local $\mathbb{G}^i$-martingale, such that

$$M_t^i = -\int_{]0,t]} e^{-\Gamma_s}\, \mathrm{d}L_s^i, \quad t \in [0,T], \tag{3.2}$$

and

$$L_t^i = 1 - \int_{]0,t]} L_{s-}^i\, \mathrm{d}M_s^i, \quad t \in [0,T].$$

Furthermore, since

$$\mathbb{E}[[M^i]_T] = \mathbb{E}[H_T^i] \leq 1 < \infty, \quad i = 1, \dots, n,$$

e.g., by Protter [31, Corollary 4 after Theorem 27 in Chapter II] M^i is a square integrable martingale. From (3.1) we have that the $\mathbb{F}$-hazard process Γ and the $(\mathbb{F}, \mathbb{G}^i)$-martingale hazard process Λ^i of τ^i coincide. By (2.5) (see, e.g., Lemma 9.1.1 of Bielecki and Rutkowski [8]) M^i is also a $\mathbb{G}$-martingale, i.e. Γ is also the $(\mathbb{F}, \mathbb{G})$-martingale hazard process of τ^i. Note that it is easily seen that for $j \neq i$, (2.5) implies that $L^i L^j$ is a local $\mathbb{G}$-martingale (see also Proposition 6.1 in Chapter 3 of Barbarin [2]), hence L^i and L^j are strongly orthogonal. Then by (3.2) we have that M^i and M^j are also strongly orthogonal. Note that since M^i are $\mathbb{G}$-martingales and

$$M_t := \sum_{i=1}^n M_t^i = N_t - \int_0^t (n - N_{s-})\mu_s\, \mathrm{d}s, \quad t \in [0,T], \tag{3.3}$$

is a $\mathbb{G}$-martingale, the process $(\int_0^t (n - N_{s-})\mu_s\, \mathrm{d}s)_{t\in[0,T]}$ is the $\mathbb{G}$-compensator of N. In the following by making use of the affine structure of $(\mu, \bar{\mu})$ as introduced in (2.2) and (2.3) we compute the dynamics of different processes related to $(\mu, \bar{\mu})$, such as the longevity bond introduced in (2.8), that will be needed for the computations in Sections 3.2 - 3.4.

Lemma 3.1. *For the longevity bond as introduced in* (2.8) *we have the dynamics:*

$$Y_t = Y_0 + \int_0^t Y_s e^{-rT} \beta^T(s) \sigma_2 \sqrt{\bar{\mu}_s}\, \mathrm{d}W_s^{\bar{\mu}}, \quad t \in [0,T],$$

where β^T is given by the following differential equation:

$$\partial_t \beta^T(t) = 1 + \gamma_2 \beta^T(t) - \frac{1}{2}\sigma_2^2(\beta^T(t))^2, \quad \beta^T(T) = 0. \tag{3.4}$$

Proof. We rewrite $(\mu, \bar{\mu})$ as introduced in (2.2) and (2.3) as

$$\mathrm{d}\begin{pmatrix}\mu_t \\ \bar{\mu}_t\end{pmatrix} = \begin{pmatrix}0 \\ \gamma_2 m(t)\end{pmatrix} + \begin{pmatrix}-\gamma_1 & \gamma_1 \\ 0 & -\gamma_2\end{pmatrix}\begin{pmatrix}\mu_t \\ \bar{\mu}_t\end{pmatrix}\mathrm{d}t + \begin{pmatrix}\sigma_1\sqrt{\mu_t} & 0 \\ 0 & \sigma_2\sqrt{\bar{\mu}_t}\end{pmatrix}\mathrm{d}\begin{pmatrix}W_t^{\mu} \\ W_t^{\bar{\mu}}\end{pmatrix}$$

for $t \in [0, T]$, i.e. $(\mu, \bar{\mu})$ is affine. By equation (B.1) in Appendix B we immediately obtain

$$\tilde{Y}_t := \mathbb{E}\left[\exp\left(-\int_t^T \bar{\mu}_s \,\mathrm{d}s\right) \,\Big|\, \mathcal{F}_t\right] = \exp(\alpha^T(t) + \beta^T(t)\bar{\mu}_t), \quad t \in [0, T],$$

where

$$\partial_t \beta^T(t) = 1 + \gamma_2 \beta^T(t) - \frac{1}{2}\sigma_2^2(\beta^T(t))^2, \quad \beta^T(T) = 0,$$

and

$$\partial_t \alpha^T(t) = -\gamma_2 m(t)\beta^T(t), \quad \alpha^T(T) = 0.$$

Then by Itô's formula we obtain

$$\begin{aligned}\mathrm{d}\tilde{Y}_t &= \tilde{Y}_t(\partial_t \alpha^T(t) + \partial_t \beta^T(t)\bar{\mu}_t)\mathrm{d}t + \tilde{Y}_t\beta^T(t)\mathrm{d}\bar{\mu}_t + \frac{1}{2}\tilde{Y}_t(\beta^T(t))^2\mathrm{d}\langle\bar{\mu}\rangle_t \\ &= \tilde{Y}_t(\bar{\mu}_t \,\mathrm{d}t + \beta^T(t)\sigma_2\sqrt{\bar{\mu}_t}\,\mathrm{d}W_t^{\bar{\mu}}),\end{aligned}$$

and by (2.8) we have that

$$\mathrm{d}Y_t = Y_t e^{-rT}\beta^T(t)\sigma_2\sqrt{\bar{\mu}_t}\,\mathrm{d}W_t^{\bar{\mu}}, \quad t \in [0, T],$$

hence the result follows. □

The following lemma will be needed in the proofs of Corollary 3.6 and 3.10.

Lemma 3.2. *Fix $u \in [0, T]$. For*

$$Z_t^u := \mathbb{E}[\exp(-\Gamma_u) \,|\, \mathcal{F}_t] = \mathbb{E}\left[\exp\left(-\int_0^u \mu_s \,\mathrm{d}s\right) \,\Big|\, \mathcal{F}_t\right], \quad t \in [0, u],$$

we have the following dynamics:

$$Z_t^u = Z_0^u + \int_0^t Z_s^u \beta_1^u(s)\sigma_1\sqrt{\mu_s}\,\mathrm{d}W_s^{\mu} + \int_0^t Z_s^u \beta_2^u(s)\sigma_2\sqrt{\bar{\mu}_s}\,\mathrm{d}W_s^{\bar{\mu}}, \tag{3.5}$$

where β_1^u and β_2^u are given by the following differential equations:

$$\partial_t \beta_1^u(t) = 1 + \gamma_1 \beta_1^u(t) - \frac{1}{2}\sigma_1^2(\beta_1^u(t))^2, \quad \beta_1^u(u) = 0, \tag{3.6}$$

$$\partial_t \beta_2^u(t) = -\gamma_1 \beta_1^u(t) + \gamma_2 \beta_2^u(t) - \frac{1}{2}\sigma_2^2(\beta_2^u(t))^2, \quad \beta_2^u(u) = 0. \tag{3.7}$$

Proof. Fix $u \in [0,T]$. With the same arguments as in the proof of Lemma 3.1, by equation (B.1) in Appendix B we have

$$\tilde{Z}_t^u := \mathbb{E}\left[\exp\left(-\int_t^u \mu_s \,\mathrm{d}s\right) \Big| \mathcal{F}_t\right] = \exp(\alpha^u(t) + \beta_1^u(t)\mu_t + \beta_2^u(t)\bar{\mu}_t) \tag{3.8}$$

for $t \in [0,u]$, where the functions α^u, β_1^u and β_2^u are given by

$$\begin{aligned}
&\partial_t \beta_1^u(t) = 1 + \gamma_1 \beta_1^u(t) - \frac{1}{2}\sigma_1^2(\beta_1^u(t))^2, \quad \beta_1^u(u) = 0,\\
&\partial_t \beta_2^u(t) = -\gamma_1 \beta_1^u(t) + \gamma_2 \beta_2^u(t) - \frac{1}{2}\sigma_2^2(\beta_2^u(t))^2, \quad \beta_2^u(u) = 0,\\
&\partial_t \alpha^u(t) = -\gamma_2 m(t)\beta_2^u(t), \quad \alpha^u(u) = 0.
\end{aligned}$$

Then, again by an application of Itô's formula, we obtain that

$$\begin{aligned}
\mathrm{d}\tilde{Z}_t^u &= \tilde{Z}_t^u\Big[(\partial_t\alpha^u(t) + \partial_t\beta_1^u(t)\mu_t + \partial_t\beta_2^u(t)\bar{\mu}_t)\,\mathrm{d}t + \beta_1^u(t)\,\mathrm{d}\mu_t + \beta_2^u(t)\,\mathrm{d}\bar{\mu}_t\\
&\qquad + \frac{1}{2}(\beta_1^u(t))^2\,\mathrm{d}\langle\mu\rangle_t + \frac{1}{2}(\beta_2^u(t))^2\,\mathrm{d}\langle\bar{\mu}\rangle_t + \beta_1^u(t)\beta_2^u(t)\,\mathrm{d}\langle\mu,\bar{\mu}\rangle_t\Big]\\
&= \tilde{Z}_t^u(\mu_t\,\mathrm{d}t + \beta_1^u(t)\sigma_1\sqrt{\mu_t}\,\mathrm{d}W_t^\mu + \beta_2^u(t)\sigma_2\sqrt{\bar{\mu}_t}\,\mathrm{d}W_t^{\bar{\mu}})
\end{aligned}$$

for $t \in [0,u]$, hence the result follows. □

The following lemma will be needed in the proof of Corollary 3.8.

