General Analysis of Long-Term Interest Rates

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Abstract

We introduce here the idea of a long-term swap rate, characterized as the fair rate of an overnight indexed swap with infinitely many exchanges. Furthermore we analyse the relationship between the long-term swap rate, the long-term yield, (Biagini et al. 2018, Biagini and Härtel 2014, El Karoui et al. 1997), and the long-term simple rate (Brody and Hughston 2016) as long-term discounting rate. Finally, we investigate the existence of these long-term rates in two term structure methodologies, the Flesaker-Hughston model and the linear-rational model. A numerical example illustrates how our results can be used to estimate the non-optional component of a CoCo bond.

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1 Introduction

The modelling of long-term interest rates is an important topic for financial institutions investing in securities with maturities that have a long time horizon, such as life- insurance products and infrastructure projects. Most articles focusing on long-term interest rate modelling take the long-term yield, defined as the continuously compounded spot rate where the maturity goes to infinity, to be the discounting rate for these products (Biagini et al. 2018, Biagini and Härtel 2014, Dybvig et al. 1996, El Karoui et al. 1997, Yao 2000). An important result which characterises the long-term yield is the Dybvig-Ingersoll-Ross (DIR) theorem, which states that the long-term yield is a non-decreasing process. It was

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first shown in Dybvig et al. [1996] and then discussed in Goldammer and Schmock [2012], Hubalek et al. [2002], Kardaras and Platen [2012], McCulloch [2000], and Schulze [2008]. According to Brody and Hughston [2016] the DIR theorem ultimately implies that discounted cashflows with higher time-to-maturity are overpenalised, so that the use of this long-term interest rate becomes unsuitable for the valuation of projects having maturity in a distant future. To overcome this issue, in Brody and Hughston [2016] the authors propose to use for discounting the long-term simple rate, which is defined as the simple spot rate with an infinite maturity. Motivated by this ongoing discussion in the literature, we investigate in this paper alternative long-term interest rates.

We introduce here the long-term swap rate, which we define as the fair fixed rate of a fixed to floating swap with infinitely many exchanges. To the best of our knowledge, there has not been any attempt in the literature to study the long-term swap rate so far. In particular, we focus our attention on swap rates because, unlike zero-coupon bonds, they are directly observable in the market. Our interest in the long term swap rate is also motivated by the observation that some financial products may involve the interchange of cashflows on a possibly unlimited time horizon. This is the case of some kind of contingent convertible (CoCo) bonds, which became popular after the financial crisis in 2008. Such products are debt instruments issued by credit institutes, which embed the option for the bank to convert debt into equity, typically in order to overcome the situation where the bank is not capitalised enough (Albul et al. 2010, Brigo et al. 2015, Duffie 2009, Flannery 2005, 2009). In the course of the crisis the importance of CoCo bonds for financial institutions to maintain a certain level of capital was pointed out in Bernanke [2009]. In Dudley [2009], the increase in their use in systemically relevant financial institutions was one of three main points that should be realised in the aftermath of the crisis to strengthen the financial system.

As reported in Brigo et al. [2015], the value of these instruments may be decomposed as a portfolio consisting of plain bonds and exotic options. A valuation method for CoCo bonds with finite maturity is presented in Brigo et al. [2015], whereas Albul et al. [2010] also considers the case of unlimited maturity. Such a result is of practical importance since some of these products offered in the market have maturity equal to infinity (see PLC 2014a). In a situation where the CoCo bond has infinite maturity and the coupons of the non-optional part are floating, it is then natural to ask for an instrument which allows to hedge the interest rate risk involved in the non-optional part of the contract. A fixed to floating interest rate swap with infinitely many exchanges could serve as a hedging product for the interest rate risk beared by CoCo bonds. The main input for defining such a swap is its fixed rate, i.e. the long-term swap rate. Furthermore the long-term swap rate may also play an important role in the context of multiple curve bootstrapping. As we shall see in the following, we will concentrate our investigations on overnight indexed swap (OIS) contracts. Such OIS contracts constitute the input quotes for bootstrapping procedures which allow for the construction of a discounting curve, according to the post-crisis market practice (see for example

Cuchiero et al. 2016a or Henrard 2014). In view of this, the long-term swap rate becomes quite a natural object, from which information on the long-end of the discounting curve can be inferred.

The main result of the paper is then the definition of the long-term swap rate R and the study of its properties and relations with the long-term yield and the long-term simple rate. In particular, we obtain that the long-term swap rate always exists finitely and that this rate is either constant or non-monotonic. In the case of a convergent infinite weighted sum S_∞ of bonds, we are able to provide an explicit model-independent formula for R, which is only dependent on S_∞ (see (4.2)). Hence the long-term swap rate could represent an alternative discounting tool for long-term investments, since it is always finite, non-monotonic, can be explicitly characterised, and can be inferred by products existing in the markets.

As a contribution to the ongoing discussion on suitable discounting factors for investments over long time horizons, we then provide a comprehensive analysis of the relations among the long-term yield, the long-term simple rate, and the long-term swap rate in a model-free approach. In particular, we study how the existence of one of these long-term rates impacts the existence and finiteness of the other ones. This analysis shows the advantage of using the long-term swap rate as discounting rate, since it always remains finite when the other rates may become zero or explode.

The paper is structured as follows. First, we introduce in Section 2 some necessary prerequisites, such as the different kinds of interest rates and interest rate swaps, in particular OISs. Then, Sections 3 and 4 describe the three asymptotic rates and some important features of the long-term swap rate like the model-free formula. In Section 5 we investigate the influence of each long-term rate on the existence and finiteness of the other rates. Finally, in Section 6 we analyse the long-term rates in some selected term structure models. We chose the Flesaker-Hughston methodology, developed in Flesaker and Hughston [1996], and the linear-rational term structure methodology, presented in Filipović et al. [2017], since they also include the wide class of affine interest models and possess some appealing features such as high tractability and simple forms of the different interest rates. In both cases we compute the long-term swap rate and the other long-term rates. We conclude with a numerical example, where we illustrate how our results can be used to estimate the non-optional component of a CoCo bond.

2 Fixed Income Setup

2.1 Interest Rates

We now introduce some notations. All quantities in the following are assumed to be associated to a risk-free curve, which, in the post-crisis market setting, can be approximated by the overnight curve used in collateralised transactions (see Section 1.1 of Cuchiero et al. 2016a).

First, we define the contract value of a zero-coupon bond at time t with maturity $T > t$ as $P(t,T)$. It guarantees its holder the payment of one unit of currency at time T, hence $P(T,T) = 1$ for all $T \geq 0$. We assume that there exists a frictionless market for zero-coupon bonds for every time $T > 0$ and that $P(t,T)$ is differentiable in T. In the following we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the filtration $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual hypothesis of right-continuity and completeness. Furthermore, we only consider finite positive zero-coupon bond prices, i.e. $0 < P(t,T) < +\infty$ $\mathbb{P}$-a.s. for all $0 \leq t \leq T$. Then, we define the yield for $[t,T]$ as the continuously compounded spot rate for $[t,T]$

$$Y(t,T) := -\frac{\log P(t,T)}{T-t}. \tag{2.1}$$

The simple spot rate for $[t,T]$ is

$$L(t,T) := \frac{1}{T-t}\left(\frac{1}{P(t,T)} - 1\right). \tag{2.2}$$

The short rate at time t is defined as

$$r_t := \lim_{T \downarrow t} Y(t,T) \quad \mathbb{P}\text{-a.s.} \tag{2.3}$$

The corresponding money-market account is denoted by $(\beta_t)_{t \geq 0}$ with

$$\beta_t := \exp\left(\int_0^t r_s \, ds\right). \tag{2.4}$$

In particular we assume an arbitrage-free setting, where the discounted bond price process $\frac{P(t,T)}{\beta_t}, t \in [0,T]$, is an $(\mathbb{F}, \mathbb{P})$-martingale for all $T > 0$. This implies that the large financial market, consisting of infinitely many bonds, is arbitrage free in the sense of *no asymptotic free lunch with vanishing risk* (Cuchiero et al. 2016b Assumption 2.2 and Cuchiero et al. 2018). This also implies that β is well-defined, i.e. $\int_0^t |r_s| \, ds < \infty$ a.s. for all $t \geq 0$. We assume to work with the càdlàg version of $\frac{P(t,T)}{\beta_t}, t \in [0,T]$, for all $T > 0$. Consequently $P(t,T), Y(t,T), L(t,T), t \in [0,T]$, are all càdlàg processes in the sequel.

2.2 Interest Rate Swaps

Swap contracts are derivatives where two counterparties exchange cashflows. There exist different kinds of swap contracts, involving cashflows deriving for example from commodities, credit risk or loans in different currencies. As far as interest rate swaps are concerned, the evaluation of such claims represents an aspect which is part of the discussion on multiple curve models, due to the recent financial crisis. While a survey of the literature on multiple-curve models would be beyond the scope of the present paper[1], we limit ourselves to note that even in the post-crisis

[1] For a complete list of references the interested reader is referred to Cuchiero et al. [2016a].

setting, there are particular types of interest rate swaps whose evaluation formulas are equivalent to the ones employed for standard interest rate swaps in the single-curve pre-crisis setting. Since such instruments, called OISs, play a pivotal role in the construction of discount curves, we concentrate our study on them, and avoid to define a full multiple-curve model.

