Risk-Minimization for Life Insurance Liabilities with Dependent Mortality Risk

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Abstract

In this paper we study the pricing and hedging of typical life insurance liabilities for an insurance portfolio with dependent mortality risk by means of the well-known risk-minimization approach. As the insurance portfolio consists of individuals of different age cohorts, in order to capture the cross-generational dependency structure of the portfolio, we introduce affine models for the mortality intensities based on Gaussian random fields that deliver analytically tractable results. We also provide specific examples consistent with historical mortality data and correlation structures. Main novelties of this work are the explicit computations of risk-minimizing strategies for life insurance liabilities written on an insurance portfolio composed of primary financial assets (a risky asset and a money market account) and a family of longevity bonds, and the simultaneous consideration of different age cohorts.

Mathematics Subject Classification (2010): 62P05; 91G80, 91G20, 62P20.
JEL Classification: C02, G19, G10.
Key words: life insurance liabilities, unit-linked life insurance, stochastic mortality, random field, dependent mortality risk, affine mortality structure, risk-minimization, martingale representation.

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The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no [228087].

1 Introduction

A large number of life insurance and pension products have mortality and longevity as primary sources of risk. Mortality risk can essentially be split into systematic risk represented by the mortality intensity, and idiosyncratic or unsystematic risk, i.e., mortality risk related to individual mortality rates. As, on a global basis, there is inadequate reinsurance capacity to effectively address the different types of mortality risks and as systematic mortality risk cannot be diversified away by pooling, securitization has emerged as a new form of risk transfer and led to the creation of a new life market (see, e.g., Blake et al. [11]). In this context, pricing and modeling of life insurance products has been studied extensively in the literature (see, e.g., Cairns et al. [14]).
Systematic mortality risk may be hedged by investing in longevity bonds (see, e.g., Cairns et al. [14]) that pay out the conditional survival probability at maturity and are based on so-called longevity or survivor indices. Survivor indices, provided by various investment banks, consist of publicly available mortality data aggregated by population, and are thus widely accepted as good proxies for the systematic component of the mortality risk. One of the main features of our approach is to allow for hedging of the risk inherent in the life insurance liabilities by investing not only in a risky asset (e.g., a stock index) and a money market account, but also in a family of longevity bonds, accounting for the systematic mortality risk.
In practice, one typically considers homogeneous classes of policyholders and then aggregates market valuations of liabilities at the portfolio level without taking dependencies between cohort classes into account. To the best of our knowledge, there are only few studies concerned with quantifying and modeling inter-age dependencies in stochastic mortality models. Based on a multivariate time series study of yearly mortality rates, Loisel and Serrant [34] propose a discrete-time multi-dimensional extension of the well-known Lee-Carter model, that takes inter-age correlations into account. Jevtic et al. [28] propose affine continuous-time factor models for the mortality surface, allowing for correlation across different generations. Biffis and Millossovich [10] model the mortality intensity surface as a random field and, with a view on the insurer's future business, consider market valuations of pure endowment contracts with deterministic survival benefit. Random fields have been employed in mathematical finance when modeling the term structure of interest rates (see, e.g., Goldstein [24] and Kennedy [30]) and have proven to be useful in our context as well. Similarly as Biffis and Millossovich [10], we model the mortality surface as a random field parameterized in time and age at inception of the contract. In a complex setting, with a portfolio consisting of different age cohorts, we study risk-minimization at an aggregate level for a general class of life insurance contracts based on the three building blocks, term insurance, annuity and pure endowment contract. By modeling the mortality intensities as a random surface we are able to look simultaneously in both the time and age direction. This is important, because there is statistical evidence that typical downward mortality improvement trends are not homogeneous across age

cohorts (see, e.g., Andreev [2] and Forfar and Smith [22]). Besides that, this approach enables us to establish a mortality model consistent with historical data that takes inter-age correlations into account in a natural and elegant way. As the mortality intensity of every age cohort is an affine process, the model is analytically tractable, allowing us to compute hedging strategies for life insurance liabilities in an immediate and parsimonious way. Affine models have become very popular in many areas of applied financial mathematics, such as exotic option pricing, or interest rate and credit risk modeling. An overview of the theory of affine processes can be found in Duffie et al. [20], as well as in Filipović and Mayerhofer [21] for the case of affine diffusions.

When modeling life insurance liabilities, we make use of the similarities between mortality and credit risk and follow the intensity-based or hazard rate approach of reduced-form modeling, see, e.g., Bielecki and Rutkowski [8]. As it is impossible to completely hedge the financial and mortality risk inherent in the liabilities of the insurance company, the market is incomplete and it is thus necessary to select one of the techniques for pricing and hedging in incomplete markets. Here we make use of the popular risk-minimization method first introduced by Föllmer and Sondermann [23]. The idea of this technique is to allow for a wide class of admissible strategies, that in general might not necessarily be self-financing, and to find an optimal hedging strategy with "minimal risk" within this class of strategies that perfectly replicates the given claim. For a survey on risk-minimization and other quadratic hedging methods we refer to Schweizer [42].

Several studies focus on applications of the risk-minimization approach in the context of mortality modeling, see, e.g., Barbarin [3], Biagini et al. [4, 5, 6], Møller et al. [16, 17, 36, 37] and Riesner [40]. However, some authors such as Møller [36, 37] and Riesner [40] assume independence between the financial market and the insurance model. Here, we work in a more general setting, i.e., we allow for mutual dependence between the times of death and the financial market. Besides that, as in Biagini et al. [4, 5, 6] and Dahl et al. [17], we allow for hedging by investing not only in the primary financial market, but also in hedging instruments related to the systematic mortality risk of the insurance liabilities.

Hence, in this paper, we extend earlier work on risk-minimization for insurance products in several directions. First, we provide explicit computations of risk-minimizing strategies for life insurance liabilities written on an insurance portfolio in a complex setting, incorporating different age cohorts simultaneously. Second, we take into account and explicitly model the dependency structure of the insurance portfolio by introducing analytically tractable affine models for the mortality intensities, consistent with historical mortality data, based on Gaussian random fields. Finally, we allow for hedging by investing in a family of longevity bonds representing the systematic mortality risk and we do not require certain technical assumptions such as the independence of the financial market and the insurance model.

The remainder of this paper is organized as follows: Section 2 introduces the general setup, including the structure of the insurance portfolio and the financial market. In Section 3, we propose two illustrative examples of intensity field models consistent with characteristics of typical mortality data. In Section 4, we compute risk-minimizing strategies of the life insurance liabilities at an aggregate portfolio level. Section 5 then provides numerical examples including illustrative plots for the case of a Gaussian intensity field. Finally, Section 6 follows with concluding remarks.

2 The setting

Let $T > 0$ be a fixed finite time horizon and $(\Omega, \mathcal{G}, \mathbb{P})$ a probability space equipped with a filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ which contains all available information. We define $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, and put $\mathcal{G} = \mathcal{G}_T$, where $\mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}$ is generated by the death counting processes of the insurance portfolio (see Subsection 2.1). The *background filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ contains all information available, except the information regarding the individual survival times. Here we define $\mathbb{F} = \mathbb{F}^X \vee \mathbb{F}^\mu$, where $\mathbb{F}^X$ is the filtration containing information regarding a risky asset, e.g., a stock (see Subsection 2.2), $\mathbb{F}^\mu$ is the filtration containing information regarding the mortality intensities (see Subsection 2.1) and we assume that $\mathbb{F}^X$ and $\mathbb{F}^\mu$ are independent. In the subsequent sections, we introduce the three components of the model: the insurance portfolio, the financial market and the combined model.

2.1 Insurance portfolio and mortality intensities

We consider an insurance portfolio consisting of n individuals belonging to a set of age cohorts $B = \{x_1, \dots, x_m\} \subseteq I$, where the interval $I = [\underline{x}, \overline{x}]$ is assumed to be a given range of possible ages of individuals at time 0, with natural lower and upper bounds $\underline{x}$, $\overline{x} > 0$. Also note that $m \leq n$, in particular if $m = 1$ all individuals belong to the same age cohort, whereas if $m = n$ all individuals belong to different age cohorts. As in Biffis and Millossovich [10], we define a function $n^\cdot : B \to \mathbb{N}$, such that the quantity n^x represents the number of insureds belonging to the age cohort x, i.e., $\sum_{i=1}^m n^{x_i} = n$. For $x \in B$ and $j = 1, \dots, n^x$, we model the *residual lifetime* of the j-th insured person within the age cohort x as a $\mathbb{G}$-stopping time $\tau^{x,j} : \Omega \to [0, T] \cup \{\infty\}$ and assume that $\mathbb{P}(\tau^{x,j} = 0) = 0$ and $\mathbb{P}(\tau^{x,j} > t) > 0$ for $t \in [0, T]$. Because the time horizon T is usually fixed as the maturity of the life insurance liabilities, in order to ensure that $\mathbb{P}(\tau^{x,j} > T) > 0$ for $x \in B$ and $j = 1, \dots, n^x$ (the remaining lifetimes are not necessarily bounded by T), it is necessary to allow $\tau^{x,j}$ to take values larger than T. The convention that $\tau^{x,j}$ can assume the value infinity is adopted for that purpose. We define $\mathcal{H}_t = \vee_{x\in B}\mathcal{H}_t^x$ with $\mathcal{H}_t^x = \mathcal{H}_t^{x,1} \vee \cdots \vee \mathcal{H}_t^{x,n^x}$, where $\mathcal{H}_t^{x,j} = \sigma\{H_s^{x,j} : 0 \leq s \leq t\}$ and $H_t^{x,j} = \mathbb{1}_{\{\tau^{x,j} \leq t\}}$ for $t \in [0, T]$, $x \in B$ and $j = 1, \dots, n^x$. Furthermore, we consider a finite measure ζ on $(B, \mathcal{P}(B))$, where $\mathcal{P}(B)$ denotes the power set of B, allowing us to weight the subsets of B differently. Then

$$\int_B n^x \, \zeta(\mathrm{d}x) = \sum_{i=1}^{m} n^{x_i} \zeta(x_i)$$

represents the weighted dimension of the portfolio B. The weights $\zeta(x_i)$, $i = 1, \dots, m$, can be interpreted as rating factors for risk classification. In a more general setting, they may depend on several factors.
The death counting process associated with the age cohort $x \in B$ is given by

$$N_t^x = \sum_{j=1}^{n^x} \mathbb{1}_{\{\tau^{x,j} \le t\}}, \quad t \in [0,T], \; x \in B.$$

Then the weighted random number of insureds alive at time t in the portfolio is

$$\int_B (n^x - N_t^x) \, \zeta(\mathrm{d}x) = \sum_{i=1}^{m} \sum_{j=1}^{n^{x_i}} \mathbb{1}_{\{\tau^{x_i,j} > t\}} \zeta(x_i), \quad t \in [0,T].$$

For $x \in B$ and $j = 1, \dots, n^x$, we assume that the times of death $\tau^{x,j}$ are totally inaccessible $\mathbb{G}$-stopping times. An important role is then played by the conditional distribution function of $\tau^{x,j}$, given by

$$F_t^{x,j} = \mathbb{P}(\tau^{x,j} \le t \,|\, \mathcal{F}_t), \quad t \in [0,T],$$

and we assume $F_t^{x,j} < 1$ for all $t \in [0,T]$. Then, the *hazard process* $\Gamma^{x,j}$ of $\tau^{x,j}$

$$\Gamma_t^{x,j} = -\ln(1 - F_t^{x,j}) = -\ln \mathbb{E}[\mathbb{1}_{\{\tau^{x,j} > t\}} \,|\, \mathcal{F}_t], \quad t \in [0,T],$$

is well-defined for every $t \in [0,T]$. For $x \in B$, we define $\Gamma^x := \Gamma^{x,j}$ for $j = 1, \dots, n^x$, i.e., all individuals of the same age cohort have the same hazard process. Moreover, we assume that Γ^x admits a *mortality intensity* μ^x, i.e.

$$\Gamma_t^x = \int_0^t \mu_s^x \, \mathrm{d}s, \quad t \in [0,T], \tag{2.1}$$

where $\mu = (\mu_{t,x})_{(t,x) \in [0,T] \times I}$, is a random field generated by a Brownian sheet $W = (W_{t,x})_{(t,x) \in [0,T] \times I}$ (see, e.g., Adler [1] for an overview of the theory of random fields). Note that, for $t \in [0,T]$ and $x \in I$, we write $\mu_{t,x}$ instead of μ_t^x to emphasize that for fixed $x \in I$ we are integrating in the t-direction (see, e.g., Lemma 3.1). For $t \in [0,T]$ and $x \in I$, the natural filtration of the Brownian sheet W is given by $\mathcal{F}_{t,x}^\mu := \sigma\{W_{s,v} : 0 \le s \le t, \; \underline{x} \le v \le x\}$, and we define $\mathbb{F}^\mu = (\mathcal{F}_t^\mu)_{t \in [0,T]}$, where $\mathcal{F}_t^\mu := \{\mathcal{F}_{t,x}^\mu : \underline{x} \le x \le \overline{x}\} = \vee_{x \in I} \mathcal{F}_{t,x}^\mu$. For fixed $x \in I$, we assume that the process $(\mu_t^x)_{t \in [0,T]}$ is an affine diffusion process (see also Section 3), which facilitates the related computations in Section 4. The process μ^x represents the mortality intensity of the age cohort $x \in I$ and can be derived by means of publicly available data of the *survivor index*

$$S_t^{\mu^x} = \exp\left(-\int_0^t \mu_s^x \, \mathrm{d}s \right), \quad t \in [0,T], \; x \in I. \tag{2.2}$$

The need for standardization in the life markets has led to the creation of such indices aggregated for different age cohorts and populations by investment banks. According to Cairns et al. [14], survivor indices can be seen as basic building blocks for many mortality-linked securities, see also the definition of a longevity bond for age cohort $x \in I$ in Subsection 2.2. This modeling approach enables us to not only capture the dependency structure in the t-direction, but also in the x-direction and additionally takes into account the cross-generational correlation of the insurance portfolio. In Section 3, we provide explicit specifications for μ that are consistent with typical characteristics of historical mortality data (see, e.g., Andreev [2]) in the sense that e.g., for fixed $x \in I$, $(\mu_{t,x})_{t\in[0,T]}$ is decreasing in t (downward mortality trend) and for fixed $t \in [0,T]$, $(\mu_{t,x})_{x\in I}$ is increasing in x.
For $x, y \in B$ and $i = 1, \dots, n^x$, $j = 1, \dots, n^y$ with $(x,i) \neq (y,j)$, we also assume

$$\mathbb{E}[\mathbb{1}_{\{\tau^{x,i}>t\}}\mathbb{1}_{\{\tau^{y,j}>s\}} \,|\, \mathcal{F}_T] = \mathbb{E}[\mathbb{1}_{\{\tau^{x,i}>t\}} \,|\, \mathcal{F}_T]\, \mathbb{E}[\mathbb{1}_{\{\tau^{y,j}>s\}} \,|\, \mathcal{F}_T], \quad 0 \leq s,t \leq T, \tag{2.3}$$

i.e., we assume conditional independence for individuals in different age cohorts as well as for individuals within the same age cohort. This assumption is well-known in the literature on credit risk modeling, see, e.g., Chapter 9 of Bielecki and Rutkowski [8]. All individuals within the insurance portfolio are subject to idiosyncratic risk factors, as well as common risk factors, given by the information within the background filtration $\mathbb{F}$. Intuitively, the assumption of conditional independence means that if the evolution of all common risk factors is known, the idiosyncratic risk factors become independent of each other.

