Local Risk-Minimization for Defaultable Markets

Francesca Biagini

Department of Mathematics,
LMU, Munich

Alessandra Cretarola

Dipartimento di Scienze Economiche e Aziendali,
LUISS Guido Carli, Rome

Abstract

We study the local risk-minimization approach for defaultable markets in a general setting where the asset price dynamics and the default time may influence each other. We find the Föllmer-Schweizer decomposition in this general setting and compute it explicitly in two particular cases, when default time depends on the risky asset's behavior and when only a dependence of discounted asset price on default time is occurring.

Key words: Defaultable markets, local risk-minimization, minimal martingale measure, pseudo-locally risk-minimizing strategy.

1 Introduction

In this paper we discuss the problem of pricing and hedging defaultable claims, i.e. options that can lose partially or totally their value if a default event occurs. We consider a simple market model with two non-defaultable primitive assets (the money market account B and the discounted risky asset X) and a (discounted) defaultable claim H. Since it is impossible to hedge against the occurrence of a default by using a portfolio consisting only of the primitive assets, the market model extended with

We thank Damir Filipović for interesting discussions and remarks and an anonymous referee for valuable comments that have contributed to improve the paper.

the defaultable claim is incomplete and it makes sense to apply some of the methods used for pricing and hedging derivatives in incomplete markets.
In particular we focus here on the *local risk-minimization* approach. Other quadratic hedging methods such as *mean-variance hedging* have been extensively studied in the context of defaultable markets by Biagini and Cretarola (2007), Bielecki and Jeanblanc (2005), Bielecki, Jeanblanc, and Rutkowski (2004b), Bielecki, Jeanblanc, and Rutkowski (2004c) and Bielecki, Jeanblanc, and Rutkowski (2004a).
The local risk-minimization method has been applied for the first time to the case of defaultable markets in Biagini and Cretarola (2007), but only in the case where the default time and the underlying Brownian motion were independent. Here we consider the more general case where the dynamics of the risky assets may be influenced by the occurring of a default event and also the default time itself may depend on the assets prices behavior. In this general setting we are able to provide the Föllmer-Schweizer decomposition of a defaultable claim with random recovery rate. In particular we focus on two cases where we compute explicitly the pseudo-locally risk-minimizing strategy and the optimal cost. First we consider the situation where the default time τ depends on the behavior of the risky asset price, but not vice versa.
In the second case we assume that drift and volatility of underlying discounted asset are affected by τ and we show how our result fits in the approach of Föllmer and Schweizer (1991) of local risk-minimization for markets affected by *incomplete information*. For local risk-minimization for defaultable markets via nonlinear filtering, we also refer to Frey, Schmidt, and Gabih (2007).

We would like to emphasize that this is the first time where local risk minimization has been applied to hedge defaultable claims in a very general setting, allowing for *dependence* between asset price behavior and default time, *stochastic* default intensity and *incomplete* non-defaultable market for the primary assets. Our results are then of general interest for computing hedging strategies in incomplete markets in presence of an additional source of randomness, that is "orthogonal" to the asset price dynamics, but not necessarily independent of them and viceversa.
In particular local risk-minimization naturally appears as suitable hedging method for the new financial instruments recently introduced to hedge against systematic mortality risk in life insurance contracts. This is the case of the so-called *mortality derivatives* (survival swaps, longevity bonds) and of the *unit-linked life insurance contracts*, i.e. contracts where the insurance benefits depend on the price of some specific traded stock. In Dahl and Mø ller (2006), Dahl, Melchior, and Mø ller (2007), Mø ller (1998), Mø ller (2001), Riesner (2006) and Riesner (2007) risk-minimizing strategy are computed for these financial insurance derivatives, but *only* when the insurance and the financial markets are *independent* (that corresponds to assume the independence of the asset price dynamics and default time in our setup). However the introduction of this new kind of financial instruments creates a link between life insurance and fi-

nancial markets, that may be reflected in the asset prices dynamics. Hence the results of our paper can be used to compute locally risk-minimizing strategies also for these kind of derivatives under the more realistic assumption that insurance and financial markets are not independent. Further applications of the results of this paper to the hedging of hybrid derivatives are also work in progress in Biagini and Sun (2007).

2 Setting

We consider a simple model of a financial market where we can find a risky asset, the money market account and *defaultable claims*, i.e. contingent agreements traded over-the-counter between default-prone parties. Each side of contract is exposed to the *counterparty risk* of the other party. Hence defaultable claims are derivatives that could fail or lose their own value.

We fix a time horizon $T \in (0, \infty)$. The random time of default is represented by a nonnegative random variable $\tau : \Omega \to [0, T] \cup \{+\infty\}$, defined on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, with $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$, for each $t \in [0, T]$. For a given default time τ, we introduce the associated *default process* $H_t = \mathbb{I}_{\{\tau \leq t\}}$, for $t \in [0, T]$ and denote by $\mathbb{H} := (\mathcal{H}_t)_{0 \leq t \leq T}$ the filtration generated by the process H_t, i.e. $\mathcal{H}_t = \sigma(H_u : u \leq t)$ for any $t \in [0, T]$.

Let W be a standard Brownian motion on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and $\mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T}$ the natural filtration of W. Let $\mathbb{G} := (\mathcal{G}_t)_{0 \leq t \leq T}$ be the filtration given by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, for every $t \in [0, T]$. We put $\mathcal{G} = \mathcal{G}_T$. We postulate that the Brownian motion W *remains a (continuous) martingale* (and then a Brownian motion) *with respect to the enlarged filtration* $\mathbb{G}$. In the sequel we refer to this assumption as the hypothesis (H). We remark that all the filtrations are assumed to satisfy the usual hypotheses of completeness and right-continuity.

- Let
$$F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t), \quad \forall t \in [0, T]$$
be the conditional distribution function of the default time τ and assume $F_t < 1$, for every $t \in [0, T]$. Then the *hazard process* of τ under $\mathbb{P}$:
$$\Gamma_t = -\ln(1 - F_t), \quad \forall t \in [0, T],$$
is well defined for every $t \in [0, T]$. Under hypothesis (H), Lemma 1.2 of Bielecki, Jeanblanc, and Rutkowski (2006) guarantees that the process Γ is increasing. We assume that the hazard process Γ admits the following representation:
$$\Gamma_t = \int_0^t \lambda_s \mathrm{d}s, \quad \forall t \in [0, T], \tag{2.1}$$

where λ is a non-negative, integrable process. The process λ is called *intensity* or *hazard rate.* By Proposition 5.1.3 of Bielecki and Rutkowski (2004) we obtain that the compensated process M given by

$$M_t := H_t - \int_0^{t\wedge\tau} \lambda_u \mathrm{d}u = H_t - \int_0^t \tilde{\lambda}_u \mathrm{d}u, \ \forall t \in [0,T] \tag{2.2}$$

follows a martingale with respect to the filtration $\mathbb{G}$. Notice that for the sake of brevity we have denoted $\tilde{\lambda}_t := \mathbb{I}_{\{\tau\geq t\}}\lambda_t$. In particular, we obtain that the existence of the intensity implies that τ is a *totally inaccessible* $\mathbb{G}$-stopping time (Dellacherie and Meyer (1982), VI.78). We note also that if Γ is an increasing process, by Lemma 5.1.6 of Bielecki and Rutkowski (2004) the stopped process $W_{t\wedge\tau}$ is always a $\mathbb{G}$-martingale, even without assuming the hypothesis (H).

- We denote the money market account by $B_t = \exp\left(\int_0^t r_s \mathrm{d}s\right)$, where r is a $\mathbb{G}$-adapted nonnegative process, and represent the risky asset price by a stochastic process Y on $(\Omega, \mathcal{G}, \mathbb{P})$, whose dynamics is given by the following equation:

$$\begin{cases} \mathrm{d}Y_t = \mu_t Y_t \mathrm{d}t + \sigma_t Y_t \mathrm{d}W_t, & \forall t \in [0,T] \\ Y_0 = y_0, \quad y_0 \in \mathbb{R}^+ \end{cases} \tag{2.3}$$

where $\sigma_t > 0$ a.s. for every $t \in [0,T]$ and μ, σ, r are $\mathbb{G}$-adapted processes such that $X_t := \dfrac{Y_t}{B_t}$ belongs to $L^2(\mathbb{P})$, $\forall t \in [0,T]$. For example, μ (respectively, σ) could be of the form $\mu_t = \mu_t^1 \mathbb{I}_{\{\tau\leq t\}} + \mu_t^2 \mathbb{I}_{\{\tau>t\}}$, where μ^1 and μ^2 are $\mathbb{F}$-adapted: in this case the influence of the default time determines a sudden change in the drift (respectively, in the volatility). Furthermore we assume that there exists an equivalent probability measure such that the discounted price process X is a local martingale. Let

$$\theta_t = \frac{\mu_t - r_t}{\sigma_t}, \quad \forall t \in [0,T] \tag{2.4}$$

be the *market price of risk.* We also assume that μ, σ and r are such that the density $\dfrac{\mathrm{d}\widehat{\mathbb{P}}}{\mathrm{d}\mathbb{P}} := \mathcal{E}\left(-\int \theta \mathrm{d}W\right)_T$ is square-integrable. In particular we have that the convex set $\mathcal{P}_e^2(X)$ of square-integrable equivalent martingale measures for X is not empty and the market model is in addition arbitrage-free.