We consider a infinite tenor structure of the form

$$0 < T_0 < T_1 < \cdots < T_n < \cdots, \tag{2.5}$$

for $n \in \mathbb{N}$. We set $\delta_i := T_i - T_{i-1}, i \in \mathbb{N} \setminus \{0\}$. In an OIS contract, floating payments are indexed to a compounded overnight rate like EONIA. The variable rate that one party has to pay every time T_i, $i = 1, 2, \dots$, is $\delta_i \bar{L}(T_{i-1}, T_i)$ with $\bar{L}(T_{i-1}, T_i)$ denoting the compounded overnight rate for $[T_{i-1}, T_i]$. This rate is given by (see equation (10) of Filipović and Trolle 2013)

$$\bar{L}(T_{i-1}, T_i) = \frac{1}{\delta_i} \left(\exp\left(\int_{T_{i-1}}^{T_i} r_s \, ds \right) - 1 \right).$$

Fixed n, the OIS rate for the period $[T_0, T_n]$, i.e. the fixed rate which makes the OIS value equal to zero at inception, is for $t \leq T_0$

$$\begin{aligned} R^{OIS}(t; T_0, T_n) &= \frac{\sum_{i=1}^n \mathbb{E}^{\mathbb{P}} \left[\exp\left(-\int_t^{T_i} r_s \, ds \right) \delta_i \bar{L}(T_{i-1}, T_i) \middle| \mathcal{F}_t \right]}{\sum_{i=1}^n P(t, T_i) \, \delta_i} \\ &= \frac{P(t, T_0) - P(t, T_n)}{\sum_{i=1}^n P(t, T_i) \, \delta_i} \end{aligned} \tag{2.6}$$

or in general

$$R^{OIS}(t; T_0, T_n) = \frac{P(t, T_0) - P(t, T_n)}{\int_{T_1}^{T_n} \exp(-(s-t) \, Y(t, s)) \, \xi(ds)},$$

where ξ is a measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ and Y is the yield, defined in (2.1). Note that (2.6) corresponds to the formula for the par swap rate in a single curve setting. In the following, we consider only OIS swaps and set

$$R(t, T_n) := R^{OIS}(t; t, T_n) \tag{2.7}$$

for all $t \leq T_n$.

Remark 1. *Note that in* (2.7) *we have set* $T_0 = t$. *This is equivalent to consider the interest rate* $R(t, T)$ *as associated to a rolling over of OIS contracts. This is possible in our model since we admit the existence of bonds for any maturity* $T > 0$.

3 Long-Term Rates

In this section we consider some possible long-term rates. In particular we focus on the long-term yield and the long-term simple rate, which have been already defined in the literature (see El Karoui et al. 1997 and Brody and Hughston 2016). The *long-term yield* can be defined in different ways. Some articles investigate interest rates with a certain time to maturity to approach the concept of "long-term", e.g. in Yao [2000] yield curves with time to maturity over thirty years are examined, Shiller [1979] considers yields with a maturity beyond twenty years to be "long-term", whereas the ECB takes ten years as a barrier, (see European Central Bank 2015). Another approach is to look at the asymptotic behaviour of the yield curve by letting the maturity go to infinity. This approach is used by Biagini et al. [2018], Biagini and Härtel [2014], Dybvig et al. [1996], El Karoui et al. [1997]. In line with the above-mentioned principle, we introduce our first object of study, and define the long-term yield $\ell := (\ell_t)_{t\geq 0}$ as

$$\ell_\cdot := \lim_{T\to\infty} Y(\,\cdot\,, T)\,, \tag{3.1}$$

if the limit exists in the sense of the uniform convergence on compacts in probability (convergence in ucp).[2] If the limit in (3.1) exists but it is infinite, positive or negative (see Definition 24), we will write $\ell = \pm\infty$ for the sake of simplicity. We will use this improper notation also for the other long-term interest rates in the sequel of the paper. We recall that the long-term yield process ℓ is a non-decreasing process by the DIR theorem (see Theorem 2 of Dybvig et al. 1996), which was first asserted by Dybvig et al. [1996] and further discussed in Goldammer and Schmock [2012], Hubalek et al. [2002], Kardaras and Platen [2012] and Brody and Hughston [2016].

In Brody and Hughston [2016] it is suggested to consider a particular model for the *long-term simple rate* for the discounting of cashflows occuring in a distant future. By using exponential discount factors the discounted value of a long-term project, that will be realised over a long time horizon, in most cases will turn out to be overdiscounted, hence too small to justify the overall project costs. To overcome this problem, the authors of Brody and Hughston [2016] make use of the concept of "social discounting", where the long-term simple rate is employed for discounting cashflows in the distant future. To integrate this interesting approach into our considerations, we now define the long-term simple rate process $L := (L_t)_{t\geq 0}$ as

$$L_\cdot := \lim_{T\to\infty} L(\,\cdot\,, T)\,,$$

if the limit exists in ucp, where $L(t,T)$ is defined in (2.2). Note that $L_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ by (2.2).

[2]For a definition of the ucp convergence and some additional results the reader is referred to Section B in the appendix.

We introduce the process $P := (P_t)_{t \geq 0}$ as

$$P_{\cdot} := \lim_{T \to \infty} P(\cdot, T)\,, \tag{3.2}$$

if the limit exists (finite or infinite) in ucp. For an alternative definition of long bond we refer to Qin and Linetsky [2018].

Remark 2. *We note that as a consequence of our assumption that the bond prices are càdlàg, we also obtain that all the long-term rates introduced above and in Section 4 are càdlàg. In the material that follows thereafter we will then use Theorem 2 of Chapter I, Section 1 of Protter [2005], which tells us that for two right-continuous stochastic processes X and Y it holds that $X_t = Y_t$ $\mathbb{P}$-a.s. for all $t \geq 0$ is equivalent to $\mathbb{P}$-a.s. for all $t \geq 0$, $X_t = Y_t$.*

We define $S_n := (S_n(t))_{t \geq 0}$ with

$$S_n(t) := \int_{T_1}^{T_n} \exp(-\,(T-t)\,Y(t,T))\,\xi(dT)\,, t \geq 0,$$

considering a tenor structure with infinite many exchange dates. It is clear that

$$S_n(t) = \sum_{i=1}^{n} \delta_i P(t, T_i)\,, t \geq 0, \tag{3.3}$$

if $\xi(dT) = \sum_{i=1}^{+\infty} \delta_i \delta_{\{T_i\}}$, with $\delta_i := T_i - T_{i-1}$ and Dirac's delta functions $\delta_{\{T_i\}}$. Then the limit

$$\lim_{n \to \infty} S_n(\cdot) \tag{3.4}$$

in ucp always exists, finite or infinite. In the sequel we denote this limit by S_∞ if it exists and is finite. All bond prices are strictly positive, therefore for all $t \geq 0, n \in \mathbb{N}$ we have $\mathbb{P}$-a.s. $S_n(t) > 0$ and $S_\infty(t) > 0$.

4 The Long-Term Swap Rate

We now introduce the *long-term swap rate* $R := (R_t)_{t \geq 0}$ as

$$R_{\cdot} := \lim_{n \to \infty} R(\,\cdot\,, T_n)$$

if the limit exists in ucp, where $R(t, T_n)$ is defined in (2.7). The long-term swap rate, defined here for the first time, can be understood as the fair fixed rate of an OIS starting in t that has a payment stream with infinitely many exchanges. This fixed rate is meant to be fair in the sense that the initial value of this OIS equals zero.

We investigate the existence and finiteness of the long-term swap rate. We first provide a model-free formula for the swap rate, when S_∞ exists and is finite.

In particular, we focus here on the case when S_n is given by (3.3) and the tenor structure is such that

$$c < \inf_{i\in\mathbb{N}\setminus\{0\}} (T_i - T_{i-1}) \tag{4.1}$$

with $c > 0$. This hypothesis avoids the degenerated case where $|T_i - T_{i-1}| \to 0$ for $i \to \infty$, and corresponds to the realistic setting of a fixed tenor (but where the number of dates may become very large).

This section relies on some properties of S_∞, which we proved in Appendix A.

Theorem 3. *Assume that S_n is defined as in (3.3) for $n \in \mathbb{N}$ and the tenor structure satisfy condition (4.1).*

(i) If $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp, then $\mathbb{P}$-a.s.

$$R_t = \frac{1}{S_\infty(t)} > 0 \tag{4.2}$$

for all $t \geq 0$.

(ii) If $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp and P, defined in (3.2), exists finitely, then it holds $R_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.

(iii) The long-term swap rate cannot explode, i.e. $\mathbb{P}(|R_t| < +\infty) = 1$ for all $t \geq 0$.

Proof. To (i): We have that in ucp

$$\begin{aligned}
\lim_{n\to\infty} R(\,\cdot\,, T_n) &\overset{(2.7)}{=} \lim_{n\to\infty} \frac{1 - P(\,\cdot\,, T_n)}{S_n(\,\cdot\,)} = \lim_{n\to\infty} \frac{1}{S_n(\,\cdot\,)} - \lim_{n\to\infty} \frac{P(\,\cdot\,, T_n)}{S_n(\,\cdot\,)} \\
&= \lim_{n\to\infty} \frac{1}{S_n(\cdot)} = \frac{1}{S_\infty(\,\cdot\,)} > 0
\end{aligned}$$

by Theorem 23.