Lemma 3.3. *Fix $u \in [0,T]$. For*

$$Z_t^{\mu,u} := \mathbb{E}\left[\exp\left(-\int_0^u \mu_s\,\mathrm{d}s\right)\mu_u \Big| \mathcal{F}_t\right], \quad t \in [0,u],$$

we have the following dynamics:

$$\begin{aligned}
Z_t^{\mu,u} = Z_0^{\mu,u} &+ \int_0^t Z_s^u\left(\hat{\beta}_1^u(s) + \beta_1^u(s)\hat{Z}_s^u\right)\sigma_1\sqrt{\mu_s}\,\mathrm{d}W_s^\mu\\
&+ \int_0^t Z_s^u\left(\hat{\beta}_2^u(s) + \beta_2^u(s)\hat{Z}_s^u\right)\sigma_2\sqrt{\bar{\mu}_s}\,\mathrm{d}W_s^{\bar{\mu}},
\end{aligned} \tag{3.9}$$

where $\hat{Z}^u$ is given by

$$\hat{Z}_t^u = \hat{\alpha}^u(t) + \hat{\beta}_1^u(t)\mu_t + \hat{\beta}_2^u(t)\bar{\mu}_t, \quad t \in [0,u], \tag{3.10}$$

and $\hat{\alpha}^u$, $\hat{\beta}_1^u$ and $\hat{\beta}_2^u$ are given by the following differential equations:

$$\partial_t\hat{\beta}_1^u(t) = \gamma_1\hat{\beta}_1^u(t) - \sigma_1^2\beta_1^u\hat{\beta}_1^u(t), \quad \hat{\beta}_1^u(u) = 1, \tag{3.11}$$

$$\partial_t\hat{\beta}_2^u(t) = -\gamma_1\hat{\beta}_1^u(t) + \gamma_2\hat{\beta}_2^u(t) - \sigma_2^2\beta_2^u(t)\hat{\beta}_2^u(t), \quad \hat{\beta}_2^u(u) = 0, \tag{3.12}$$

$$\partial_t\hat{\alpha}^u(t) = -\gamma_2 m(t)\hat{\beta}_2^u(t), \quad \hat{\alpha}^u(u) = 0,$$

and β_1^u, β_2^u, and Z_t^u are given in (3.5) - (3.7).

Proof. Fix $u \in [0,T]$. With equation (B.1) in Appendix B we immediately obtain

$$\mathbb{E}\left[\exp\left(-\int_t^u \mu_s \,\mathrm{d}s\right)\mu_u \,\Big|\, \mathcal{F}_t\right] = \tilde{Z}_t^u \hat{Z}_t^u, \quad t \in [0,u],$$

where $\tilde{Z}_t^u$ is given in (3.8) and

$$\hat{Z}_t^u = \hat{\alpha}^u(t) + \hat{\beta}_1^u(t)\mu_t + \hat{\beta}_2^u(t)\bar{\mu}_t, \quad t \in [0,u],$$

with

$$\begin{aligned}
\partial_t \hat{\beta}_1^u(t) &= \gamma_1 \hat{\beta}_1^u(t) - \sigma_1^2 \beta_1^u \hat{\beta}_1^u(t), \quad \hat{\beta}_1^u(u) = 1,\\
\partial_t \hat{\beta}_2^u(t) &= -\gamma_1 \hat{\beta}_1^u(t) + \gamma_2 \hat{\beta}_2^u(t) - \sigma_2^2 \beta_2^u(t)\hat{\beta}_2^u(t), \quad \hat{\beta}_2^u(u) = 0,\\
\partial_t \hat{\alpha}^u(t) &= -\gamma_2 m(t) \hat{\beta}_2^u(t), \quad \hat{\alpha}^u(u) = 0.
\end{aligned}$$

Then, again by an application of Itô's formula, we obtain

$$\begin{aligned}
\mathrm{d}\hat{Z}_t^u &= [\partial_t \hat{\alpha}^u(t) + \partial_t \hat{\beta}_1^u(t)\mu_t + \partial_t \hat{\beta}_2^u(t)\bar{\mu}_t]\,\mathrm{d}t + \hat{\beta}_1^u(t)\,\mathrm{d}\mu_t + \hat{\beta}_2^u(t)\,\mathrm{d}\bar{\mu}_t\\
&= [-\beta_1^u(t)\hat{\beta}_1^u(t)\sigma_1^2\mu_t - \beta_2^u(t)\hat{\beta}_2^u(t)\sigma_2^2\bar{\mu}_t]\,\mathrm{d}t\\
&\quad + \hat{\beta}_1^u(t)\sigma_1\sqrt{\mu_t}\,\mathrm{d}W_t^{\mu} + \hat{\beta}_2^u(t)\sigma_2\sqrt{\bar{\mu}_t}\,\mathrm{d}W_t^{\bar{\mu}},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{d}(\tilde{Z}_t^u \hat{Z}_t^u) &= \tilde{Z}_t^u\,\mathrm{d}\hat{Z}_t^u + \hat{Z}_t^u\,\mathrm{d}\tilde{Z}_t^u + \langle \tilde{Z}^u, \hat{Z}^u\rangle_t\\
&= \tilde{Z}_t^u[\mu_t \hat{Z}_t^u\,\mathrm{d}t + (\hat{\beta}_1^u(t) + \beta_1^u(t)\hat{Z}_t^u)\sigma_1\sqrt{\mu_t}\,\mathrm{d}W_t^{\mu}\\
&\quad + (\hat{\beta}_2^u(t) + \beta_2^u(t)\hat{Z}_t^u)\sigma_2\sqrt{\bar{\mu}_t}\,\mathrm{d}W_t^{\bar{\mu}}]
\end{aligned}$$

for $t \in [0,u]$, hence the result follows. □

The following remark elaborates more in detail on the technical assumptions regarding the model choice for $(\mu, \bar{\mu})$ in (2.2) - (2.3).

Remark 3.4. *As stated in Biffis [9], from a technical point of view for the existence and uniqueness of a solution $(\mu, \bar{\mu})$ of the set of stochastic differential equations* (2.2) - (2.3) *it is not necessarily required that the Brownian motions W^{μ} and $W^{\bar{\mu}}$ are independent. In fact from an intuitive point of view it is plausible to actually allow for correlation between the two Brownian motions driving μ and $\bar{\mu}$. However, we would like to remark that relaxing the independence assumption destroys the affine structure of $(\mu, \bar{\mu})$ (see, e.g., Dai and Singleton [17], Duffie et al. [21] or Filipović and Mayerhofer [22]), hence in order to obtain analytical expressions for the conditional expectations in Lemma 3.2 and 3.3 it is in fact necessary to assume that W^{μ} and $W^{\bar{\mu}}$ are independent. Also note that in* (3.23), (3.29) *and* (3.35) *we will make use of the fact that the Brownian motions W driving the asset price S as introduced in* (2.6) *and $(W^{\mu}, W^{\bar{\mu}})$ driving $(\mu, \bar{\mu})$ are independent. Of course it is possible to relax this independence, however then in order to evaluate* *the conditional expectations in* (3.23)*,* (3.29) *and* (3.35) *it is necessary to define* $(S, \mu, \bar{\mu})$ *as a multi-dimensional affine diffusion (with respect to correlated Brownian motions). This is only possible if the diffusion coefficients are constants for all three processes, in which case* μ *and* $\bar{\mu}$ *are no longer non-negative and for the volatility of the asset price we have* $\sigma(t, S_t) \equiv \sigma$*,* $t \in [0, T]$*, for a constant* $\sigma > 0$*.*

In the following we calculate the prices and hedging strategies of insurance payment streams as introduced in (2.9) - (2.11) by means of the risk-minimization approach (see Appendix A). We start with the pure endowment contract.

3.2 Pure endowment contract

For the pure endowment contract introduced in (2.9) we define the payment process

$$A_t^{pe} = (n - N_t)\,\frac{C^{pe}}{B_t}\mathbb{1}_{\{t=T\}}, \quad t \in [0, T], \tag{3.13}$$

where C^{pe} is a non-negative $\mathcal{F}_T$-measurable random variable and $\mathbb{E}[(C^{pe})^2] < \infty$.

Proposition 3.5. *In the setting of Section 2 the payment process* A^{pe} *introduced in* (3.13) *admits a risk-minimizing strategy* $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ *given by*

$$\xi_t = (\xi_t^X, \xi_t^Y) = \left(\frac{(n - N_t)e^{\Gamma_t}\psi_t}{\sigma(t, X_t)X_t}, \frac{(n - N_t)e^{rT+\Gamma_t}\psi_t^{\bar{\mu}}}{Y_t \beta^T(t)\sigma_2\sqrt{\bar{\mu}_t}} \right),$$
$$\xi_t^0 = V_t^{pe}(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t$$

for $t \in [0, T]$*, with discounted value process*

$$V_t^{pe}(\varphi) = nU_0^{pe} + \int_0^t \xi_s^X \,\mathrm{d}X_s + \int_0^t \xi_s^Y \,\mathrm{d}Y_s + L_t^{pe} - A_t^{pe},$$

where

$$L_t^{pe} = \int_0^t (n - N_s)e^{\Gamma_s}\psi_s^{\mu} \,\mathrm{d}W_s^{\mu} - \int_{]0,t]} U_s^{pe} e^{\Gamma_s} \,\mathrm{d}M_s$$

for $t \in [0, T]$*, where* $\beta^T(t)$ *and* M_t *are defined in* (3.3) *and* (3.4) *and* U^{pe}*,* ψ*,* ψ^{μ}*, and* $\psi^{\bar{\mu}}$ *are given by*

$$U_t^{pe} = \mathbb{E}\left[e^{-\Gamma_T}\frac{C^{pe}}{B_T} \,\Big|\, \mathcal{F}_t \right] = U_0^{pe} + \int_0^t \psi_s \,\mathrm{d}W_s + \int_0^t \psi_s^{\mu} \,\mathrm{d}W_s^{\mu} + \int_0^t \psi_s^{\bar{\mu}} \,\mathrm{d}W_s^{\bar{\mu}}, \tag{3.14}$$

where ψ*,* ψ^{μ} *and* $\psi^{\bar{\mu}}$ *are* $\mathbb{F}$*-predictable processes satisfying*

$$\mathbb{E}\left[\int_0^T (\psi_s)^2 \,\mathrm{d}s \right], \; \mathbb{E}\left[\int_0^T (\psi_s^{\mu})^2 \,\mathrm{d}s \right], \; \mathbb{E}\left[\int_0^T (\psi_s^{\bar{\mu}})^2 \,\mathrm{d}s \right] < \infty.$$