- We assume that the information at time t available to the agent is given by

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t, \ \forall t \in [0,T]. \tag{2.5}$$

Remark 2.1. *That the information filtration is supposed to be of the form* (2.5) *is a standard assumption in the literature on the modelling of defaultable markets (Bielecki and Jeanblanc (2005), Bielecki, Jeanblanc, and Rutkowski (2004b), Bielecki, Jeanblanc, and Rutkowski (2004c), Bielecki, Jeanblanc, and Rutkowski (2004a), Bielecki, Jeanblanc, and Rutkowski (2006), Bielecki and Rutkowski (2004)). Nevertheless, one might argue that this choice appears artificial and that a more natural filtration for the market information would be* $\mathbb{F}^Y \vee \mathbb{H} := (\mathcal{F}_t^Y \vee \mathcal{H}_t)_{t\in[0,T]}$*, the filtration generated by the observed asset price* Y *and by the default time process* H*. Note however, that in our setting it is easily seen that* $\mathcal{G}_t = \mathcal{F}_t^Y \vee \mathcal{H}_t$*, for every* $t \in [0,T]$*, as soon as* μ *and* σ *are* $\mathbb{F}^Y \vee \mathbb{H}$*-adapted. In particular, one can show that* σ *is* $\mathbb{F}^Y \vee \mathbb{H}$*-adapted if* σ *has a right continuous version.*

As mentioned above, in this market model we can find defaultable claims, which are represented by a triplet $(\bar{X}, \delta\bar{X}, \tau)$, where:

- the *promised contingent claim* $\bar{X}$ represents the payoff received by the owner of the claim at time T, if there was no default prior to or at time T. In particular we assume it is represented by a $\mathcal{G}_T$-measurable random variable $\bar{X} \in L^2(\mathbb{P})$;
- the *recovery claim* $\delta\bar{X}$ represents the recovery payoff at time T, if default occurs prior to or at the maturity date T. Here δ is supposed to be a *random* recovery rate.

In particular we assume that δ is represented by a $\mathcal{H}_T$-measurable random variable in $L^2(\Omega, \mathcal{G}, \mathbb{P})$, i.e.

$$\delta = h(\tau \wedge T) \tag{2.6}$$

for some Borel function $h : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $0 \le h \le 1$. Here we focus on the case when an agent recovers a part of the promised claim at time of maturity. However our approach works in the same way for a general recovery payment.
In our setting the discounted value of a defaultable claim can be represented as follows:

$$\begin{aligned} H &= \frac{\bar{X}}{B_T}\mathbb{I}_{\{\tau > T\}} + \frac{\delta\bar{X}}{B_T}\mathbb{I}_{\{\tau \le T\}} \\ &= \frac{\bar{X}}{B_T}\left(1 + (h(\tau \wedge T) - 1)H_T\right). \end{aligned} \tag{2.7}$$

In particular we obtain that $H \in L^2(\mathcal{G}_T, \mathbb{P})$. In this framework, we study the problem of a trader wishing to price and hedge the defaultable claim H. Since the presence of default makes the market incomplete, we choose to apply the *local risk-minimization* approach, one of the methods used for pricing and hedging derivatives in incomplete markets. We wish to solve analytically the problem of finding the pseudo locally risk-minimizing strategy and the portfolio with minimal cost. The local risk-minimization method has been already applied to the case of defaultable markets in Biagini and Cretarola (2007) under the assumption that the default process H_t and the underlying Brownian motion W_t are independent for every $t \in [0,T]$. Here we extend the results of Biagini and Cretarola (2007) to the case of a reciprocal dependence of τ and the risky asset trend. In the next section we provide a short review of the main results of the theory of local risk-minimization (see Föllmer and Schweizer (1991), Heath, Platen, and Schweizer (2001), Schweizer (1995)), that we reformulate in terms of our context.

3 Local Risk-Minimization

Problem: in the financial market outlined in Section 2, we look for a *hedging strategy with minimal cost* for the defaultable contingent claim H in (2.7).

We introduce the basic framework and some definitions. We recall that the asset price dynamics is given by (2.3) and that for every $t \in [0,T]$

$$X_t = \frac{Y_t}{B_t}$$

denotes the discounted risky asset price.

- We remark that in our model X belongs to the space $\mathcal{S}^2_{loc}(\mathbb{P})$ of semimartingales decomposable as the sum of a locally square-integrable local martingale and of a $\mathbb{G}$-predictable process of finite variation null at 0. Indeed it can be decomposed as follows:

$$X_t = X_0 + \int_0^t (\mu_s - r_s) X_s \mathrm{d}s + \int_0^t \sigma_s X_s \mathrm{d}W_s, \qquad t \in [0,T],$$

  where $\int_0^t \sigma_s X_s \mathrm{d}W_s$ is a locally square-integrable local martingale null at 0 and $\int_0^t (\mu_s - r_s) X_s \mathrm{d}s$ is a $\mathbb{G}$-predictable process of finite variation null at 0. Moreover, in our case X is a continuous process.

- In our model we have that the so-called **Structure Condition (SC)** is satisfied, i.e. the *mean-variance tradeoff*

$$\widehat{K}_t(\omega) := \int_0^t \theta_s^2(\omega) \mathrm{d}s$$

is almost surely finite for each $t \in [0,T]$, where θ is the market price of risk defined in (2.4), since X is continuous and $\mathcal{P}_e^2(X) \neq \emptyset$ by hypothesis (see Schweizer (2001)).
In what follows, we assume that $\widehat{K}$ is uniformly bounded in t and ω, i.e. there exists K such that

$$\widehat{K}_t(\omega) \leq K, \qquad \forall t \in [0,T], \ a.s. \tag{3.1}$$

We denote by Θ_S the space of $\mathbb{G}$-predictable processes ξ on Ω such that

$$E\left[\int_0^T (\xi_s \sigma_s X_s)^2 \mathrm{d}s\right] + E\left[\left(\int_0^T |\xi_s(\mu_s - r_s)X_s| \mathrm{d}s\right)^2\right] < \infty. \tag{3.2}$$

Definition 3.1. *An* L^2-strategy *is a pair $\varphi = (\xi, \zeta)$ such that*

1. *ξ is a $\mathbb{G}$-predictable process belonging to Θ_S.*
2. *ζ is a real-valued $\mathbb{G}$-adapted process such that the discounted value process $\bar{V}(\varphi) := \dfrac{V(\varphi)}{B} = \xi X + \zeta$ is right-continuous and square-integrable.*

Here we assume to work with strategies that are $\mathbb{G}$-adapted, i.e. the trader can invest in the risky asset according to the information relative both to the asset prices and the occurrence of a default.
The *cost process* is defined by:

$$C_t = \bar{V}_t - \int_0^t \xi_s \mathrm{d}X_s, \quad t \in [0,T].$$

In particular, the component invested in the money market account is given by:

$$\zeta_t = \bar{V}_0(\varphi) + \int_0^t \xi_s \mathrm{d}X_s + C_t(\varphi) - \xi_t X_t, \quad t \in [0,T].$$

We want to find a hedging strategy φ with "minimal" cost C and (discounted) value process

$$\bar{V}_t(\varphi) = \bar{V}_0(\varphi) + \int_0^t \xi_s \mathrm{d}X_s + C_t(\varphi), \quad t \in [0,T],$$

such that

$$\bar{V}_T(\varphi) = H \qquad \mathbb{P} - a.s.$$

In which sense is the cost minimal?

Definition 3.2. *An L^2-strategy φ is called* mean-self-financing *if its cost process $C(\varphi)$ is a $\mathbb{P}$-martingale.*

Following Schweizer (1995), we introduce an optimal strategy:

Definition 3.3. *Let $H \in L^2(\mathcal{G}_T, \mathbb{P})$. An L^2-strategy φ with $\bar{V}_T(\varphi) = H$ $\mathbb{P}$-a.e. is* pseudo-locally risk-minimizing *(in short* plrm*) for H if φ is mean-self-financing and the martingale $C(\varphi)$ is strongly orthogonal to the martingale part of X.*

Note that under our assumptions on σ, to be strongly orthogonal to the martingale part of X is equivalent to be strongly orthogonal to W.
In general how to characterize a pseudo-locally risk-minimizing strategy is shown in the next result (see Föllmer and Schweizer (1991)). Let $\mathcal{M}_0^2(\mathbb{P})$ be the space all square-integrable $\mathbb{P}$-martingales null at 0.

Proposition 3.4. *A contingent claim $H \in L^2(\mathbb{P})$ admits a plrm strategy $\varphi = (\xi, \zeta)$ if and only if H can be written as*

$$H = H_0 + \int_0^T \xi_s^H \mathrm{d}X_s + L_T^H \quad \mathbb{P} - a.s. \tag{3.3}$$

with $H_0 \in \mathbb{R}$, $\xi^H \in \Theta_S$, $L^H \in \mathcal{M}_0^2(\mathbb{P})$ strongly orthogonal to the martingale part of X. The plrm strategy is given by

$$\xi_t = \xi_t^H, \quad t \in [0,T]$$

with minimal cost

$$C_t(\varphi) = H_0 + L_t^H, \quad t \in [0,T].$$

If (3.3) *holds, the optimal portfolio value is*

$$\bar{V}_t(\varphi) = C_t(\varphi) + \int_0^t \xi_s \mathrm{d}X_s = H_0 + \int_0^t \xi_s^H \mathrm{d}X_s + L_t^H,$$

and

$$\zeta_t = \zeta_t^H = \bar{V}_t(\varphi) - \xi_t^H X_t,$$

for $t \in [0,T]$.