To (ii): We have that in ucp

$$\begin{aligned}
\lim_{n\to\infty} R(\,\cdot\,, T_n) &\overset{(2.7)}{=} \lim_{n\to\infty} \frac{1 - P(\,\cdot\,, T_n)}{S_n(\,\cdot\,)} \\
&= \lim_{n\to\infty} \frac{1}{S_n(\,\cdot\,)} - \lim_{n\to\infty} \frac{P(\,\cdot\,, T_n)}{S_n(\,\cdot\,)} \\
&= \lim_{n\to\infty} \frac{1}{S_n(\cdot)} - \lim_{n\to\infty} \frac{P(\,\cdot\,, T_n)}{S_n(\,\cdot\,)} \\
&= -\lim_{n\to\infty} \frac{P(\,\cdot\,, T_n)}{S_n(\,\cdot\,)} = -P. \lim_{n\to\infty} \frac{1}{S_n(\cdot)} = 0
\end{aligned}$$

by Theorem 23.

To (iii): Since (i) and (ii) hold, we need only to study the case when $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp and $P = +\infty$. We have that in this case

$$\lim_{n\to\infty} R(\,\cdot\,, T_n) = -\lim_{n\to\infty} \frac{P(\,\cdot\,, T_n)}{S_n(\,\cdot\,)}$$

in ucp. We note that $\mathbb{P}$-a.s.

$$0 \leq \sup_{0\leq s\leq t} \frac{P(s,T_n)}{S_n(s)} = \sup_{0\leq s\leq t} \frac{1}{\delta_n}\left(1-\frac{S_{n-1}(s)}{S_n(s)}\right) \leq 1/c$$

for all $t \geq 0$ with $\delta_n = T_n - T_{n-1}$. Hence

$$\mathbb{P}\left(\inf_{0\leq s\leq t} \frac{P\left(s,T_n\right)}{S_n(s)} > M\right) \overset{n\to\infty}{\longrightarrow} 0$$

for all $M > 1/c$. This contradicts Definition 24 of ucp convergence to $+\infty$ applied to $|R|$, so the long-term swap rate always exists and is finite $\mathbb{P}$-a.s.. $\square$

Remark 4. *If now we consider a tenor structure with $T_i - T_{i-1} = \delta$ for all $i \in \mathbb{N}\backslash\{0\}$, then $S_n(t) = \delta\sum_{i=1}^{n} P(t,T_i)$ and (4.2) boils down to*

$$R_t = \frac{1}{\delta\sum_{i=1}^{\infty} P(t,T_i)}\,,$$

i.e. R_t is proportional to the consol rate of a perpetual bond. For more details on consol bonds, we refer to Delbaen [1993], Duffie et al. [1995], and the references therein. However, our construction is more general and goes beyond the existence of the consol bond rate. Our approach has the advantage of being consistent with the multi-curve theory of interest rate modelling as well as of delivering a long-term interest rate which is always finite.

By Theorem 3 (i) and (ii) we obtain the existence of the long-term swap rate as a finite limit if P exists finitely. However this result always holds as shown by Theorem 3 (iii).

Corollary 5. *If $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp, then it holds $-\infty < R_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. This is a consequence of Theorem 3 (ii) and (iii). $\square$

Proposition 6. *Suppose $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. If for all $n \in \mathbb{N}$ $P(t,T_n) \geq P(t,T_{n+1})$ for all $t \in [0,T_n]$, then*

$$R_t = -k_t$$

for a process $(k_t)_{t\geq 0}$ with $0 \leq k_t \leq 1$ $\mathbb{P}$-a.s. for all $t \geq 0$.

Proof. Since for all $n \in \mathbb{N}$, $S_n(t) \leq S_{n+1}(t)$ $\mathbb{P}$-a.s. for all $t \geq 0$, for all $n \in \mathbb{N}$ we have $\mathbb{P}$-a.s.

$$\frac{P(t,T_n)}{S_n(t)} = \frac{P\left(t,T_n\right)}{\beta_t}\frac{\beta_t}{S_n(t)} \geq \frac{P\left(t,T_{n+1}\right)}{\beta_t}\frac{\beta_t}{S_{n+1}(t)} = \frac{P(t,T_{n+1})}{S_{n+1}(t)}$$

for all $t \geq 0$. This implies that $\mathbb{P}$-a.s.

$$1 \geq \sup_{0 \leq s \leq t} \frac{P(s, T_n)}{S_n(s)} \geq \sup_{0 \leq s \leq t} \frac{P(s, T_{n+1})}{S_{n+1}(s)}$$

for all $t \geq 0$. Hence $\frac{P(\cdot, T_n)}{S_n(\cdot)} \overset{n\to\infty}{\longrightarrow} k.$ in ucp, with $0 \leq k_t \leq 1$ $\mathbb{P}$-a.s. for all $t \geq 0$. In particular by Theorem 3 (ii), we get $k_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ if $P_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$. □

Remark 7. 1. *Note that if $r_t \geq 0$ for all $t \geq 0$, then bond prices are decreasing with respect to time to maturity.*

2. *We remark that the process k in Proposition 6 may not necessarily be identically zero if $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. In particular consider the (unrealistic) case when $P(t, T_n) := 1 + (T - t)^n$ for some $T > 0$. Then*

$$\frac{P(t, T_n)}{S_n(t)} = \frac{1 + (T - t)^n}{\sum_{i=1}^{n} (1 + (T - t)^i)} = \frac{1}{1 + \sum_{i=1}^{n-1} \frac{1+(T-t)^i}{1+(T-t)^n}} \overset{n\to\infty}{\longrightarrow} 1.$$

If we assume that there exists a liquid market for perpetual OIS, meaning OIS with infinitely many exchanges with the fixed rate corresponding to the long-term swap rate, we can state the following theorem. We recall that we are working under the hypothesis that $\mathbb{P}$ is an equivalent martingale measure for the bond market, i.e that the bond market is arbitrage-free in the sense of *no asymptotic free lunch with vanishing risk* (Cuchiero et al. 2016b).

Theorem 8. *In the setting outlined in Section 2.1, the long-term swap rate is either constant or non-monotonic.*

Proof. First, we assume that $R_s \geq R_t$ $\mathbb{P}$-a.s. with $\mathbb{P}(R_s > R_t) > 0$ for $0 \leq t < s$. Then, let us consider the following investment strategy. At time t we enter a payer OIS with perpetual annuity, nominal value N, fixed-rate R_t and the following tenor structure

$$t < s \leq T_1 < \cdots < T_n \tag{4.3}$$

where $n \to \infty$. This investment has zero value in t, so there is no net investment so far. We receive the following payoff in each T_i, $i \in \mathbb{N}\backslash\{0\}$:

$$\left(\bar{L}(T_{i-1}, T_i) - R_t\right) \delta_i N \, .$$

Then at time s we enter a receiver OIS with a perpetual annuity, nominal value N, a fixed-rate of R_s and the same tenor structure as in (4.3). The value of this OIS is zero in s, hence there is still no net investment, and the payoff in each T_i, $i \in \mathbb{N}$, resulting from this OIS is:

$$\left(R_s - \bar{L}(T_{i-1}, T_i)\right) \delta_i N \, .$$

This strategy leads to the payoff at T_i

$$H_i := \left(\bar{L}(T_{i-1},T_i) - R_t\right)\delta_i N + \left(R_s - \bar{L}(T_{i-1},T_i)\right)\delta_i N = \delta_i N\,(R_s - R_t) \geq 0$$

with $\mathbb{P}(H_i > 0) > 0$, i.e. to an arbitrage.

If we assume that $R_s \leq R_t$ $\mathbb{P}$-a.s. with $\mathbb{P}(R_s < R_t) > 0$ for $0 \leq t \leq s \leq T_1$, we use an analogue arbitrage strategy with the only difference that we invest in t in a receiver OIS and in s in a payer OIS.

It follows that in an arbitrage-free market setting the long-term swap rate cannot be non-decreasing or non-increasing, i.e. it can only be monotonic if it is constant. $\square$

5 Relation between Long-Term Rates

We now study the relation among the long-term rates introduced in Sections 3 and 4 in terms of their existence. For further details, we also refer to Härtel [2015].

For the sake of simplicity we now assume a tenor structure with $T_i - T_{i-1} = \delta$ for all $i \in \mathbb{N}\setminus\{0\}$. This is of course the case when we extrapolate the long-term swap rate by OIS contracts existing in the market. We choose this setting in order to focus on the behaviour of the bond price for $T \to \infty$, i.e. of P defined in (3.2), independently of the maturity distances in the tenor structure.

5.1 Influence of the Long-Term Yield on Long-Term Rates

In this section we study the influence of the existence of the long-term yield on the existence of the long-term swap and simple rates. Since typical market data indicate positive long-term yields[3], we restrict ourselves to the cases of $\ell \geq 0$. For a more general analysis which also takes into account the possibility of a negative long-term yield, we refer to Härtel [2015].

Theorem 9. *If $0 < \ell_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $0 < R_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $L = +\infty$.*

Proof. First, we show that $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp. For this, it is sufficient to show that for all $t \geq 0$, $\lim_{n\to\infty} \sup_{0\leq s\leq t} S_n(s) < +\infty$ $\mathbb{P}$-a.s., since this also implies $\lim_{n\to\infty} \sup_{0\leq s\leq t} S_n(s) < +\infty$ in probability.