The optimal cost and risk processes are given by

$$C_t^{pe}(\varphi) = nU_0^{pe} + L_t^{pe},$$
$$R_t^{pe}(\varphi) = \mathbb{E}[(L_T^{pe} - L_t^{pe})^2 \,|\, \mathcal{G}_t],$$

for $t \in [0, T]$*.*

Proof. Let $t \in [0,T]$. Then we have that

$$\mathbb{E}[A_T^{pe} \mid \mathcal{G}_t] = \sum_{i=1}^{n} \mathbb{E}\left[\mathbb{1}_{\{\tau^i > T\}} \frac{C^{pe}}{B_T} \,\Big|\, \mathcal{G}_t\right],$$

and by Proposition 4.10 and 5.11 of Barbarin [2, Chapter 3], as well as Corollary 5.1.1 of Bielecki and Rutkowski [8] and (2.5) we have

$$\begin{aligned} J_t^{pe,i} &:= \mathbb{E}\left[\mathbb{1}_{\{\tau^i > T\}} \frac{C^{pe}}{B_T} \,\Big|\, \mathcal{G}_t\right] \\ &= U_0^{pe} + \int_0^t L_s^i \,\mathrm{d}U_s^{pe} - \int_{]0,t]} U_s^{pe} e^{\Gamma_s} \,\mathrm{d}M_s^i, \end{aligned} \tag{3.15}$$

where

$$U_t^{pe} = \mathbb{E}\left[e^{-\Gamma_T} \frac{C^{pe}}{B_T} \,\Big|\, \mathcal{F}_t\right], \quad t \in [0,T],$$

is a square integrable martingale, since $\mathbb{E}[(C^{pe})^2] < \infty$. By the martingale representation theorem for Brownian filtrations (see, e.g., Theorem 43 of Protter [31, Chapter IV.3]) we have that

$$U_t^{pe} = U_0^{pe} + \int_0^t \psi_s \,\mathrm{d}W_s + \int_0^t \psi_s^{\mu} \,\mathrm{d}W_s^{\mu} + \int_0^t \psi_s^{\bar{\mu}} \,\mathrm{d}W_s^{\bar{\mu}}, \quad t \in [0,T],$$

where ψ, ψ^{μ} and $\psi^{\bar{\mu}}$ are $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T (\psi_s)^2 \,\mathrm{d}s\right], \; \mathbb{E}\left[\int_0^T (\psi_s^{\mu})^2 \,\mathrm{d}s\right], \; \mathbb{E}\left[\int_0^T (\psi_s^{\bar{\mu}})^2 \,\mathrm{d}s\right] < \infty.$$

Hence by (2.7) and Lemma 3.1 for $t \in [0,T]$ we have that

$$\begin{aligned} \mathbb{E}[A_T^{pe} \mid \mathcal{G}_t] = \sum_{i=1}^{n} \Big(U_0^{pe} &+ \int_0^t \mathbb{1}_{\{\tau^i \geq s\}} e^{\Gamma_s} \psi_s \,\mathrm{d}W_s + \int_0^t \mathbb{1}_{\{\tau^i \geq s\}} e^{\Gamma_s} \psi_s^{\mu} \,\mathrm{d}W_s^{\mu} \\ &+ \int_0^t \mathbb{1}_{\{\tau^i \geq s\}} e^{\Gamma_s} \psi_s^{\bar{\mu}} \,\mathrm{d}W_s^{\bar{\mu}} - \int_{]0,t]} U_s^{pe} e^{\Gamma_s} \,\mathrm{d}M_s^i \Big) \end{aligned} \tag{3.16}$$

$$= nU_0^{pe} + \int_0^t \xi_s^X \,\mathrm{d}X_s + \int_0^t \xi_s^Y \,\mathrm{d}Y_s + L_t^{pe}, \tag{3.17}$$

where

$$\xi_t^X = \frac{(n - N_t) e^{\Gamma_t} \psi_t}{\sigma(t, X_t) X_t},$$

$$\xi_t^Y = \frac{(n - N_t) e^{rT + \Gamma_t} \psi_t^{\bar{\mu}}}{Y_t \beta^T(t) \sigma_2 \sqrt{\bar{\mu}_t}},$$

and

$$L_t^{pe} = \int_0^t (n - N_s) e^{\Gamma_s} \psi_s^{\mu} \,\mathrm{d}W_s^{\mu} - \int_{]0,t]} U_s^{pe} e^{\Gamma_s} \,\mathrm{d}M_s. \tag{3.18}$$

It remains to prove that (3.17) is indeed the GKW decomposition of $\mathbb{E}[A_T^{pe} \mid \mathcal{G}_t]$, i.e. that all integrals are square integrable and that

$$\left(\int_0^t \tilde{\xi}_s^X \, \mathrm{d}X_s + \int_0^t \tilde{\xi}_s^Y \, \mathrm{d}Y_s\right) \cdot L_t^{pe}, \quad t \in [0,T],$$

is a (uniformly integrable) martingale for all $\mathbb{G}$-predictable processes $\tilde{\xi}^X \in L^2(X)$, $\tilde{\xi}^Y \in L^2(Y)$. To this end note that since $J^{pe,i}$ is a square integrable martingale, we have that $\mathbb{E}[[J^{pe,i}]_T] < \infty$. Then from (3.15) we follow that

$$\mathbb{E}[[J^{pe,i}]_T] = \mathbb{E}\left[\int_0^T (L_s^i)^2 \, \mathrm{d}[U^{pe}]_s\right] + \mathbb{E}\left[\int_0^T (U_s^{pe} e^{\Gamma_s})^2 \, \mathrm{d}[M^i]_s\right] < \infty,$$

$i = 1, \ldots, n$, since $[U^{pe}, M^i]_t \equiv 0$, $t \in [0,T]$. Hence by Lemma 2.1 of Schweizer [33] we have that

$$\int_0^t L_s^i \, \mathrm{d}U_s^{pe}, \quad t \in [0,T],$$

and

$$\int_0^t (-U_s^{pe} e^{\Gamma_s}) \, \mathrm{d}M_s^i, \quad t \in [0,T],$$

are square integrable martingales. Again by the martingale representation theorem we have that

$$\int_0^t L_s^i \, \mathrm{d}U_s^{pe} = \int_0^t \tilde{\psi}_s^i \, \mathrm{d}W_s + \int_0^t \tilde{\psi}_s^{\mu,i} \, \mathrm{d}W_s^\mu + \int_0^t \tilde{\psi}_s^{\bar{\mu},i} \, \mathrm{d}W_s^{\bar{\mu}}, \quad t \in [0,T], \tag{3.19}$$

where $\tilde{\psi}^i$, $\tilde{\psi}^{\mu,i}$ and $\tilde{\psi}^{\bar{\mu},i}$ are $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T (\tilde{\psi}_s^i)^2 \, \mathrm{d}s\right], \ \mathbb{E}\left[\int_0^T (\tilde{\psi}_s^{\mu,i})^2 \, \mathrm{d}s\right], \ \mathbb{E}\left[\int_0^T (\tilde{\psi}_s^{\bar{\mu},i})^2 \, \mathrm{d}s\right] < \infty.$$

Hence by comparing (3.16) with (3.19) for $i = 1, \ldots, n$ and $t \in [0,T]$ we have that

$$\tilde{\psi}_t^i = \mathbb{1}_{\{\tau^i \geq t\}} e^{\Gamma_t} \psi_t, \quad \tilde{\psi}_t^{\mu,i} = \mathbb{1}_{\{\tau^i \geq t\}} e^{\Gamma_t} \psi_t^\mu, \quad \tilde{\psi}_t^{\bar{\mu},i} = \mathbb{1}_{\{\tau^i \geq t\}} e^{\Gamma_t} \psi_t^{\bar{\mu}},$$

since W, W^μ and $W^{\bar{\mu}}$ are independent. Hence $(\xi^X, \xi^Y) \in L^2(X,Y)$ and L^{pe} as defined in (3.18) is a square integrable martingale as the sum of square integrable martingales. It remains to prove that

$$\left(\int_0^t \tilde{\xi}_s^X \, \mathrm{d}X_s + \int_0^t \tilde{\xi}_s^Y \, \mathrm{d}Y_s\right) \cdot L_t^{pe}, \quad t \in [0,T],$$

is a (uniformly integrable) martingale for all $\mathbb{G}$-predictable processes $\tilde{\xi}^X \in L^2(X)$, $\tilde{\xi}^Y \in L^2(Y)$. However, this follows directly from the fact that for $t \in [0,T]$, $[W, M]_t = [W^{\bar{\mu}}, M]_t \equiv 0$ and $[W, W^\mu]_t = [W^{\bar{\mu}}, W^\mu]_t \equiv 0$ and by using Proposition 4.50 of Jacod and Shiryaev [24, Chapter I]. □

Note that the cost process is the sum of two orthogonal martingales, the first of which is related to the fact that due to the structure of $(\mu, \bar{\mu})$ as defined in (2.2) - (2.3) the financial market given by the filtration $\mathbb{F}$ is not complete. The second integral is related to the unpredictability of the times of death.
In the following (see Corollary 3.6, 3.8 and 3.10) we now consider special payoff structures in the context of unit-linked life insurance products, where the life insurance liabilities are given in terms of a non-negative Borel measurable function $f(S_t)$ of the asset price S_t, $t \in [0, T]$. Then following Møller [29] for fixed $u \in [0, T]$ the arbitrage-free price process

$$F^u(t, S_t) = \mathbb{E}\left[\exp\left(-r\left(u-t\right)\right) f(S_u) | \mathcal{F}_t\right], \quad t \in [0, u], \tag{3.20}$$

associated with the payoff $f(S_u)$ at time u can be be characterized by the partial differential equation

$$-rF^u(t, s) + F^u_t(t, s) + rsF^u_s(t, s) + \frac{1}{2}\sigma(t, s)^2 s^2 F^u_{ss}(t, s) = 0, \tag{3.21}$$

with boundary value $F^u(u, s) = f(s)$, where we denote by $F^u_t(t, s)$, $F^u_s(t, s)$ and $F^u_{ss}(t, s)$ the partial first and second order derivatives of F^u with respect to t and s.
The next corollary provides an application of Proposition 3.5 where we set

$$C^{pe} = f(S_T)$$

in (2.9) and (3.13), where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that

$$\mathbb{E}\left[f(S_T)^2\right] < \infty,$$

i.e. we define the payment process

$$A^{pe,f}_t = (n - N_t)\, \frac{f(S_t)}{B_t} \mathbb{1}_{\{t=T\}}, \quad t \in [0, T]. \tag{3.22}$$

