The Second Fundamental Theorem of Asset Pricing

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Abstract

The second fundamental theorem of asset pricing (in short, **sft**) concerns the mathematical characterization of the economic concept of *market completeness* for liquid and frictionless markets with an arbitrary number of assets. The theorem establishes the mathematical necessary and sufficient conditions in order to guarantee that every contingent claim on the market can be duplicated with a portfolio of primitive assets. For finite assets economies, completeness (i.e. *perfect replication* of every claim on the market by admissible self-financing strategies) is equivalent to uniqueness of the equivalent martingale measure. This result can be extended to market models with an infinite number of assets by defining completeness in terms of *approximate replication* of claims by attainable ones.

Keywords: market completeness, equivalent martingale measure, predictable representation property.

The **second fundamental theorem of asset pricing** (in short, **sft**) concerns the mathematical characterization of the economic concept of *market completeness* for liquid and frictionless markets with an arbitrary number of assets. The theorem establishes the mathematical necessary and sufficient conditions in order to guarantee that every contingent claim on the market can be duplicated with a portfolio of primitive assets. For finite assets

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economies, completeness (i.e. *perfect replication* of every claim on the market by admissible self-financing strategies) is equivalent to uniqueness of the equivalent martingale measure. This result can be extended to market models with an infinite number of assets by defining completeness in terms of *approximate replication* of claims by attainable ones. Hence several definitions of completeness are possible and in the sequel we will present and discuss them extensively.

1 Markets with a finite number of asset prices

The first form of the **sft** appeared in [9] under the assumption of zero interest rate and that the agent employs only simple trading strategies in order to address the following issue, raised in the economic literature ([1], [19], [22]): given a financial market, which contingent claims are "spanned" by a given set of market securities?

In the seminal paper [7] it was already observed that in the idealized Black-Scholes market the cash flow of an option can be duplicated by managing a portfolio containing only stock and bond. A natural question is then: for which contingent claim does this result hold in more general markets? When does it hold for *all* contingent claims on the market?

For markets with a finite number of asset prices, the answer to this problem was provided for the first time in [10] and in [11]. Here we follow the notation of [11] in order to state the **sft**.

Let $T < \infty$ be a fixed time horizon and consider a probability space $(\Omega, \mathcal{F}, P)$ endowed with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions and such that $\mathcal{F}_0$ contains only Ω and the null sets of P and with $\mathcal{F}_T = \mathcal{F}$. Let $S = (S_t^0, \cdots, S_t^d)_{t\in[0,T]}$ be a $(d+1)$-dimensional strictly positive semimartingale, whose components $S^0, \cdots, S^d$ are right-continuous with left limits. Moreover we assume that $S_0^0 = 1$. Here the stochastic process S_t^k represents the value at time t of the kth security on the market. The discounted price process $Z = (Z_t^1, \cdots, Z_t^d)_{t\in[0,T]}$ is then defined by setting $Z^k = S^k/S^0$, for $k = 1, \cdots, d$. Let $\mathbb{P}$ be the set of probability measures Q on $(\Omega, \mathcal{F})$ that are equivalent to P and such that Z is a (vector) martingale under Q. We assume that $\mathbb{P}$ is not empty, i.e. that the market is arbitrage-free (see Eqf04/002). We fix an element P^* in $\mathbb{P}$ and denote by E^* the expectation under P^*. Let $L(Z)$ denote the set of all vector-valued, predictable processes $H = (H_t^1, \cdots, H_t^d)_{t\in[0,T]}$ that are *integrable* with respect to the semimartingale Z. For further details on $L(Z)$, we refer to Remark 1.3.

Definition 1.1. *A stochastic process $\phi \in L(Z)$ is said to be an* admissible self-financing strategy *if*

(i) the discounted value process $V^*(\phi) := \sum_{k=1}^{d} \phi^k Z^k$ *is almost surely non-negative;*

(ii) $V^*(\phi)$ *satisfies the* self-financing condition

$$V_t^*(\phi) = V_0^*(H) + \int_0^t \sum_{k=1}^{d} \phi_s^k \mathrm{d}Z_s^k, \ t \in [0,T];$$

(iii) $V^*(\phi)$ *is a martingale under* P^*.