We know that for all $t \geq 0$ and all $\epsilon > 0$ it holds

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |Y(s,T_n) - \ell_s| \leq \epsilon\right) \overset{(2.1)}{=} \mathbb{P}\left(\sup_{0\leq s\leq t} \left|\frac{\log P(s,T_n)}{T_n - s} + \ell_s\right| \leq \epsilon\right) \overset{n\to\infty}{\longrightarrow} 1,$$

i.e. for all $t \geq 0$ and all $\epsilon > 0$ there exists $N_\epsilon^t \in \mathbb{N}$ such that for all $n \geq N_\epsilon^t$

$$\mathbb{P}\left(\sup_{0\leq s\leq t} \left|\frac{\log P(s,T_n)}{T_n - s} + \ell_s\right| \leq \epsilon\right) > 1 - \delta(\epsilon) \tag{5.1}$$

[3]For long-term interest rate market data please refer to European Central Bank [2015] for the EUR market and to Board of Governors of the Federal Reserve System [2015] for the USD market.

with $\delta(\epsilon) \to 0$ for $\epsilon \to 0$. Define for $\epsilon > 0$, $u \geq 0$ and $n \in \mathbb{N}$

$$A_1^{\epsilon,u,n} := \left\{ \omega \in \Omega : \sup_{0 \leq s \leq u} \left| \frac{\log P(s,T_n)}{T_n - s} + \ell_s \right| \leq \epsilon \right\}. \tag{5.2}$$

Then for $n \geq N_\epsilon^u$ with $u > t$ we have $\mathbb{P}(A_1^{\epsilon,u,n}) > 1 - \delta(\epsilon)$ by (5.1) and

$$A_1^{\epsilon,u,n} \subseteq \left\{ \omega \in \Omega : \left| \log P(t,T_n) + (T_n - t)\,\ell_t \right| \leq \epsilon\,(T_n - t) \right\}.$$

Consequently for $n \geq N_\epsilon^u$ on $A_1^{\epsilon,u,n}$ we have

$$\exp[-\,(\epsilon + \ell_t)\,(T_n - t)] \leq P(t,T_n) \leq \exp[(\epsilon - \ell_t)\,(T_n - t)] \tag{5.3}$$

for all $t \in [0,u]$ and since $\ell_0 \leq \ell_t \leq \ell_u$ for all $t \in [0,u]$ by the DIR theorem (see for example Hubalek et al. 2002), we have that for $n \geq N_\epsilon^u$ on $A_1^{\epsilon,u,n}$ it holds

$$\exp[-\,(\epsilon + \ell_u)\,(T_n - u)] \leq \sup_{0 \leq s \leq t} P(s,T_n) \leq \exp[(\epsilon - \ell_0)\,T_n]\,. \tag{5.4}$$

For $t \geq 0$ we define

$$B_1(t) := \left\{ \omega \in \Omega : \lim_{n \to \infty} \sup_{0 \leq s \leq t} S_n(s) < +\infty \right\}. \tag{5.5}$$

We then obtain for $t < u$ and $n \geq N_\epsilon^u$

$$\begin{aligned}
\mathbb{P}(B_1(t)) &= \mathbb{P}\left(\left\{ \sup_{0 \leq s \leq t} S_{N_\epsilon^u - 1}(s) < +\infty \right\} \cap \left\{ \lim_{n \to +\infty} \sup_{0 \leq s \leq t} \sum_{i=N_\epsilon^u}^{n} P(s,T_i) < +\infty \right\} \right) \\
&= \mathbb{P}\left(\lim_{n \to \infty} \sup_{0 \leq s \leq t} \sum_{i=N_\epsilon^u}^{n} P(s,T_i) < +\infty \right) \\
&= \mathbb{P}\left(\lim_{n \to \infty} \sup_{0 \leq s \leq t} \sum_{i=N_\epsilon^u}^{n} P(s,T_i) < +\infty \,\middle|\, A_1^{\epsilon,u,n} \right) \mathbb{P}(A_1^{\epsilon,u,n}) \\
&\quad + \mathbb{P}\left(\lim_{n \to \infty} \sup_{0 \leq s \leq t} \sum_{i=N_\epsilon^u}^{n} P(s,T_i) < +\infty \,\middle|\, \Omega \backslash A_1^{\epsilon,u,n} \right) \mathbb{P}(\Omega \backslash A_1^{\epsilon,u,n}) \\
&\geq \mathbb{P}\left(\lim_{n \to \infty} \sup_{0 \leq s \leq t} \sum_{i=N_\epsilon^u}^{n} P(s,T_i) < +\infty \,\middle|\, A_1^{\epsilon,u,n} \right) \mathbb{P}(A_1^{\epsilon,u,n}) \\
&\geq \mathbb{P}\left(\lim_{n \to \infty} \sum_{i=N_\epsilon^u}^{n} \sup_{0 \leq s \leq t} P(s,T_i) < +\infty \,\middle|\, A_1^{\epsilon,u,n} \right) \mathbb{P}(A_1^{\epsilon,u,n}) \\
&\overset{(5.4)}{\geq} \mathbb{P}\left(\lim_{n \to \infty} \sum_{i=N_\epsilon^u}^{n} \exp[(\epsilon - \ell_0)\,T_i] < +\infty \,\middle|\, A_1^{\epsilon,u,n} \right) \mathbb{P}(A_1^{\epsilon,u,n}) \\
&\geq (1 - \delta(\epsilon)) \to 1
\end{aligned}$$

for $\epsilon \to 0$ since it holds $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \frac{\exp\left(-\ell_0 T_{n+1}\right)}{\exp\left(-\ell_0 T_n\right)} = \exp(-\ell_0 \delta) \in (0,1)\,,$$

which implies by the ratio test that $\lim_{n\to\infty} \sum_{i=0}^{n} \exp[(\epsilon - \ell_0)\, T_i] < +\infty$ $\mathbb{P}$-a.s. for $\epsilon \to 0$. That means, it holds $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp.

Hence by Theorem 3 (i) and (iii) we get for all $t \geq 0$ that $0 < R_t < +\infty$ $\mathbb{P}$-a.s. with

$$R_t = \frac{1}{S_\infty(t)}.$$

The exploding long-term simple rate, $L = +\infty$, is a result of Proposition 5.4 of Brody and Hughston [2016]. □

Now, let us investigate what happens to the long-term rates if the long-term yield either vanishes or explodes. We see that besides the asymptotic behaviour of the yield, information about the long-term zero-coupon bond price is needed to state the consequences on the other long-term rates. For the analysis of the cases when ℓ is negative, we refer to Härtel [2015].

Proposition 10. *Let $\ell_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. If P exists finitely with $\inf_{0\leq s\leq t} P_s > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $R_t = 0$ and $L_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. From Corollary 20 follows that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp, hence by applying Theorem 3 (ii) we get that $R_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.

To show that the long-term simple rate vanishes $\mathbb{P}$-a.s., we prove that for all $t \geq 0$ it holds that $\mathbb{P}(B_2(t)) = 1$ with $B_2(t)$ defined for $t \geq 0$ as follows

$$B_2(t) := \left\{ \omega \in \Omega : \lim_{n\to\infty} \sup_{0\leq s\leq t} L(s, T_n) = 0 \right\}.$$

We have for all $t \geq 0$

$$\begin{aligned} \mathbb{P}(B_2(t)) &\overset{(2.2)}{=} \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\leq s\leq t} \frac{1}{(T_n - s)\, P(s, T_n)} = 0 \right) \\ &\geq \mathbb{P}\left(\lim_{n\to\infty} \frac{1}{(T_n - t) \inf_{0\leq s\leq t} P_s} = 0 \right) = 1. \end{aligned}$$

□

In the following, we investigate exploding long-term yields.

Theorem 11. *If $\ell = +\infty$, then $0 < R_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ and $L = +\infty$.*

Proof. First, we show that $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp. We know by (B.6) that for all $t \geq 0$ and all $\epsilon > 0$ it holds

$$\mathbb{P}\left(\inf_{0\leq s\leq t} |Y(s, T_n)| > \epsilon \right) \overset{(2.1)}{\geq} \mathbb{P}\left(\inf_{0\leq s\leq t} |\log P(s, T_n)| > \epsilon\, T_n \right) \overset{n\to\infty}{\longrightarrow} 1,$$

i.e. for all $t \geq 0$ and all $\epsilon > 0$ there exists a $N_\epsilon^t \in \mathbb{N}$ such that for all $n \geq N_\epsilon^t$

$$\mathbb{P}\left(\inf_{0\leq s\leq t} |\log P(s,T_n)| > \epsilon T_n\right) > 1 - \delta(\epsilon) \tag{5.6}$$

with $\delta(\epsilon) \to 0$ for $\epsilon \to +\infty$. Define for $\epsilon > 0$, $u \geq 0$ and $n \in \mathbb{N}$

$$A_2^{\epsilon,u,n} := \left\{\omega \in \Omega : \inf_{0\leq s\leq u} |\log P(s,T_n)| > \epsilon T_n\right\}. \tag{5.7}$$

Then for $n \geq N_\epsilon^u$, $t < u$ and $B_1(t)$ defined as in (5.5), we obtain

$$\begin{aligned}
\mathbb{P}(B_1(t)) &= \mathbb{P}\left(\lim_{n\to\infty} \sup_{0\leq s\leq t} \sum_{i=N_\epsilon^u}^{n} P(s,T_i) < +\infty\right) \\
&\geq \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\epsilon^u}^{n} \sup_{0\leq s\leq t} P(s,T_i) < +\infty \,\middle|\, A_2^{\epsilon,u,n}\right) \mathbb{P}(A_2^{\epsilon,u,n}) \\
&\overset{(5.7)}{\geq} \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\epsilon^u}^{n} \exp(-\epsilon T_n) < +\infty \,\middle|\, A_2^{\epsilon,u,n}\right) \mathbb{P}(A_2^{\epsilon,u,n}) \\
&\geq (1-\delta(\epsilon)) \to 1
\end{aligned}$$

for $\epsilon \to +\infty$ due to the ratio test. That means $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp and consequently $0 < R_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ due to Theorem 3 (i).