Corollary 3.6. *In the setting of Section 2 the payment process $A^{pe,f}$ introduced in* (3.22) *admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ given by*

$$\begin{aligned}
\xi^X_t &= (n - N_t)e^{\Gamma_t} Z^T_t F^T_s(t, S_t), \\
\xi^Y_t &= \frac{(n - N_t)e^{\Gamma_t + r(T-t)} \beta^T_2(t) F^T(t, S_t) Z^T_t}{Y_t \beta^T(t)}, \\
\xi^0_t &= V^{pe,f}_t(\varphi) - \xi^X_t X_t - \xi^Y_t Y_t
\end{aligned}$$

for $t \in [0, T]$, with discounted value process

$$V^{pe,f}_t(\varphi) = nZ^T_0 F^T(0, S_0) + \int_0^t \xi^X_s \, \mathrm{d}X_s + \int_0^t \xi^Y_s \, \mathrm{d}Y_s + L^{pe,f}_t - A^{pe,f}_t,$$

where

$$L_t^{pe,f} = \int_0^t (n - N_s) e^{\Gamma_s - rs} \beta_1^T(s) \sigma_1 \sqrt{\mu_s} F^T(s, S_s) Z_s^T \, \mathrm{d}W_s^\mu$$
$$- \int_{]0,t]} e^{\Gamma_s - rs} F^T(s, S_s) Z_s^T \, \mathrm{d}M_s,$$

for $t \in [0,T]$*, where* $\beta^T(t)$*,* $\beta_1^T(t)$*,* $\beta_2^T(t)$*,* $F^T(t,S_t)$*,* $F_s^T(t,S_t)$*,* Z_t^T *and* M_t *are defined in* (3.3) - (3.7)*,* (3.20) *and* (3.21)*.*

Proof. By the independence of the underlying driving processes, for U^{pe} as defined in (3.14) we have

$$U_t^{pe} = \mathbb{E}\left[e^{-\Gamma_T} \frac{f(S_T)}{B_T} \,\Big|\, \mathcal{F}_t \right] = \mathbb{E}[e^{-\Gamma_T} \mid \mathcal{F}_t] \mathbb{E}\left[\frac{f(S_T)}{B_T} \,\Big|\, \mathcal{F}_t \right], \tag{3.23}$$

for $t \in [0,T]$, and by (2.6) - (2.7), (3.20) - (3.21) and Itô's formula the discounted arbitrage-free price process $\frac{F^T(t,S_t)}{B_t}$, $t \in [0,T]$, follows the dynamics

$$\mathrm{d}\left(\frac{F^T(t,S_t)}{B_t} \right) = F_s^T(t,S_t) \sigma(t,S_t) X_t \, \mathrm{d}W_t = F_s^T(t,S_t) \, \mathrm{d}X_t, \quad t \in [0,T], \tag{3.24}$$

and by integration by parts and (3.5) and (3.24) we obtain that

$$U_t^{pe} = Z_0^T F^T(0,S_0) + \int_0^t \sigma(s,X_s) X_s F_s^T(s,S_s) Z_s^T \, \mathrm{d}W_s$$
$$+ \int_0^t \beta_1^T(s) \sigma_1 \sqrt{\mu_s} \frac{F^T(s,S_s)}{B_s} Z_s^T \, \mathrm{d}W_s^\mu + \int_0^t \beta_2^T(s) \sigma_2 \sqrt{\bar{\mu}_s} \frac{F^T(s,S_s)}{B_s} Z_s^T \, \mathrm{d}W_s^{\bar{\mu}}$$

for $t \in [0,T]$, hence the result follows by using Proposition 3.5. □

3.3 Term insurance contract

For the term insurance contract introduced in (2.10) we define the payment process

$$A_t^{ti} = \int_0^t \frac{C_s^{ti}}{B_s} \, \mathrm{d}N_s = \sum_{i=1}^n \int_0^t \frac{C_s^{ti}}{B_s} \, \mathrm{d}H_s^i = \sum_{i=1}^n \mathbb{1}_{\{\tau^i \le t\}} \frac{C_{\tau^i}^{ti}}{B_{\tau^i}}, \quad t \in [0,T], \tag{3.25}$$

where C^{ti} is assumed to be a non-negative $\mathbb{F}$-predictable process such that

$$\mathbb{E}\left[\sup_{t \in [0,T]} (C_t^{ti})^2 \right] < \infty.$$

Proposition 3.7. *In the setting of Section 2 the payment process* A^{ti} *introduced in* (3.25) *admits a risk-minimizing strategy* $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ *given by*

$$\xi_t = (\xi_t^X, \xi_t^Y) = \left(\frac{(n - N_t) e^{\Gamma_t} \psi_t}{\sigma(t, X_t) X_t}, \frac{(n - N_t) e^{\Gamma_t + rT} \psi_t^{\bar{\mu}}}{Y_t \beta^T(t) \sigma_2 \sqrt{\bar{\mu}_t}} \right),$$
$$\xi_t^0 = V_t^{ti}(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t$$

for $t \in [0,T]$, with discounted value process

$$V_t^{ti}(\varphi) = nU_0^{ti} + \int_0^t \xi_s^X \,\mathrm{d}X_s + \int_0^t \xi_s^Y \,\mathrm{d}Y_s + L_t^{ti} - A_t^{ti},$$

where

$$L_t^{ti} = \int_0^t (n - N_s)e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu + \int_{]0,t]} \left(\frac{C_s^{ti}}{B_s} - \mathbb{E}\left[\int_s^T \frac{C_u^{ti}}{B_u} e^{\Gamma_s - \Gamma_u} \,\mathrm{d}\Gamma_u \,\Big|\, \mathcal{F}_s \right] \right) \mathrm{d}M_s,$$

for $t \in [0,T]$, where $\beta^T(t)$ and M_t are defined in (3.3) *and* (3.4) *and where U^{ti}, ψ, ψ^μ and $\psi^{\bar{\mu}}$ are given by*

$$\begin{aligned} U_t^{ti} &= \mathbb{E}\left[\int_0^T \frac{C_s^{ti}}{B_s} e^{-\Gamma_s} \,\mathrm{d}\Gamma_s \,\Big|\, \mathcal{F}_t \right] \\ &= U_0^{ti} + \int_0^t \psi_s \,\mathrm{d}W_s + \int_0^t \psi_s^\mu \,\mathrm{d}W_s^\mu + \int_0^t \psi_s^{\bar{\mu}} \,\mathrm{d}W_s^{\bar{\mu}}, \end{aligned} \tag{3.26}$$

where ψ, ψ^μ and $\psi^{\bar{\mu}}$ are $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T (\psi_s)^2 \,\mathrm{d}s \right],\ \mathbb{E}\left[\int_0^T (\psi_s^\mu)^2 \,\mathrm{d}s \right],\ \mathbb{E}\left[\int_0^T (\psi_s^{\bar{\mu}})^2 \,\mathrm{d}s \right] < \infty.$$

The optimal cost and risk processes are given by

$$\begin{aligned} C_t^{ti}(\varphi) &= nU_0^{ti} + L_t^{ti}, \\ R_t^{ti}(\varphi) &= \mathbb{E}[(L_T^{ti} - L_t^{ti})^2 \,|\, \mathcal{G}_t], \end{aligned}$$

for $t \in [0,T]$.

Proof. By Proposition 4.11 and 5.12 of Barbarin [2, Chapter 3], as well as Corollary 5.1.3 of Bielecki and Rutkowski [8] and (2.5) we have

$$\begin{aligned} \mathbb{E}[A_T^{ti} \,|\, \mathcal{G}_t] = nU_0^{ti} &+ \int_0^t (n - N_s)e^{\Gamma_s} \,\mathrm{d}U_s^{ti} \\ &+ \int_{]0,t]} \left(\frac{C_s^{ti}}{B_s} - \mathbb{E}\left[\int_s^T \frac{C_u^{ti}}{B_u} e^{\Gamma_s - \Gamma_u} \,\mathrm{d}\Gamma_u \,\Big|\, \mathcal{F}_s \right] \right) \mathrm{d}M_s, \end{aligned}$$

where

$$U_t^{ti} = \mathbb{E}\left[\int_0^T \frac{C_s^{ti}}{B_s} e^{-\Gamma_s} \,\mathrm{d}\Gamma_s \,\Big|\, \mathcal{F}_t \right], \quad t \in [0,T],$$

is a square integrable martingale, since by Jensen's inequality for any $t \in [0,T]$ we have

$$\mathbb{E}[(U_t^{ti})^2] \le \mathbb{E}\left[\sup_{t \in [0,T]} (C_t^{ti})^2 \left(\int_0^T \mathrm{d}e^{-\Gamma_s} \right)^2 \right] \le \mathbb{E}\left[\sup_{t \in [0,T]} (C_t^{ti})^2 \right],$$

and $\mathbb{E}\left[\sup_{t\in[0,T]}(C_t^{ti})^2\right] < \infty$. By the martingale representation theorem for Brownian filtrations we have that

$$U_t^{ti} = U_0^{ti} + \int_0^t \psi_s \,\mathrm{d}W_s + \int_0^t \psi_s^\mu \,\mathrm{d}W_s^\mu + \int_0^t \psi_s^{\bar{\mu}} \,\mathrm{d}W_s^{\bar{\mu}}, \quad t \in [0,T],$$

where ψ, ψ^μ and $\psi^{\bar{\mu}}$ are $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T (\psi_s)^2 \,\mathrm{d}s\right],\ \mathbb{E}\left[\int_0^T (\psi_s^\mu)^2 \,\mathrm{d}s\right],\ \mathbb{E}\left[\int_0^T (\psi_s^{\bar{\mu}})^2 \,\mathrm{d}s\right] < \infty.$$

Hence by (2.7) and Lemma 3.1 we have that

$$\mathbb{E}\left[A_T^{ti}\,\middle|\,\mathcal{G}_t\right] = nU_0^{ti} + \int_0^t \xi_s^X \,\mathrm{d}X_s + \int_0^t \xi_s^Y \,\mathrm{d}Y_s + L_t^{ti}, \tag{3.27}$$

where

$$\xi_t^X = \frac{(n-N_t)e^{\Gamma_t}\psi_t^X}{\sigma(t,X_t)X_t},$$
$$\xi_t^Y = \frac{(n-N_t)e^{\Gamma_t+rT}\psi_t^{\bar{\mu}}}{Y_t\beta^T(t)\sigma_2\sqrt{\bar{\mu}_t}},$$

and

$$L_t^{ti} = \int_0^t (n-N_s)e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu + \int_{]0,t]} \left(\frac{C_s^{ti}}{B_s} - \mathbb{E}\left[\int_s^T \frac{C_u^{ti}}{B_u} e^{\Gamma_s-\Gamma_u} \,\mathrm{d}\Gamma_u\,\middle|\,\mathcal{F}_s\right]\right) \mathrm{d}M_s.$$