Condition *(iii)* is here introduced to rule out "certain foolish strategies that throw out money" ([11]), i.e. for no-arbitrage reasons. Note also that in the above definition only the last condition may depend on the choice of the reference measure P^*.

A *contingent claim* X with maturity T is then represented by a non-negative ($\mathcal{F}_T$-measurable) random variable. Such a claim is said to be *attainable* if there exists an admissible trading strategy ϕ such that $V_T^*(\phi) = X/S_T^0$. The model is said to be **complete** if every claim[1] is attainable.

Theorem 1.2. *(***The second fundamental theorem of asset pricing***, [11]) Let $\mathbb{P} \neq \emptyset$. Then the following statements are equivalent:*

1. *The model is complete under P^*.*
2. *Every P^*-martingale M can be represented in the form*

$$M_t = M_0 + \int_0^t \sum_{k=1}^{d} H_s^k \mathrm{d}Z_s^k, \ t \in [0,T], \tag{1}$$

*for some $H \in L(Z)$ (*predictable representation property*).*

3. *$\mathbb{P}$ is a singleton, i.e. there exists a* unique *equivalent martingale measure for Z.*

The proof of this theorem relies on some results of [12] and [13], Chapter XI, relating the representation property (1) to a condition involving a certain set of probability measures.

[1] We say that a contingent claim is integrable if $E^*\left[X/S_T^0\right] < \infty$. By Definition 1.1, it follows that an attainable contingent claim is necessarily integrable. Hence we can restate the definition of market completeness as follows. The model is said to be **complete** if every integrable claim is attainable.

Remark 1.3. *In Theorem 1.2 the definition of the space $L(Z)$ is crucial, as shown by a counterexample in [20]. By [15] we obtain that $L(Z)$ must be the largest class of integrands over which multidimensional integrals with respect to Z can be defined, as done implicitly in [11]. Hence by Theorem 4.6 of [13] we have that $L(Z)$ is the space of the vector-valued, predictable processes $H = (H_t^1, \cdots, H_t^d)_{t\in[0,T]}$ such that*

$$\int_0^t \sum_{i,j=1}^{d} H_s^i H_s^j \mathrm{d}[Z^i, Z^j]_s, \ t \in [0,T], \tag{2}$$

is locally integrable.

Completeness can be easily characterized in some particular cases, as shown by the following examples.

Example 1.4. Consider a market with a finite number of assets in discrete times $\{0, \cdots, T\}$ and let $\mathcal{P}_t$ be the partition of Ω underlying $\mathcal{F}_t$. For each cell A of $\mathcal{P}_t$, $t \in \{0, \cdots, T-1\}$, we define as *splitting index of A* the number $K_t(A)$ of cells of $\mathcal{P}_{t+1}$, which are contained in A. Then completeness can be characterized as follows.

Proposition 1.5. *(Proposition 2.12 of [10]) Let $\mathbb{P} \neq \emptyset$ and suppose that the securities are not redundant*[2]*. Then the model is complete if and only if $K_t(A) = d + 1$ for all $A \in \mathcal{P}_t$ and $t = 0, \cdots, T-1$.*

Hence completeness is a matter of dimension. Corollary 4.2 of [23] shows that if the market is complete, then the splitting index $K_t(A)$ is determined by the price process S only, i.e. for every $t = 0, \cdots, T$ and each $A \in \mathcal{P}_t$, we have $K_t(A) = \dim(\mathrm{span}\,\{\mathrm{S_{t+1}}(\omega) : \omega \in \mathrm{A}\})$. Hence it is sufficient to check if the rank of the matrix with columns formed by the vectors $S_{t+1}(\omega), \omega \in A$, equals the splitting index $K_t(A)$ of A. By using this geometric property of the sample paths of the price process, an algorithm is then provided in [23], to check if finite securities markets in discrete times are complete.