Proposition 5.4 of Brody and Hughston [2016] leads to $L = +\infty$. □

The following table summarises the influence of the long-term yield on the long-term swap rate and long-term simple rate.

If the long-term yield is	With P	Then the long-term swap rate is	Then the long-term simple rate is
$\ell = 0$	$0 < P < +\infty$	$R = 0$	$L = 0$
$\ell > 0$	$P = 0$	$0 < R < +\infty$	$L = +\infty$
$\ell = +\infty$	$P = 0$	$0 < R < +\infty$	$L = +\infty$

Table 1: Influence of the long-term yield on long-term rates.

5.2 Influence of the Long-Term Swap Rate on Long-Term Rates

After we investigated the influence of the long-term yield on the long-term swap rate and long-term simple rate, we are also interested in the other direction of this relation.

Proposition 12. *If $R_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $\ell_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. First, we show that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. For this, let us assume S_n converges in ucp. Then, according to Theorem 3 (i) it is $0 < R_t$ $\mathbb{P}$-a.s. for all $t \geq 0$, but this is a contradiction and therefore S_n converges to $+\infty$ in ucp.

Consequently $\ell_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ due to Theorems 9 and 11. □

Now, we investigate the behaviour of the long-term rates if the long-term swap rate is strictly positive.

Proposition 13. *If $0 < R_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $\ell_t \geq 0$ and $L_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. We know from Corollary 5 that $R_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ if S_n converges to $+\infty$ in ucp. Hence if $R_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, we have $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp. Then, according to Propositions 3.2.3 and 3.2.9 of Härtel [2015] it holds $\mathbb{P}$-a.s. $\ell_t \geq 0$ for all $t \geq 0$.

Further, $L_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ is a consequence of Proposition 3.2.11 of Härtel [2015].

□

The only case left now is a strictly negative long-term swap rate.

Proposition 14. *If $-\infty < R_t < 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $\ell_t \leq 0$ and $L_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. First, we show that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. We know from Theorem 3 (i) that $R_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ if S_n converges to S_∞ in ucp, but this is a contradiction to $R_t < 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. As a consequence of Theorems 9 and 11 it is $\ell_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.

Since $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp and $R < 0$, by Theorem 3 (ii) we get that P cannot exist finitely, hence $L_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ because of Lemma 21. □

In the table below we summarise the influence of the long-term swap rate on the other long-term rates by using the previous results as well as Lemma 21. Note, that $-\infty < R_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ by Theorem 3 (iii). Hence, only three different cases have to be distinguished, $R_t = 0$, $0 < R_t < +\infty$, and $-\infty < R_t < 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.

If the long-term swap rate is	With P	Then the long-term yield is	Then the long-term simple rate is
$R = 0$	$0 \leq P < +\infty$	$\ell \leq 0$	$0 \leq L \leq +\infty$
$0 < R < +\infty$	$P = 0$	$\ell \geq 0$	$0 < L \leq +\infty$
$-\infty < R < 0$	$P = +\infty$	$\ell \leq 0$	$L = 0$

Table 2: Influence of the long-term swap rate on long-term rates.

5.3 Influence of the Long-Term Simple Rate on Long-Term Rates

Finally, we want to know about the influence of the long-term simple rate on long-term yields and long-term swap rates. Since $L_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, it is sufficient to investigate the three different cases, where $L_t = 0$, or $0 < L_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$, or $L = +\infty$.

Theorem 15. *If $L_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$, then $\ell_t \leq 0$ and $R_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. Furthermore, $R_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ if $P_t < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. For the sake of simplicity we prove the result for the case $L_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. The general case for $0 \leq L < +\infty$ is completely analogous. First, we show that $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp. We know that for all $t \geq 0$ and all $\epsilon > 0$ it holds

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |L(s,T_n)| \leq \epsilon\right) \overset{n\to\infty}{\longrightarrow} 1,$$

i.e. by (2.2) for all $t \geq 0$ and all $\epsilon > 0$ there exists $N_\epsilon^t \in \mathbb{N}$ such that for all $n \geq N_\epsilon^t$

$$\mathbb{P}\left(\sup_{0\leq s\leq t}\left|\frac{1}{T_n - s}\left(\frac{1}{P(s,T_n)} - 1\right)\right| \leq \epsilon\right) > 1 - \delta(\epsilon) \tag{5.8}$$

with $\delta(\epsilon) \to 0$ for $\epsilon \to 0$. Define for $\epsilon > 0$, $u \geq 0$ and $n \in \mathbb{N}$

$$A_3^{\epsilon,u,n} := \left\{\omega \in \Omega : \sup_{0\leq s\leq u}\left|\frac{1}{T_n - s}\left(\frac{1}{P(s,T_n)} - 1\right)\right| \leq \epsilon\right\}. \tag{5.9}$$

Let us define for $t \geq 0$

$$B_3(t) := \left\{\omega \in \Omega : \lim_{n\to\infty} \inf_{0\leq s\leq t} S_n(s) = +\infty\right\}.$$

For $t < u$ and $n \geq N_\epsilon^u$ we then obtain

$$\begin{aligned}
\mathbb{P}(B_3(t)) &\geq \mathbb{P}\left(\lim_{n\to\infty}\sum_{i=N_\epsilon^u}^{n}\inf_{0\leq s\leq t} P(s,T_i) = +\infty \,\middle|\, A_3^{\epsilon,u,n}\right)\mathbb{P}(A_3^{\epsilon,u,n}) \\
&\overset{(5.9)}{\geq} \mathbb{P}\left(\lim_{n\to\infty}\sum_{i=N_\epsilon^u}^{n}\frac{1}{1+\epsilon T_i} = +\infty \,\middle|\, A_3^{\epsilon,u,n}\right)\mathbb{P}(A_3^{\epsilon,u,n}) \\
&\geq (1 - \delta(\epsilon)) \to 1
\end{aligned}$$

for $\epsilon \to 0$. That means $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp and consequently $\ell_t \leq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ due to Theorems 9 and 11.

The behaviour of the long-term swap rate is a direct consequence of Theorem 3 (ii) and Corollary 5. $\square$

Lastly, we are interested in the influence of an exploding long-term simple rate on the long-term yield and long-term swap rate.

Theorem 16. *If $L = +\infty$, then $\ell_t \geq 0$ and $R_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. We show that $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp. We know by (B.6) that for all $t \geq 0$ and all $M > 0$

$$\mathbb{P}\left(\inf_{0\leq s\leq t} L(s,T_n) > M\right) \overset{n\to\infty}{\longrightarrow} 1.$$

Hence it holds by (2.2) for all $t \geq 0$ and all $\epsilon > 0$ that there exists $N_\epsilon^t \in \mathbb{N}$ such that for all $n \geq N_\epsilon^t$

$$\mathbb{P}\left(T_n \sup_{0\leq s\leq t} P(s,T_n) \leq \epsilon\right) > 1 - \delta(\epsilon) \tag{5.10}$$

with $\delta(\epsilon) \to 0$ for $\epsilon \to 0$. Then, let us define for $\epsilon > 0$, $u \geq 0$ and $n \in \mathbb{N}$

$$A_5^{\epsilon,u,n} := \left\{\omega \in \Omega : T_n \sup_{0\leq s\leq u} P(s,T_n) \leq \epsilon\right\}. \tag{5.11}$$

For $t < u$ and $n \geq N_\epsilon^u$ we obtain by (5.10) with $B_1(t)$ defined as in (5.5) that

$$\begin{aligned}\mathbb{P}(B_1(t)) &\geq \mathbb{P}\left(\lim_{n\to\infty} \sum_{i=N_\epsilon^u}^{n} \sup_{0\leq s\leq t} P(s,T_i) < +\infty \,\middle|\, A_5^{\epsilon,u,n}\right) \mathbb{P}(A_5^{\epsilon,u,n}) \\ &\geq (1-\delta(\epsilon)) \to 1\end{aligned}$$

for $\epsilon \to 0$. ∎

Table 3 summarises the influence of the long-term simple rate on the other long-term rates.

If the long-term simple rate is	With P	Then the long-term yield is	Then the long-term swap rate is
$0 \leq L < +\infty$	$0 \leq P < +\infty$	$\ell \leq 0$	$R = 0$
$0 \leq L < +\infty$	$P = +\infty$	$\ell \leq 0$	$-\infty < R \leq 0$
$L = +\infty$	$P = 0$	$\ell \geq 0$	$0 < R < +\infty$

Table 3: Influence of the long-term simple rate on long-term rates.

6 Long-Term Rates in Specific Term Structure Models

In this section we compute the long-term interest rates in two specific models, the Flesaker-Hughston model and the linear-rational model. We note that this class of models also includes affine interest rate models. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be the filtered probability space introduced in Section 2.1.