By the same arguments as in the proof of Proposition 3.5 we obtain that all integrals in (3.27) are square integrable and strongly orthogonal, hence (3.27) is indeed the GKW decomposition of $\mathbb{E}[A_T^{ti}\,|\,\mathcal{G}_t]$. □

Note that Corollary 5.1.3 of Bielecki and Rutkowski [8] requires C^{ti} to be bounded. However, it can be easily seen that this result also holds if $\mathbb{E}[\sup_{t\in[0,T]}(C_t^{ti})^2] < \infty$ and we may therefore apply it in our setting.
The next corollary provides an application of Proposition 3.7 where we set

$$C_t^{ti} = f(S_t), \quad t \in [0,T],$$

in (2.10) and (3.25), where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} f(S_t)^2\right] < \infty,$$

i.e. we define the payment process

$$A_t^{ti,f} = \int_0^t \frac{f(S_s)}{B_s} \,\mathrm{d}N_s = \sum_{i=1}^n \mathbb{1}_{\{\tau^i \le t\}} \frac{f(S_{\tau^i})}{B_{\tau^i}}, \quad t \in [0,T]. \tag{3.28}$$

Corollary 3.8. *In the setting of Section 2 the payment process $A^{ti,f}$ introduced in* (3.28) *admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ given by*

$$\begin{aligned}
\xi_t^X &= (n - N_t)e^{\Gamma_t} \int_t^T F_s^u(t, S_t) Z_t^{\mu,u} \,\mathrm{d}u \\
\xi_t^Y &= (n - N_t)e^{\Gamma_t + r(T-t)} Y_t^{-1} (\beta^T(t))^{-1} \int_t^T F^u(t, S_t) Z_t^u (\hat{\beta}_2^u(t) + \beta_2^u(t)\hat{Z}_t^u) \,\mathrm{d}u, \\
\xi_t^0 &= V_t^{ti}(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t
\end{aligned}$$

for $t \in [0, T]$, with discounted value process

$$V_t^{ti,f}(\varphi) = n \int_0^T Z_0^{\mu,u} F^u(0, S_0) \,\mathrm{d}u + \int_0^t \xi_s^X \,\mathrm{d}X_s + \int_0^t \xi_s^Y \,\mathrm{d}Y_s + L_t^{ti,f} - A_t^{ti,f},$$

where

$$\begin{aligned}
L_t^{ti,f} = &\int_0^t (n - N_s)e^{\Gamma_s} \int_s^T \frac{F^u(s, S_s)}{B_s} Z_s^u (\hat{\beta}_1^u(s) + \beta_1^u(s)\hat{Z}_s^u)\sigma_1 \sqrt{\mu_s} \,\mathrm{d}u \,\mathrm{d}W_s^\mu \\
&+ \int_{]0,t]} \left(\frac{f(S_s)}{B_s} - \mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u} e^{\Gamma_s - \Gamma_u} \,\mathrm{d}\Gamma_u \,|\, \mathcal{F}_s \right] \right) \mathrm{d}M_s
\end{aligned}$$

for $t \in [0, T]$, where $\beta^T(t)$, $\beta_1^u(t)$, $\beta_2^u(t)$, $\hat{\beta}_1^u(t)$, $\hat{\beta}_2^u(t)$, $F^u(t, S_t)$, $F_s^u(t, S_t)$, Z_t^u, $Z_t^{\mu,u}$, $\hat{Z}_t^u$ and M_t are defined in (3.3) - (3.7), (3.9) - (3.12), (3.20) *and* (3.21).

Proof. For U^{ti} as defined in (3.26) we have

$$\begin{aligned}
U_t^{ti} &= \mathbb{E}\left[\int_0^T \frac{f(S_u)}{B_u} e^{-\Gamma_u} \mu_u \,\mathrm{d}u \,\middle|\, \mathcal{F}_t \right] \\
&= \int_0^T \mathbb{E}\left[\frac{f(S_u)}{B_u} \middle|\, \mathcal{F}_t \right] \mathbb{E}\left[e^{-\Gamma_u} \mu_u \,\middle|\, \mathcal{F}_t \right] \mathrm{d}u, \quad t \in [0, T], \qquad (3.29)
\end{aligned}$$

where we have used Fubini's theorem and the independence of the underlying driving processes. By the same arguments as in the proof of Corollary 3.6 we have that

$$\mathbb{E}\left[\frac{f(S_u)}{B_u} \,\middle|\, \mathcal{F}_t \right] = F^u(0, S_0) + \int_0^t F_s^u(s, S_s)\sigma(s, X_s) X_s \mathbb{1}_{\{s \le u\}} \,\mathrm{d}W_s \qquad (3.30)$$

for $0 \le t$, $u \le T$, where $F^u(u, S_u) = f(S_u)$. Furthermore, by (3.9) we have

$$\begin{aligned}
Z_t^{\mu,u} = \mathbb{E}\left[e^{-\Gamma_u} \mu_u \,\middle|\, \mathcal{F}_t \right] = Z_0^{\mu,u} &+ \int_0^t Z_s^u \left(\hat{\beta}_1^u(s) + \beta_1^u(s)\hat{Z}_s^u \right) \sigma_1 \sqrt{\mu_s} \mathbb{1}_{\{s \le u\}} \,\mathrm{d}W_s^\mu \\
&+ \int_0^t Z_s^u \left(\hat{\beta}_2^u(s) + \beta_2^u(s)\hat{Z}_s^u \right) \sigma_2 \sqrt{\bar{\mu}_s} \mathbb{1}_{\{s \le u\}} \,\mathrm{d}W_s^{\bar{\mu}}, \quad 0 \le t,\, u \le T,
\end{aligned}$$

where β_1^u, β_2^u, $\hat{\beta}_1^u$, $\hat{\beta}_2^u$, Z^u and $\hat{Z}^u$ are given in (3.5) - (3.7) and (3.10) - (3.12). Then since all integrands are continuous (see Theorem 15 in Chapter IV of Protter

[31]), once again by Itô's formula and by the stochastic Fubini theorem (see, e.g., Theorem 65 in Chapter IV of Protter [31]) we obtain

$$
\begin{aligned}
U_t^{ti} = &\int_0^T Z_0^{\mu,u} F^u(0, S_0)\,\mathrm{d}u + \int_0^t \int_s^T F_s^u(s, S_s) Z_s^{\mu,u} \sigma(s, X_s) X_s\,\mathrm{d}u\,\mathrm{d}W_s \\
&+ \int_0^t \int_s^T \frac{F^u(s, S_s)}{B_s} Z_s^u (\hat{\beta}_1^u(s) + \beta_1^u(s)\hat{Z}_s^u)\sigma_1 \sqrt{\mu_s}\,\mathrm{d}u\,\mathrm{d}W_s^{\mu} \\
&+ \int_0^t \int_s^T \frac{F^u(s, S_s)}{B_s} Z_s^u (\hat{\beta}_2^u(s) + \beta_2^u(s)\hat{Z}_s^u)\sigma_2 \sqrt{\bar{\mu}_s}\,\mathrm{d}u\,\mathrm{d}W_s^{\bar{\mu}}
\end{aligned}
$$

for $t \in [0, T]$, hence the result follows by using Proposition 3.7. □

3.4 Annuity contract

For the annuity contract introduced in (2.11) we define the payment process

$$
A_t^a = \int_0^t (n - N_s) \frac{1}{B_s}\,\mathrm{d}C_s^a = \sum_{i=1}^n \int_0^t \mathbb{1}_{\{\tau^i > s\}} \frac{1}{B_s}\,\mathrm{d}C_s^a, \quad t \in [0, T], \tag{3.31}
$$

where C^a is assumed to be a right-continuous, non-negative increasing $\mathbb{F}$-adapted process such that

$$
\mathbb{E}\left[\sup_{t\in[0,T]} (C_t^a)^2\right] < \infty.
$$

Proposition 3.9. *In the setting of Section 2 the payment process A^a introduced in* (3.31) *admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ given by*

$$
\begin{aligned}
\xi_t &= (\xi_t^X, \xi_t^Y) = \left(\frac{(n - N_t)e^{\Gamma_t}\psi_t}{\sigma(t, X_t)X_t}, \frac{(n - N_t)e^{\Gamma_t + rT}\psi_t^{\bar{\mu}}}{Y_t \beta^T(t)\sigma_2\sqrt{\bar{\mu}_t}} \right), \\
\xi_t^0 &= V_t^a(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t
\end{aligned}
$$

for $t \in [0, T]$, with discounted value process

$$
V_t^a(\varphi) = nU_0^a + \int_0^t \xi_s^X\,\mathrm{d}X_s + \int_0^t \xi_s^Y\,\mathrm{d}Y_s + L_t^a - A_t^a,
$$

where

$$
L_t^a = \int_0^t (n - N_s)e^{\Gamma_s}\psi_s^{\mu}\,\mathrm{d}W_s^{\mu} - \int_{]0,t]} \mathbb{E}\left[\int_s^T \frac{e^{\Gamma_s - \Gamma_u}}{B_u}\,\mathrm{d}C_u^a \,\middle|\, \mathcal{F}_s\right] \mathrm{d}M_s,
$$

for $t \in [0, T]$, where $\beta^T(t)$ and M_t are defined in (3.3) *and* (3.4) *and U^a, ψ, ψ^{μ} and $\psi^{\bar{\mu}}$ are given by*

$$
U_t^a = \mathbb{E}\left[\int_0^T \frac{e^{-\Gamma_s}}{B_s}\,\mathrm{d}C_s^a \,\middle|\, \mathcal{F}_t\right] = U_0^a + \int_0^t \psi_s\,\mathrm{d}W_s + \int_0^t \psi_s^{\mu}\,\mathrm{d}W_s^{\mu} + \int_0^t \psi_s^{\bar{\mu}}\,\mathrm{d}W_s^{\bar{\mu}}, \tag{3.32}
$$

where ψ, ψ^{μ} and $\psi^{\bar{\mu}}$ are $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T (\psi_s)^2 \,\mathrm{d}s\right],\ \mathbb{E}\left[\int_0^T (\psi_s^{\mu})^2 \,\mathrm{d}s\right],\ \mathbb{E}\left[\int_0^T (\psi_s^{\bar{\mu}})^2 \,\mathrm{d}s\right] < \infty.$$

The optimal cost and risk processes are given by

$$\begin{aligned} C_t^a(\varphi) &= nU_0^a + L_t^a, \\ R_t^a(\varphi) &= \mathbb{E}[(L_T^a - L_t^a)^2 \,|\, \mathcal{G}_t], \end{aligned} \tag{3.33}$$

for $t \in [0, T]$.