Example 1.6. In the case when security prices follow Itô processes on a multidimensional Brownian filtration, completeness of the market can be characterized in terms of the volatility matrix of the underlying asset prices, as shown in [3], [15] and [18]. Consider a market with d risky assets given by Itô processes of the form

$$S_t^i = S_0^i \exp\left[\int_0^{\mathrm{t}} \alpha_{\mathrm{s}}^{\mathrm{i}} \mathrm{ds} - 1/2 \sum_{\mathrm{j=1}}^{\mathrm{n}} \int_0^{\mathrm{t}} (\sigma_{\mathrm{s}}^{\mathrm{ij}})^2 \mathrm{ds} + \sum_{\mathrm{j=1}}^{\mathrm{n}} \int_0^{\mathrm{t}} \sigma_{\mathrm{s}}^{\mathrm{ij}} \mathrm{dW}_{\mathrm{s}}^{\mathrm{j}}\right], \ \mathrm{t} \in [0, \mathrm{T}],$$

[2]The price process is said to contain a redundancy if $P(\alpha \cdot S_{t+1} = 0|A) = 1$ for some nontrivial vector α, some $t < T$ and some $A \in \mathcal{P}_t$.

$i = 1, \cdots, d$, on the probability space $(\Omega, \mathcal{F}, P)$ endowed with the (augmented) natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$ generated by the n-dimensional Brownian motion $W = (W_t^1, \cdots, W_t^n)_{t\in[0,T]}$ with $\mathcal{F}_T = \mathcal{F}$. Here S^0 can be assumed constantly equal to 1 for the sake of simplicity. For $t \in [0,T]$ we denote by $\Sigma_t(\omega)$ the (random) volatility matrix, whose entries are given by

$$[\Sigma_t(\omega)]_{ij} = \sigma_t^{ij}(\omega),\ i = 1, \cdots, d,\ j = 1, \cdots, n.$$

If for all $i = 1, \cdots, d$, S_0^i is a positive constant, $(\alpha_t^i)_{t\in[0,T]}$ an adapted stochastic process with

$$\int_0^T |\alpha_s^i| \mathrm{d}s < \infty,\ \text{a.s.} \tag{3}$$

and $(\sigma_t^{ij})_{t\in[0,T]}$ are adapted stochastic processes with

$$\int_0^T (\sigma_s^{ij})^2 \mathrm{d}s < \infty,\ \text{a.s.} \tag{4}$$

for $j = 1, \cdots, n$, then following characterization of market completeness holds.

Theorem 1.7. *(Theorem 4 of [3], Theorem 2.2 and 3.2 of [15]) Let $\mathbb{P} \neq \emptyset$. Then the market is complete if and only if* $P(\mathrm{rank}(\Sigma_t) = \mathrm{d}$ *for almost all* $\mathrm{t} \in [0, \mathrm{T}]) = 1$.

For further references, see also Theorem 4.1 of [18]. Since there are n sources of randomness represented by the Brownian motions, it is natural to expect that n sufficiently independent asset prices are needed for completeness. Clearly, if $d < n$ the market cannot be complete.

Example 1.8. If price processes have a finite number of jumps, then we obtain again a characterization of completeness in terms of the volatility matrix, as shown by the following theorem due to [3]. We set again $S^0 = 1$ and consider price processes driven by a multivariate point process[3] μ with compensator $\nu(\mathrm{d}t, \mathrm{d}x) = K_t(\mathrm{d}x)\mathrm{d}t$ such that

$$S_t^i = S_0^i \mathcal{E}\left(R^i\right)_t,\ t \in [0,T],\ i = 1, \cdots, d,$$

with

$$R_t^i = \int_0^t \alpha_s^i \mathrm{d}s + \int_{[0,t]\times E} \sigma^i(u,x)(\mu(\mathrm{d}u, \mathrm{d}x) - \nu(\mathrm{d}u, \mathrm{d}x)),\ t \in [0,T],\ i = 1, \cdots, d,$$

[3] Let E be a Blackwell space. An E- multivariate point process is an integer-valued random measure on $[0,T] \times E$ with $\mu([0,t] \times E) < \infty$ for every $\omega, t \in [0,T]$ (see Definition III.1.23 of [14]).

where the $\sigma^i(t,x)$'s are bounded $\mathrm{d}\mu \otimes \mathrm{d}P$-a.e., $\mathcal{E}$ is the Doléans exponential (for the definition, we refer to Theorem I.4.61 of [14]) and $E \subset \mathbb{R}$. Note that here σ^i, μ and ν may depend on ω, but for the sake of simplicity we do not indicate this dependence. In this context asset prices may have jumps, that can be thought as the result of possible shocks which may trigger the market. If the cardinality $|E|$ of E is finite, we denote again by Σ_t the volatility matrix, whose row vectors are given by $(\sigma^i(t,x))_{x\in E}$, $i = 1, \cdots, d$.