6.1 Long-Term Rates in the Flesaker-Hughston Model

We now derive the long-term swap rate in the Flesaker-Hughston interest rate model. The model was introduced in Flesaker and Hughston [1996] and further developed in Musiela and Rutkowski [2005] and Rutkowski [1997]; see also Rogers [1997]. The main advantages of this approach are that it specifies non-negative interest rates only and has a high degree of tractability. Another appealing feature is that besides relatively simple models for bond prices, short and forward rates, there are closed-form formulas for caps, floors and swaptions available. In the following, we first shortly outline the generalised Flesaker-Hughston model that is explained in detail in Rutkowski [1997] and then consider two specific cases. The basic input of the model is a strictly positive supermartingale A on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ which represents the state price density, so that the zero-coupon bond price can be expressed as

$$P(t,T) = \frac{\mathbb{E}^{\mathbb{P}}[\, A_T |\, \mathcal{F}_t]}{A_t}\,, \; 0 \leq t \leq T\,, \tag{6.1}$$

for all $T \geq 0$. It immediately follows $P(T,T) = 1$ for all $T \geq 0$ and $P(t,U) \leq P(t,T)$ for all $0 \leq t \leq T \leq U$, i.e. the zero-coupon bond price is a decreasing process in the maturity. This choice guarantees positive forward and short rates for all maturities (see equations (10) and (11) of Flesaker and Hughston 1996). To model the long-term yield and swap rate in this methodology a specific choice of A is needed. For this matter, we focus on two special cases presented in Section 2.3 of Rutkowski [1997]. See also Goldberg [1996], Hunt and Kennedy [2000], section 8.3.8, and Musiela and Rutkowski [2005], section 13.8

Example 17. *The supermartingale A is given by*

$$A_t = f(t) + g(t)\, M_t\,, \; t \geq 0\,, \tag{6.2}$$

where $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ are strictly positive decreasing functions and M is a strictly positive martingale defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with $M_0 = 1$. We shall consider in the sequel a càdlàg version of M. It follows from (6.1) *that for all* $0 \leq t \leq T$

$$P(t,T) = \frac{f(T) + g(T)\, M_t}{f(t) + g(t)\, M_t}\,. \tag{6.3}$$

Flesaker and Hughston [1996] refer to models of this type as "rational" models since the discount bond prices are given as rational functions of the underlying martingale M. The initial yield curve can easily be fitted by choosing strictly positive decreasing functions f and g in such a way that

$$P(0,T) = \frac{f(T) + g(T)}{f(0) + g(0)} \tag{6.4}$$

for all $T \geq 0$.

For the calculations of the long-term yield and swap rate, we assume that the following conditions on the asymptotic behaviour of f and g hold:

$$F := \sum_{i=1}^{\infty} f(T_i) < +\infty \,, \qquad G := \sum_{i=1}^{\infty} g(T_i) < +\infty \,, \tag{6.5}$$

with $F, G \in \mathbb{R}_+$.

From (6.5) *it follows immediately that* $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0$ *and hence* $P_t = 0$ *for all* $t \geq 0$. *This condition is assumed in Flesaker and Hughston [1996], whereas here it follows from* (6.5)*. We also get for all* $t \geq 0$

$$S_\infty(t) = \delta \, \frac{F + GM_t}{f(t) + g(t)\, M_t} \quad \mathbb{P}\text{-}a.s.$$

since

$$\begin{aligned} &\sup_{0\le s\le t} \left| \sum_{i=1}^{n} \frac{f(T_i) + g(T_i)\, M_s}{f(s) + g(s)\, M_s} - \frac{F + GM_s}{f(s) + g(s)\, M_s} \right| \\ &= \sup_{0\le s\le t} \left| \frac{M_s}{f(s) + g(s)\, M_s} \left(\sum_{i=1}^{n} g(T_i) - G \right) + \frac{\sum_{i=1}^{n} f(T_i) - F}{f(s) + g(s)\, M_s} \right| \\ &\to 0 \quad \mathbb{P}\text{-}a.s. \end{aligned}$$

for all $t \geq 0$, hence in probability because

$$\sup_{0\le s\le t} \frac{M_s}{f(s) + g(s)\, M_s} \le \sup_{0\le s\le t} \frac{M_s}{g(s)\, M_s} \le \frac{1}{g(t)} < +\infty.$$

Then, by Proposition 3 it holds $\mathbb{P}$-a.s.

$$R_t = \frac{f(t) + g(t)\, M_t}{\delta\,(F + GM_t)}, \ t \geq 0. \tag{6.6}$$

Now, we also want to compute the long-term yield in this model specification. It is for all $t \geq 0$

$$\ell_{\cdot} \overset{(3.1)}{=} \lim_{T\to\infty} Y(\,\cdot\,, T) \underset{(6.3)}{\overset{(2.1)}{=}} - \lim_{T\to\infty} T^{-1} \log(f(T) + g(T)\, M_{\cdot}) \quad \textit{in ucp.} \tag{6.7}$$

We know from Proposition 13 that $\ell_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ since the long-term swap rate is strictly positive due to (6.6) *and P vanishes.*

Let us consider a simple example, where $f(t) = \exp(-\alpha t)$, $g(t) = \exp(-\beta t)$ *with* $0 < \alpha < \beta$. *Then f and g are decreasing strictly positive functions and the ratio test shows that the infinite sums of f and g exist. We denote them by*

$$\alpha_\infty := \sum_{i=1}^{\infty} \exp(-\alpha T_i) \ , \qquad \beta_\infty := \sum_{i=1}^{\infty} \exp(-\beta T_i)$$

with $0 < \beta_\infty \leq \alpha_\infty$. *Hence all required conditions are fulfilled and we get the following equations for the long-term swap rate and the long-term yield, respectively*

$$R_t \overset{(6.6)}{=} \frac{\exp(-\alpha t) + \exp(-\beta t)\, M_t}{\delta\,(\alpha_\infty + \beta_\infty M_t)},\ t \geq 0,$$

and

$$\begin{aligned}
\ell. &\overset{(6.7)}{=} -\lim_{T\to\infty} T^{-1} \log(f(T) + g(T)\, M.) \\
&= -\lim_{T\to\infty} T^{-1} \log(\exp(-\alpha T)\,(1 + \exp\,(-\,(\beta-\alpha)\,T)\, M.)) \\
&= \alpha + \lim_{T\to\infty} T^{-1} \log(1 + \exp(-\,(\beta-\alpha)\,T)\, M.) = \alpha \quad \textit{in ucp.}
\end{aligned}$$

It follows by Theorem 9 that $L(t, T_n) \overset{n\to\infty}{\longrightarrow} +\infty$ *in ucp. This result can also be obtained by direct computation since for all* $t \geq 0$

$$\sup_{0\leq s\leq t} \frac{\exp(-\alpha s) + \exp(-\beta s)\, M_s}{T\exp(-\alpha T) + T\exp(-\beta T)\, M_s} \overset{T\to\infty}{\longrightarrow} +\infty \ \ \mathbb{P}\text{-}a.s.,$$

i.e. in probability, since M *is càdlàg.*

Example 18. *In the second special case of the Flesaker-Hughston model the supermartingale* A *is defined as*

$$A_t = \int_t^\infty \phi(s)\, M(t,s)\; ds\,,\ t \geq 0\,,$$

where for every $s > 0$ *the process* $M(t,s)\,, t \leq s$, *is a strictly positive martingale on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ *with* $M(0,s) = 1$ *such that* $\int_0^\infty \phi(s)\, M(t,s)\; ds < +\infty$ $\mathbb{P}$-*a.s. for all* $t \geq 0$ *and* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a strictly positive continuous function. From* (6.1) *follows for* $0 \leq t \leq T$

$$P(t,T) = \frac{\int_T^\infty \phi(s)\, M(t,s)\; ds}{\int_t^\infty \phi(s)\, M(t,s)\; ds}\,, \tag{6.8}$$

for all $T \geq 0$. *By differentiation of the zero-coupon bond price with respect to the maturity date, we see that the initial term structure satisfies* $\phi(t) = -\frac{\partial P(0,t)}{\partial t}$ *(see equation (6) of Flesaker and Hughston 1996).*

According to (6.8) *we get that* $P_t = 0$ $\mathbb{P}$-*a.s. for all* $t \geq 0$. *We define* $Q_n := (Q_n(t))_{t\geq 0}$ *for all* $n \geq 0$ *with*

$$Q_n(t) := \sum_{i=1}^{n} \int_{T_i}^\infty \phi(s)\, M(t,s)\; ds$$

and assume that for $Q := (Q(t))_{t\geq 0}$ *we have*

$$Q(t) := \sum_{i=1}^{\infty} \int_{T_i}^\infty \phi(s)\, M(t,s)\; ds \ < +\infty$$

for all $t \geq 0$, and that $Q_n \overset{n\to\infty}{\longrightarrow} Q$ in ucp. Then, we get $S_n \overset{n\to\infty}{\longrightarrow} S_\infty < +\infty$ in ucp and hence the convergences of the long-term swap rate and the long-term yield hold also in ucp. Due to Theorem 3 (i) the long-term swap rate is

$$R_t = \frac{\int_t^\infty \phi(s)\, M(t,s)\; ds}{\delta \sum_{i=1}^\infty \int_{T_i}^\infty \phi(s)\, M(t,s)\; ds}\,, t \geq 0\,. \tag{6.9}$$

Now, we again want to know the long-term yield in this case. It holds

$$\ell_\cdot = -\lim_{T\to\infty} T^{-1} \log\left(\int_T^\infty \phi(s)\, M(\,\cdot\,, s)\; ds\right) \quad \text{in ucp.}$$

From Proposition 13 we know that $\ell_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ since $R_t > 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ due to (6.9) *and P vanishes. Further, $L_t \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ by Proposition 13.*

6.2 Long-Term Rates in the Linear-Rational Methodology

In Filipović et al. [2017], a class of linear-rational term structure models is introduced based on a multivariate state process satisfying an Ornstein-Uhlenbeck-type SDE driven by an underlying martingale M. This class presents several advantages: it is highly tractable and offers a very good fit to interest rate swaps and swaptions data since 1997. Further, non-negative interest rates are guaranteed, unspanned factors affecting volatility and risk premia are accommodated, and analytical solutions to swaptions are admitted.