Proof. By Proposition 4.12 and 5.13 of Barbarin [2, Chapter 3], as well as Proposition 5.1.2 of Bielecki and Rutkowski [8] and (2.5) we have

$$\begin{aligned} \mathbb{E}[A_T^a \,|\, \mathcal{G}_t] = nU_0^a &+ \int_0^t (n - N_s) e^{\Gamma_s} \,\mathrm{d}U_s^a \\ &- \int_{]0,t]} \mathbb{E}\left[\int_s^T \frac{e^{\Gamma_s - \Gamma_u}}{B_u} \,\mathrm{d}C_u^a \,\Big|\, \mathcal{F}_s\right] \mathrm{d}M_s, \end{aligned}$$

where

$$U_t^a = \mathbb{E}\left[\int_0^T \frac{e^{-\Gamma_s}}{B_s} \,\mathrm{d}C_s^a \,\Big|\, \mathcal{F}_t\right] = U_0^a + \int_0^t \psi_s \,\mathrm{d}W_s + \int_0^t \psi_s^{\mu} \,\mathrm{d}W_s^{\mu} + \int_0^t \psi_s^{\bar{\mu}} \,\mathrm{d}W_s^{\bar{\mu}},$$

$t \in [0, T]$, is a square integrable martingale, since $\mathbb{E}\left[\sup_{t\in[0,T]} (C_t^a)^2\right] < \infty$ and where ψ, ψ^{μ} and $\psi^{\bar{\mu}}$ are $\mathbb{F}$-predictable processes satisfying

$$\mathbb{E}\left[\int_0^T (\psi_s)^2 \,\mathrm{d}s\right],\ \mathbb{E}\left[\int_0^T (\psi_s^{\mu})^2 \,\mathrm{d}s\right],\ \mathbb{E}\left[\int_0^T (\psi_s^{\bar{\mu}})^2 \,\mathrm{d}s\right] < \infty.$$

The result follows by the same arguments as in the proofs of Proposition 3.5 and 3.7. ☐

Note that Proposition 5.1.2 of Bielecki and Rutkowski [8] requires C^a to be bounded. However, it can be easily seen that this result also holds if

$$\mathbb{E}\left[\sup_{t\in[0,T]} (C_t^a)^2\right] < \infty.$$

The next corollary provides an application of Proposition 3.9 where we set

$$C_t^a = \int_0^t f(S_s) \,\mathrm{d}s, \quad t \in [0, T],$$

in (2.11) and (3.31), where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} f(S_t)^2\right] < \infty,$$

i.e. we define the payment process

$$A_t^{a,f} = \int_0^t (n - N_s) \frac{f(S_s)}{B_s} \,\mathrm{d}s = \sum_{i=1}^n \int_0^t \mathbb{1}_{\{\tau^i > s\}} \frac{f(S_s)}{B_s} \,\mathrm{d}s, \quad t \in [0,T]. \tag{3.34}$$

Corollary 3.10. *In the setting of Section 2 the payment process $A^{a,f}$ introduced in (3.34) admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ given by*

$$\begin{aligned}
\xi_t^X &= (n - N_t) e^{\Gamma_t} \int_t^T F_s^u(t, S_t) Z_t^u \,\mathrm{d}u \\
\xi_t^Y &= (n - N_t) e^{\Gamma_t + r(T-t)} Y_t^{-1} (\beta^T(t))^{-1} \int_t^T \beta_2^u(t) F^u(t, S_t) Z_t^u \,\mathrm{d}u, \\
\xi_t^0 &= V_t^a(\varphi) - \xi_t^X X_t - \xi_t^Y Y_t
\end{aligned}$$

for $t \in [0,T]$, with discounted value process

$$V_t^{a,f}(\varphi) = n \int_0^T Z_0^u F^u(0, S_0) \,\mathrm{d}u + \int_0^t \xi_s^X \,\mathrm{d}X_s + \int_0^t \xi_s^Y \,\mathrm{d}Y_s + L_t^{a,f} - A_t^{a,f},$$

where

$$\begin{aligned}
L_t^{a,f} = &\int_0^t (n - N_s) e^{\Gamma_s} \int_s^T \beta_1^u(s) \sigma_1 \sqrt{\mu_s} \frac{F^u(s, S_s)}{B_s} Z_s^u \,\mathrm{d}u \,\mathrm{d}W_s^\mu \\
&- \int_{]0,t]} \mathbb{E}\left[\int_s^T \frac{e^{\Gamma_s - \Gamma_u}}{B_u} \,\mathrm{d}C_u^a \,\Big|\, \mathcal{F}_s \right] \mathrm{d}M_s,
\end{aligned}$$

for $t \in [0,T]$, where $\beta^T(t)$, $\beta_1^u(t)$, $\beta_2^u(t)$, $F^u(t, S_t)$, $F_s^u(t, S_t)$, Z_t^u and M_t are defined in (3.3) - (3.7), (3.20) *and* (3.21).

Proof. For U^a as defined in (3.32) we have

$$\begin{aligned}
U_t^a &= \mathbb{E}\left[\int_0^T e^{-\Gamma_u} \frac{f(S_u)}{B_u} \,\mathrm{d}u \,\Big|\, \mathcal{F}_t \right] \\
&= \int_0^T \mathbb{E}\left[\frac{f(S_u)}{B_u} \Big|\, \mathcal{F}_t \right] \mathbb{E}\left[e^{-\Gamma_u} \,\Big|\, \mathcal{F}_t \right] \mathrm{d}u, \quad t \in [0,T] \qquad (3.35)
\end{aligned}$$

where we have used Fubini's theorem and the independence of the underlying driving processes. By (3.5) we have

$$\begin{aligned}
Z_t^u := \mathbb{E}\left[e^{-\Gamma_u} \,\Big|\, \mathcal{F}_t \right] = Z_0^u &+ \int_0^t Z_s^u \beta_1^u(s) \sigma_1 \sqrt{\mu_s} \mathbb{1}_{\{s \le u\}} \,\mathrm{d}W_s^\mu \\
&+ \int_0^t Z_s^u \beta_2^u(s) \sigma_2 \sqrt{\bar{\mu}_s} \mathbb{1}_{\{s \le u\}} \,\mathrm{d}W_s^{\bar{\mu}} \qquad (3.36)
\end{aligned}$$

for $0 \leq t,\, u \leq T$, where $\beta_1(t)$ and $\beta_2(t)$ are given in (3.6) - (3.7), and by the same arguments as in the proof of Corollary 3.8 we have that

$$U_t^a = \int_0^T Z_0^u F^u(0, S_0)\,\mathrm{d}u + \int_0^t \int_s^T \sigma(s, X_s) X_s F_s^u(s, S_s) Z_s^u \,\mathrm{d}u\,\mathrm{d}W_s + \int_0^t \int_s^T \beta_1^u(s)\sigma_1\sqrt{\mu_s}\frac{F^u(s, S_s)}{B_s} Z_s^u\,\mathrm{d}u\,\mathrm{d}W_s^\mu + \int_0^t \int_s^T \beta_2^u(s)\sigma_2\sqrt{\bar{\mu}_s}\frac{F^u(s, S_s)}{B_s} Z_s^u\,\mathrm{d}u\,\mathrm{d}W_s^{\bar{\mu}}, \quad t \in [0, T],$$

where we used (3.30), (3.36), Itô's formula and the stochastic Fubini theorem. Then the result follows by using Proposition 3.9. □

We conclude this section with a remark regarding the hedging error of the risk-minimizing strategies as computed in Propositions 3.5, 3.7 and 3.9. Following Barbarin [2], Møller [29] and Riesner [32] we take the initial intrinsic risk $R_0(\varphi)$ as a measure of the total risk associated with the non-hedgeable part of the insurance claims. In the case of the annuity contract, for $R_0^a(\varphi)$ as defined in (3.33) we have

$$R_0^a(\varphi) = \mathbb{E}[(L_T^a - L_0^a)^2] = \mathbb{E}\left[\left(\int_0^T (n - N_s)e^{\Gamma_s}\psi_s^\mu\,\mathrm{d}W_s^\mu\right)^2\right] + \mathbb{E}\left[\left(\int_0^T \zeta_s\,\mathrm{d}M_s\right)^2\right] - 2\mathbb{E}\left[\left(\int_0^T (n - N_s)e^{\Gamma_s}\psi_s^\mu\,\mathrm{d}W_s^\mu\right)\left(\int_0^T \zeta_s\,\mathrm{d}M_s\right)\right],$$

where $\zeta_t = \mathbb{E}[\int_t^T \frac{e^{\Gamma_t} - e^{\Gamma_u}}{B_u}\,\mathrm{d}C_u^a \,|\, \mathcal{F}_t]$, $t \in [0, T]$, and since W^μ and M are strongly orthogonal, the square integrable martingales

$$\left(\int_0^t (n - N_s)e^{\Gamma_s}\psi_s^\mu\,\mathrm{d}W_s^\mu\right), \quad \left(\int_0^t \zeta_s\,\mathrm{d}M_s\right), \quad t \in [0, T]$$

are strongly orthogonal, and e.g., by Proposition 4.50 in Chapter I of Jacod and Shiryaev [24], we have that

$$\mathbb{E}\left[\left(\int_0^T (n - N_s)e^{\Gamma_s}\psi_s^\mu\,\mathrm{d}W_s^\mu\right)\left(\int_0^T \zeta_s\,\mathrm{d}M_s\right)\right] = 0.$$