2.2 The financial market

We consider a financial market defined on $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ consisting of a bank account or numéraire B with constant short rate $r > 0$, i.e.

$$B_t = \exp\{rt\}, \quad t \in [0,T], \tag{2.4}$$

as well as a risky asset, e.g., a stock, with asset price S and a family of *longevity bonds* (P^x), $x \in I$. We assume that S follows the $\mathbb{P}$-dynamics

$$\mathrm{d}S_t = S_t\left(r\mathrm{d}t + \sigma(t,S_t)\,\mathrm{d}W_t^X\right), \quad t \in [0,T], \tag{2.5}$$

for a Brownian motion $W^X = (W_t^X)_{t\in[0,T]}$ with $S_0 = s$, and that σ satisfies certain regularity conditions that ensure the existence and uniqueness of a solution to (2.5). We denote by $X = S/B$ the discounted asset price, i.e., the dynamics of X are given by

$$\mathrm{d}X_t = \mathrm{d}\left(\frac{S_t}{B_t}\right) = \sigma(t,S_t)X_t\,\mathrm{d}W_t^X, \quad t \in [0,T], \tag{2.6}$$

and define $\mathbb{F}^X = \left(\mathcal{F}_t^X\right)_{t\in[0,T]}$, with $\mathcal{F}_t^X := \sigma\{W_s^X : 0 \leq s \leq t\}$. Following Cairns et al. [14], for $x \in I$, we introduce the *longevity bond* P^x with maturity T in order to hedge the systematic mortality risk inherent in the life insurance contracts for the age cohort x, i.e., P^x is defined as a zero-coupon bond that pays out the value of the survivor or longevity index as defined in (2.2) at T. This means the discounted value process $Y^x = P^x/B$ is given by

$$Y_t^x = \mathbb{E}\left[\frac{S_T^{\mu^x}}{B_T} \,\middle|\, \mathcal{G}_t\right], \quad t \in [0,T],\ x \in I. \tag{2.7}$$

Thus, the discounted asset prices $X, (Y^x)_{x\in I}$ are (local) $(\mathbb{P}, \mathbb{F})$-martingales, i.e., the financial market is arbitrage-free and the physical measure $\mathbb{P}$ belongs to the set of equivalent local martingale measures.

2.3 The combined model

We consider the extended market $\mathbb{G} = \mathbb{F} \vee \mathbb{H} = \mathbb{F}^X \vee \mathbb{F}^\mu \vee \mathbb{H}$, such that the information available at time $t \in [0,T]$ is assumed to be $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \mathcal{F}_t^X \vee \mathcal{F}_t^\mu \vee \mathcal{H}_t$. All filtrations are assumed to satisfy the usual hypotheses of completeness and right-continuity. We postulate that all $\mathbb{F}$-local martingales are also $\mathbb{G}$-local martingales, and in the sequel, we refer to this hypothesis as Hypothesis (H). This hypothesis is well-known in the literature on enlargements of filtrations, see Blanchet-Scalliet and Jeanblanc [12] and Bielecki and Rutkowski [8, Chapter 6]. In this setting, we study unit-linked life insurance liabilities in the form of insurance payment streams as introduced by Møller [37]. It is now widely acknowledged (see, e.g., Barbarin [3], Biffis [9] and Møller [36]) that most payment streams of practical relevance are covered by the three building blocks consisting of term insurance, annuity and pure endowment contracts. Following Barbarin [3] and Møller [36], the *term insurance contract* is defined by the following portfolio payoff structure

$$\int_B \sum_{j=1}^{n^x} f(S_{\tau^{x,j}}) \mathbb{1}_{\{\tau^{x,j} \le T\}} \zeta(\mathrm{d}x) = \sum_{i=1}^{m} \sum_{j=1}^{n^{x_i}} f(S_{\tau^{x_i,j}}) \mathbb{1}_{\{\tau^{x_i,j} \le T\}} \zeta(x_i), \tag{2.8}$$

where f is a non-negative function fulfilling certain regularity conditions, i.e., the amount $f(S_{\tau^{x_i,j}})$ weighted by the measure ζ is payed at the time of death $\tau^{x_i,j}$ to policyholder j within the age cohort x_i, $i = 1,\ldots,m$, $j = 1,\ldots,n^{x_i}$. The *annuity contract* consists of multiple payoffs as functions of the asset price, which the insurer has to pay as long as the policyholders are alive, i.e.

$$\int_B \left(\int_0^T (n^x - N_s^x)\, f(S_s)\, \mathrm{d}s \right) \zeta(\mathrm{d}x) = \sum_{i=1}^{m} \sum_{j=1}^{n^{x_i}} \int_0^T \mathbb{1}_{\{\tau^{x_i,j} > s\}} f(S_s)\, \mathrm{d}s\, \zeta(x_i), \tag{2.9}$$

where weighting of the different age cohorts is again enabled through the measure ζ. The *pure endowment contract* consists of the following portfolio payoff

$$\int_B \sum_{j=1}^{n^x} f(S_T) \mathbb{1}_{\{\tau^{x,j} > T\}} \zeta(\mathrm{d}x) = \sum_{i=1}^{m} \sum_{j=1}^{n^{x_i}} f(S_T) \mathbb{1}_{\{\tau^{x_i,j} > T\}} \zeta(x_i) \tag{2.10}$$

at time T, i.e., the insurer pays the aggregate amount $f(S_T)$ to every policyholder in the portfolio B who has survived until T, weighted by the measure ζ. As the payoff functions in (2.8), (2.9) and (2.10) are homogeneous with respect to the amounts insured for every age cohort, the weighting measure ζ allows for payouts that differ between different age cohorts. We would also like to refer to Section 5 for specific choices of ζ. In the following section, we specify examples for the intensity field model introduced in (2.1), consistent with characteristics of historical mortality data.

3 Intensity field model

It is now widely acknowledged (see, e.g., Andreev [2] and Forfar and Smith [22]) that downward mortality trends are not uniform across ages. In this context, modeling mortality intensities by means of random fields appears as a natural modeling choice. This approach enables us to look simultaneously at the evolution of death probabilities over time for a given age, death probabilities across all ages at a given time and death probabilities over time for people born in the same year. This section provides two specific examples of affine intensity field models for μ defined in (2.1), a Gaussian random field (see Subsection 3.1) with nice analytical properties and intuitive interpretation, as well as a χ^2-random field (see Subsection 3.2) that has the advantage of restraining the mortality intensities to non-negative values.

3.1 Gaussian intensity field

Define

$$\mu_{t,x} = \bar{\mu}(t,x) + O_{t,x}, \quad t \in [0,T],\ x \in I, \tag{3.1}$$

where $\bar{\mu}$ is a deterministic function, differentiable in x and t, and O is a space-time changed Brownian sheet, i.e.,

$$O_{t,x} = \frac{\sigma}{\sqrt{2\theta\alpha}} e^{-\theta t} e^{-\alpha x} W_{\nu_1(t),\nu_2(x)}, \quad t \in [0,T],\ x \in I, \tag{3.2}$$

with

$$\nu_1(t) = e^{2\theta t}, \quad \nu_2(x) = e^{2\alpha x}, \quad t \in [0,T],\ x \in I, \tag{3.3}$$

for α, $\theta > 0$. In particular, we assume that $\nu_1 : [0,T] \to [1, e^{2\theta T}]$ and $\nu_2 : [\underline{x}, \overline{x}] \to [e^{2\alpha\underline{x}}, e^{2\alpha\overline{x}}]$. Intuitively, $\mu_{t,x}$ fluctuates around the deterministic mortality level $\bar{\mu}$. We would like to obtain a model for μ that is consistent with typical characteristics of historical mortality data, i.e., for fixed x, $(\mu_{t,x})_{t\in[0,T]}$ is decreasing in t and for fixed t, $(\mu_{t,x})_{x\in I}$ is increasing in x. These properties can be directly imposed on the deterministic function $\bar{\mu}(t,x)$. For example, $\bar{\mu}(t,x)$ could be given by the well-known Lee-Carter model (see, e.g., Lee and Carter [33] and Lee [32])

$$\bar{\mu}(t,x) = \exp\left(a(x) + b(x)k(t)\right), \quad t \in [0,T],\ x \in I. \tag{3.4}$$

For further extensions of the Lee-Carter model, we refer to Renshaw and Haberman [39], Brouhns et al. [13] and Denuit and Dhaene [18]. Note that, in (3.4), a is a negative increasing function such that $e^{a(x)}$ represents the general shape of the mortality curve at age x, k is a real-valued decreasing function representing the downward trend in time of the logarithm of the force of mortality. The non-negative function b represents the sensitivity of the logarithm of the force of mortality at age x to variations in k and allows us to model this trend heterogeneously over cohorts (see, e.g., Andreev [2]). For example, if b is decreasing for high values of x, it implies that mortality improvements are lower for older ages, as suggested by Forfar and Smith [22]. Note that, for $t \in [0,T]$, $x \in I$,

$$\mathbb{E}[\mu_{t,x}] = \bar{\mu}(t,x), \quad t \in [0,T],\ x \in I,$$

and as the covariance function of a Brownian sheet $W = (W_t)_{t\in\mathbb{R}_+^N}$ is given by

$$c(s,t) = \prod_{i=1}^{N} (s_i \wedge t_i), \quad t = (t_1,\dots,t_N),\ s = (s_1,\dots,s_N) \in \mathbb{R}_+^N, \tag{3.5}$$

we obtain

$$\begin{aligned} \mathrm{Cov}(\mu_{t,x},\mu_{s,y}) &= \frac{\sigma^2}{2\alpha\theta} e^{-\theta(t+s)} e^{-\alpha(x+y)} \mathrm{Cov}\left(W_{\nu_1(t),\nu_2(x)}, W_{\nu_1(s),\nu_2(y)}\right) \\ &= \frac{\sigma^2}{2\alpha\theta} e^{-\theta(t+s)} e^{2\theta(t\wedge s)} e^{-\alpha(x+y)} e^{2\alpha(x\wedge y)} \\ &= \frac{\sigma^2}{2\alpha\theta} e^{-\theta|t-s|} e^{-\alpha|x-y|}, \quad s,t \in [0,T],\ x,y \in I, \end{aligned} \tag{3.6}$$

as well as

$$\mathrm{Corr}(\mu_{t,x},\mu_{s,y}) = e^{-\theta|t-s|} e^{-\alpha|x-y|}, \quad s,t \in [0,T],\ x,y \in I, \tag{3.7}$$

i.e., correlation is positive, symmetric and exponentially declining. The next lemma provides a stochastic representation of μ in the t-direction.