We assume the existence of a state price density, i.e. of a positive adapted process $A := (A_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that the price $\Pi(t,T)$ at time t of any time T cashflow C_T is given by

$$\Pi(t,T) = \frac{\mathbb{E}^{\mathbb{P}}[A_T C_T | \mathcal{F}_t]}{A_t}\,,\ 0 \leq t \leq T\,, \tag{6.10}$$

for all $T \geq 0$. In particular we suppose that the state price density A is driven by a multivariate factor process $X := (X_t)_{t\geq 0}$ with state space $E \subseteq \mathbb{R}^d$, $d \geq 1$, where

$$dX_t = k\,(\theta - X_t)\,dt + dM_t\,,\ t \geq 0\,, \tag{6.11}$$

for some $k \in \mathbb{R}_+$, $\theta \in \mathbb{R}^d$, and some martingale $M := (M_t)_{t\geq 0}$ on E. We assume to work with the càdlàg version of X. Next, A is defined as

$$A_t := \exp(-\alpha t)\left(\phi + \psi^\top X_t\right)\,,\ t \geq 0\,, \tag{6.12}$$

with $\phi \in \mathbb{R}$ and $\psi \in \mathbb{R}^d$ such that $\phi + \psi^\top x > 0$ for all $x \in E$, and $\alpha \in \mathbb{R}$. It holds $\alpha = \sup_{x\in E} \frac{k\,\psi^\top(\theta - x)}{\phi + \psi^\top x}$ to guarantee non-negative short rates (see equation (6) of Filipović et al. 2017). Then, equations (6.10), (6.11), (6.12), together with the fact that $P(T,T) = 1$ for all $T \geq 0$, lead to

$$P(t,T) = \frac{\left(\phi + \psi^\top\theta\right)\exp(-\alpha\,(T-t)) + \psi^\top(X_t - \theta)\exp(-\,(\alpha+k)\,(T-t))}{\phi + \psi^\top X_t} \tag{6.13}$$

for all $0 \leq t \leq T$. Hence, $P_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$ and we know by the ratio test that for all $t \geq 0$

$$\alpha_\infty(t) := \sum_{i=1}^{\infty} \exp(-\alpha\,(T_i - t)) < +\infty\,, \qquad \beta_\infty(t) := \sum_{i=1}^{\infty} \exp(-\,(\alpha + k)\,(T_i - t)) < +\infty\,.$$

Then for all $t \geq 0$ $\mathbb{P}$-a.s.

$$S_\infty(t) \overset{(6.13)}{=} \frac{\left(\phi + \psi^\top \theta\right) \alpha_\infty(t) + \psi^\top (X_t - \theta)\, \beta_\infty(t)}{\phi + \psi^\top X_t} < +\infty\,. \tag{6.14}$$

It follows by Proposition 3 that for all $t \geq 0$ $\mathbb{P}$-a.s.

$$R_t \underset{(6.14)}{\overset{(6.13)}{=}} \frac{\phi + \psi^\top X_t}{\delta\left(\left(\phi + \psi^\top \theta\right) \alpha_\infty(t) + \psi^\top (X_t - \theta)\, \beta_\infty(t)\right)}\,.$$

Finally, we want to know the form of the long-term yield in the linear-rational term structure methodology. We define $y := \phi + \psi^\top \theta$ and see that for all $t \geq 0$ holds

$$\log\left[y + \psi^\top \left(\sup_{0 \leq s \leq t} X_s - \theta\right) e^{-k(T-t)}\right] \geq \log\left[y + \psi^\top \left(X_t - \theta\right) e^{-k(T-t)}\right]$$

as well as

$$\log\left[y + \psi^\top \left(X_t - \theta\right) e^{-k(T-t)}\right] \geq \log\left[y + \psi^\top \left(\inf_{0 \leq s \leq t} X_s - \theta\right) e^{-k\,T}\right].$$

This yields $\mathbb{P}$-a.s. for all $t \geq 0$

$$\begin{aligned}
\sup_{0 \leq s \leq t} \left|\alpha + \frac{\log P(s,T)}{T}\right| &\overset{(6.13)}{=} \sup_{0 \leq s \leq t} \left|\alpha \frac{s}{T} + \frac{1}{T} \log\left[y + \psi^\top \left(X_s - \theta\right) e^{-k(T-s)}\right]\right| \\
&\leq \sup_{0 \leq s \leq t} \left|\alpha \frac{s}{T}\right| + \sup_{0 \leq s \leq t} \frac{1}{T} \left|\log\left[y + \psi^\top \left(X_s - \theta\right) e^{-k(T-s)}\right]\right| \\
&\overset{T \to \infty}{\longrightarrow} 0
\end{aligned}$$

because $\sup_{0 \leq s \leq t} X_s < \infty$ $\mathbb{P}$-a.s. for all $t \geq 0$ since X is càdlàg. Hence, we have for all $t \geq 0$ that $\lim_{T \to \infty} \sup_{0 \leq s \leq t} Y(s,T) = \alpha$ $\mathbb{P}$-a.s., consequently in probability, i.e. we get $\ell_t = \alpha$ $\mathbb{P}$-a.s. for all $t \geq 0$. In case of α positive, the long-term simple rate explodes due to Theorem 9.

7 Application: Valuation of the Non-Optional Component of a CoCo Bond

In this section we present an application of our results on the long-term swap rate to evaluate the non-optional component of a CoCo bond.

Several banks have issued CoCo bonds with perpetual characteristics in recent years. They are perpetual in the sense that the time to maturity is unbounded if the option for conversion is not executed. Hence, this kind of financial product can be understood as a perpetual floating rate bond combined with an embedded option. For investors it is crucial to know the value of the option and the non-optional component to make an informed investment decision. We calibrate a model for the long-term swap rate, and use the resulting specification to compute the price of the perpetual floating rate bond corresponding to the non-optional component of the CoCo bond. In particular, we consider the CoCo bond with ISIN $XS1002801758$ issued by Barclays; seePLC [2014b].

In the same setting as in Section 6.1, we assume that the strictly positive martingale $M = (M_t)_{t \geq 0}$ in (6.2) satisfies

$$dM_t = \sigma_t M_t dW_t, \qquad M_0 = 1, \tag{7.1}$$

where $W = (W_t)_{t \geq 0}$ is a one-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and $\sigma = (\sigma_t)_{t \geq 0}$ is a deterministic continuous function such that $\int_0^{+\infty} \sigma_s^2 ds < \infty$. In Musiela and Rutkowski [2005], term structure models of this type are called "rational log-normal models", since the bond prices at each instant of time are given by linear-rational functions of a log-normally distributed random variable. For the functions f, g in (6.3), we set $f(t) = k_1 e^{-\alpha t}, g(t) = k_2 e^{-\beta t}$, $t \geq 0$. We consider $T_0, T_1, \cdots$ such that $T_i - T_{i-1} = \delta$ for all $i = 1, \cdots$. In particular we choose here $\delta = 3$ months, which is a typical interval between maturities for swaps on real markets.

As first step, we estimate a term structure for the discount factors. This is achieved by considering market data of overnight indexed swap on 22 December 2016. From the market prices of overnight indexed swap we bootstrap a term structure of discount factors by relying on the Finmath Java library (see Fries 2018). Secondly, we estimate the parameters for the functions f, g, F, and G in (6.4) by minimizing the squared distance between the term structure of zero coupon bonds obtained from the bootstrap and the right hand side of (6.4). We obtain $k_1 = 0.4894723$ and $\alpha = 0.1536072$ for the function f, and $k_2 = 8.6235042$ and $\beta = 0.0117588$ for the function g. By using this result, we compute $F = 165.95163$ and $G = 11742.367$ by evaluating the sums in (6.5) along a time discretization with a horizon of one thousand years.
Concerning the volatility function σ in (7.1), we set $\sigma_t = e^{-\lambda t}$. For the estimation of the parameter λ, we rely on Remark 4. Since the long term swap rate is proportional to the consol rate of a perpetual bond, we use time series data of a consol bond to estimate the missing volatility parameter. We consider the yield of the perpetual Bond with Isin $BMG7498P3093$ and perform a maximum likelihood estimation, obtaining $\lambda = 0.0748829$.

With the given full specification of the process R, we then estimate the value of the non-optional part of the CoCo Bond $XS1002801758$. Let $P_{NO}(T_0)$ denote the price of the non-optional part at time T_0. From the term sheet of the CoCo, we observe that the investor initially receives an 8% fixed coupon up to 2020, where no conversion is possible. On the other side, when the claim starts exhibiting an optionality feature, the investor receives 6.75% plus the mid market swap rate.

A simple estimate for the non-optional component of the stream of payments involved in the CoCo is simply given by considering the following perpetual bond with unit notional:

$$P_{NO}(T_0) = \sum_{i=0}^{\mathcal{N}} P(T_0, T_i)\,(R_{T_i} + S)$$

where we truncate the infinite sum up to $\mathcal{N} = 50$. The long term estimate of the Euribor-Eonia spread $S = 0.0011$ is obtained from the bootstrapped curves by assuming a constant Euribor-Eonia Spread for maturities larger than fifty years. Using (6.6) we simulate the dynamics of the long term swap rate R and obtain $P_{NO}(T_0) = 0.1969$ by a Monte Carlo simulation. By using the market price $P_{MKT}(T_0) = 1.05386$ of $XS1002801758$ observed on 22 December 2016, we immediately deduce our estimate of the value of the embedded optional part, which is equal to 0.85695. Such a result shows that the price of the CoCo bond is mainly driven by the embedded option. As the option is written on a unit notional, it can be concluded that the option tends to a position on the underlying.