Theorem 1.9. *(Theorem 5 of [3]) Let* $\mathbb{P} \neq \emptyset$, $|E| < \infty$ *and* $K_t(\{x\}) > 0$ *for every* $x \in E$. *Then the market is complete if and only if* $P(\mathrm{rank}(\Sigma_t) = |\mathrm{E}|$ *for almost all* $\mathrm{t} \in [0, \mathrm{T}]) = 1$.

Also in the case of a finite number of jumps that may trigger the economy, the characterization of market completeness is similar to the Itô price process case, i.e. one needs $|E|$ sufficiently independent processes for completeness in presence of $|E|$ sources of randomness, given by the $|E|$ different possible shocks.

We have seen that the key to completeness is the predictable representation property. Hence a natural question concerns for which kind of martingales the predictable representation property is satisfied. For the continuous case, we have that the predictable representation property holds for diffusion processes that are martingales and have either Lipschitz coefficients ([24]) or non-degenerate diffusion matrix and continuous coefficients ([12]). The only one-dimensional martingales with stationary and independent increments that satisfy the predictable representation property are the Wiener and the Poisson martingales ([25]). Hence the representation property holds for finite Lévy measures, but it fails for infinite Lévy measures. In the next Section we discuss the **sft** in the case of infinite dimensional financial markets.

2 Markets with an infinite number of asset prices

Many applications of hedging involve dynamic trading in principle in infinitely many securities, for example in pricing of interest rate derivatives by using pure discount bonds, or in the use of the term and strike structure of European put and call options to hedge exotic derivatives, when asset prices are driven by Lévy measures. Hence it is natural to develop infinite dimensional market models to address this kind of issues. The problem is now to establish if the **sft** still holds, if the market is endowed with an infinite number of assets.

By defining a complete market via the density of a vector space, the **sft** is in [8] proved to hold true for (infinitely many) continuous and bounded asset price processes, if all the martingales with respect to the reference filtration $\mathcal{F}_t$ are continuous ([8], Theorem 6.7). In the case of a general filtration, Theorem 6.5 of [8] states that completeness is equivalent for P^* to be an extreme point of $\mathbb{P}$, i.e. a weaker version of the **sft** holds.

The hypothesis of continuity cannot be dropped and in the presence of jump discontinuities and infinitely many assets, a counterexample to the **sft** is provided in [2], where an economy with infinitely many assets is constructed, in which the market is complete, but yet there exists an infinity of equivalent martingale measures.

Since the formulation of this counterexample, many papers have studied the problem of extending the result of the **sft** to markets with infinitely many assets. Since many definitions of completeness are possible, the solution to the counterexample of [2] relies on the choice of the definition of completeness that is adopted. A first answer to this problem was provided in 1997 by [6] and [5], where Theorem 6.11 shows that in presence of infinitely many assets and a continuum of jump sizes, the uniqueness of the equivalent martingale measure is equivalent to the market being *approximately complete*, i.e. every bounded contingent claim can be approached in $L^2(Q)$ for some $Q \in \mathbb{P}$ by a sequence of hedgeable claims.