Furthermore

$$\mathbb{E}\left[\left(\int_0^T (n - N_s)e^{\Gamma_s}\psi_s^\mu\,\mathrm{d}W_s^\mu\right)^2\right] = \sum_{i=1}^n \mathbb{E}\left[\left(\int_0^T \mathbb{1}_{\{\tau^i > s\}} e^{\Gamma_s}\psi_s^\mu\,\mathrm{d}W_s^\mu\right)^2\right] + \sum_{i \neq j} \mathbb{E}\left[\left(\int_0^T \mathbb{1}_{\{\tau^i > s\}} e^{\Gamma_s}\psi_s^\mu\,\mathrm{d}W_s^\mu\right)\left(\int_0^T \mathbb{1}_{\{\tau^j > s\}} e^{\Gamma_s}\psi_s^\mu\,\mathrm{d}W_s^\mu\right)\right],$$

and by (2.1), (2.5) and Fubini's theorem we have that

$$\sum_{i=1}^n \mathbb{E}\left[\left(\int_0^T \mathbb{1}_{\{\tau^i>s\}} e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)^2\right] = \sum_{i=1}^n \mathbb{E}\left[\int_0^T \mathbb{1}_{\{\tau^i>s\}} e^{2\Gamma_s}(\psi_s^\mu)^2 \,\mathrm{d}s\right]$$
$$= \sum_{i=1}^n \int_0^T \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{\{\tau^i>s\}} \mid \mathcal{F}_s] e^{2\Gamma_s}(\psi_s^\mu)^2\right] \mathrm{d}s = n\mathbb{E}\left[\int_0^T e^{\Gamma_s}(\psi_s^\mu)^2 \,\mathrm{d}s\right],$$

as well as

$$\sum_{i\neq j} \mathbb{E}\left[\left(\int_0^T \mathbb{1}_{\{\tau^i>s\}} e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)\left(\int_0^T \mathbb{1}_{\{\tau^j>s\}} e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)\right]$$
$$= \sum_{i\neq j} \mathbb{E}\left[\int_0^T \mathbb{1}_{\{\tau^i>s\}}\mathbb{1}_{\{\tau^j>s\}} e^{2\Gamma_s}(\psi_s^\mu)^2 \,\mathrm{d}s\right]$$
$$= \sum_{i\neq j} \int_0^T \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{\{\tau^i>s\}}\mathbb{1}_{\{\tau^j>s\}} \mid \mathcal{F}_s] e^{2\Gamma_s}(\psi_s^\mu)^2\right] \mathrm{d}s = (n^2-n)\mathbb{E}\left[\int_0^T (\psi_s^\mu)^2 \,\mathrm{d}s\right].$$

Hence

$$\mathbb{E}\left[\left(\int_0^T (n-N_s)e^{\Gamma_s}\psi_s^\mu \,\mathrm{d}W_s^\mu\right)^2\right] = n\mathbb{E}\left[\int_0^T e^{\Gamma_s}(\psi_s^\mu)^2 \,\mathrm{d}s\right] + (n^2-n)\mathbb{E}\left[\int_0^T (\psi_s^\mu)^2 \,\mathrm{d}s\right].$$

Besides that

$$\mathbb{E}\left[\left(\int_0^T \zeta_s \,\mathrm{d}M_s\right)^2\right] = \sum_{i=1}^n \mathbb{E}\left[\left(\int_0^T \zeta_s \,\mathrm{d}M_s^i\right)^2\right] + \sum_{i\neq j} \mathbb{E}\left[\left(\int_0^T \zeta_s \,\mathrm{d}M_s^i\right)\left(\int_0^T \zeta_s \,\mathrm{d}M_s^j\right)\right],$$

and since M^i and M^j are strongly orthogonal for $i \neq j$, by Proposition 4.15 in Chapter I of Jacod and Shiryaev [24] it follows that

$$\sum_{i\neq j} \mathbb{E}\left[\left(\int_0^T \zeta_s \,\mathrm{d}M_s^i\right)\left(\int_0^T \zeta_s \,\mathrm{d}M_s^j\right)\right] = 0,$$

hence

$$\mathbb{E}\left[\left(\int_0^T \zeta_s \,\mathrm{d}M_s\right)^2\right] = \sum_{i=1}^n \mathbb{E}\left[\int_0^T \zeta_s^2 \,\mathrm{d}\langle M^i\rangle_s\right] = \sum_{i=1}^n \int_0^T \mathbb{E}\left[\zeta_s^2 \mathbb{1}_{\{\tau^i>s\}}\mu_s\right] \mathrm{d}s$$
$$= n\mathbb{E}\left[\int_0^T \zeta_s^2 e^{-\Gamma_s}\mu_s \,\mathrm{d}s\right].$$

Putting the results together we obtain that

$$R_0^a(\varphi) = n\mathbb{E}\left[\int_0^T e^{\Gamma_s}(\psi_s^\mu)^2 \,\mathrm{d}s\right] + (n^2-n)\mathbb{E}\left[\int_0^T (\psi_s^\mu)^2 \,\mathrm{d}s\right] + n\mathbb{E}\left[\int_0^T \zeta_s^2 e^{-\Gamma_s}\mu_s \,\mathrm{d}s\right],$$

hence

$$\lim_{n\to\infty} \frac{\sqrt{R_0^a(\varphi)}}{n} = \sqrt{\mathbb{E}\left[\int_0^T (\psi_s^\mu)^2 \, \mathrm{d}s\right]}. \tag{3.37}$$

The analogous results hold for the pure endowment and term insurance contract. Therefore, in contrast to the setting in Møller [29], with increasing portfolio size the hedging error cannot be fully eliminated. As noted already in Barbarin [2] the non-diversifiable term in (3.37) is related to the incompleteness of the market given by the filtration $\mathbb{F}$.

Appendices

A Risk-minimization for payment processes

The (local) risk-minimization method is a quadratic hedging approach that was first introduced by Föllmer and Sondermann [23] in the case of European type contingent claims and later extended to the case of payment processes by Møller [30] and later Schweizer [34] and Barbarin [2, Chapter 4]. In this section of the appendix for the readers convenience we briefly review all aspects of the theoretical background that are relevant for our purposes. Note that this borrows extensively from Møller [30] and Schweizer [33].

For a finite time horizon $T > 0$ consider a financial market defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ fulfills the usual conditions, consisting of one risk-free asset or numéraire $B = (B_t)_{t\in[0,T]}$, as well as d risky assets $S^i = (S^i_t)_{t\in[0,T]}$, $i = 1, \dots, d$. We denote by $X = (X^1, \dots, X^d)$ the discounted asset prices, where $X^i = S^i/B$, $i = 1, \dots, d$, and we assume that X is a local $\mathbb{P}$-martingale. In particular we assume that the market is arbitrage-free and we are working under a risk-neutral measure, i.e., the measure $\mathbb{P}$ itself belongs to the set of equivalent local martingale measures. In this setting we would like to find a hedging strategy for an $\mathbb{F}$-adapted, square integrable payment process $A = (A_t)_{t\in[0,T]}$, representing cumulative discounted payments up to time t, $t \in [0, T]$. Since the market is not necessarily complete, it is in general not possible to find a self-financing hedging strategy that perfectly replicates the payment process A. In this context the idea of risk-minimization is to relax the self-financing assumption, allowing for a wider class of admissible strategies, and to find an optimal hedging strategy with "minimal risk" within this class of strategies that perfectly replicates A. In the following we now explain how to find the risk-minimizing strategy and explain in what sense this strategy is optimal. We begin with some definitions.

Definition A.1. *An L^2-*strategy *is a pair $\varphi = (\xi, \xi^0)$, such that ξ is a d-dimensional* *process belonging to* $L^2(X)$*, with*

$$L^2(X) := \left\{ \xi \,\middle|\, \xi \ \mathbb{F}\text{-predictable}, \ \left(\mathbb{E}\left[\int_0^T \xi_s' \, \mathrm{d}[X,X]_s \, \xi_s \right] \right)^{1/2} < \infty \right\},$$

and ξ^0 *is a real-valued* $\mathbb{F}$*-adapted process, such that the discounted value process*

$$V_t(\varphi) = \xi_t X_t + \xi_t^0, \quad t \in [0,T],$$

is right-continuous and square integrable.

For an L^2-strategy φ the *discounted (cumulative) cost process* $C(\varphi)$ is defined as

$$C_t(\varphi) = V_t(\varphi) - \int_0^t \xi_s \, \mathrm{d}X_s + A_t, \quad t \in [0,T],$$

describing the accumulated costs of the trading strategy φ during $[0,t]$ including the payments A_t. Note that $V_t(\varphi)$ should therefore be interpreted as the discounted value of the portfolio φ_t held at time t *after* the payments A_t have been made. In particular, $V_T(\varphi)$ is the value of the portfolio upon settlement of all liabilities, and a natural condition is then to restrict to 0-*admissible strategies* satisfying

$$V_T(\varphi) = 0 \quad \mathbb{P}\text{-a.s.}$$

The *risk process* of φ is given by the conditional expected value of the squared future costs

$$R_t(\varphi) = \mathbb{E}[(C_T(\varphi) - C_t(\varphi))^2 \,|\, \mathcal{F}_t], \quad t \in [0,T], \tag{A.1}$$

and is taken as a measure of the hedger's remaining risk. We would like to find a trading strategy that minimizes the risk in a sense we define now.

Definition A.2. *An* L^2*-strategy* $\varphi = (\xi, \xi^0)$ *is called* risk-minimizing *for the payment stream A, if for any* L^2*-strategy* $\tilde{\varphi} = (\tilde{\xi}, \tilde{\xi}^0)$ *such that* $V_T(\tilde{\varphi}) = V_T(\varphi) = 0$ $\mathbb{P}$*-a.s., we have*

$$R_t(\varphi) \leq R_t(\tilde{\varphi}), \quad t \in [0,T],$$

i.e., φ *pointwise minimizes the risk process introduced in* (A.1)*.*

The key to finding the strategy with minimal risk is the well-known Galtchouk-Kunita-Watanabe (GKW) decomposition, see Ansel and Stricker [1]. Since A is square integrable, the expected accumulated total payments may be decomposed by use of the GKW decomposition as

$$\mathbb{E}[A_T \,|\, \mathcal{F}_t] = \mathbb{E}[A_T \,|\, \mathcal{F}_0] + \int_{]0,t]} \xi_s^A \, \mathrm{d}X_s + L_t^A, \quad t \in [0,T], \tag{A.2}$$

where $\xi^A \in L^2(X)$ and L^A is a square integrable martingale null at 0 that is strongly orthogonal to the space of stochastic integrals with respect to X

$$\mathcal{I}^2(X) := \left\{ \int \psi \, \mathrm{d}X \,\middle|\, \psi \in L^2(X) \right\},$$

i.e., for $\psi \in L^2(X)$, $L_t^A \int_0^t \psi \, \mathrm{d}X$, $t \in [0,T]$, is a (uniformly integrable) martingale.