Proof. For the proof, see Föllmer and Schweizer (1991). □

Decomposition (3.3) is well known in literature as the *Föllmer-Schweizer decomposition* (in short *FS decomposition*). If X is a $\mathbb{P}$-martingale, (3.3) coincides with the *Galtchouk-Kunita-Watanabe*[1](in short GKW) *decomposition*. We see now how one can obtain the FS decomposition by choosing a convenient martingale measure for X following Föllmer and Schweizer (1991). We remark that assumption (3.1) implies the existence of a FS decomposition.

Definition 3.5 (**The Minimal Martingale Measure**). *A martingale measure $\widehat{\mathbb{P}}$ equivalent to $\mathbb{P}$ with square-integrable density is called **minimal** if* any square-integrable $\mathbb{P}$-martingale which is strongly orthogonal to the martingale part of X under $\mathbb{P}$ remains a martingale under $\widehat{\mathbb{P}}$.

The minimal measure is the equivalent martingale measure that modifies the martingale structure as little as possible. Hypothesis (3.1) is sufficient to guarantee the existence of $\widehat{\mathbb{P}}$.

Theorem 3.6. *Let $H \in L^2(\mathcal{G}_T, \mathbb{P})$. Define the process $\widehat{V}^H$ as follows*

$$\widehat{V}_t^H := \widehat{E}[H|\mathcal{G}_t], \quad t \in [0,T],$$

where $\widehat{E}[\,\cdot\,|\mathcal{G}_t]$ denotes the conditional expectation under $\widehat{\mathbb{P}}$. Let

$$\widehat{V}_T^H = \widehat{E}[H|\mathcal{G}_T] = \widehat{V}_0^H + \int_0^T \widehat{\xi}_s^H \mathrm{d}X_s + \widehat{L}_T^H \tag{3.4}$$

be the GKW decomposition of $\widehat{V}^H$ with respect to X under $\widehat{\mathbb{P}}$. If either H admits a FS decomposition or $\hat{\xi}^H \in \Theta_S$ and $\hat{L}^H \in \mathcal{M}_0^2(\mathbb{P})$, then (3.4) *gives the FS decomposition of H and $\hat{\xi}^H$ gives a plrm strategy for H.*

Proof. Since in our model (SC) is satisfied and the existence of $\widehat{\mathbb{P}}$ is a consequence of assumption (3.1), the proof follows by Theorem 3.5 of Schweizer (1995). □

We apply these results to the case of defaultable claims.

4 Local Risk-Minimization for Defaultable Claims

Under the hypotheses of Section 2, we study now the local risk-minimization approach for a defaultable claim H defined in (2.7). The next result guarantees the existence of a pseudo-locally risk-minimizing strategy for H.

Proposition 4.1. *For any $\mathbb{G}$-martingale N under $\mathbb{P}$ we have*

$$N_t = N_0 + \int_0^t \xi_u^N \mathrm{d}W_u + \int_{]0,t]} \zeta_u^N \mathrm{d}M_u = N_0 + M_t^N + L_t^N, \tag{4.1}$$

where ξ^N and ζ^N are $\mathbb{G}$-predictable processes. The continuous $\mathbb{G}$-martingale M^N and the purely discontinuous $\mathbb{G}$-martingale L^N are mutually orthogonal.

Proof. Since Γ is continuous, the proof follows from Corollary 5.2.4 of Bielecki and Rutkowski (2004). □

By hypothesis (3.1), Definition 3.5 and Proposition 4.1, we have

$$\frac{\mathrm{d}\widehat{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \mathcal{E}\left(-\int \theta \mathrm{d}W\right)_T . \tag{4.2}$$

By (4.2) we have that $\hat{W}_t = W_t + \int_0^t \theta_s \mathrm{d}s$ is a $\mathbb{G}$-Brownian motion under $\widehat{\mathbb{P}}$ and the results of Proposition 4.1 can be reformulated in terms of $(\hat{W}, M)$. In fact $M_t = H_t - \int_0^t \tilde{\lambda}_s \mathrm{d}s$ is also a $\widehat{\mathbb{P}}$-martingale since the orthogonal martingale structure is not affected by the change of measure from $\mathbb{P}$ to $\widehat{\mathbb{P}}$. Hence by the representation property (4.1), every $\mathbb{G}$-martingale $\hat{N}$ under $\widehat{\mathbb{P}}$ is of the form

$$\hat{N}_t = \hat{N}_0 + \int_0^t \xi_u^{\hat{N}} \mathrm{d}\hat{W}_u + \int_{]0,t]} \zeta_u^{\hat{N}} \mathrm{d}M_u, \quad t \in [0,T]. \tag{4.3}$$

We now find a plrm strategy for H by computing the decomposition (4.3) for $\widehat{E}\left[H \middle| \mathcal{G}_t\right]$ under $\widehat{\mathbb{P}}$. Theorem 3.6 and our hypothesis (3.1) guarantee that this is indeed the FS decomposition for H.

Under the equivalent martingale probability measure $\widehat{\mathbb{P}}$, the discounted optimal portfolio value $\hat{V}$ of the defaultable claim H at time t, is given by:

$$\begin{aligned}
\hat{V}_t &= \widehat{E}\left[H \middle| \mathcal{G}_t\right] \\
&= \widehat{E}\left[\frac{\bar{X}}{B_T}\left(1 + (h(\tau \wedge T) - 1)H_T\right) \middle| \mathcal{G}_t\right] \\
&= \underbrace{\widehat{E}\left[\frac{\bar{X}}{B_T} \middle| \mathcal{G}_t\right]}_{a)} + \underbrace{\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau \wedge T) - 1)H_T \middle| \mathcal{G}_t\right]}_{b)} .
\end{aligned} \tag{4.4}$$

a) Since $\bar{X} \in L^1(\mathcal{G}_T, \widehat{\mathbb{P}})$, by (4.3) we have

$$\widehat{E}\left[\frac{\bar{X}}{B_T} \middle| \mathcal{G}_t\right] = \widehat{E}\left[\frac{\bar{X}}{B_T}\right] + \int_0^t \bar{\xi}_s \mathrm{d}\hat{W}_s + \int_{]0,t]} \bar{\eta}_s \mathrm{d}M_s, \tag{4.5}$$

where $\bar{\xi}$ and $\bar{\eta}$ are $\mathbb{G}$-predictable process.

b) It remains to compute the term $\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau \wedge T) - 1)H_T \middle| \mathcal{G}_t\right]$. First by Corollary

5.1.2 of Bielecki and Rutkowski (2004) we can obtain the following decomposition:

$$
\begin{aligned}
\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle|\,\mathcal{G}_t\right] &= H_t\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle|\mathcal{F}_t\vee\mathcal{H}_T\right] \\
&+(1-H_t)\widehat{E}\left[(1-H_t)e^{\int_0^t\lambda_s\mathrm{d}s}\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle|\,\mathcal{F}_t\right]= \\
&= H_t(h(\tau\wedge T)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{F}_t\vee\mathcal{H}_T\right]+(1-H_t)e^{\int_0^t\lambda_s\mathrm{d}s}. \\
&\cdot\underbrace{\widehat{E}\left[\mathbb{I}_{\{t<\tau\leq T\}}(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\middle|\,\mathcal{F}_t\right]}_{c)}
\end{aligned}
\tag{4.6}
$$

We focus now on the conditional expectation c). We introduce here the σ-algebra

$$\mathcal{G}_{\tau-}=\sigma\left(A\cap\{\tau>t\},\ A\in\mathcal{G}_t,\ 0\leq t\leq T\right)$$

of the *events strictly prior* to τ. We set

$$N:=\widehat{E}\left[(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\middle|\,\mathcal{G}_{\tau-}\right] \tag{4.7}$$

and note that

$$N=(h(\tau\wedge T)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{G}_{\tau-}\right]$$

since the $\mathbb{G}$-stopping time τ is $\mathcal{G}_{\tau-}$-measurable by Theorem 5.6 on page 118 of Dellacherie and Meyer (1978).