Lemma 3.1. *For μ as defined in* (3.1) *and fixed $x \in I$, we have that*

$$\mu_t^x = \bar{\mu}(t,x) + e^{-\theta t}(\mu_0^x - \bar{\mu}(0,x)) + \frac{\sigma}{\sqrt{\alpha}} \int_0^t e^{-\theta(t-s)} \mathrm{d}\tilde{W}_s^{\nu_2(x)}, \quad t \in [0,T],$$

where

$$\tilde{W}_t^{\nu_2(x)} := \frac{W_{t,\nu_2(x)}}{\sqrt{\nu_2(x)}}, \quad t \in [0,T], \tag{3.8}$$

is a standard Brownian motion. The set $(\mu^x)_{x\in I}$ is a family of affine diffusion processes, i.e., for fixed $x \in I$, the dynamics of $(\mu_t^x)_{t\in[0,T]}$ are given by

$$\mathrm{d}\mu_t^x = \theta\left[\left(\bar{\mu}(t,x) + \frac{\partial_t \bar{\mu}(t,x)}{\theta}\right) - \mu_t^x\right]\mathrm{d}t + \frac{\sigma}{\sqrt{\alpha}} \mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad t \in [0,T], \tag{3.9}$$

Proof. Fix $x \in I$ and define

$$O_t^x = O_{t,x} = \frac{\sigma}{\sqrt{2\theta\alpha}} e^{-\theta t} \tilde{W}_{\nu_1(t)}^{\nu_2(x)}, \quad t \in [0, T].$$

Here, as for the random field μ introduced in (2.1), for $t \in [0, T]$ and $x \in I$, we write $O_{t,x}$ interchangeably with O_t^x, to emphasize that, for fixed $x \in I$, we are integrating in the t-direction. Then, $\mathbb{E}[O_t^x] = 0$ and, from (3.6), we have $\mathrm{Cov}(O_t^x, O_s^x) = \frac{\sigma^2}{2\theta\alpha} e^{-\theta|t-s|}$. We now show that O^x is a stationary Ornstein-Uhlenbeck (OU) process with dynamics

$$\mathrm{d}O_t^x = -\theta O_t^x \mathrm{d}t + \frac{\sigma}{\sqrt{\alpha}} \mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad t \in [0, T], \tag{3.10}$$

and $O_0^x \sim \mathcal{N}(0, \frac{\sigma^2}{2\theta\alpha})$. To this end, consider a process $\bar{O}^x$ that solves the stochastic differential equation (3.10) such that $\bar{O}_0^x \sim \mathcal{N}(0, \frac{\sigma^2}{2\theta\alpha})$. It is easily seen (see, e.g., Example 6.8 in Chapter 5 of Karatzas and Shreve [29]), that $\bar{O}^x$ is given by

$$\bar{O}_t^x = e^{-\theta t} \bar{O}_0^x + \frac{\sigma}{\sqrt{\alpha}} \int_0^t e^{-\theta(t-s)} \mathrm{d}\tilde{W}_s^{\nu_2(x)}, \quad t \in [0, T],$$

i.e., $\mathbb{E}[\bar{O}_t^x] = 0$ and

$$\mathrm{Cov}(\bar{O}_t^x, \bar{O}_s^x) = e^{-\theta(t+s)} \left(\frac{\sigma^2}{2\theta\alpha} \left(e^{2\theta(t \wedge s)} - 1 \right) + \mathrm{Var}(\bar{O}_0^x) \right) = \frac{\sigma^2}{2\theta\alpha} e^{-\theta|t-s|}$$

for $s, t \in [0, T]$. As O^x and $\bar{O}^x$ are Gaussian processes with the same covariance structure, it follows that O^x solves (3.10) and

$$O_t^x = e^{-\theta t} O_0^x + \frac{\sigma}{\sqrt{\alpha}} \int_0^t e^{-\theta(t-s)} \mathrm{d}\tilde{W}_s^{\nu_2(x)}, \quad t \in [0, T],$$

with $O_0^x \sim \mathcal{N}(0, \frac{\sigma^2}{2\alpha\theta})$. Then, by (3.1), it follows that

$$\mu_t^x = \bar{\mu}(t, x) + e^{-\theta t}(\mu_0^x - \bar{\mu}(0, x)) + \frac{\sigma}{\sqrt{\alpha}} \int_0^t e^{-\theta(s-t)} \mathrm{d}\tilde{W}_s^{\nu_2(x)}, \quad t \in [0, T],$$

and, by (3.1), (3.10), and Itô's lemma we get the following dynamics for μ^x:

$$\begin{aligned} \mathrm{d}\mu_t^x &= \partial_t \bar{\mu}(t, x) \mathrm{d}t + \mathrm{d}O_t^x \\ &= \theta \left[\left(\bar{\mu}(t, x) + \frac{\partial_t \bar{\mu}(t, x)}{\theta} \right) - \mu_t^x \right] \mathrm{d}t + \frac{\sigma}{\sqrt{\alpha}} \mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad t \in [0, T]. \end{aligned}$$

□

In the next lemma we compute the sharp bracket or quadratic covariation process of μ^x and μ^y for x, $y \in I$.

Lemma 3.2. *Let μ be given by the Gaussian intensity field model introduced in (3.1). Then, for fixed x, $y \in I$, the sharp bracket process of $\mu^x = (\mu_t^x)_{t\in[0,T]}$ and $\mu^y = (\mu_t^y)_{t\in[0,T]}$ is*

$$\langle \mu^x, \mu^y \rangle_t = \frac{\sigma^2}{\alpha} e^{-\alpha|x-y|} t, \quad t \in [0,T].$$

Proof. Fix $x, y \in I$ and let $\nu_2(\cdot)$ be given as in (3.3). For $0 \le s \le t \le T$, we have

$$\begin{aligned}
\mathbb{E}\left[W_t^{\nu_2(x)} W_t^{\nu_2(y)} \Big| \mathcal{F}_s^\mu \right] &= \mathbb{E}\Big[\left(W_t^{\nu_2(x)} - W_s^{\nu_2(x)} \right) \left(W_t^{\nu_2(y)} - W_s^{\nu_2(y)} \right) \Big| \mathcal{F}_s^\mu \Big] \\
&\quad + \mathbb{E}\Big[\left(W_t^{\nu_2(x)} - W_s^{\nu_2(x)} \right) W_s^{\nu_2(y)} + W_s^{\nu_2(x)} \left(W_t^{\nu_2(y)} - W_s^{\nu_2(y)} \right) \Big| \mathcal{F}_s^\mu \Big] \\
&\quad + \mathbb{E}\Big[W_s^{\nu_2(x)} W_s^{\nu_2(y)} \Big| \mathcal{F}_s^\mu \Big] \\
&= \mathbb{E}\left[\left(W_t^{\nu_2(x)} - W_s^{\nu_2(x)} \right) \left(W_t^{\nu_2(y)} - W_s^{\nu_2(y)} \right) \right] + W_s^{\nu_2(x)} W_s^{\nu_2(y)} \qquad (3.11) \\
&= t(\nu_2(x) \wedge \nu_2(y)) - s(\nu_2(x) \wedge \nu_2(y)) + W_s^{\nu_2(x)} W_s^{\nu_2(y)}, \qquad (3.12)
\end{aligned}$$

where in (3.11), we used the fact that, for fixed $z \in I$, $W_s^{\nu_2(z)}$ is $\mathcal{F}_s^\mu$-measurable and $W_t^{\nu_2(z)} - W_s^{\nu_2(z)}$ is independent of $\mathcal{F}_s^\mu$, and (3.12) is a consequence of the covariance structure of the Brownian sheet, see also equation (3.5). Here again, for $t \in [0,T]$ and $x \in I$, we write $W_{t,x}$ interchangeably with W_t^x. It follows that

$$\mathbb{E}\left[W_t^{\nu_2(x)} W_t^{\nu_2(y)} - t\left(\nu_2(x) \wedge \nu_2(y)\right) \Big| \mathcal{F}_s^\mu \right] = W_s^{\nu_2(x)} W_s^{\nu_2(y)} - s\left(\nu_2(x) \wedge \nu_2(y)\right),$$

i.e., for the two martingales $\tilde{W}^{\nu_2(x)}$ and $\tilde{W}^{\nu_2(y)}$ introduced in (3.8), we have that

$$\mathbb{E}\left[\tilde{W}_t^{\nu_2(x)} \tilde{W}_t^{\nu_2(y)} - \frac{\nu_2(x) \wedge \nu_2(y)}{\sqrt{\nu_2(x)\nu_2(y)}}\, t \Bigg| \mathcal{F}_s^\mu \right] = \tilde{W}_s^{\nu_2(x)} \tilde{W}_s^{\nu_2(y)} - \frac{\nu_2(x) \wedge \nu_2(y)}{\sqrt{\nu_2(x)\nu_2(y)}}\, s.$$

Hence, by Theorem 4.2 in Section 4 of Jacod and Shiryaev [27, Chapter I], we have

$$\left\langle \tilde{W}^{\nu_2(x)}, \tilde{W}^{\nu_2(y)} \right\rangle_t = \frac{\nu_2(x) \wedge \nu_2(y)}{\sqrt{\nu_2(x)\nu_2(y)}}\, t = e^{-\alpha|x-y|} t, \qquad (3.13)$$

and, by (3.9), for fixed x, $y \in I$, we obtain that

$$\langle \mu^x, \mu^y \rangle_t = \frac{\sigma^2}{\alpha} \left\langle \tilde{W}^{\nu_2(x)}, \tilde{W}^{\nu_2(y)} \right\rangle_t = \frac{\sigma^2}{\alpha} e^{-\alpha|x-y|} t.$$

□

We would like to conclude this subsection with a short remark on a drawback of the Gaussian framework. Mortality intensities are (by definition) non-negative processes. Unfortunately, the Gaussian intensity model, although very convenient due to its simplicity, analytical tractability and intuitive interpretation, allows for negative values with positive probability. However, although one cannot exclude negative mortality rates, the probability of negative values tends to be very small for some choices of parameters (see Section 5). A detailed discussion on this issue is in Luciano and Vigna [35], where a statistical study shows that, in practical applications, the probability of negative values turns out to be negligible when using calibrated parameters. In the next section, we introduce a χ^2-intensity field model that overcomes this drawback by restricting the mortality intensities to non-negative values.

3.2 χ^2-field

Recall that a χ^2-*field* $Y = (Y_t)_{t\in J}$, $J \subseteq \mathbb{R}_+^N$, with parameter $d \in \mathbb{N}$, is generated by Gaussian random fields by means of a positive transformation:

$$Y_t := (Z_t^1)^2 + \cdots + (Z_t^d)^2, \quad t \in J,$$

where $Z^1, \ldots, Z^d$ are independent, stationary centered Gaussian random fields with common covariance function $c(h)$, $h \in J$, and variance $c(0) = \sigma^2$, $\sigma > 0$ (see, e.g., Adler [1]). For each $t \in J$, the random variable Y_t has χ^2-distribution with d degrees of freedom. It is easily seen that

$$\mathbb{E}[Y_t] = d\sigma^2, \quad t \in J. \tag{3.14}$$

The covariance structure of Y is given by

$$\mathrm{Cov}(Y_s, Y_t) = 2dc^2(s,t) \quad \text{and} \quad \mathrm{Var}(Y_t) = 2d\sigma^4, \tag{3.15}$$

for $s, t \in J$, where $c(s,t)$ is the covariance function of the Gaussian fields Z^i, $i = 1, \ldots, d$ (see, e.g., Adler [1]). In order to overcome the drawback of negative mortality intensities, this subsection models the mortality intensities as a non-negative χ^2-random field, by defining

$$\mu_{t,x} = \left(c(t,x)O_{t,x}\right)^2, \quad t \in [0,T],\ x \in I, \tag{3.16}$$

where $c(t,x)$ is a continuously differentiable function in both t and x and O is defined in (3.2), $t \in [0,T]$, $x \in I$. From (3.14) and (3.15), we have that

$$\mathbb{E}[\mu_{t,x}] = c^2(t,x)\mathrm{Cov}(O_{t,x}, O_{t,x}) = c^2(t,x)\frac{\sigma^2}{2\theta\alpha},$$

for $t \in [0,T]$ and $x \in I$ and

$$\begin{aligned}\mathrm{Cov}(\mu_{t,x}, \mu_{s,y}) &= 2c^2(t,x)c^2(s,y)\mathrm{Cov}(O_{t,x}, O_{s,y})^2 \\ &= \frac{\sigma^4}{2\theta^2\alpha^2}c^2(t,x)c^2(s,y)e^{-2\theta|t-s|}e^{-2\alpha|x-y|},\end{aligned}$$

i.e.,

$$\mathrm{Corr}(\mu_{t,x}, \mu_{s,y}) = e^{-2\theta|t-s|}e^{-2\alpha|x-y|},$$

for $s, t \in [0,T]$ and $x, y \in I$. In particular, the correlation function of the χ^2-field is the square of the correlation function of the Gaussian field, thus featuring the same properties as discussed in Subsection 3.1.

Lemma 3.3. *For μ as defined in* (3.16)*, the set $(\mu^x)_{x\in I}$ is a family of affine diffusion processes, i.e., for fixed $x \in I$ the dynamics of $(\mu_t^x)_{t\in[0,T]}$ are given by*

$$\mathrm{d}\mu_t^x = 2\left(\theta - \frac{\partial_t c(t,x)}{c(t,x)}\right)\left(\frac{\sigma^2}{2\alpha}\bar{c}(t,x) - \mu_t^x\right)\mathrm{d}t + \sqrt{\frac{4}{\alpha}\sigma^2 c^2(t,x)\mu_t^x}\,\mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad (3.17)$$

for $t \in [0,T]$, where $\bar{c}(t,x) = \frac{c^3(t,x)}{(\theta c(t,x) - \partial_t c(t,x))}$.