Theorem A.3. *There exists a unique* 0*-admissible risk-minimizing* L^2*-strategy* $\varphi = (\xi, \xi^0)$*, given by*

$$\xi_t := \xi_t^A,$$
$$\xi_t^0 := V_t(\varphi) - \xi_t X_t,$$

with discounted value process

$$V_t(\varphi) = \mathbb{E}[A_T \mid \mathcal{F}_t] - A_t = \mathbb{E}[A_T \mid \mathcal{F}_0] + \int_{]0,t]} \xi_s \, \mathrm{d}X_s + L_t^A - A_t,$$

discounted optimal cost process

$$C_t(\varphi) = \mathbb{E}[A_T \mid \mathcal{F}_0] + L_t^A = C_0(\varphi) + L_t^A,$$

and minimal risk process

$$R_t(\varphi) = \mathbb{E}[(L_T^A - L_t^A)^2 \mid \mathcal{F}_t],$$

$t \in [0, T]$*, where* ξ^A *and* L^A *are given by* (A.2)*.*

Proof. See Schweizer [33] for the single payoff case or Møller [30] and Schweizer [34] for the extension to the case of payment streams. □

Note that the preceding approach relies heavily on the fact that the discounted asset prices are local martingales under the original measure $\mathbb{P}$. In a more general setting, when the discounted asset price is merely required to be a semimartingale under $\mathbb{P}$, one finds the price by following the *local* risk-minimization technique, see Schweizer [34] or Barbarin [2, Chapter 4]. For more information on (local) risk-minimization and other quadratic hedging approaches we would like to refer the interested reader to the survey paper of Schweizer [33].

B Affine diffusion processes

In this section of the appendix we give a brief review of some aspects of the theory of affine processes that are relevant for this work. Note that this borrows extensively from Section 3 and Appendix A of Biffis [9]. We would also like to refer the interested reader to Duffie et al. [21] and Filipović and Mayerhofer [22]. An affine diffusion process $X = (X_t)_{t\in[0,T]}$ with values in $\mathbb{R}^n$ is a Markov process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ fulfills the usual conditions, solving (in the strong sense) the stochastic differential equation

$$\mathrm{d}X_t = \delta(t, X_t)\, \mathrm{d}t + \sigma(t, X_t)\, \mathrm{d}W_t, \quad t \in [0, T],$$

where W is an n-dimensional standard Brownian motion, and $\delta(t, X_t)$ and $\sigma(t, X_t)$ are "affine" in X in the sense that

$$\delta(t, x) = d_0(t) + d_1(t)x,$$

where $d_0 : [0,T] \to \mathbb{R}^n$ and $d_1 : [0,T] \to \mathbb{R}^{n\times n}$ are continuous functions and

$$\left(\sigma(t,x)\sigma(t,x)'\right)_{ij} = (v_0(t))_{ij} + (v_1(t))'_{ij}\, x, \quad i,j = 1,\dots,n,$$

for continuous functions $v_0 : [0,T] \to \mathbb{R}^{n\times n}$ and $v_1 : [0,T] \to \mathbb{R}^{n\times n\times n}$. Let $c \in \mathbb{C}$, $a, b \in \mathbb{C}^n$ and

$$\Lambda(t,x) = \lambda_0(t) + \lambda_1(t)'x,$$

for $\lambda_0 : [0,T] \to \mathbb{R}$ and $\lambda_1 : [0,T] \to \mathbb{R}^n$ continuous. Under certain technical conditions (see, e.g., Duffie et al. [20]) for $0 \le t \le u \le T$ the following expression holds:

$$\mathbb{E}\left[e^{-\int_t^u \Lambda(s,X_s)\mathrm{d}s} e^{a'X_u} \left(b'X_u + c\right) \,\middle|\, \mathcal{F}_t \right] = e^{\alpha^u(t)+\beta^u(t)'X_t}\left[\hat{\alpha}^u(t) + \hat{\beta}^u(t)'X_t\right] \tag{B.1}$$

where α^u and β^u are functions uniquely solving the following ordinary differential equations:

$$\begin{aligned}
\partial_t\beta^u(t) &= \lambda_1(t) - d_1(t)'\beta^u(t) - \frac{1}{2}\beta^u(t)'v_1(t)\beta^u(t),\\
\partial_t\alpha^u(t) &= \lambda_0(t) - d_0(t)'\beta^u(t) - \frac{1}{2}\beta^u(t)'v_0(t)\beta^u(t),
\end{aligned}$$

and $\hat{\alpha}^u$ and $\hat{\beta}^u$ are functions uniquely solving the following ordinary differential equations:

$$\begin{aligned}
\partial_t\hat{\beta}^u(t) &= -d_1(t)'\hat{\beta}^u(t) - \beta^u(t)'v_1(t)\hat{\beta}^u(t),\\
\partial_t\hat{\alpha}^u(t) &= -d_0(t)'\hat{\beta}^u(t) - \beta^u(t)'v_0(t)\hat{\beta}^u(t),
\end{aligned}$$

for $t \in [0,u]$ with boundary conditions $\alpha^u(u) = 0$, $\beta^u(u) = a$ and $\hat{\beta}^u(u) = b$, $\hat{\alpha}^u(u) = c$.

References

[1] J. P. Ansel and C. Stricker. Décomposition de Kunita-Watanabe. In *Séminaire de Probabilités XXVII*, Lecture Notes in Mathematics Volume 1557. Springer, 1993.

[2] J. Barbarin. *Valuation, Hedging and the Risk Management of Insurance Contracts.* PhD thesis, Catholic University of Louvain, 2008.

[3] F. Biagini and A. Cretarola. Quadratic hedging methods for defaultable claims. *Applied Mathematics & Optimization*, 56(3):425–443, 2007.

[4] F. Biagini and A. Cretarola. Local risk-minimization for defaultable markets. *Mathematical Finance*, 19(4):669–689, 2009.

[5] F. Biagini and A. Cretarola. Local risk-minimization with recovery process. *Applied Mathematics & Optimization*, 65(3):293–314, 2012.

[6] F. Biagini and I. Schreiber. Risk-minimization for mortality derivatives. Preprint, University of Munich, 2012.

[7] F. Biagini, T. Rheinländer, and J. Widenmann. Hedging mortality claims with longevity bonds. *To appear in ASTIN Bulletin*, 2012.

[8] T. R. Bielecki and M. Rutkowski. *Credit Risk: Modeling, Valuation and Hedging*. Springer Finance, 2004.

[9] E. Biffis. Affine processes for dynamic mortality and actuarial valuations. *Insurance: Mathematics & Economics*, 37(3):443–468, 2005.

[10] D. Blake, A. J. G. Cairns, and K. Dowd. The birth of the life market. *Asia-Pacific Journal of Risk and Insurance*, 3(1):6–36, 2008.

[11] C. Blanchet-Scalliet and M. Jeanblanc. Hazard rate for credit risk and hedging defaultable contingent claims. *Finance & Stochastics*, 8(1):145–159, 2004.

[12] A. J. G. Cairns, D. Blake, and K. Dowd. Pricing death: Frameworks for the valuation and securitization of mortality risk. *ASTIN Bulletin*, 36(1):79–120, 2006.

[13] A. J. G. Cairns, D. Blake, K. Dowd, G. D. Coughlan, and M. Khalaf-Allah. Bayesian stochastic mortality modelling for two populations. *ASTIN Bulletin*, 41(1):29–59, 2011.

[14] G. D. Coughlan, M. Khalaf-Allah, Y. Ye, S. Kumar, A. J. G. Cairns, D. Blake, and K. Dowd. Longevity hedging 101: A framework for longevity basis risk analysis and hedge effectiveness. *North American Actuarial Journal*, 15(2): 150–176, 2011.

[15] M. Dahl and T. Møller. Valuation and hedging of life insurance liabilities with systematic mortality risk. *Insurance: Mathematics & Economics*, 39(2): 193–217, 2006.

[16] M. Dahl, M. Melchior, and T. Møller. On systematic mortality risk and risk-minimization with survivor swaps. *Scandinavian Actuarial Journal*, 2-3: 114–146, 2008.

[17] Q. Dai and K. Singleton. Specification analysis of affine term structure models. *Journal of Finance*, 55(5):1943–1978, 2000.

[18] G. Deelstra and F. Delbaen. Convergence of discretized stochastic (interest rate) processes with stochastic drift term. *Applied Stochastic Models and Data Analysis*, 14(1):77–84, 1998.

[19] K. Dowd, A. J. G. Cairns, D. Blake, G. D. Coughlan, and M. Khalaf-Allah. A gravity model of mortality rates for two related populations. *North American Actuarial Journal*, 15(2):334–356, 2011.

[20] D. Duffie, J. Pan, and K. Singleton. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica*, 68:1343–1376, 2000.

[21] D. Duffie, D. Filipović, and W. Schachermayer. Affine processes and applications in finance. *Annals of Applied Probability*, 13(3):984–1053, 2003.

[22] D. Filipović and E. Mayerhofer. Affine diffusion processes: Theory and applications. *Radon Series Comp. Appl. Math*, 8:1–40, 2009.

[23] H. Föllmer and D. Sondermann. Hedging of non-redundant contingent claims. In A. Mas-Colell and W. Hildenbrand, editors, *Contributions to Mathematical Economics*, pages 205–223. North Holland, 1986.

[24] J. Jacod and A. N. Shiryaev. *Limit Theorems for Stochastic Processes.* Springer, 2002.

[25] S. F. Jarner and E. M. Kryger. Modelling adult mortality in small populations: The SAINT model. Pensions Institute Working Paper PI-0902, Cass Business School, London, 2009.

[26] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus.* Springer, 1991.

[27] J. S.-H. Li and M. R. Hardy. Measuring basis risk in longevity hedges. *North American Actuarial Journal*, 15(2):177–200, 2011.

[28] N. Li and R. Lee. Coherent mortality forecasts for a group of populations: An extension to the Lee-Carter method. *Demography*, 42(3):575–594, 2005.

[29] T. Møller. Risk-minimizing hedging strategies for unit-linked life insurance contracts. *ASTIN Bulletin*, 28(1):17–47, 1998.

[30] T. Møller. Risk-minimizing hedging strategies for insurance payment processes. *Finance and Stochastics*, 4(5):419–446, 2001.

[31] P. E. Protter. *Stochastic Integration and Differential Equations.* Springer, 2003.

[32] M. Riesner. Hedging life insurance contracts in a lévy process financial market. *Insurance: Mathematics and Economics*, 38(3):599–608, 2006.

[33] M. Schweizer. A guided tour through quadratic hedging approaches. In E. Jouini, J. Cvitanic, and M. Musiela, editors, *Option Pricing, Interest Rates and Risk Management*, pages 538–574. Cambridge University Press, 2001.

[34] M. Schweizer. Local risk-minimization for multidimensional assets and payment streams. *Banach Center Publications*, 83:213–229, 2008.