In 1999 a certain number of papers have appeared ([3], [4], [16], [17]) at the same time, where new definitions of market completeness were proposed in order to maintain the **sft**, even in complex economies. The equivalence between market completeness and uniqueness of the pricing measure is maintained by introducing a notion of market completeness, that is independent both of the notion of no-arbitrage and of a chosen equivalent martingale measure. In finite-dimensional markets, the definition of market completeness is given in terms of replicating value processes in economies without arbitrage possibilities and with respect to a given equivalent martingale measure. However the issue of completeness is about the ability of replicating certain cash flows, and not about how these cash flows are valued or whether these values are arbitrage-free. From this perspective, the appropriate measure to address the issue of completeness is the statistical probability measure P, and not an equivalent martingale measure, that may also not exist. In [17] this new approach was also motivated by the empirical asset pricing literature. Moreover an example in [3] shows an economy, where the existence of an equivalent martingale measure precludes the possibility of market completeness. Hence in [3], [4], [16] and [17], the concept of exact (almost everywhere) replication of a contingent claim via an admissible portfolio is substituted by the notion of *approximation* of a contingent claim. The main outlines of this approach

are the following.
Let $\mathbb{M}$ denote the space of the P-absolutely continuous signed measures on $\mathcal{F}_T$. Then $Q \in \mathbb{M}$ can be interpreted as a market agent's personal way of assigning values to claims, i.e. the set $\mathbb{M}$ represents the possible contingent claims valuation measures held by traders. An agent using the valuation measure $Q \in \mathbb{M}$ assigns to a contingent claim H the value $\int H \mathrm{d}Q$. The fact that $\mathbb{M}$ is given by the P-absolutely continuous signed measures on $\mathcal{F}_T$ has two particular meanings: first that all traders agree on null events, and secondly, that there can be strictly positive random variables with negative personal value. For a given trader, represented by $Q \in \mathbb{M}$, two contingent claims H_1 and H_2 are approximately equal if

$$|\int (H_1 - H_2)\mathrm{d}Q| < \epsilon \text{ for small } \epsilon > 0.$$

Denote by $\mathcal{C}$ the space of all bounded contingent claims. The finite intersections of the sets of the form $B(H_1, \epsilon) = \left\{H_2 \in \mathcal{C} \mid |\int (H_1 - H_2)\mathrm{d}Q| < \epsilon\right\}$, $H_1 \in \mathcal{C}$ and $\epsilon > 0$, give a basis for a topology τ^Q on $\mathcal{C}$. We endow $\mathcal{C}$ with the coarsest topology τ finer than all of the τ^Q, $Q \in \mathbb{M}$. This topology is now agent-independent, i.e. two claims are approximately equal if all the agents believe that their values are close. The topology τ is usually referred as the weak* topology on $\mathcal{C}$ (see [21]).
An agent is then allowed to trade in a finite number of assets via self-financing, bounded, stopping time simple strategies that yield a bounded payoff at T. As in the previous Section, a (bounded) claim is said to be attainable if it can be replicated by one of such strategies. In this setting the market is said to be *quasicomplete* if any contingent claim $H \in \mathcal{C}$ can be approximated by attainable claims in the weak* topology induced by $\mathbb{M}$ on $\mathcal{C}$. Since the weak* topology as well as the trading strategies are agent measure independent, the same is true for this notion of completeness. Consider now the space $\mathbb{P}_\pm$ of the P-absolutely continuous signed martingale measures. Then the following generalized version of the **sft** holds.

Theorem 2.1. *(***The second fundamental theorem of asset pricing***, Theorem 2 of [3], Theorem 1 of [4], Theorem 5 of [17]) Let $\mathbb{P}_\pm \neq \emptyset$. Then there exists a unique P-absolutely continuous signed martingale measures if and only if the market is quasicomplete.*

The proof of this theorem relies on the theory of linear operators between locally convex topological vector spaces.
Since the market is endowed with an infinite number of assets, in principle trading in infinitely many assets may be possible. To take in account this possibility, in [5], [6], [16] and [17] portfolios consisting of infinitely many

assets are allowed by considering measure-valued strategies. The result of Theorem 2.1 still holds in the case of market models, where measure-valued strategies are allowed, as shown in Theorem 6.11 of [5] and Theorem 2.1 of [16].

This approach resolves the paradox of the counterexample of [2], since the economy considered in [2] is incomplete under this new definition of market completeness. Moreover if $\mathbb{P} \neq \emptyset$ and the number of assets is finite or the asset prices are given by continuous processes, then Theorem 5 of [4] shows that the market model is quasicomplete if and only if it is complete.

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