Proof. Fix $x \in I$. By Itô's formula and (3.10), we have that

$$\mathrm{d}(c(t,x)O_t^x) = O_t^x\left(\partial_t c(t,x) - \theta c(t,x)\right)\mathrm{d}t + \frac{\sigma}{\sqrt{\alpha}}c(t,x)\mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad t \in [0,T],$$

and by (3.16) it follows that

$$\begin{aligned}\mathrm{d}\mu_t^x &= \mathrm{d}\left(c(t,x)O_t^x\right)^2 = 2c(t,x)O_t^x\mathrm{d}(c(t,x)O_t^x) + \frac{\sigma^2}{\alpha}c^2(t,x)\mathrm{d}t \\ &= \left(2c^2(t,x)\left(O_t^x\right)^2\left(\frac{\partial_t c(t,x)}{c(t,x)} - \theta\right) + \frac{\sigma^2}{\alpha}c^2(t,x)\right)\mathrm{d}t \\ &+ 2\frac{\sigma}{\sqrt{\alpha}}c^2(t,x)O_t^x\mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad t \in [0,T].\end{aligned}$$

The assertion follows by rearranging the terms. □

Lemma 3.4. *Let μ be given by the χ^2-intensity field model introduced in* (3.16)*. Then, for fixed $x,\ y \in I$, the sharp bracket process of $\mu^x = (\mu_t^x)_{t\in[0,T]}$ and $\mu^y = (\mu_t^y)_{t\in[0,T]}$ is given by*

$$\langle \mu^x, \mu^y\rangle_t = \frac{4\sigma^2}{\alpha}e^{-\alpha|x-y|}\int_0^t \mu_s^x\mu_s^y c(s,x)c(s,y)\,\mathrm{d}s, \quad t \in [0,T].$$

Proof. Fix $x, y \in I$. By (3.13) and (3.17), we immediately obtain

$$\begin{aligned}\mathrm{d}\left\langle \mu^x, \mu^y\right\rangle_t &= \frac{4\sigma^2}{\alpha}c(t,x)c(t,y)\mu_t^x\mu_t^y\mathrm{d}\left\langle \tilde{W}^{\nu_2(x)}, \tilde{W}^{\nu_2(y)}\right\rangle_t \\ &= \frac{4\sigma^2}{\alpha}\mu_t^x\mu_t^y c(t,x)c(t,y)e^{-\alpha|x-y|}\mathrm{d}t, \quad t \in [0,T].\end{aligned}$$

□

The function c needs to be specified such that μ, as defined in (3.16), is consistent with typical characteristics of historical mortality data, e.g., we define

$$c(t,x) = \exp\left(\frac{1}{2}[a(x) - kb(x)t]\right)\frac{\sqrt{2\theta\alpha}}{\sigma}, \quad t \in [0,T],\ x \in I, \quad (3.18)$$

where a and b are given in (3.4) and $k > 0$. Then

$$\mathbb{E}[\mu_t^x] = \exp(a(x) - kb(x)t)$$

and for the mean reversion level we have

$$\frac{\sigma^2}{2\alpha}\bar{c}(t,x) = \frac{\theta \exp(a(x) - kb(x)t)}{\theta + \frac{kb(x)}{2}},$$

for the mean reversion speed

$$2\left(\theta - \frac{\partial_t c(t,x)}{c(t,x)}\right) = 2\left(\theta + \frac{kb(x)}{2}\right)$$

and for the volatility

$$\sqrt{\frac{4}{\alpha}\sigma^2 c^2(t,x)\mu_t^x} = \sqrt{8\theta \exp(a(x) - kb(x)t)\mu_t^x},$$

for $t \in [0,T]$, $x \in I$.

3.3 Model comparison

With the notation of Subsections 3.1 and 3.2, in Table 1 we compare the Gaussian and the χ^2-intensity field models specified in (3.1), (3.16) and (3.18). When comparing (3.10) with (3.17), we observe that, while in the Gaussian model we have an age- and time dependent mean reversion level with constant mean reversion speed and volatility, in the χ^2-intensity field model all three parameters are age- and time dependent. Both models are affine (see also Example 4.1), thus facilitating the computation of conditional survival probabilities, and allow us to model an inhomogeneous downward mortality trend, while taking into account a realistic dependency structure. In spite of allowing for negative values with positive probability, the Gaussian intensity model is attractive due to its simplicity and intuitive interpretation.

criteria	Gaussian intensity field	χ^2-intensity field
affine	yes	yes
closed form solution	yes	no
mean reverting	yes	yes
mean	$\bar{\mu}(t,x)$	$\exp(a(x) - kb(x)t)$
mean reversion level	$\bar{\mu}(t,x) + \partial_t\bar{\mu}(t,x)/\theta$	$\theta \exp(a(x) - kb(x)t)/(\theta + kb(x)/2)$
mean reversion speed	θ	$2(\theta + kb(x)/2)$
volatility	$\sigma/\sqrt{\alpha}$	$\sqrt{8\theta \exp(a(x) - kb(x)t)\mu_t^x}$
correlation	$e^{-\theta\|t-s\|}e^{-\alpha\|x-y\|}$	$e^{-2\theta\|t-s\|}e^{-2\alpha\|x-y\|}$
non-negative	no	yes

Table 1: Comparison between the intensity field models

4 Risk-minimization for life insurance liabilities

Recall that the financial market defined in Subsection 2.2 is arbitrage-free, however, the market is not complete because the times of death occur as surprises to the market and hence represent a kind of "orthogonal" risk. Therefore, in this section, in order to find a price and hedging strategy for the insurance payment processes, we make use of a quadratic hedging method for incomplete markets, the well-known risk-minimization approach, first introduced by Föllmer and Sondermann [23] for European type contingent claims and later extended to payment processes by Møller [37], Schweizer [43] and Barbarin [3, Chapter 4]. For more information on (local) risk-minimization and other quadratic hedging approaches, we refer the reader to the survey paper of Schweizer [42]. We now start with some preliminary results.

4.1 Preliminary results

Recall that for fixed $x \in I$, $\mu^x = (\mu^x_t)_{t\in[0,T]}$ is assumed to be an affine diffusion process (see, e.g., Duffie et al. [20] or Filipović and Mayerhofer [21]), i.e., μ^x follows the dynamics

$$\mathrm{d}\mu^x_t = \delta(t,\mu^x_t)\mathrm{d}t + \sigma(t,\mu^x_t)\mathrm{d}\,\tilde{W}^{\nu_2(x)}_t, \quad t \in [0,T], \tag{4.1}$$

with $\delta(t,r) = d_0(t) + d_1(t)r$ and $\sigma^2(t,r) = v_0(t) + v_1(t)r =: (\sigma^x_t)^2$.

Example 4.1. *Note that, for fixed $x \in I$, both the Gaussian and the χ^2-field models are affine. In particular, for the Gaussian intensity model defined in* (3.1) *by* (3.9)*, we have that*

$$d_0(t) = \theta\bar{\mu}(t,x) + \partial_t\bar{\mu}(t,x), \quad d_1(t) = -\theta,$$
$$v_0(t) = \frac{\sigma^2}{\alpha}, \quad v_1(t) = 0.$$

for $t \in [0,T]$, $x \in I$. For the χ^2-intensity model defined in (3.16) *by* (3.17)*, we have*

$$d_0(t) = \frac{\sigma^2}{\alpha}c^2(t,x), \quad d_1(t) = 2\left(\frac{\partial_t c(t,x)}{c(t,x)} - \theta\right),$$
$$v_0(t) = 0, \quad v_1(t) = \frac{4}{\alpha}\sigma^2 c^2(t,x),$$

for $t \in [0,T]$, $x \in I$.

Lemma 4.2. *Fix $u \in [0,T]$ and $x \in I$. If μ^x is an affine diffusion satisfying* (4.1)*, then under the hypothesis of Section 2, the process*

$$Z^{x,u}_t = \mathbb{E}[\exp(-\Gamma^x_u) \,|\, \mathcal{F}_t] = \mathbb{E}\left[\exp\left(-\int_0^u \mu^x_s \,\mathrm{d}s\right) \,\Big|\, \mathcal{F}^\mu_t\right],$$

$t \in [0, u]$*, has the following dynamics*

$$Z_t^{x,u} = Z_0^{x,u} + \int_0^t Z_s^{x,u} \sigma_s^x \beta^{x,u}(s) \mathrm{d}\tilde{W}_s^{\nu_2(x)}, \quad t \in [0, u], \tag{4.2}$$

where $\beta^{x,u}$ *is given by the following differential equation*

$$\partial_t \beta^{x,u}(t) = 1 - d_1(t)\beta^{x,u}(t) - \frac{1}{2} v_1(t) \left(\beta^{x,u}(t)\right)^2, \quad \beta^{x,u}(u) = 0. \tag{4.3}$$

Proof. Fix $u \in [0, T]$ and $x \in I$. As μ^x is affine, we immediately obtain that

$$\tilde{Z}_t^{x,u} := \mathbb{E}\left[e^{-\int_t^u \mu_s^x \mathrm{d}s} \middle| \mathcal{F}_t \right] = \mathbb{E}\left[e^{-\int_t^u \mu_s^x \mathrm{d}s} \middle| \mathcal{F}_t^\mu \right] = e^{\alpha^{x,u}(t) + \beta^{x,u}(t)\mu_t^x}, \tag{4.4}$$

for $t \in [0, u]$ (see Duffie et al. [19]), where the functions $\alpha^{x,u}$ and $\beta^{x,u}$ are given by

$$\begin{aligned} \partial_t \beta^{x,u}(t) &= 1 - d_1(t)\beta^{x,u}(t) - \frac{1}{2} v_1(t) \left(\beta^{x,u}(t)\right)^2, \quad \beta^{x,u}(u) = 0, \\ \partial_t \alpha^{x,u}(t) &= -d_0(t)\beta^{x,u}(t) - \frac{1}{2} v_0(t) \left(\beta^{x,u}(t)\right)^2, \quad \alpha^{x,u}(u) = 0. \end{aligned} \tag{4.5}$$

Then, by Itô's formula, we have that

$$\begin{aligned} \mathrm{d}\tilde{Z}_t^{x,u} &= \tilde{Z}_t^{x,u} \left(\partial_t \alpha^{x,u}(t) + \partial_t \beta^{x,u}(t) \mu_t^x \right) \mathrm{d}t + \tilde{Z}_t^{x,u} \beta^{x,u}(t) \mathrm{d}\mu_t^x \\ &\quad + \frac{1}{2} \tilde{Z}_t^{x,u} \left(\beta^{x,u}(t)\right)^2 \mathrm{d}\left\langle \mu^x \right\rangle_t \\ &= \tilde{Z}_t^{x,u} \left(\mu_t^x \mathrm{d}t + \beta^{x,u}(t) \sigma_t^x \mathrm{d}\tilde{W}_t^{\nu_2(x)} \right), \end{aligned} \tag{4.6}$$

as well as

$$\begin{aligned} \mathrm{d}Z_t^{x,u} &= e^{-\Gamma_t^x} \mathrm{d}\tilde{Z}_t^{x,u} - e^{-\Gamma_t^x} \tilde{Z}_t^{x,u} \mu_t^x \mathrm{d}t \\ &= Z_t^{x,u} \beta^{x,u}(t) \sigma_t^x \mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad t \in [0, u]. \end{aligned}$$

The result follows. □

Lemma 4.3. *Fix* $u \in [0, T]$ *and* $x \in I$*. If* μ^x *is an affine diffusion satisfying* (4.1)*, then under the hypothesis of Section 2, the process*

$$\bar{Z}_t^{x,u} := \mathbb{E}\left[\exp\left(-\int_0^u \mu_s^x \,\mathrm{d}s \right) \mu_u^x \,\middle|\, \mathcal{F}_t^\mu \right], \quad t \in [0, u],$$

has the following dynamics

$$\bar{Z}_t^{x,u} = \bar{Z}_0^{x,u} + \int_0^t Z_s^{x,u} \sigma_s^x \left[\beta^{x,u}(s) \hat{Z}_s^{x,u} + \hat{\beta}^{x,u}(s) \right] \mathrm{d}\tilde{W}_s^{\nu_2(x)}, \quad t \in [0, u], \tag{4.7}$$

where $\hat{Z}^{x,u}$ *is given by*

$$\hat{Z}_t^{x,u} = \hat{\alpha}^{x,u}(t) + \hat{\beta}^{x,u}(t) \mu_t^x, \quad t \in [0, u], \tag{4.8}$$

and $\hat{\alpha}^{x,u}$ *and* $\hat{\beta}^{x,u}$ *are given by the following differential equations*

$$\begin{aligned}\partial_t\hat{\beta}^{x,u}(t) &= -d_1(t)\hat{\beta}^{x,u}(t) - \beta^{x,u}(t)\hat{\beta}^{x,u}(t)v_1(t), \quad \hat{\beta}^{x,u}(u) = 1, \\ \partial_t\hat{\alpha}^{x,u}(t) &= -d_0(t)\hat{\beta}^{x,u}(t) - \beta^{x,u}(t)\hat{\beta}^{x,u}(t)v_0(t), \quad \hat{\alpha}^{x,u}(u) = 0,\end{aligned} \tag{4.9}$$

and $Z^{x,u}$ *and* $\beta^{x,u}$ *are given by* (4.2) *and* (4.3).