Lemma 4.2. *Let N be defined in* (4.7). *Then*

$$\widehat{E}\left[\mathbb{I}_{\{t<\tau\leq T\}}(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\middle|\,\mathcal{F}_t\right]=\widehat{E}\left[\mathbb{I}_{\{t<\tau\leq T\}}N\middle|\,\mathcal{F}_t\right],\quad\forall t\in[0,T].$$

Proof. Consider an arbitrary event $A\in\mathcal{F}_t$. By using the definition of the conditional expectation, we have

$$
\begin{aligned}
\int_A\mathbb{I}_{\{t<\tau\leq T\}}(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\mathrm{d}\widehat{\mathbb{P}} &= \int_{A\cap\{\tau>t\}}\mathbb{I}_{\{\tau\leq T\}}(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\mathrm{d}\widehat{\mathbb{P}} \\
&= \int_{A\cap\{\tau>t\}}\widehat{E}\left[\mathbb{I}_{\{\tau\leq T\}}(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\middle|\,\mathcal{G}_{\tau-}\right]\mathrm{d}\widehat{\mathbb{P}} \\
&= \int_{A\cap\{\tau>t\}}\mathbb{I}_{\{\tau\leq T\}}N\mathrm{d}\widehat{\mathbb{P}} \\
&= \int_A\mathbb{I}_{\{t<\tau\leq T\}}N\mathrm{d}\widehat{\mathbb{P}},
\end{aligned}
$$

since the event $\{\tau\leq T\}$ is in $\mathcal{G}_{\tau-}$. $\square$

Since $\mathcal{G}_{\tau-} = \mathcal{F}_{\tau-}$, we know that there exists an $\mathbb{F}$-predictable[2] process Z such that

$$Z_\tau = N. \tag{4.8}$$

Hence we obtain

$$\begin{aligned}
\widehat{E}\left[\mathbb{I}_{\{t<\tau\le T\}}(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\middle|\mathcal{F}_t\right] &= \widehat{E}\left[\mathbb{I}_{\{t<\tau\le T\}}N\middle|\mathcal{F}_t\right] \\
&= \widehat{E}\left[\mathbb{I}_{\{t<\tau\le T\}}Z_\tau\middle|\mathcal{F}_t\right] \\
&= \widehat{E}\left[\int_t^T Z_s e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s\middle|\mathcal{F}_t\right],
\end{aligned} \tag{4.9}$$

where the last equality holds in view of Proposition 5.1.1 (ii) of Bielecki and Rutkowski (2004) and the $\mathbb{F}$-predictability of Z (see page 148 of Bielecki and Rutkowski (2004)). Hence we can rewrite (4.6) as follows:

$$\begin{aligned}
\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle|\mathcal{G}_t\right] &= H_t(h(\tau\wedge T)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{F}_t\vee\mathcal{H}_T\right] \\
&\quad + (1-H_t)e^{\int_0^t \lambda_s \mathrm{d}s}\widehat{E}\left[\int_t^T Z_s e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s\middle|\mathcal{F}_t\right].
\end{aligned} \tag{4.10}$$

We put for each $t\in[0,T]$

$$D_t := e^{\int_0^t \lambda_s \mathrm{d}s}\widehat{E}\left[\int_t^T Z_s e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s\middle|\mathcal{F}_t\right] \tag{4.11}$$

and we introduce the $\mathbb{F}$-martingale m by setting for each $t\in[0,T]$

$$m_t = \widehat{E}\left[\int_0^T Z_s e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s\middle|\mathcal{F}_t\right]. \tag{4.12}$$

Following the proof of Proposition 5.2.1 of Bielecki and Rutkowski (2004), we write D in terms of the $\mathbb{F}$-martingale m and by applying the Itô integration by parts formula, we obtain

$$D_t = m_0 + \int_0^t e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s + \int_0^t (D_s - Z_s)\lambda_s \mathrm{d}s.$$

Furthermore, since D is a continuous process, we have

$$(1-H_t)D_t = m_0 + \int_{]0,t\wedge\tau]} \mathrm{d}D_s - \mathbb{I}_{\{\tau\le t\}}D_\tau.$$

Hence

$$(1-H_t)D_t = m_0 + \int_0^{t\wedge\tau} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s + \int_0^{t\wedge\tau}(D_s - Z_s)\lambda_s \mathrm{d}s - \mathbb{I}_{\{\tau\le t\}}D_\tau$$
$$= m_0 + \int_0^{t\wedge\tau} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s - \int_{]0,t]} D_s \mathrm{d}M_s - \int_0^{t\wedge\tau} Z_s\lambda_s \mathrm{d}s.$$

Consequently we can rewrite (4.10) as follows:

$$\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle|\mathcal{G}_t\right] = H_t(h(\tau\wedge T)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{F}_t\vee\mathcal{H}_T\right]$$
$$+m_0 + \int_0^{t\wedge\tau} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s - \int_{]0,t]} D_s \mathrm{d}M_s - \int_0^{t\wedge\tau} Z_s\lambda_s \mathrm{d}s.$$

Lemma 4.3. *Let Z be the $\mathbb{F}$-predictable process given by* (4.8)*. Then the following equality holds:*

$$H_t Z_\tau = H_t(h(\tau\wedge T)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{F}_t\vee\mathcal{H}_T\right],\quad \forall t\in[0,T]. \tag{4.13}$$

Proof. It is clear that

$$H_t Z_\tau = \widehat{E}\left[H_t(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\middle|\mathcal{G}_{\tau-}\right].$$

Hence we need only to show that

$$\widehat{E}\left[H_t(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\middle|\mathcal{G}_{\tau-}\right] = H_t(h(\tau\wedge T)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{F}_t\vee\mathcal{H}_T\right]. \tag{4.14}$$

By using the definition of conditional expectation and the fact that conditioning with respect to $\mathcal{G}_t$ can be replaced by conditioning with respect to $\mathcal{F}_t\vee\mathcal{H}_T$ on the event $\{\tau\le t\}$ (see Lemma 5.1.5 of Bielecki and Rutkowski (2004)), given an arbitrary event A in $\mathcal{F}_s$, with $0<s\le t$, for any $t\in[0,T]$, we have

$$\int_{A\cap\{\tau>s\}} H_t(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\mathrm{d}\widehat{\mathbb{P}} = \int_{A\cap\{s<\tau\le t\}}(h(\tau\wedge T)-1)\frac{\bar{X}}{B_T}\mathrm{d}\widehat{\mathbb{P}}$$
$$= \int_{A\cap\{\tau>s\}} H_t(h(\tau\wedge t)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{G}_t\right]\mathrm{d}\widehat{\mathbb{P}}$$
$$= \int_{A\cap\{\tau>s\}} H_t(h(\tau\wedge T)-1)\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\mathcal{F}_t\vee\mathcal{H}_T\right]\mathrm{d}\widehat{\mathbb{P}},$$

since

$$H_t(h(\tau\wedge T)-1) = H_t(h(\tau\wedge t)-1),\quad \forall t\in[0,T].$$

Then the statement is proved since (4.14) is verified on the generators. □

Finally gathering the results, by using (4.13) we obtain:

$$
\begin{aligned}
&\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle|\mathcal{G}_t\right]\\
&\qquad = H_t Z_\tau + m_0 + \int_0^{t\wedge\tau} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s - \int_{]0,t]} D_s \mathrm{d}M_s - \int_0^{t\wedge\tau} Z_s\lambda_s \mathrm{d}s\\
&\qquad = m_0 + \int_0^{t\wedge\tau} e^{\int_0^s \lambda_u \mathrm{d}u}\mathrm{d}m_s + \int_{]0,t]}(Z_s - D_s)\mathrm{d}M_s\\
&\qquad = m_0 + \int_0^{t}(1-H_s) e^{\int_0^s \lambda_u \mathrm{d}u}\xi^m_s\mathrm{d}\hat{W}_s + \int_{]0,t]}(Z_s - D_s)\mathrm{d}M_s, \qquad (4.15)
\end{aligned}
$$

where in particular we have used the fact that the continuous $\mathbb{F}$-martingale m admits the following integral representation with respect to the Brownian motion $\hat{W}$:

$$
m_t = m_0 + \int_0^t \xi^m_s \mathrm{d}\hat{W}_s, \quad t\in[0,T], \qquad (4.16)
$$

for some $\mathbb{F}$-predictable process ξ^m, such that $\forall t\in[0,T]$, $\int_0^t(\xi^m_s)^2\mathrm{d}s<\infty$.

Proposition 4.4. *In the market model outlined in Section 2, the FS decomposition for H defined in* (2.7) *is given by*

$$
\begin{aligned}
H = \widehat{E}\left[\frac{\bar{X}}{B_T}\right] + m_0 + \int_0^T \frac{1}{\sigma_s X_s}\left(\bar{\xi}_s + \mathbb{I}_{\{\tau\geq s\}}\xi^m_s e^{\int_0^s \lambda_u \mathrm{d}u}\right)\mathrm{d}X_s\\
+ \int_{]0,T]}(Z_s - D_s + \bar{\eta}_s)\mathrm{d}M_s, \qquad (4.17)
\end{aligned}
$$

where the processes m, Z, D, $\bar{\xi}$, $\bar{\eta}$, ξ^m and M are defined in (4.12), (4.7), (4.11), (4.5), (4.16) *and* (2.2). *In particular we have that a* plrm strategy $\varphi=(\xi,\zeta)$ *is given by*

$$
\xi_t = \xi^H_t = \frac{1}{\sigma_t X_t}\left(\bar{\xi}_t + \mathbb{I}_{\{\tau\geq t\}}\xi^m_t e^{\int_0^t \lambda_s \mathrm{d}s}\right), \qquad (4.18)
$$

$$
\eta_t = \hat{V}_t - \frac{1}{\sigma_t}\left(\bar{\xi}_t + \mathbb{I}_{\{\tau\geq t\}}\xi^m_t e^{\int_0^t \lambda_s \mathrm{d}s}\right) \qquad (4.19)
$$

for $t\in[0,T]$ and the minimal cost *is*

$$
C^H_t = \widehat{E}\left[\frac{\bar{X}}{B_T}\right] + m_0 + \int_{]0,t]}(Z_s - D_s + \bar{\eta}_s)\mathrm{d}M_s, \quad t\in[0,T]. \qquad (4.20)
$$