A Behaviour of S_∞

For the study of the long-term swap rate in Section 3 as well as of the relations among the different long-term interest rates in Section 5 we need to obtain some results on the infinite sum S_∞ of bond prices defined in (3.4). We recall that we consider a tenor structure with $c < \sup_{i\in\mathbb{N}\setminus\{0\}}(T_i - T_{i-1}) < C$ for some $c, C \in \mathbb{R}_+, c < C$. The next two statements give insight about the relation between the long-term zero-coupon bond prices and the asymptotic behaviour of the sum of these prices, whereas Lemma 21 tells us that the long-term simple rate vanishes if P explodes.

Proposition 19. *If $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp, then $P_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. From $S_n \overset{n\to\infty}{\longrightarrow} S_\infty$ in ucp it follows that $S_\infty(t) < +\infty$ $\mathbb{P}$-a.s. for all $t \geq 0$. We get for all $\epsilon > 0$ and $t \geq 0$ with $C^{\epsilon,t,n} := \{\omega \in \Omega : \sup_{0\leq s\leq t} |P(s,T_n)| > \epsilon\}$

$$\begin{aligned}
\mathbb{P}\big(C^{\epsilon,t,n}\big) &\leq \mathbb{P}\bigg(\sup_{0\leq s\leq t} |S_n(s) - S_{n-1}(s)| > \epsilon\, c\bigg) \\
&= \mathbb{P}\bigg(\sup_{0\leq s\leq t} |S_n(s) - S_\infty(s) + S_\infty(s) - S_{n-1}(s)| > \epsilon\, c\bigg) \\
&\leq \mathbb{P}\bigg(\sup_{0\leq s\leq t} \big(|S_n(s) - S_\infty(s)| + |S_{n-1}(s) - S_\infty(s)|\big) > \epsilon\, c\bigg) \\
&\leq \mathbb{P}\bigg(\sup_{0\leq s\leq t} |S_n(s) - S_\infty(s)| + \sup_{0\leq s\leq t} |S_{n-1}(s) - S_\infty(s)| > \epsilon\, c\bigg) \\
&\leq \mathbb{P}\bigg(\bigg\{\sup_{0\leq s\leq t} |S_n(s) - S_\infty(s)| > \frac{\epsilon\, c}{2}\bigg\} \cup \bigg\{\sup_{0\leq s\leq t} |S_{n-1}(s) - S_\infty(s)| > \frac{\epsilon\, c}{2}\bigg\}\bigg) \\
&\overset{(*)}{\leq} \mathbb{P}\bigg(\sup_{0\leq s\leq t} |S_n(s) - S_\infty(s)| > \frac{\epsilon\, c}{2}\bigg) + \mathbb{P}\bigg(\sup_{0\leq s\leq t} |S_{n-1}(s) - S_\infty(s)| > \frac{\epsilon\, c}{2}\bigg) \\
&\overset{n\to\infty}{\longrightarrow} 0
\end{aligned}$$

due to the ucp convergence of S_n. Hence $P_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. □

Corollary 20. *If $\mathbb{P}(P_t > 0) > 0$ for some $t \geq 0$, then $S_n \overset{n\to\infty}{\longrightarrow} +\infty$ in ucp.*

Proof. This is a direct consequence of Proposition 19. □

Lemma 21. *If $P = +\infty$, it follows $L_t = 0$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. It follows $L(\,\cdot\,, T_n) \overset{n\to\infty}{\longrightarrow} 0$ in ucp by (2.2) and the definition of convergence to $+\infty$ in ucp (see Definition 24). □

B UCP Convergence

The definition of uniform convergence on compacts in probability (ucp convergence) can be found in Chapter II, Section 4 of Protter [2005]. We repeat this here in a slightly rephrased form for the reader's convenience. As before we consider a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ with $\mathcal{F}_\infty \subseteq \mathcal{F}$ satisfying the usual hypothesis. All processes are adapted to $\mathbb{F}$.

Definition 22. *[Protter [2005], Chapter II, Section 4] A sequence of processes $(X^n)_{n\in\mathbb{N}}$ with values in $\mathbb{R}^d$ converges to a process X uniformly on compacts in probability if, for each $t > 0$, $\sup_{0\leq s\leq t} \|X_s^n - X_s\|$ converges to 0 in probability, i.e. for all $\epsilon > 0$ it holds*

$$\mathbb{P}\bigg(\sup_{0\leq s\leq t} \|X_s^n - X_s\| > \epsilon\bigg) \overset{n\to\infty}{\longrightarrow} 0\,. \tag{B.1}$$

We write $X^n \overset{n\to\infty}{\longrightarrow} X$ in ucp.

We now show the following result needed in the proof of Theorem 3.

Theorem 23. *Let $(X^n)_{n\in\mathbb{N}}$ and $(Y^n)_{n\in\mathbb{N}}$ be sequences of real-valued processes . If $(X^n, Y^n) \overset{n\to\infty}{\longrightarrow} (X, Y)$ in ucp with $\sup_{0\le s\le t}|X_s| < +\infty$ and $\sup_{0\le s\le t}|Y_s| < +\infty$ $\mathbb{P}$-a.s. for all $t \ge 0$, then $f(X^n, Y^n) \overset{n\to\infty}{\longrightarrow} f(X, Y)$ in ucp for all $f : \mathbb{R}^2 \to \mathbb{R}$ continuous.*

Proof. Let us define $\nu_s^n := (X_s^n, Y_s^n)$, $\nu_s := (X_s, Y_s)$, and let $\|\cdot\|$ be the Euclidean norm on $\mathbb{R}^2$. We have to show that for all $t \ge 0$ and $\epsilon > 0$ it holds

$$\mathbb{P}\left(\sup_{0\le s\le t} |f(\nu_s^n) - f(\nu_s)| > \epsilon\right) \overset{n\to\infty}{\longrightarrow} 0. \tag{B.2}$$

Let $k \ge 0$. Then for all $t \ge 0$ it holds

$$\begin{aligned}\left\{\sup_{0\le s\le t} |f(\nu_s^n) - f(\nu_s)| > \epsilon\right\} &\subseteq \left\{\sup_{0\le s\le t} |f(\nu_s^n) - f(\nu_s)| > \epsilon,\ \sup_{0\le s\le t} \|\nu_s\| \le k\right\} \\ &\quad \cup \left\{\sup_{0\le s\le t} \|\nu_s\| > k\right\}. \end{aligned}\tag{B.3}$$

By the Heine-Cantor theorem (see Theorem A.1.1 of Canuto and Tabacco 2015) it follows from f continuous that f is uniformly continuous on any bounded interval and therefore there exists for the given $\epsilon > 0$ a $\delta > 0$ such that

$$\begin{aligned}\left\{\sup_{0\le s\le t} |f(\nu_s^n) - f(\nu_s)| > \epsilon,\ \sup_{0\le s\le t} \|\nu_s\| \le k\right\} &\subseteq \left\{\sup_{0\le s\le t} \|\nu_s^n - \nu_s\| > \delta,\ \sup_{0\le s\le t} \|\nu_s\| \le k\right\} \\ &\subseteq \left\{\sup_{0\le s\le t} \|\nu_s^n - \nu_s\| > \delta\right\} \end{aligned}\tag{B.4}$$

Substituting (B.4) into (B.3) gives us

$$\left\{\sup_{0\le s\le t} |f(\nu_s^n) - f(\nu_s)| > \epsilon\right\} \subseteq \left\{\sup_{0\le s\le t} \|\nu_s^n - \nu_s\| > \delta\right\} \cup \left\{\sup_{0\le s\le t} \|\nu_s\| > k\right\}.$$

Hence

$$\mathbb{P}\left(\sup_{0\le s\le t} |f(\nu_s^n) - f(\nu_s)| > \epsilon\right) \le \mathbb{P}\left(\sup_{0\le s\le t} \|\nu_s^n - \nu_s\| > \delta\right) + \mathbb{P}\left(\sup_{0\le s\le t} \|\nu_s\| > k\right). \tag{B.5}$$

Since $\sup_{0\le s\le t}|X_s| < +\infty$ and $\sup_{0\le s\le t}|Y_s| < +\infty$ $\mathbb{P}$-a.s. for all $t \ge 0$, it holds for all $t \ge 0$ that $\mathbb{P}\big(\sup_{0\le s\le t}\|\nu_s\| > k\big) \overset{k\to\infty}{\longrightarrow} 0$. Let first $k \to \infty$ and then $n \to \infty$, to obtain (B.2) from (B.5). $\square$

In order to treat the case of exploding long-term interest rates, we now provide a definition of convergence to $\pm\infty$ in ucp.

Definition 24. *A sequence of real-valued processes $(X^n)_{n\in\mathbb{N}}$ converges to $+\infty$ uniformly on compacts in probability if, for each $t > 0$ and $M > 0$ it holds*

$$\mathbb{P}\left(\inf_{0\le s\le t} X_s^n > M\right) \overset{n\to\infty}{\longrightarrow} 1\,. \tag{B.6}$$

We write $X^n \overset{n\to\infty}{\longrightarrow} +\infty$ *in ucp.*

Accordingly the sequence of real-valued processes $(X^n)_{n\in\mathbb{N}}$ *converges to* $-\infty$ *uniformly on compacts in probability if, for each* $t > 0$ *and* $M > 0$ *it holds*

$$\mathbb{P}\left(\sup_{0\le s\le t} X^n_s < -M\right) \overset{n\to\infty}{\longrightarrow} 1\,. \tag{B.7}$$

Then, we write $X^n \overset{n\to\infty}{\longrightarrow} -\infty$ *in ucp.*

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