Proof. Fix $u \in [0,T]$ and $x \in I$. As μ^x is affine, as in the proof of Lemma 4.2 and following Duffie et al. [19], we obtain that

$$\mathbb{E}\left[e^{-\int_t^u \mu_s^x \mathrm{d}s}\mu_u^x \,\middle|\, \mathcal{F}_t\right] = \tilde{Z}_t^{x,u}\hat{Z}_t^{x,u}, \quad t \in [0,u],$$

where $\tilde{Z}_t^{x,u}$ is given in (4.4) and

$$\hat{Z}_t^{x,u} = \hat{\alpha}^{x,u}(t) + \hat{\beta}^{x,u}(t)\mu_t^x, \quad t \in [0,u],$$

with

$$\begin{aligned}\partial_t\hat{\beta}^{x,u}(t) &= -d_1(t)\hat{\beta}^{x,u}(t) - \beta^{x,u}(t)\hat{\beta}^{x,u}(t)v_1(t), \quad \hat{\beta}^{x,u}(u) = 1, \\ \partial_t\hat{\alpha}^{x,u}(t) &= -d_0(t)\hat{\beta}^{x,u}(t) - \beta^{x,u}(t)\hat{\beta}^{x,u}(t)v_0(t), \quad \hat{\alpha}^{x,u}(u) = 0.\end{aligned}$$

Then, again by an application of Itô's formula, we obtain

$$\begin{aligned}\mathrm{d}\hat{Z}_t^{x,u} &= \left(\partial_t\hat{\alpha}^{x,u}(t) + \partial_t\hat{\beta}^{x,u}(t)\mu_t^x\right)\mathrm{d}t + \hat{\beta}^{x,u}(t)\mathrm{d}\mu_t^x \\ &= \hat{\beta}^{x,u}(t)\sigma_t^x\left(-\beta^{x,u}(t)\sigma_t^x\mathrm{d}t + \mathrm{d}\tilde{W}_t^{\nu_2(x)}\right),\end{aligned} \tag{4.10}$$

and by (4.6) and (4.10), we have

$$\begin{aligned}\mathrm{d}(\tilde{Z}_t^{x,u}\hat{Z}_t^{x,u}) &= \tilde{Z}_t^{x,u}\mathrm{d}\hat{Z}_t^{x,u} + \hat{Z}_t^{x,u}\mathrm{d}\tilde{Z}_t^{x,u} + \mathrm{d}\left\langle\tilde{Z}^{x,u},\hat{Z}_t^{x,u}\right\rangle_t \\ &= \tilde{Z}_t^{x,u}\hat{\beta}^{x,u}(t)\sigma_t^x\left(-\beta^{x,u}(t)\sigma_t^x\mathrm{d}t + \mathrm{d}\tilde{W}_t^{\nu_2(x)}\right) \\ &\quad + \hat{Z}_t^{x,u}\tilde{Z}_t^{x,u}\left(\mu_t^x\mathrm{d}t + \beta^{x,u}(t)\sigma_t^x\mathrm{d}\tilde{W}_t^{\nu_2(x)}\right) + \tilde{Z}_t^{x,u}\left(\sigma_t^x\right)^2\beta^{x,u}(t)\hat{\beta}^{x,u}(t)\mathrm{d}t \\ &= \tilde{Z}_t^{x,u}\left(\hat{Z}_t^{x,u}\mu_t^x\mathrm{d}t + \sigma_t^x\left(\hat{Z}_t^{x,u}\beta^{x,u}(t) + \hat{\beta}^{x,u}(t)\right)\mathrm{d}\tilde{W}_t^{\nu_2(x)}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{d}\bar{Z}_t^{x,u} &= e^{-\Gamma_t^x}\mathrm{d}(\tilde{Z}_t^{x,u}\hat{Z}_t^{x,u}) - e^{-\Gamma_t^x}\tilde{Z}_t^{x,u}\hat{Z}_t^{x,u}\mu_t^x\mathrm{d}t \\ &= Z_t^{x,u}\sigma_t^x\left(\hat{Z}_t^{x,u}\beta^{x,u}(t) + \hat{\beta}^{x,u}(t)\right)\mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad t \in [0,u],\end{aligned}$$

and the result follows. □

Remark 4.4. *For the Gaussian field model specified in* (3.1) *and* (3.9)*, for fixed* $u \in [0,T]$ *and* $x \in I$*, we can easily compute the functions* $\alpha^{x,u}$*,* $\beta^{x,u}$*,* $\hat{\alpha}^{x,u}$ *and* $\hat{\beta}^{x,u}$ *analytically. The closed forms for* $\alpha^{x,u}$ *and* $\beta^{x,u}$ *are given by*

$$\beta^{x,u}(t) = \frac{e^{-\theta(u-t)} - 1}{\theta}, \tag{4.11}$$

$$\alpha^{x,u}(t) = \int_t^u \beta^{x,u}(s) \left(\theta\bar{\mu}(s,x) + \partial_s \bar{\mu}(s,x) + \frac{\sigma}{2\sqrt{\alpha}} \beta^{x,u}(s) \right) \mathrm{d}s, \tag{4.12}$$

for $t \in [0,u]$*, as can be verified by substitution in* (4.3) *and* (4.5)*. The closed forms for* $\hat{\alpha}^{x,u}$ *and* $\hat{\beta}^{x,u}$ *are given by*

$$\hat{\beta}^{x,u}(t) = e^{-\theta(u-t)},$$

$$\hat{\alpha}^{x,u}(t) = \int_t^u \hat{\beta}^{x,u}(s) \left(\theta\bar{\mu}(s,x) + \partial_s \bar{\mu}(s,x) + \frac{\sigma}{\sqrt{\alpha}} \beta^{x,u}(s) \right) \mathrm{d}s.$$

for $t \in [0,u]$*. For the* χ^2*-field model in* (3.16)*, if* c *is a function of age only, i.e.,* $c(t,x) \equiv c(x)$*, we obtain the well-known time-homogeneous Cox-Ingersoll-Ross model for* μ^x *for any given fixed* x*, and* $\alpha^{x,u}$*,* $\beta^{x,u}$*,* $\hat{\alpha}^{x,u}$ *and* $\hat{\beta}^{x,u}$ *are explicitly computable (see Cox et al. [15]). In general, if the model has time-dependent parameters, closed form solutions are not available (see Heath et al. [25] and Hull and White [26]) and the differential equations determining* $\alpha^{x,u}$*,* $\beta^{x,u}$*,* $\hat{\alpha}^{x,u}$ *and* $\hat{\beta}^{x,u}$ *have to be solved by numerical methods.*

In the following, we calculate the prices and hedging strategies of the insurance payment streams introduced in (2.8) - (2.10), by means of the risk-minimization approach. An important role is played by the $\mathbb{G}$-(local) martingales

$$M_t^{x_i,j} = H_t^{x_i,j} - \Gamma^{x_i}_{t \wedge \tau^{x_i,j}} \quad \text{and} \quad M_t^{x_i} := \sum_{j=1}^{n^{x_i}} M_t^{x_i,j}, \tag{4.13}$$

as well as

$$L_t^{x_i,j} = \mathbb{1}_{\{\tau^{x_i,j} > t\}} e^{\Gamma_t^{x_i}} = 1 - \int_{]0,t]} L_{s-}^{x_i,j} \, \mathrm{d}M_s^{x_i,j} = 1 - \int_{]0,t]} e^{\Gamma_s^{x_i}} \, \mathrm{d}M_s^{x_i,j}, \tag{4.14}$$

for $t \in [0,T]$, $i = 1, \ldots, m$, $j = 1, \ldots, n^{x_i}$ (see, e.g., Chapter 5 and Chapter 9 of Bielecki and Rutkowski [8]). Recall that we consider unit-linked life insurance products, i.e., the insurance liabilities defined in (2.8) - (2.10) are given in terms of a non-negative Borel measurable function $f(S_t)$ of the asset price S_t, $t \in [0,T]$. Then, following Møller [36], for fixed $u \in [0,T]$, the arbitrage-free price process

$$F^u(t, S_t) = \mathbb{E}\left[\exp\left(-r\,(u-t)\right) f(S_u) | \mathcal{F}_t^X \right], \quad t \in [0,u], \tag{4.15}$$

associated with the payoff $f(S_u)$ at time u can be characterized by the partial differential equation

$$-rF^u(t,s) + F_t^u(t,s) + rsF_s^u(t,s) + \frac{1}{2}\sigma(t,s)^2 s^2 F_{ss}^u(t,s) = 0, \tag{4.16}$$

with boundary value $F^u(u,s) = f(s)$, where $F^u_t(t,s)$, $F^u_s(t,s)$ and $F^u_{ss}(t,s)$ are the partial first and second order derivatives of F^u with respect to t and s. Also recall that we assume that trading in the (discounted) risky asset X introduced in (2.6), as well as in the family of (discounted) longevity bonds Y^x, $x \in I$, defined in (2.7) is possible (see Subsection 2.2). However, the insurance portfolio introduced in Subsection 2.1 only consists of individuals belonging to the age cohorts $\{x_1, \dots, x_m\} \subset I$. Therefore, the risk-minimizing strategies will be given in terms of investments in X as well as the portfolio of longevity bonds $Y := (Y^{x_1}, \dots, Y^{x_m})$ corresponding to the age cohorts $x_1, \dots, x_m$ of the insurance portfolio, see, e.g., (4.33) and (4.34). In the following, let $\int_0^t \xi_s \,\mathrm{d}Y_s := \sum_{i=1}^m \int_0^t \xi^i_s \,\mathrm{d}Y^{x_i}_s$, for any m-dimensional $\mathbb{G}$-predictable process $\xi = (\xi^1, \dots, \xi^m)$, and $\xi \cdot Y := \sum_{i=1}^m \xi^i Y^{x_i}$.

4.2 Term insurance contract

For the term insurance contract introduced in (2.8), define the payment process

$$A^{ti}_t = \sum_{i=1}^m \zeta(x_i) \sum_{j=1}^{n^{x_i}} \frac{f(S_{\tau^{x_i,j}})}{B_{\tau^{x_i,j}}} \mathbb{1}_{\{\tau^{x_i,j} \le t\}}, \quad t \in [0,T], \tag{4.17}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} f(S_t)^2\right] < \infty. \tag{4.18}$$

Proposition 4.5. *In the setting of Section 2, the payment process A^{ti} introduced in* (4.17) *admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ with discounted value process*

$$V^{ti}_t(\varphi) = \mathbb{E}[A^{ti}_T \,|\, \mathcal{G}_0] + \int_0^t \xi^X_s \,\mathrm{d}X_s + \int_0^t \xi^Y_s \,\mathrm{d}Y_s + L^{ti}_t - A^{ti}_t, \tag{4.19}$$

and

$$\xi^0_t = V^{ti}_t(\varphi) - \xi^X_t X_t - \xi^Y_t \cdot Y_t$$

for $t \in [0,T]$, where the investment in the (discounted) risky asset X is given by

$$\xi^X_t = \sum_{i=1}^m \zeta(x_i)(n^{x_i} - N^{x_i}_t) e^{\Gamma^{x_i}_t} \int_t^T \bar{Z}^{x_i,u}_t F^u_s(t, S_t) \,\mathrm{d}u,$$

and the investment in the family of (discounted) longevity bonds $Y = (Y^{x_1}, \dots, Y^{x_m})$ is given by $\xi^Y_t = (\xi^{Y^{x_1}}_t, \dots, \xi^{Y^{x_m}}_t)$, with

$$\xi^{Y^{x_i}}_t = \zeta(x_i)(n^{x_i} - N^{x_i}_t) \frac{e^{\Gamma^{x_i}_t} e^{r(T-t)}}{Z^{x_i,T}_t \beta^{x_i,T}(t)} \int_t^T F^u(t, S_t) Z^{x_i,u}_t (\beta^{x_i,u}(t) \hat{Z}^{x_i,u}_t + \hat{\beta}^{x_i,u}(t)) \,\mathrm{d}u,$$

and

$$L^{ti}_t = \sum_{i=1}^m \zeta(x_i) \int_{]0,t]} \left(\frac{f(S_s)}{B_s} - \mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u} e^{\Gamma^{x_i}_s - \Gamma^{x_i}_u} \,\mathrm{d}\Gamma^{x_i}_u \,\middle|\, \mathcal{F}_s \right] \right) \mathrm{d}M^{x_i}_s,$$

$t \in [0,T]$, *where* $F^u(t,S_t)$, $F^u_s(t,S_t)$, $\beta^{x_i,u}(t)$, $\hat{\beta}^{x_i,u}(t)$, $Z^{x_i,u}_t$, $\bar{Z}^{x_i,u}_t$, $\hat{Z}^{x_i,u}_t$ *and* $M^{x_i}_t$ *are defined in* (4.2) - (4.3), (4.7) - (4.9), (4.13) *and* (4.15) - (4.16). *The optimal cost and risk processes are given by*

$$\begin{aligned} C^{ti}_t(\varphi) &= \mathbb{E}[A^{ti}_T \,|\, \mathcal{G}_0] + L^{ti}_t, \\ R^{ti}_t(\varphi) &= \mathbb{E}[(L^{ti}_T - L^{ti}_t)^2 \,|\, \mathcal{G}_t], \end{aligned}$$

for $t \in [0,T]$.