Proof. It follows by hypothesis (3.1) and Theorem 3.6. □

Remark 4.5. *It is possible to choose different hypotheses that guarantee that decomposition (4.17) gives the FS decomposition. Assumption (3.1) is the simplest condition that can be assumed. For a complete survey and a discussion of the other sufficient conditions, we refer to Schweizer (1995).*

Proposition 4.4 extends the main result of Biagini and Cretarola (2007), where decomposition (4.17) was already proved in the case when the trajectories of X are $\mathbb{F}$-adapted and $\mathcal{F}_t$ and $\mathcal{H}_t$ are independent for every $t \in [0, T]$.
In general if $\frac{\bar{X}}{B_T}$ is $\mathcal{F}_T$-measurable, we have $\bar{\eta} = 0$ in decomposition (4.5) and

$$Z_t = (h(t \wedge T) - 1) \left(\widehat{E}\left[\frac{\bar{X}}{B_T}\right] + \int_0^{t\wedge T} \bar{\xi}_s \mathrm{d}\hat{W}_s \right) \tag{4.21}$$

in equation (4.9). In fact by (4.5) and Theorem 67 page 125 in Dellacherie and Meyer (1978), we get

$$\begin{aligned} \widehat{E}\left[\frac{\bar{X}}{B_T}\bigg|\mathcal{G}_{\tau-}\right] &= \widehat{E}\left[\widehat{E}\left[\frac{\bar{X}}{B_T}\bigg|\mathcal{G}_{\tau}\right]\bigg|\mathcal{G}_{\tau-}\right] \\ &= \widehat{E}\left[\widehat{E}\left[\frac{\bar{X}}{B_T}\right] + \int_0^{\tau} \bar{\xi}_s \mathrm{d}\hat{W}_s \bigg| \mathcal{G}_{\tau-}\right] \\ &= \widehat{E}\left[\frac{\bar{X}}{B_T}\right] + \int_0^{\tau\wedge T} \bar{\xi}_s \mathrm{d}\hat{W}_s. \end{aligned} \tag{4.22}$$

Note that here we are using implicitly hypothesis (H) under $\widehat{\mathbb{P}}$.

Remark 4.6. *The introduction of the process Z in* (4.8) *may appear artificial. However it is necessary to find decomposition* (4.9). *We have already seen that Z can be explicitly calculated if $\frac{\bar{X}}{B_T}$ is $\mathcal{F}_T$-measurable. This is already a quite general case since we do not require the trajectories of X to be $\mathbb{F}$-adapted or the independence of τ from $\mathcal{F}_t$, for each $t \in [0, T]$.*

Another example is the following. We suppose that under $\widehat{\mathbb{P}}$, the discounted asset price X is of the form

$$X_t = x_0 e^{\int_0^t \sigma(\tau\wedge s)\mathrm{d}\hat{W}_s - \frac{1}{2}\int_0^t \sigma(\tau\wedge s)^2 \mathrm{d}s}, \quad x_0 > 0,$$

where σ is a bounded Borel function, and $\frac{\bar{X}}{B_T} = X_T^2$. In this case $\frac{\bar{X}}{B_T}$ is (strictly)

$\mathcal{G}_T$-measurable. We obtain

$$\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\,\mathcal{G}_{\tau-}\right] = \widehat{E}\left[x_0^2 e^{2\int_0^T \sigma(\tau\wedge s)\mathrm{d}\hat{W}_s - \int_0^T \sigma(\tau\wedge s)^2\mathrm{d}s}\middle|\,\mathcal{G}_{\tau-}\right]$$
$$= x_0^2 e^{\int_0^T \sigma(\tau\wedge s)^2\mathrm{d}s}\widehat{E}\left[e^{2\int_0^T \sigma(\tau\wedge s)\mathrm{d}\hat{W}_s - 2\int_0^T \sigma(\tau\wedge s)^2\mathrm{d}s}\middle|\,\mathcal{G}_{\tau-}\right]$$
$$= x_0^2 e^{\sigma(\tau)^2(T-\tau\wedge T)} e^{2\int_0^{\tau\wedge T} \sigma(s)\mathrm{d}\hat{W}_s - \int_0^{\tau\wedge T}\sigma(s)^2\mathrm{d}s},$$

and

$$\bar{Z}_t = x_0^2(h(t\wedge T)-1)e^{\sigma(t)^2(T-t\wedge T)}e^{2\int_0^{t\wedge T}\sigma(s)\mathrm{d}\hat{W}_s - \int_0^{t\wedge T}\sigma(s)^2\mathrm{d}s}.$$

We remark that Z is not uniquely defined. However in the case that there exist several possible $\mathbb{F}$-predictable processes Z satisfying equation (4.8)*, they all provide the same conditional expectation* (4.9)*. We refer also to Bielecki and Rutkowski (2004), page 148, for a further discussion of this issue.*

We compute decomposition (4.17) in two particular cases.

5 Example 1: τ dependent on X

We consider first the case where the default process may depend on the evolution of the asset price, but the dynamics of the money market account and of the stock are *not* influenced by the presence of the default in the market. We represent this fact by assuming that the interest rate, the drift and volatility in (2.3) are $\mathbb{F}$-adapted processes.
Since the promised contingent claim $\bar{X}$ is written on the underlying non-defaultable assets Y and B, in this setting $\bar{X}$ is $\mathcal{F}_T$-measurable and we have

$$\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\,\mathcal{G}_t\right] = \widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\,\mathcal{F}_t\right], \quad \forall t\in[0,T],$$

as a consequence of our hypothesis (H) under $\widehat{\mathbb{P}}$. Hence we get $\bar{\eta}=0$ in (4.5).
We show now how to hedge a Corporate bond with a Treasury bond by using the local risk-minimizing approach, i.e. we compute a plrm strategy for a defaultable claim H whose promised contingent claim $\bar{X}$ is equal to 1, i.e. $\bar{X}=p(T,T)=1$, where the process $p(t,T)$ represents the price of a Treasury bond that expires at time T. For the sake of simplicity we put

$$B_t \equiv 1, \quad \forall t\in[0,T].$$

Hence the discounted value of H can be represented as follows:

$$H = 1 + (h(\tau\wedge T)-1)H_T. \tag{5.1}$$

In addition we assume the following hypotheses:

- λ is an affine process, in particular it satisfies the following equation under $\widehat{\mathbb{P}}$:

$$\left\{ \begin{array}{ll} \mathrm{d}\lambda_t & = (b+\beta\lambda_t)\mathrm{d}t + \alpha\sqrt{\lambda_t}\mathrm{d}\hat{W}_t \\ \lambda_0 & = 0, \end{array} \right. \tag{5.2}$$

where $b, \alpha \in \mathbb{R}^+$ and β is arbitrary. It is the Cox-Ingersoll-Ross model and we know it has a unique strong solution $\lambda \geq 0$ for every $\lambda_0 \geq 0$. You can see Duffie (2004) for further details.

- The Borel function $h : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$h(x) = \alpha_0 \mathbb{I}_{\{x \leq T_0\}} + \alpha_1 \mathbb{I}_{\{x > T_0\}}, \tag{5.3}$$

where $\alpha_0, \alpha_1 \in \mathbb{R}^+$ with $0 \leq \alpha_0 < \alpha_1$ and T_0 is a fixed date before the maturity T.

Under the equivalent martingale probability measure $\widehat{\mathbb{P}}$, the discounted optimal portfolio value $\hat{V}_t$ of the defaultable claim H given in (5.1) at time t, is given by:

$$\begin{aligned} \hat{V}_t &= \widehat{E}\left[H \middle| \mathcal{G}_t\right] \\ &= 1 + \widehat{E}\left[(h(\tau \wedge T) - 1)H_T \middle| \mathcal{G}_t\right] \\ &= 1 + m_0 + \int_0^t \mathbb{I}_{\{\tau \geq s\}} e^{\int_0^s \lambda_u \mathrm{d}u} \mathrm{d}m_s + \int_{]0,t]} (h(s) - 1 - D_s)\mathrm{d}M_s, \end{aligned} \tag{5.4}$$

where h is given in (5.3) and m, D and M are the processes introduced in (4.12), (4.11) and (2.2) respectively. Here we have $Z_t = h(t) - 1$ by (4.21). Note that in this case (5.4) can be recovered directly by Corollary 5.2.2 of Bielecki and Rutkowski (2004).
We now need only to compute the decomposition of the $\mathbb{F}$-martingale m, i.e. the conditional expectation $\widehat{E}\left[\int_0^T (h(s) - 1)e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s \middle| \mathcal{F}_t\right]$. We obtain