Proof. Let $t \in [0,T]$. Then,

$$\mathbb{E}[A^{ti}_T \,|\, \mathcal{G}_t] = \sum_{i=1}^{m} \zeta(x_i) \sum_{j=1}^{n^{x_i}} J^{ij}_t,$$

where

$$J^{ij}_t = \mathbb{E}\left[\frac{f(S_{\tau^{x_i,j}})}{B_{\tau^{x_i,j}}} \mathbb{1}_{\{\tau^{x_i,j} \le T\}} \,\Big|\, \mathcal{G}_t \right], \tag{4.20}$$

$t \in [0,T]$, $i = 1,\dots,m$, $j = 1,\dots,n^{x_i}$. Then, by Proposition 4.11 and 5.12 of Barbarin [3, Chapter 3], as well as Corollary 5.1.3 of Bielecki and Rutkowski [8] and (2.3), we have

$$\begin{aligned} J^{ij}_t = U^{x_i,ti}_0 &+ \int_0^t L^{x_i,j}_s \, \mathrm{d}U^{x_i,ti}_s \\ &+ \int_{]0,t]} \underbrace{\left(\frac{f(S_s)}{B_s} - \mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u} e^{\Gamma^{x_i}_s - \Gamma^{x_i}_u} \, \mathrm{d}\Gamma^{x_i}_u \,\Big|\, \mathcal{F}_s \right] \right)}_{=:\psi_s} \mathrm{d}M^{x_i,j}_s, \end{aligned} \tag{4.21}$$

where $M^{x_i,j}_t$ and $L^{x_i,j}_t$ are defined in (4.13) and (4.14). Furthermore

$$\begin{aligned} U^{x_i,ti}_t &:= \mathbb{E}\left[\int_0^T \frac{f(S_u)}{B_u} e^{-\Gamma^{x_i}_u} \, \mathrm{d}\Gamma^{x_i}_u \,\Big|\, \mathcal{F}_t \right] \\ &= \int_0^T \mathbb{E}\left[\frac{f(S_u)}{B_u} \Big|\, \mathcal{F}^X_t \right] \mathbb{E}\left[e^{-\Gamma^{x_i}_u} \mu^{x_i}_u \,\Big|\, \mathcal{F}^\mu_t \right] \mathrm{d}u, \end{aligned}$$

for $t \in [0,T]$ and $i = 1,\dots,m$, $j = 1,\dots,n^{x_i}$, where we have used Fubini's theorem and the independence of the underlying driving processes. By (2.5) - (2.6), (4.15) - (4.16) and Itô's formula, the discounted arbitrage-free price process $\frac{F^u(t,S_t)}{B_t}$, $0 \le u \le T$, follows the dynamics

$$\mathrm{d}\left(\frac{F^u(t,S_t)}{B_t} \right) = F^u_s(t,S_t)\sigma(t,S_t)X_t \, \mathrm{d}W^X_t = F^u_s(t,S_t) \, \mathrm{d}X_t, \quad t \in [0,u],$$

and

$$\mathbb{E}\left[\frac{f(S_u)}{B_u} \Big|\, \mathcal{F}^X_t \right] = F^u(0,S_0) + \int_0^t F^u_s(s,S_s)\sigma(s,S_s)X_s \mathbb{1}_{\{s \le u\}} \, \mathrm{d}W^X_s$$

for $0 \leq t,\, u \leq T$, where $F^u(u, S_u) = f(S_u)$. Furthermore, by (4.7), we have

$$\begin{aligned}\bar{Z}_t^{x_i,u} &= \mathbb{E}\left[e^{-\Gamma_u^{x_i}}\mu_u^{x_i}\,\middle|\,\mathcal{F}_t^{\mu}\right]\\ &= \bar{Z}_0^{x_i,u} + \int_0^t Z_s^{x_i,u}\sigma_s^{x_i}\left[\beta^{x_i,u}(s)\hat{Z}_s^{x_i,u} + \hat{\beta}^{x_i,u}(s)\right]\mathbb{1}_{\{s\leq u\}}\mathrm{d}\tilde{W}_s^{\nu_2(x_i)},\end{aligned}$$

for $0 \leq t,\, u \leq T$ and $i = 1, \ldots, m$, where $\beta^{x_i,u}$, $\hat{\beta}^{x_i,u}$, $Z^{x_i,u}$ and $\hat{Z}^{x_i,u}$ are given in (4.2), (4.3) and (4.8) - (4.9). Then, for $u \in [0,T]$, integration by parts gives

$$\begin{aligned}\frac{F^u(t,S_t)}{B_t}\bar{Z}_t^{x_i,u} &= F^u(0,S_0)\bar{Z}_0^{x_i,u} + \int_0^t \bar{Z}_s^{x_i,u}F_s^u(s,S_s)\sigma(s,S_s)X_s\mathbb{1}_{\{s\leq u\}}\,\mathrm{d}W_s^X\\ &\quad + \int_0^t \frac{F^u(s,S_s)}{B_s}Z_s^{x_i,u}\sigma_s^{x_i}\left(\beta^{x_i,u}(s)\hat{Z}_s^{x_i,u} + \hat{\beta}^{x_i,u}(s)\right)\mathbb{1}_{\{s\leq u\}}\,\mathrm{d}\tilde{W}_s^{\nu_2(x_i)}, \quad t\in[0,T].\end{aligned}$$

By Lemma 4.2, for each $x \in I$, the dynamics of the (discounted) longevity bond with maturity T associated to the age cohort x defined in (2.7) are

$$\mathrm{d}Y_t^x = \frac{Z_t^{x,T}}{B_T}\sigma_t^x\beta^{x,T}(t)\mathrm{d}\tilde{W}_t^{\nu_2(x)}, \quad t\in[0,T]. \tag{4.22}$$

As all integrands are continuous (see Theorem 15 in Chapter IV of Protter [38]), by the stochastic Fubini theorem (see, e.g., Theorem 65 in Chapter IV of Protter [38]) and integration by parts, we obtain

$$\begin{aligned}U_t^{x_i,ti} &= \int_0^T F^u(0,S_0)\bar{Z}_0^{x_i,u}\,\mathrm{d}u + \int_0^t \sigma(s,S_s)X_s\int_s^T F_s^u(s,S_s)\bar{Z}_s^{x_i,u}\,\mathrm{d}u\,\mathrm{d}W_s^X\\ &\quad + \int_0^t \frac{\sigma_s^{x_i}}{B_s}\int_s^T F^u(s,S_s)Z_s^{x_i,u}(\beta^{x_i,u}(s)\hat{Z}_s^{x_i,u} + \hat{\beta}^{x_i,u}(s))\,\mathrm{d}u\,\mathrm{d}\tilde{W}_s^{\nu_2(x_i)} \qquad (4.23)\\ &= \int_0^T F^u(0,S_0)\bar{Z}_0^{x_i,u}\,\mathrm{d}u + \int_0^t\int_s^T F_s^u(s,S_s)\bar{Z}_s^{x_i,u}\,\mathrm{d}u\,\mathrm{d}X_s\\ &\quad + \int_0^t \frac{e^{r(T-s)}}{Z_s^{x_i,T}\beta^{x_i,T}(s)}\int_s^T F^u(s,S_s)Z_s^{x_i,u}(\beta^{x_i,u}(s)\hat{Z}_s^{x_i,u} + \hat{\beta}^{x_i,u}(s))\,\mathrm{d}u\,\mathrm{d}Y_s^{x_i},\end{aligned}$$

$t \in [0,T]$, where in the second equation, we used (2.6) and (4.22). Finally,

$$\mathbb{E}[A_T^{ti}\,|\,\mathcal{G}_t] = \mathbb{E}[A_T^{ti}\,|\,\mathcal{G}_0] + \int_0^t \xi_s^X\,\mathrm{d}X_s + \int_0^t \xi_s^Y\,\mathrm{d}Y_s + L_t^{ti}, \tag{4.24}$$

for $t \in [0,T]$, where the investment in the (discounted) risky asset X is given by

$$\xi_t^X = \sum_{i=1}^m \zeta(x_i)(n^{x_i} - N_t^{x_i})e^{\Gamma_t^{x_i}}\int_t^T \bar{Z}_t^{x_i,u}F_s^u(t,S_t)\,\mathrm{d}u,$$

and the investment in the family of (discounted) longevity bonds $Y = (Y^{x_1}, \ldots, Y^{x_m})$ is given by $\xi_t^Y = (\xi_t^{Y^{x_1}}, \ldots, \xi_t^{Y^{x_m}})$, with

$$\xi_t^{Y^{x_i}} = \zeta(x_i)(n^{x_i} - N_t^{x_i})\frac{e^{\Gamma_t^{x_i}}e^{r(T-t)}}{Z_t^{x_i,T}\beta^{x_i,T}(t)}\int_t^T F^u(t,S_t)Z_t^{x_i,u}(\beta^{x_i,u}(t)\hat{Z}_t^{x_i,u} + \hat{\beta}^{x_i,u}(t))\,\mathrm{d}u,$$

and

$$L_t^{ti} = \sum_{i=1}^m \zeta(x_i) \int_{]0,t]} \left(\frac{f(S_s)}{B_s} - \mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u} e^{\Gamma_s^{x_i} - \Gamma_u^{x_i}} \,\mathrm{d}\Gamma_u^{x_i} \,\Big|\, \mathcal{F}_s \right] \right) \mathrm{d}M_s^{x_i},$$

$t \in [0,T]$. It remains to prove that (4.24) is indeed the GKW decomposition of $\mathbb{E}[A_T^{ti} \,|\, \mathcal{G}_t]$, $t \in [0,T]$. To this end, define $\overline{S} = (X,Y) = (X, Y^{x_1}, \ldots, Y^{x_m})$ and $\xi = \left(\xi^X, \xi^Y\right)$. As $\mathbb{E}\left[\sup_{t\in[0,T]} f(S_t)^2\right] < \infty$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n^{x_i}$, we have that J^{ij} introduced in (4.20) is a square integrable martingale, hence $\mathbb{E}[[J^{ij}]_T] < \infty$, and from (4.21) it follows that

$$\mathbb{E}\left[\int_0^T (L_s^{x_i,j})^2 \,\mathrm{d}[U^{x_i,ti}]_s\right], \quad \mathbb{E}\left[\int_0^T \psi_s^2 \,\mathrm{d}[M^{x_i,j}]_s\right] < \infty, \tag{4.25}$$

because $\mathrm{d}[U^{x_i,ti}, M^{x_i,j}]_t \equiv 0$, $t \in [0,T]$, $i = 1, \ldots, m$ and $j = 1, \ldots, n^{x_i}$. As for $i = 1, \ldots, m$, $\mathrm{d}[W^X, \tilde{W}^{\nu_2(x_i)}]_t \equiv 0$, $t \in [0,T]$, because W^X and $\tilde{W}^{\nu_2(x_i)}$ are independent, by (4.23), (4.24) and (4.25) and by the Kunita-Watanabe Inequality (see, e.g., Theorem 25 in Chapter II.6 of Protter [38]), we obtain that

$$\mathbb{E}\left[\int_0^T \xi_s' \mathrm{d}[\overline{S}]_s \xi_s\right] < \infty \quad \text{and} \quad \mathbb{E}[[L^{ti}]_T] < \infty,$$

i.e., L^{ti} is a square integrable martingale and $\xi \in L^2(\overline{S})$, where

$$L^2(\overline{S}) := \left\{ \xi \,\middle|\, \xi \ \mathbb{G}\text{-predictable}, \ \left(\mathbb{E}\left[\int_0^T \xi_s' \,\mathrm{d}[\overline{S}, \overline{S}]_s \,\xi_s\right]\right)^{1/2} < \infty \right\}.$$

As L^{ti} is strongly orthogonal to all continuous $\mathbb{F}$-local martingales, it follows that

$$\left(\int_0^t \tilde{\xi}_s \mathrm{d}\overline{S}_s\right) \cdot L_t^{ti}, \quad t \in [0,T],$$

is a (uniformly integrable) martingale for any $\tilde{\xi} \in L^2(\overline{S})$, i.e., (4.24) is the GKW decomposition of $\mathbb{E}[A_T^{ti} \,|\, \mathcal{G}_t]$, $t \in [0,T]$ (see, e.g., Møller [37] or Schweizer [42]). □

4.3 Annuity contract

For the annuity contract introduced in (2.9), we define the payment process

$$A_t^a = \sum_{i=1}^m \zeta(x_i) \sum_{j=1}^{n^{x_i}} \int_0^t \mathbb{1}_{\{\tau^{x_i,j} > s\}} \frac{f(S_s)}{B_s} \mathrm{d}s, \quad t \in [0,T], \tag{4.26}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} f(S_t)^2\right] < \infty. \tag{4.27}$$

Proposition 4.6. *In the setting of Section 2, the payment process A^a introduced in* (4.26) *admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ with discounted value process*

$$V_t^a(\varphi) = \mathbb{E}[A_T^a \,|\, \mathcal{G}_0] + \int_0^t \xi_s^X \,\mathrm{d}X_s + \int_0^t \xi_s^Y \,\mathrm{d}Y_s + L_t^a - A_t^a, \tag{4.28}$$

and

$$\xi_t^0 = V_t^a(\varphi) - \xi_t^X X_t - \xi_t^Y \cdot Y_t$$

for $t \in [0,T]$, where the investment in the (discounted) risky asset X is given by

$$\xi_t^X = \sum_{i=1}^m \zeta(x_i)(n^{x_i} - N_t^{x_i}) e^{\Gamma_t^{x_i}} \int_t^T Z_t^{x_i,u} F_s^u(t, S_t) \,\mathrm{d}u,$$

and the investment in the family of (discounted) longevity bonds $Y = (Y^{x_1}, \ldots, Y^{x_m})$ is given by $\xi_t^Y = (\xi_t^{Y^{x_1}}, \ldots, \xi_t^{Y^{x_m}})$, with

$$\xi_t^{Y^{x_i}} = \zeta(x_i)(n^{x_i} - N_t^{x_i}) \frac{e^{\Gamma_t^{x_i}} e^{r(T-t)}}{Z_t^{x_i,T} \beta^{x_i,T}(t)} \int_t^T F^u(t, S_t) Z_t^{x_i,u} \beta^{x_i,u}(t) \,\mathrm{d}u,$$

and

$$L_t^a = -\sum_{i=1}^m \zeta(x_i) \int_{]0,t]} \mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u} e^{\Gamma_s^{x_i} - \Gamma_u^{x_i}} \,\mathrm{d}u \,\Big|\, \mathcal{F}_s \right] \mathrm{d}M_s^{x_i},$$

$t \in [0,T]$, where $F^u(t, S_t)$, $F_s^u(t, S_t)$, $\beta^{x_i,u}(t)$, $Z_t^{x_i,u}$ and $M_t^{x_i}$ are defined in (4.2) *-* (4.3)*,* (4.13) *and* (4.15) *-* (4.16)*. The optimal cost and risk processes are*

$$\begin{aligned} C_t^a(\varphi) &= \mathbb{E}[A_T^a \,|\, \mathcal{G}_0] + L_t^a, \\ R_t^a(\varphi) &= \mathbb{E}[(L_T^a - L_t^a)^2 \,|\, \mathcal{G}_t], \end{aligned}$$

for $t \in [0,T]$.