$$\begin{aligned} & m_t \\ &= \widehat{E}\left[\int_0^T e^{-\int_0^s \lambda_u \mathrm{d}u} \left((\alpha_0 - \alpha_1)\mathbb{I}_{\{s \leq T_0\}} + (\alpha_1 - 1)\mathbb{I}_{\{s \leq T\}}\right) \lambda_s \mathrm{d}s \middle| \mathcal{F}_t\right] \\ &= (\alpha_0 - \alpha_1)\widehat{E}\left[\int_0^{T_0} e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s \middle| \mathcal{F}_t\right] + (\alpha_1 - 1)\widehat{E}\left[\int_0^T e^{-\int_0^s \lambda_u \mathrm{d}u}\lambda_s \mathrm{d}s \middle| \mathcal{F}_t\right] \\ &= (\alpha_1 - \alpha_0)\underbrace{\widehat{E}\left[e^{-\int_0^{T_0} \lambda_s \mathrm{d}s} \middle| \mathcal{F}_t\right]}_{a)} + (1 - \alpha_1)\underbrace{\widehat{E}\left[e^{-\int_0^T \lambda_s \mathrm{d}s} \middle| \mathcal{F}_t\right]}_{b)} + \alpha_0 - 1. \end{aligned}$$

b) Since λ is an affine process whose dynamics is given in (5.2) , we have

$$\begin{aligned}\widehat{E}\left[e^{-\int_0^T \lambda_s \mathrm{d}s}\middle|\,\mathcal{F}_t\right] &= e^{-\int_0^t \lambda_s \mathrm{d}s}\widehat{E}\left[e^{-\int_t^T \lambda_s \mathrm{d}s}\middle|\,\mathcal{F}_t\right]\\ &= e^{-\int_0^t \lambda_s \mathrm{d}s}\cdot e^{-A(t,T)-B(t,T)\lambda_t},\end{aligned}$$

where the functions $A(t,T)$, $B(t,T)$ satisfy the following equations:

$$\partial_t B(t,T) = \frac{\alpha^2}{2}B^2(t,T) - \beta B(t,T) - 1, \quad B(T,T) = 0 \tag{5.5}$$

$$\partial_t A(t,T) = -bB(t,T), \quad A(T,T) = 0, \tag{5.6}$$

that admit explicit solutions (see for instance Filipović (2006)). Since $\widehat{E}\left[e^{-\int_0^T \lambda_s \mathrm{d}s}\middle|\,\mathcal{F}_t\right]$ must be of the form

$$\widehat{E}\left[e^{-\int_0^T \lambda_s \mathrm{d}s}\middle|\,\mathcal{F}_t\right] = \widehat{E}\left[e^{-\int_0^T \lambda_s \mathrm{d}s}\right] + \int_0^t \varphi_s \mathrm{d}\hat{W}_s,$$

for a suitable φ, by applying Itô formula and by (5.5) and (5.6), we obtain

$$\mathrm{d}(e^{-\int_0^t \lambda_s \mathrm{d}s}\cdot e^{-A(t,T)-B(t,T)\lambda_t}) = -e^{-\int_0^t \lambda_s \mathrm{d}s - A(t,T)-B(t,T)\lambda_t}(\alpha B(t,T)\sqrt{\lambda_t}\mathrm{d}\hat{W}_t).$$

Hence

$$\begin{aligned}&\widehat{E}\left[e^{-\int_0^T \lambda_s \mathrm{d}s}\middle|\mathcal{F}_t\right]\\ &= e^{-A(0,T)} - \int_0^t \alpha e^{-\int_0^s \lambda_u \mathrm{d}u - A(s,T)-B(s,T)\lambda_s}B(s,T)\sqrt{\lambda_s}\mathrm{d}\hat{W}_s.\end{aligned} \tag{5.7}$$

Similarly we can compute a) and we get

$$\begin{aligned}&\widehat{E}\left[e^{-\int_0^{T_0} \lambda_s \mathrm{d}s}\middle|\mathcal{F}_t\right]\\ &= e^{-A(0,T_0)} - \int_0^t \alpha \mathbb{I}_{\{s\leq T_0\}} e^{-\int_0^s \lambda_u \mathrm{d}u - A(s,T_0)-B(s,T_0)\lambda_s}B(s,T_0)\sqrt{\lambda_s}\mathrm{d}\hat{W}_s.\end{aligned} \tag{5.8}$$

Finally gathering the results, we obtain

$$\begin{aligned}m_t &= \alpha_0 - 1 + (\alpha_1-\alpha_0)e^{-A(0,T_0)} + (1-\alpha_1)e^{-A(0,T)}\\ &\quad - \int_0^t \alpha e^{-\int_0^s \lambda_u \mathrm{d}u}\Big((\alpha_1-\alpha_0)\mathbb{I}_{\{s\leq T_0\}}e^{-A(s,T_0)-B(s,T_0)\lambda_s}B(s,T_0)\\ &\quad + (1-\alpha_1)e^{-A(s,T)-B(s,T)\lambda_s}B(s,T)\Big)\sqrt{\lambda_s}\mathrm{d}\hat{W}_s.\end{aligned} \tag{5.9}$$

Consequently D is given by

$$
\begin{aligned}
D_t &= e^{\int_0^t \lambda_s \mathrm{d}s} m_t - e^{\int_0^t \lambda_s \mathrm{d}s} \int_0^t (h(s)-1) e^{-\int_0^s \lambda_u \mathrm{d}u} \lambda_s \mathrm{d}s \\
&= e^{\int_0^t \lambda_s \mathrm{d}s} m_t + (\alpha_0 - \alpha_1)(1 - e^{-\int_t^{T_0} \lambda_s \mathrm{d}s}) \mathbb{I}_{\{t \le T_0\}} \\
&\quad + [(\alpha_0 - \alpha_1) e^{-\int_t^{T_0} \lambda_s \mathrm{d}s} - (\alpha_0 - 1) e^{\int_0^t \lambda_s \mathrm{d}s} + \alpha_1 - 1]
\end{aligned} \tag{5.10}
$$

Finally by plugging (5.9) and (5.10) into (5.4), we can write explicitly the FS decomposition for H:

$$
\begin{aligned}
H &= \alpha_0 + (\alpha_1 - \alpha_0) e^{-A(0,T_0)} + (1-\alpha_1) e^{-A(0,T)} + \\
&- \int_0^T \mathbb{I}_{\{\tau \ge s\}} \frac{\alpha}{\sigma_s X_s} \Big((\alpha_1 - \alpha_0) \mathbb{I}_{\{s \le T_0\}} e^{-A(s,T_0) - B(s,T_0)\lambda_s} B(s,T_0) + \\
&(1-\alpha_1) e^{-A(s,T) - B(s,T)\lambda_s} B(s,T) \Big) \sqrt{\lambda_s} \mathrm{d}X_s + \int_{]0,T]} (h(s) - 1 - D_s) \mathrm{d}M_s,
\end{aligned} \tag{5.11}
$$

where A, B, h, D and M are given in (5.6), (5.5), (5.3), (5.10) and (2.2) respectively.

6 Example 2: X dependent on τ

We study now the case when the default time may influence the dynamics of the asset price but not vice versa. We suppose then that the default time $\tau = \tau(\eta)$ and the underlying Brownian motion $W = W(\tilde{\omega})$ are independent and defined on the product space $\Omega = \tilde{\Omega} \times E$, endowed with the product filtration $\mathbb{G} = \mathbb{F} \otimes \mathbb{H}$, given by $\mathcal{G}_t = \mathcal{F}_t \otimes \mathcal{H}_t$, for every $t \in [0,T]$ and the product probability $\mathbb{P} = \mathbb{P}^W \otimes \nu$, where $\mathbb{P}^W$ is the Wiener measure and ν is the law of $H_t = \mathbb{I}_{\{\tau \le t\}}$. Note that now with respect to the previous setting we have $\omega = (\tilde{\omega}, \eta)$. In particular following Biagini and Pratelli (1999), we assume that the dynamics of Y are of the form

$$
\mathrm{d}Y_t = Y_t \left[\mu_t(\eta) \mathrm{d}t + \sigma_t(\eta) \mathrm{d}W_t \right],
$$

and that the hypotheses outlined in Section 2 still hold. Note that here we are focusing on the case where drift and volatility *depend only* on η, seen as an *exterior source of randomness.*

Consider now the larger filtration $\tilde{\mathbb{G}}$ given by $\tilde{\mathcal{G}}_t = \mathcal{F}_t \otimes \mathcal{H}_T$, for every $t \in [0,T]$, obtained by adding to $\mathcal{G}_t$ the full information about η since the initial instant $t = 0$: it follows that $\mathcal{G}_t \subset \tilde{\mathcal{G}}_t,\ 0 \le t < T$. Since τ and W are independent, we note that W is a Brownian motion also with respect to $\tilde{\mathbb{G}}$.