Proof. Let $t \in [0,T]$. Then, we have that

$$\mathbb{E}[A_T^a \,|\, \mathcal{G}_t] = \sum_{i=1}^m \zeta(x_i) \sum_{j=1}^{n^{x_i}} \mathbb{E}\left[\int_0^T \mathbb{1}_{\{\tau^{x_i,j} > s\}} \frac{f(S_s)}{B_s} \mathrm{d}s \,\Big|\, \mathcal{G}_t \right],$$

and by Proposition 4.12 and 5.13 of Barbarin [3, Chapter 3], as well as Proposition 5.1.2 of Bielecki and Rutkowski [8] and (2.3), we have

$$\begin{aligned} \mathbb{E}\left[\int_0^T \mathbb{1}_{\{\tau^{x_i,j} > s\}} \frac{f(S_s)}{B_s} \mathrm{d}s \,\Big|\, \mathcal{G}_t \right] &= U_0^{x_i,a} + \int_0^t L_s^{x_i,j} \,\mathrm{d}U_s^{x_i,a} \\ &\quad - \int_{]0,t]} \mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u} e^{\Gamma_s^{x_i} - \Gamma_u^{x_i}} \,\mathrm{d}u \,\Big|\, \mathcal{F}_s \right] \mathrm{d}M_s^{x_i,j}, \end{aligned}$$

and

$$U_t^{x_i,a} := \mathbb{E}\left[\int_0^T \frac{f(S_u)}{B_u} e^{-\Gamma_u^{x_i}}\,\mathrm{d}u \,\Big|\, \mathcal{F}_t\right]$$
$$= \int_0^T \mathbb{E}\left[\frac{f(S_u)}{B_u}\Big|\, \mathcal{F}_t^X\right] \mathbb{E}\left[e^{-\Gamma_u^{x_i}}\,\Big|\, \mathcal{F}_t^\mu\right] \mathrm{d}u,$$

for $t \in [0,T]$ and $i = 1,\dots,m$, $j = 1,\dots,n^{x_i}$, where we have used Fubini's theorem and the independence of the underlying driving processes. We proceed as in the proof of Proposition 4.5. By (4.2), we have that

$$Z_t^{x_i,u} = \mathbb{E}\left[e^{-\Gamma_u^{x_i}}\,\Big|\, \mathcal{F}_t^\mu\right] = Z_0^{x_i,u} + \int_0^t Z_s^{x_i,u}\sigma_s^{x_i}\beta^{x_i,u}(s)\mathbb{1}_{\{s\le u\}}\mathrm{d}\tilde{W}_s^{\nu_2(x_i)},$$

for $0 \le t$, $u \le T$ and $i = 1,\dots,m$, where $\beta^{x_i,u}$, is given in (4.3). Then, by the stochastic Fubini theorem (see, e.g., Theorem 65 in Chapter IV of Protter [38]) and integration by parts, we obtain

$$\begin{aligned}
U_t^{x_i,a} &= \int_0^T F^u(0,S_0)Z_0^{x_i,u}\,\mathrm{d}u + \int_0^t \sigma(s,S_s)X_s \int_s^T F_s^u(s,S_s)Z_s^{x_i,u}\,\mathrm{d}u\,\mathrm{d}W_s^X \\
&\quad + \int_0^t \frac{\sigma_s^{x_i}}{B_s}\int_s^T F^u(s,S_s)Z_s^{x_i,u}\beta^{x_i,u}(s)\,\mathrm{d}u\,\mathrm{d}\tilde{W}_s^{\nu_2(x_i)} \\
&= \int_0^T F^u(0,S_0)Z_0^{x_i,u}\,\mathrm{d}u + \int_0^t \int_s^T F_s^u(s,S_s)Z_s^{x_i,u}\,\mathrm{d}u\,\mathrm{d}X_s \\
&\quad + \int_0^t \frac{e^{r(T-s)}}{Z_s^{x_i,T}\beta^{x_i,T}(s)}\int_s^T F^u(s,S_s)Z_s^{x_i,u}\beta^{x_i,u}(s)\,\mathrm{d}u\,\mathrm{d}Y_s^{x_i},
\end{aligned}$$

where in the second equation, we used (2.6) and (4.22). Finally,

$$\mathbb{E}[A_T^a \,|\, \mathcal{G}_t] = \mathbb{E}[A_T^a \,|\, \mathcal{G}_0] + \int_0^t \xi_s^X\,\mathrm{d}X_s + \int_0^t \xi_s^Y\,\mathrm{d}Y_s + L_t^a, \tag{4.29}$$

for $t \in [0,T]$, where the investment in the (discounted) risky asset X is given by

$$\xi_t^X = \sum_{i=1}^m \zeta(x_i)(n^{x_i} - N_t^{x_i})e^{\Gamma_t^{x_i}}\int_t^T Z_t^{x_i,u}F_s^u(t,S_t)\,\mathrm{d}u,$$

and the investment in the family of (discounted) longevity bonds $Y = (Y^{x_1},\dots,Y^{x_m})$ is given by $\xi_t^Y = (\xi_t^{Y^{x_1}},\dots,\ \xi_t^{Y^{x_m}})$, with

$$\xi_t^{Y^{x_i}} = \zeta(x_i)(n^{x_i} - N_t^{x_i})\frac{e^{\Gamma_t^{x_i}}e^{r(T-t)}}{Z_t^{x_i,T}\beta^{x_i,T}(t)}\int_t^T F^u(t,S_t)Z_t^{x_i,u}\beta^{x_i,u}(t)\,\mathrm{d}u,$$

and

$$L_t^a = -\sum_{i=1}^m \zeta(x_i)\int_{]0,t]}\mathbb{E}\left[\int_s^T \frac{f(S_u)}{B_u}e^{\Gamma_s^{x_i}-\Gamma_u^{x_i}}\,\mathrm{d}u\,\Big|\,\mathcal{F}_s\right]\mathrm{d}M_s^{x_i},$$

$t \in [0,T]$. By the same arguments as in the proofs of Propositions 4.7 and 4.5, the terms in (4.29) are square integrable and strongly orthogonal, hence (4.29) is indeed the GKW decomposition of $\mathbb{E}[A_T^a \,|\, \mathcal{G}_t]$, $t \in [0,T]$. □

Note that Proposition 5.1.2 and Corollary 5.1.3 of Bielecki and Rutkowski [8] require the process $f(S_t)$, $t \in [0,T]$, to be bounded. However, it can be seen that this result also holds if $\mathbb{E}[\sup_{t\in[0,T]} f(S_t)^2] < \infty$ and we may therefore apply it in our setting.

4.4 Pure endowment contract

For the pure endowment contract introduced in (2.10), we define the payment process

$$A_t^{pe} = \frac{f(S_t)}{B_t} \sum_{i=1}^{m} \zeta(x_i) \sum_{j=1}^{n^{x_i}} \mathbb{1}_{\{\tau^{x_i,j} > t\}} \mathbb{1}_{\{t=T\}}, \quad t \in [0,T], \tag{4.30}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a Borel measurable function such that

$$\mathbb{E}\left[f(S_T)^2\right] < \infty. \tag{4.31}$$

Proposition 4.7. *In the setting of Section 2, the payment process A^{pe} introduced in* (4.30) *admits a risk-minimizing strategy $\varphi = (\xi, \xi^0) = (\xi^X, \xi^Y, \xi^0)$ with discounted value process*

$$V_t^{pe}(\varphi) = \mathbb{E}[A_T^{pe} \mid \mathcal{G}_0] + \int_0^t \xi_s^X \, \mathrm{d}X_s + \int_0^t \xi_s^Y \, \mathrm{d}Y_s + L_t^{pe} - A_t^{pe}, \tag{4.32}$$

and

$$\xi_t^0 = V_t^{pe}(\varphi) - \xi_t^X X_t - \xi_t^Y \cdot Y_t$$

for $t \in [0,T]$, where the investment in the (discounted) risky asset X is given by

$$\xi_t^X = F_s^T(t, S_t) \sum_{i=1}^{m} \zeta(x_i)(n^{x_i} - N_t^{x_i}) e^{\Gamma_t^{x_i}} Z_t^{x_i,T}, \tag{4.33}$$

and the investment in the family of (discounted) longevity bonds $Y = (Y^{x_1}, \ldots, Y^{x_m})$ is given by $\xi_t^Y = (\xi_t^{Y^{x_1}}, \ldots, \xi_t^{Y^{x_m}})$, with

$$\xi_t^{Y^{x_i}} = F^T(t, S_t) e^{r(T-t)} \zeta(x_i)(n^{x_i} - N_t^{x_i}) e^{\Gamma_t^{x_i}}, \tag{4.34}$$

and

$$L_t^{pe} = -\sum_{i=1}^{m} \zeta(x_i) \int_{]0,t]} \frac{F^T(s, S_s)}{B_s} e^{\Gamma_s^{x_i}} Z_s^{x_i,T} \mathrm{d}M_s^{x_i},$$

$t \in [0,T]$, where $F^T(t, S_t)$, $F_s^T(t, S_t)$, $Z_t^{x_i,T}$ and $M_t^{x_i}$ are defined in (4.2), (4.13), (4.15) *and* (4.16)*. The optimal cost and risk processes are given by*

$$\begin{aligned} C_t^{pe}(\varphi) &= \mathbb{E}[A_T^{pe} \mid \mathcal{G}_0] + L_t^{pe}, \\ R_t^{pe}(\varphi) &= \mathbb{E}[(L_T^{pe} - L_t^{pe})^2 \mid \mathcal{G}_t], \end{aligned} \tag{4.35}$$

for $t \in [0,T]$.

Proof. For the proof, we refer to Proposition 4.4.5 of Schreiber [41]. □

Remark 4.8. *If the function f satisfies additional regularity conditions, then decomposition* (4.32) *can be obtained from decompositions* (4.19) *and* (4.28) *for the term insurance and the annuity contract by using Itô's formula, as we show in the following. However our method is more general as it only requires assumption* (4.31)*. W.l.o.g. suppose $n = m = 1$, $B = x$ and $\zeta(x) = 1$. Furthermore let $f \in \mathcal{C}^2(\mathbb{R})$. We define $\tau := \tau^x$ and $H_t := \mathbb{1}_{\{\tau \le t\}}$, $t \in [0, T]$. Then by Itô's formula we have that*

$$\begin{aligned} A_T^{pe} = \frac{f(S_T)}{B_T}\mathbb{1}_{\{\tau > T\}} &= \int_0^T \frac{f(S_s)}{B_s}\,\mathrm{d}(1 - H_s) + \int_0^T (1 - H_s)\,\mathrm{d}\left(\frac{f(S_s)}{B_s}\right) \\ &= -\frac{f(S_\tau)}{B_\tau}\mathbb{1}_{\{\tau \le T\}} + \int_0^T \mathbb{1}_{\{\tau > s\}} f'(S_s)\,\mathrm{d}X_s \\ &\quad + \int_0^T \mathbb{1}_{\{\tau > s\}} \frac{1}{B_s} \underbrace{\left(rS_s f'(S_s) - rf(S_s) + \frac{\sigma^2(s, S_s)}{2} S_s^2 f''(S_s)\right)}_{=:\eta_s} \mathrm{d}s, \end{aligned} \tag{4.36}$$

i.e. the pure endowment contract can be seen as the sum of a term insurance contract, an investment in the discounted asset price and an annuity contract. If now f and η satisfy respectively (4.18) *and* (4.27)*, and $\mathbb{E}[(\int_0^T f'(S_t)\mathrm{d}X_t)^2] < \infty$, then the risk-minimizing strategy can be computed via decomposition* (4.36) *by using the results of Propositions 4.5 and 4.6.*

5 Simulation study

In this section, we perform a simulation study for the case of the pure endowment contract introduced in (2.10) and $m = n = 2$, i.e., we have two individuals belonging to two different cohort classes x_1 and x_2, and we set $x_1 = 25$ and $x_2 = 40$. Based on numerical simulations, we compute the paths of the optimal risk-minimizing strategies. We work with the Gaussian intensity field model introduced in Subsection 3.1, where $\bar{\mu}$ as in (3.4) is generated by the standard forecasting method of the Lee-Carter model (see Lee and Carter [33] and Lee [32]) based on historical data. The dataset is comprised of historical death rates for the US population from 1933 to 2010, with one-year age groups from the Human Mortality Database. Following Luciano and Vigna [35], for the parameters in (3.2) we choose $\sigma = 0.001$ and $\alpha = \theta = 0.05$. Then, as μ_t^x is normally distributed for $x \in [\underline{x}, \overline{x}]$ and $t \in [0, T]$, we have that

$$\mathbb{P}(\mu_t^x < 0) = \Phi\left(-\frac{\mathbb{E}[\mu_t^x]}{\sqrt{\mathrm{Var}(\mu_t^x)}}\right) = \Phi\left(-\frac{\bar{\mu}(t, x)\sqrt{2\alpha\theta}}{\sigma}\right) \approx 0.02, \tag{5.1}$$

See www.mortality.org.

where $\bar{\mu}(t,x) \approx 0.03$, based on historical data. With this parametrization, we obtain a realistic correlation structure between different age cohorts, as shown by Table 2. At the same time, we obtain a non-trivial random distortion of the