Proposition 6.1. *Under the hypotheses outlined above the process* ξ^H *given in* (4.18) *coincides with the* $\mathbb{G}$*-predictable projection of the* $\tilde{\mathbb{G}}$*-predictable process* $\tilde{\xi}^H$ *such that* $\int_0^T (\tilde{\xi}_s^H)^2 \mathrm{d}s < \infty$ *a.s. and*

$$H = \widehat{E}\left[H \middle| \tilde{\mathcal{G}}_0\right] + \int_0^T \tilde{\xi}_s^H \mathrm{d}\hat{W}_s.$$

Proof. Since $\mathcal{G}_T = \mathcal{F}_T \vee \mathcal{H}_T$, we may prove the Proposition in the case when the $\mathcal{G}_T$-measurable random variable $\dfrac{\bar{X}}{B_T}$ is of the form $\dfrac{\bar{X}}{B_T} = (1 - H_s)F$, for some fixed $s \leq T$ and some $\mathcal{F}_T$-measurable integrable random variable F. We compute first decomposition (4.5) for $\dfrac{\bar{X}}{B_T}$. We note that

$$\frac{\bar{X}}{B_T} = (1 - H_s)F = (1 - H_s)e^{\int_0^T \lambda_u \mathrm{d}u}\bar{F} = L_s\bar{F},$$

where the process $L_t = (1 - H_t)e^{\int_0^t \lambda_u \mathrm{d}u}$, for $t \leq s$, is a $\mathbb{G}$-martingale (see Lemma 5.1.7 of Bielecki and Rutkowski (2004) for further details) and $\bar{F} = e^{-\int_0^s \lambda_u \mathrm{d}u}F$ is an $\mathcal{F}_T$-measurable, integrable random variable.
First by the martingale representation property of the Brownian filtration, we have

$$\bar{F} = \widehat{E}\left[\bar{F}\right] + \int_0^T \xi_u \mathrm{d}\hat{W}_u,$$

where ξ is an $\mathbb{F}$-predictable. Then

$$\frac{\bar{X}}{B_T} = L_s\left(\widehat{E}\left[\bar{F}\right] + \int_0^T \xi_u \mathrm{d}\hat{W}_u\right) = L_s\widehat{E}\left[\bar{F}\right] + \int_0^T L_s\xi_t \mathrm{d}\hat{W}_t,$$

i.e. $\dfrac{\bar{X}}{B_T}$ is attainable with respect to the larger filtration $\tilde{\mathbb{G}}$. If we put $G_t := \widehat{E}\left[\bar{F}\middle| \mathcal{F}_t\right]$, for $t \in [0, T]$, we have

$$\frac{\bar{X}}{B_T} = L_s\bar{F} = L_s\widehat{E}\left[F|\mathcal{F}_T\right] = L_sG_T.$$

By Proposition 5.1.3 of Bielecki and Rutkowski (2004) we have $L_t = \mathcal{E}(-M)_t$, for $t \leq s$, where $M_t = H_t - \int_0^{t\wedge\tau} \lambda_u \mathrm{d}u$. Hence $[L, G]_t = 0$, for every $t \in [0, s]$ and the Itô integration by parts formula yields:

$$
\begin{aligned}
\frac{\bar{X}}{B_T} &= L_0G_0 + \int_0^T L_t \mathrm{d}G_t + \int_{]0,s]} G_t \mathrm{d}L_t + [L,G]_s \\
&= \widehat{E}\left[\bar{F}\right] + \int_0^T L_t\xi_t \mathrm{d}\hat{W}_t + \int_{]0,T]} \mathbb{I}_{\{s\geq t\}} \widehat{E}\left[\bar{F}\middle|\mathcal{F}_t\right] \mathrm{d}L_t \\
&= \widehat{E}\left[\bar{F}\right] + \int_0^T L_t\xi_t \mathrm{d}\hat{W}_t - \int_{]0,T]} \mathbb{I}_{\{s\geq t\}} \widehat{E}\left[\bar{F}\middle|\mathcal{F}_t\right] L_{t-}\mathrm{d}M_t, \qquad (6.1)
\end{aligned}
$$

since $G_t = \widehat{E}\left[\bar{F}\middle|\mathcal{F}_t\right]$ is continuous. On the other hand by (4.3), we get

$$
\frac{\bar{X}}{B_T} = L_s\bar{F} = \widehat{E}\left[L_s\bar{F}\right] + \int_0^T \bar{\xi}_t \mathrm{d}\hat{W}_t + \int_0^T \bar{\eta}_t \mathrm{d}M_t,
$$

and the uniqueness of the decomposition implies that

$$
\bar{\xi}_t = L_{t-}\xi_t = (L_s\xi_\cdot)^p_t,
$$

i.e. $\bar{\xi}$ coincides with the $\mathbb{G}$-predictable projection of the process $L_s\xi_t$.
Analogously we compute the decomposition of $\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle|\tilde{\mathcal{G}}_t\right]$, that is given by

$$
\begin{aligned}
&\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle|\tilde{\mathcal{G}}_t\right] \\
&\qquad = (h(\tau\wedge T)-1)H_T\left(\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\tilde{\mathcal{G}}_0\right] + \int_0^T L_s\xi_u \mathrm{d}\hat{W}_u\right) \\
&\qquad = (h(\tau\wedge T)-1)H_T\widehat{E}\left[\frac{\bar{X}}{B_T}\middle|\tilde{\mathcal{G}}_0\right] + \int_0^T \underbrace{L_s\xi_u(h(\tau\wedge T)-1)H_T}_{\tilde{\Psi}_u} \mathrm{d}\hat{W}_u.
\end{aligned}
$$

With a similar argument as before we can conclude that the integrand

$$
\Psi_t = (1-H_t)e^{\int_0^t \lambda_s \mathrm{d}s}\xi^m_t
$$

appearing in decomposition (4.15) of $\widehat{E}\left[\frac{\bar{X}}{B_T}(h(\tau\wedge T)-1)H_T\middle|\mathcal{G}_t\right]$ is the $\mathbb{G}$-predictable projection of $\tilde{\Psi}$. □

In particular we note that we obtain again the results of Theorem 4.6 and Theorem 4.16 of Föllmer and Schweizer (1991). Hence (6.1) is the FS decomposition in the case of *incomplete information.* Namely if the trader would have access to the larger filtration $\tilde{\mathbb{G}}$ which contains at any time the information on past and future behavior of the default time, the market would be complete because the volatility and drift are deterministic with respect to $\tilde{\mathbb{G}}$.

Example 6.2. *We apply these results to find a plrm strategy for a defaultable claim H whose promised contingent claim $\bar{X}$ is given by the standard payoff of a call option, i.e. $\bar{X} = (Y_T - K)^+$, where $K \in \mathbb{R}_+$ represents the exercise price. Hence the discounted value of H can be represented as follows:*

$$H = \frac{(Y_T - K)^+}{B_T}(1 + (h(\tau \wedge T) - 1)H_T) \tag{6.2}$$

and with respect to $\tilde{\mathbb{G}}$, the discounted replicating portfolio $\tilde{V}$ for H, at time $t \in [0, T]$, is given by:

$$\begin{aligned}
&\tilde{V}_t \\
&= \widehat{E}[H|\tilde{\mathcal{G}}_t] \\
&= \widehat{E}\left[\frac{(Y_T - K)^+}{B_T}(1 + (h(\tau \wedge T) - 1)H_T)\middle|\tilde{\mathcal{G}}_t\right] \\
&= (1 + (h(\tau \wedge T) - 1)H_T)\widehat{E}\left[\frac{(Y_T - K)^+}{B_T}\middle|\tilde{\mathcal{G}}_t\right] \\
&= (1 + (h(\tau \wedge T) - 1)H_T)\left(X_t\widehat{E}^X[\mathbb{I}_A|\tilde{\mathcal{G}}_t] - \frac{K}{B_T}\widehat{E}[\mathbb{I}_A|\tilde{\mathcal{G}}_t]\right)
\end{aligned}$$

(6.3)

$$= (1 + (h(\tau \wedge T) - 1)H_T)\widehat{E}^X[\mathbb{I}_A|\tilde{\mathcal{G}}_t]X_t - (1 + (h(\tau \wedge T) - 1)H_T)\frac{K}{B_T}\widehat{E}[\mathbb{I}_A|\tilde{\mathcal{G}}_t],$$

where A denotes the event $\{Y_T \geq K\}$ and by Biagini and Pratelli (1999) we have that the minimal martingale measure under the numéraire X satisfies

$$\left.\frac{d\widehat{\mathbb{P}}^X}{d\widehat{\mathbb{P}}}\right|_{\tilde{\mathcal{G}}_t} = \frac{X_T}{X_0}$$

since X is a square-integrable $\tilde{\mathbb{G}}$-martingale under $\widehat{\mathbb{P}}$. By standard delta-hedging arguments the process $\tilde{\xi}_t^H = (1 + (h(\tau \wedge T) - 1)H_T)\widehat{E}^X[\mathbb{I}_A|\tilde{\mathcal{G}}_t]$ represents the component invested in the discounted risky asset X of the replicating portfolio with respect to the filtration $\tilde{\mathbb{G}}$.
By Proposition 6.1 we only need to compute the $\mathbb{G}$-predictable projection ξ^H of the process $\tilde{\xi}^H$.

By Theorem VI.43 of Dellacherie and Meyer (1982), we need to check that for every $\mathbb{G}$-predictable $\mathbb{G}$-stopping time $\hat{\tau}$

$$\xi_{\hat{\tau}}\mathbb{I}_{\{\hat{\tau}<\infty\}} = \widehat{E}\left[(h(\tau\wedge T)-1)H_T\widehat{E}^X[\mathbb{I}_A|\tilde{\mathcal{G}}_{\hat{\tau}}]\mathbb{I}_{\{\hat{\tau}<\infty\}}\Big|\mathcal{G}_{\hat{\tau}-}\right],$$

i.e.

$$\begin{aligned}
\xi_{\hat{\tau}}\mathbb{I}_{\{\hat{\tau}<\infty\}} &= \widehat{E}\left[(h(\tau\wedge T)-1)H_T\frac{\widehat{E}[X_T\mathbb{I}_A|\tilde{\mathcal{G}}_{\hat{\tau}}]}{\widehat{E}[X_T|\tilde{\mathcal{G}}_{\hat{\tau}}]}\mathbb{I}_{\{\hat{\tau}<\infty\}}\Bigg|\mathcal{G}_{\hat{\tau}-}\right]\\
&= \widehat{E}\left[(h(\tau\wedge T)-1)H_T\frac{1}{X_{\hat{\tau}}}\widehat{E}[X_T\mathbb{I}_A|\tilde{\mathcal{G}}_{\hat{\tau}}]\mathbb{I}_{\{\hat{\tau}<\infty\}}\Bigg|\mathcal{G}_{\hat{\tau}-}\right]\\
&= \widehat{E}\left[\widehat{E}\left[(h(\tau\wedge T)-1)H_T\frac{X_T}{X_{\hat{\tau}}}\mathbb{I}_A\Bigg|\tilde{\mathcal{G}}_{\hat{\tau}}\right]\mathbb{I}_{\{\hat{\tau}<\infty\}}\Bigg|\mathcal{G}_{\hat{\tau}-}\right]\\
&= \widehat{E}\left[(h(\tau\wedge T)-1)H_T\frac{X_T}{X_{\hat{\tau}}}\mathbb{I}_A\mathbb{I}_{\{\hat{\tau}<\infty\}}\Bigg|\mathcal{G}_{\hat{\tau}-}\right]\\
&= \widehat{E}^X\left[(h(\tau\wedge T)-1)H_T\mathbb{I}_A\mathbb{I}_{\{\hat{\tau}<\infty\}}\Big|\mathcal{G}_{\hat{\tau}-}\right].
\end{aligned}$$

If we suppose that the process $\widehat{E}^X[(h(\tau\wedge T)-1)H_T\mathbb{I}_A|\mathcal{G}_{t-}]$ has a left-continuous version, then it coincides with the $\mathbb{G}$-predictable projection under the probability $\widehat{\mathbb{P}}^X$. Hence a plrm strategy for H, whose promised contingent claim $\bar{X}$ is given by the standard payoff of a call option, is given by

$$\xi_t^H = \widehat{E}^X\left[\mathbb{I}_A\left(1+\left(h(\tau\wedge T)-1\right)H_T\right)\Big|\mathcal{G}_{t-}\right],\quad t\in[0,T]. \tag{6.4}$$

Appendix

We recall briefly the definition of $\mathbb{F}$-predictable projection of a measurable process endowed with some suitable integrability properties.

Theorem 6.3 (Predictable Projection)**.** *Let X be a measurable process either positive or bounded. There exists a $\mathbb{F}$-predictable process Y such that*

$$E\left[X_\tau\mathbb{I}_{\{\tau<\infty\}}|\mathcal{F}_{\tau-}\right] = Y_\tau\mathbb{I}_{\{\tau<\infty\}}\quad a.s.$$

for every $\mathbb{F}$-predictable stopping time τ.

The process Y is called the *predictable projection* of X.

Proof. See Dellacherie and Meyer (1982) or Revuz and Yor (2005) for the proof. □

References

BIAGINI, F., AND A. CRETAROLA (2007): "Quadratic Hedging Methods for Defaultable Claims," *Applied Mathematics and Optimization*, 56(3), 425–443.

BIAGINI, F., AND M. PRATELLI (1999): "Local Risk Minimization and Numéraire," *Journal of Applied Probability*, 35(4), 1–14.

BIAGINI, F., AND Y. SUN (2007): "Local Risk Minimization for Hybrid Derivatives," Work in progress.

BIELECKI, T., AND M. JEANBLANC (2005): "Indifference Pricing of Defaultable Claims," in *Indifference Pricing, Theory and Applications*, ed. by R. Carmona. University Press, Princeton.

BIELECKI, T., M. JEANBLANC, AND M. RUTKOWSKI (2004a): "Hedging of Defaultable Claims," in *Paris-Princeton Lectures on Mathematical Finance 2003*, ed. by R. A. Carmona, E. Cinlar, I. Ekeland, E. Jouini, and N. Touzi, Lecture Notes in Mathematics, pp. 1–132. Springer-Verlag.

——— (2004b): "Pricing and Hedging of Credit Risk: Replication and Mean-Variance Approaches I," in *Mathematics of Finance*, ed. by G. Yin, and Q. Zhang, Contemporary Mathematics, pp. 37–53. AMS, Providence.

——— (2004c): "Pricing and Hedging of Credit Risk: Replication and Mean-Variance Approaches II," in *Mathematics of Finance*, ed. by G. Yin, and Q. Zhang, pp. 55–64. AMS, Providence.

——— (2006): "Hedging of Credit Derivatives in Models with Totally Unexpected Default," in *Stochastic Processes and Applications to Mathematical Finance*, ed. by J. Akahori, S. Ogawa, and S. Watanabe, pp. 35–100, Ritsumeikan University, Japan. World Scientific.

BIELECKI, T., AND M. RUTKOWSKI (2004): *Credit Risk: Modelling, Valuation and Hedging*. Springer, second edn.

DAHL, M., M. MELCHIOR, AND T. MØ LLER (2007): "On Systematic Mortaliy Risk and Risk-Minimization with Survivor Swaps," Preprint.

DAHL, M., AND T. MØ LLER (2006): "Valuation and Hedging of Life Insurance Liabilities with Systematic Mortaliy Risk," Preprint.

DELLACHERIE, C., AND P. A. MEYER (1978): *Probabilities and Potential. Chapitres XVII-XXIV*. North Holland, Amsterdam.

——— (1982): *Probabilities and Potential B: Theory of Martingales.* North Holland, Amsterdam.

DUFFIE, D. (2004): "Credit Risk Modeling with Affine Process," .

FILIPOVIĆ, D. (2006): "Interest Rate models," Lecture Notes.

FÖLLMER, H., AND M. SCHWEIZER (1991): "Hedging of Contingent Claims under Incomplete Information," in *Applied Stochastic Analysis*, ed. by S. J. Elliot, and M. Davis, pp. 389–414. Gordon and Breach.

FREY, R., T. SCHMIDT, AND A. GABIH (2007): "Pricing and Hedging of Credit Derivatives via Nonlinear Filtering," Preprint.

HEATH, D., E. PLATEN, AND M. SCHWEIZER (2001): "A Comparison of Two Quadratic Approaches to Hedging in Incomplete Markets," *Mathematical Finance*, 11, 385–413.

MØ LLER, T. (1998): "Risk-minimizing Hedging Strategies for Unit-linked Life Insurance Contracts," *ASTIN Bulletin*, 28, 17–47.

——— (2001): "Risk-Minimizing Hedging Strategies for Insurance Payment Processes," *Finance and Stochastics*, 5, 419–446.

REVUZ, D., AND M. YOR (2005): *Continuous Martingales and Brownian Motion.* Springer, third edn.

RIESNER, M. (2006): "Hedging Life Insurance Contracts in a Lévy Process Financial Market," *Insurance: Mathematics and Economics*, 38, 599–608.

——— (2007): "Locally Risk-Minimizing Hedging of Insurance Payment Streams," *ASTIN Bulletin*, 32, 67–92.

SCHWEIZER, M. (1995): "On the Minimal Martingale Measure and the Foellmer-Schweizer Decomposition," *Stochastic Analysis and Applications*, 13, 573–599.

——— (2001): "A Guided Tour through Quadratic Hedging Approaches," in *Option Pricing, Interest Rates and Risk Management*, ed. by E. Jouini, J. Cvitanic, and M. Musiela, Handbooks in Mathematical Finance, pp. 538–574. Cambridge University Press, Cambridge.

Notes

[1] We recall for reader's convenience the definition of Galtchouk-Kunita-Watanabe (GKW) decomposition: if X is a $\mathbb{P}$-martingale, any $H \in L^2(\mathcal{G}_T, \mathbb{P})$ admits a GKW decomposition with respect to X, i.e. it can be uniquely written as

$$H = E[H] + \int_0^T \xi_s^H \mathrm{d}X_s + L_T^H, \quad \mathbb{P} - a.s.,$$

for some $\mathbb{G}$-predictable process ξ^H that satisfies $E\left[\int_0^T (\xi_s^H)^2 \sigma_s^2 X_s^2 \mathrm{d}s\right] < \infty$, and some $L^H \in \mathcal{M}_0^2(\mathbb{P})$ which is strongly orthogonal to X.

[2] Since $\mathcal{G}_{\tau-} = \mathcal{F}_{\tau-} := \sigma(A \cap \{\tau > t\},\ A \in \mathcal{F}_t,\ 0 \leq t \leq T)$, we have that N is also $\mathcal{F}_{\tau-}$-measurable. Hence by Dellacherie and Meyer (1978), (68.1) page 126, there exists a $\mathbb{F}$-predictable process Z such that $Z_\tau = N$.