	$\lvert t-s\rvert$								
$\lvert x-y\rvert$	0	1	2	3	4	5	10	20	30
0	1.00	0.95	0.90	0.86	0.82	0.78	0.61	0.37	0.22
1	0.95	0.90	0.86	0.82	0.78	0.74	0.58	0.35	0.21
2	0.90	0.86	0.82	0.78	0.74	0.70	0.55	0.33	0.20
3	0.86	0.82	0.78	0.74	0.70	0.67	0.52	0.32	0.19
4	0.82	0.78	0.74	0.70	0.67	0.64	0.50	0.30	0.18
5	0.78	0.74	0.70	0.67	0.64	0.61	0.47	0.29	0.17
10	0.61	0.58	0.55	0.52	0.50	0.47	0.37	0.22	0.14
20	0.37	0.35	0.33	0.32	0.30	0.29	0.22	0.14	0.08
30	0.22	0.21	0.20	0.19	0.18	0.17	0.14	0.08	0.05

Table 2: Correlation structure for $\mathrm{Corr}(\mu_{t,x}, \mu_{s,y}) = e^{-\theta|t-s|}e^{-\alpha|x-y|}$, for $s,t \in [0,T]$, $x,y \in I$, as introduced in (3.7) with $\alpha = \theta = 0.05$.

deterministic driving part of equation (3.1). Following Kroese and Botev [31], we use the circulant embedding method for Toeplitz covariance matrices, in order to efficiently simulate the space-time changed Brownian sheet. Figure 1 depicts $\bar{\mu}(t,x)$ introduced in (3.4) for $x \in [25,40]$, $t \in [0,50]$, as well as an exemplary realization of $O_{t,x}$ and $\mu_{t,x}$ defined in (3.1) and (3.2). We set

$$\zeta(x_i) = \frac{x_i}{\sum_{i=1}^m x_i}$$

for $i = 1,2$, i.e., the weighting function is increasing in age. For the asset price S, we assume $\sigma(t,S_t) \equiv 0.3$, $t \in [0,T]$, and $S_0 = 100$ in (2.5). Furthermore, $f(x) = x$, $x \in \mathbb{R}_+$, hence for the payment process in (4.30), we obtain

$$A_t^{pe} = X_t \sum_{i=1}^{2} \zeta(x_i) \mathbb{1}_{\{\tau^{x_i} > t\}} \mathbb{1}_{\{t=T\}}, \quad t \in [0,T],$$

where $\tau^{x_i} := \tau^{x_i,1}$ for $i = 1,2$. Condition (4.31) is then automatically satisfied for this choice of S and f. From (4.33), the investment in the (discounted) risky asset X is given by

$$\xi_t^X = \sum_{i=1}^{2} \mathbb{1}_{\{\tau^{x_i} > t\}} \zeta(x_i) e^{\Gamma_t^{x_i}} Z_t^{x_i,T}, \tag{5.2}$$

and, by (4.34), the investment in the family of (discounted) longevity bonds $Y = (Y^{x_1}, Y^{x_2})$ is determined by

$$\xi_t^Y = (\xi_t^{Y^{x_1}}, \xi_t^{Y^{x_2}}) \quad \text{with} \quad \xi_t^{Y^{x_i}} = S_t e^{r(T-t)} \mathbb{1}_{\{\tau^{x_i} > t\}} \zeta(x_i) e^{\Gamma_t^{x_i}}, \tag{5.3}$$

for $t \in [0,T]$, $i = 1,2$, where $\Gamma_t^{x_i}$ and $Z_t^{x_i,T}$ are given in (2.1) and (4.2). The simulation of $Z^{x_i,T}$ uses (4.2), where $\beta^{x,T}$ is determined by (4.3) and (4.11). The stopping times τ^{x_1} and τ^{x_2} are generated by the canonical construction method following Bielecki and Rutkowski [8, Chapter 8]. Figure 2 shows exemplary paths of the hedging strategies. Note that we can observe jumps in the strategies at the time, when an insured person has died. In particular, we can observe up to two jumps in the investment in the risky asset, because it depends on the hazard process of both insured individuals and up to one jump in the investments in the two longevity bonds. One can also see that positions are automatically closed when the respective risk no longer exists, i.e., in case of a death before the maturity of the contract. Investments in the longevity bonds also appear more irregular than the investment in asset S, due to the influence of the asset price's volatility on the strategies in (5.3). More details and further examples can be found in Biagini et al. [7].

6 Conclusion

The main contribution of this work is to provide a first step in the modeling of cross-generational dependencies in an insurance portfolio consisting of different age cohorts by using Gaussian and χ^2-random fields. This approach provides a flexible framework, where analytical results can be derived and easily implemented. Unfortunately, the Gaussian intensity model, although very convenient due to its simplicity, analytical tractability and intuitive interpretation, allows for negative values with positive probability. However, although one cannot exclude negative mortality rates, within our simulation study, we demonstrate that in applications the probability of negative values turns out to be very small, when using calibrated parameters. In this setting, we computed risk-minimizing strategies for typical building blocks of life insurance liabilities, thereby taking into account the dependency structure between different age cohorts. We deliberately kept model complexity low in order to obtain explicit results that can be implemented, calibrated and backtested on data. From the point of view of controling model risk, our setting presents the advantages of providing a clear correlation structure, low complexity and high applicability. Implementation and calibration can be easily performed, so that the model can be tested and possible deficiences detected. After this initial, nevertheless technically non-trivial analysis, additional empirical studies must follow to validate the model. This will provide an indication on the goodness of fit of our approach and on possible extensions and improvements.

Acknowledgment

We wish to thank Rebecca Declara and Ran Zhang for interesting discussions and remarks, as well as their help with the simulations.

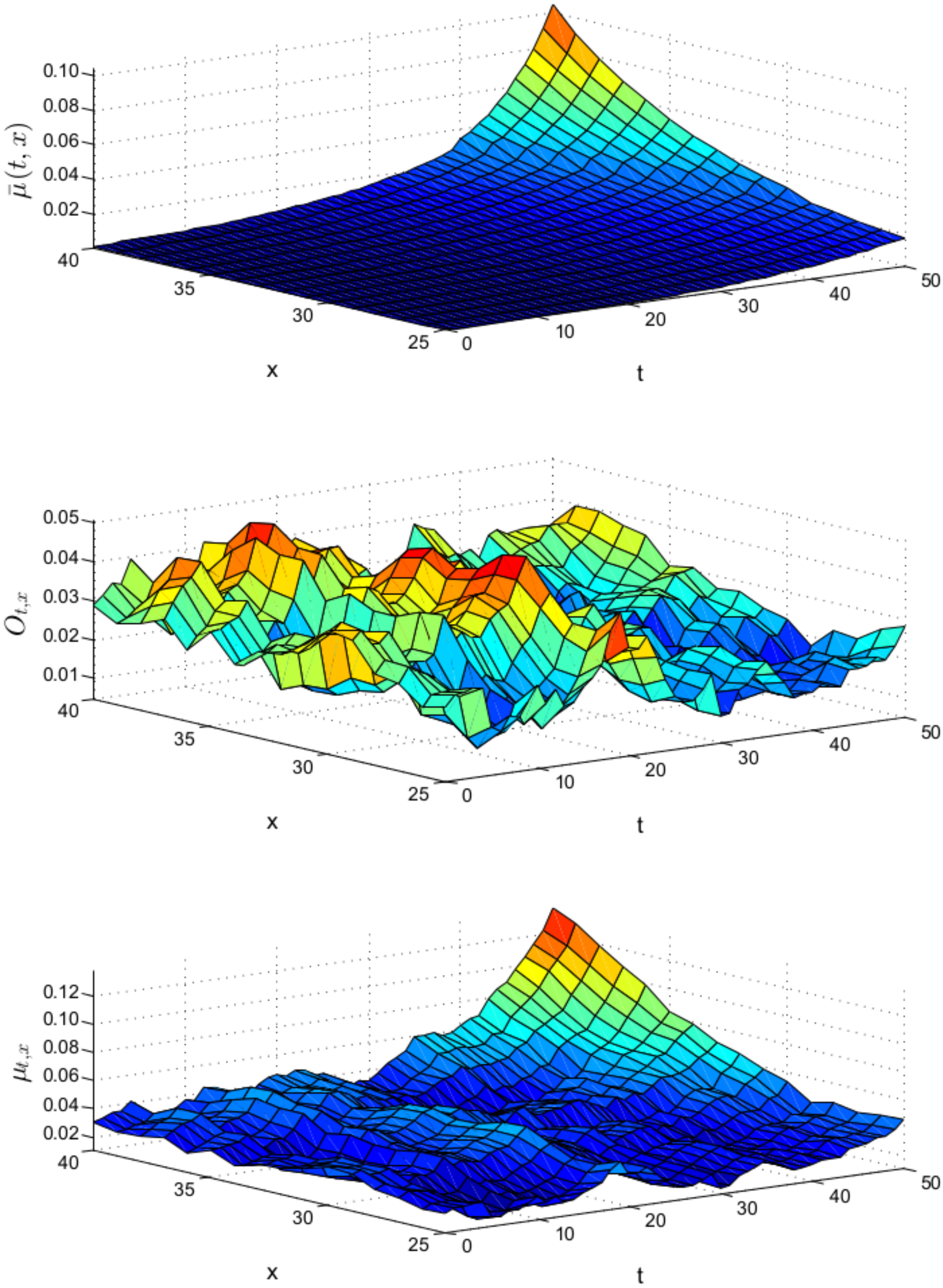

Figure 1: This plot depicts $\bar{\mu}(t,x)$, $O_{t,x}$ and $\mu_{t,x}$ as defined in (3.1), (3.2) and (3.4) for $x \in [25,40]$, $t \in [0,50]$ and with parametrization $\sigma = 0.001$, $\alpha = \theta = 0.05$.

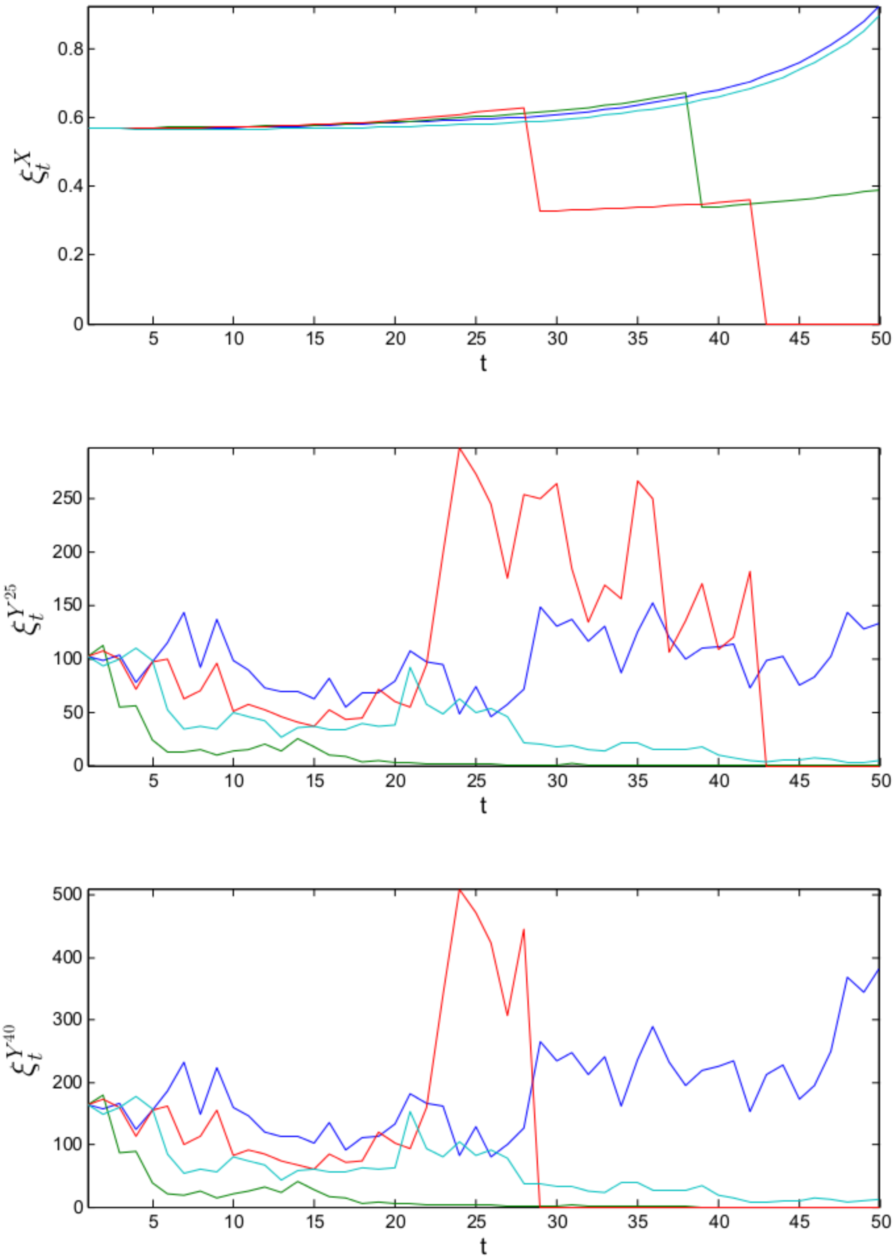

Figure 2: Exemplary paths of the optimal risk-minimizing hedging strategy $\xi = (\xi^X, \xi^{Y^{25}}, \xi^{Y^{40}})$ for $t \in [0, 50]$ and with parametrization $\sigma = 0.001$, $\alpha = \theta = 0.05$.

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