Optional projection under equivalent local martingale measures

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Abstract

In this paper we study optional projections of $\mathbb{G}$-adapted strict local martingales into a smaller filtration $\mathbb{F}$ under changes of equivalent martingale measures. Some general results are provided as well as a detailed analysis of two specific examples given by the inverse three dimensional Bessel process and a class of stochastic volatility models. This analysis contributes to clarify some properties, for example absence of arbitrage opportunities of market models under restricted information.

1 Introduction

In this paper we study optional projections of $\mathbb{G}$-adapted strict local martingales into a smaller filtration $\mathbb{F}$ under changes of equivalent local martingale measures.

It is a well known fact that, when projecting a stochastic process into a filtration with respect to which it is not adapted, some attributes of its dynamics may change, see for example Föllmer and Protter [2011] and Bielecki et al. [2018], where the authors study the semimartingale characteristics of projections of special semimartingales. Moreover, some basic properties of the process can be lost. Most notably, the optional projection of a local

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martingale may fail to be a local martingale, see for example [Föllmer and Protter, 2011, Theorem 3.7] and [Larsson, 2014, Corollary 1], where conditions under which this happen are stated, and Kardaras and Ruf [2019a] for a study of optional projections of local martingale deflators.

In the literature it is a classical problem to investigate financial market models under restricted information, where full information on the asset prices is not available and agent's decisions are based on a restricted information flow. This could be due to a delay in the diffusion of information or to an incomplete data flow, as it may happen for example when investors only see an asset value when it crosses certain levels, see Jarrow et al. [2007]. In general this is represented by assuming that trading strategies are adapted to a filtration which can be smaller than the filtration $\mathbb{G}$ generated by the asset prices.

In Cuchiero et al. [2020] and Kabanov and Stricker [2006] it is shown that the analysis of characteristics and properties of financial markets under restricted or delayed information boils down to study optional projections of the asset prices on the smaller filtration. In particular in both papers it is shown how a certain "no-arbitrage" condition is equivalent to the existence of an equivalent measure Q such that the Q-optional projection of the asset price on $\mathbb{F}$ is a martingale. This setting can be applied to trading with delay, trading under restricted information, semistatic hedging and trading under transaction costs.

A further, relevant application of the study of optional projections is provided by the works of Cetin et al. [2004], Jarrow et al. [2007] and Sezer [2007] in the field of credit risk modeling: taking inspiration from Jarrow and Protter [2004], the authors characterise reduced form models as optional projections of structural models into a smaller filtration. In particular, the cash balance of a firm, represented by a process $X = (X_t)_{t \geq 0}$, is adapted to the filtration $\mathbb{G}$ of the firm's management, but not necessarily to the filtration $\mathbb{F}$ representing the information available to the market. In this setting, the price of a zero-coupon bond issued by the firm is the optional projection of the value X estimated by the company's management.

An interesting question to analyse is then if traders with partial information $\mathbb{F} \subset \mathbb{G}$ may perceive different characteristics of the market they observe with respect to the individuals with access to the complete information $\mathbb{G}$, for example for what concerns the existence of perceived arbitrages. As already noted in Jarrow and Protter [2013], traders with limited information may interpret the presence of a bubble on the price process in the larger filtration as an arbitrage opportunity. This happens if X is a $(P, \mathbb{G})$-strict local martingale, ${}^oX$ fails to be a $(P, \mathbb{F})$-strict local martingale and, in addition,

there exists no measure $Q \sim P$ under which ${}^oX$ is a local martingale.
This short literature's overview shows that it is important to assess the properties of optional projections under change of equivalent martingale measures.
In particular we study the relation among the set $\mathcal{M}_{loc}$ of equivalent local martingale measures (ELMMs) for a process X and the set $\mathcal{M}^o_{loc}$ of measures $Q \sim P$ such that the optional projection under Q is a Q-local martingale. We obtain a chacterization of the relations between $\mathcal{M}_{loc}$ and $\mathcal{M}^o_{loc}$ for two main cases: the inverse three-dimensional Bessel process and an extension of the stochastic volatility model of Sin [1998]. We also consider the optional projection of a $\mathbb{G}$-adapted process into the delayed filtration $(\mathcal{G}_{t-\epsilon})_{t\geq 0}$, $\epsilon > 0$: this is a case with interesting consequences for financial applications, as it represents the scenario where investors in the market have access to the information with a given positive time delay. Delayed information has been extensively studied in the literature, see among others Guo et al. [2009], Hillairet and Jiao [2012], Jeanblanc and Lecam [2008], Xing and Yiyun [2012] in the setting of credit risk models, Dolinsky and Zouari [2020] under the model uncertainty framework and Bank and Dolinsky [2020] in the context of option pricing.
Moreover, we provide an invariance theorem about local martingales which are solutions of a one-dimensional SDE in the natural filtration of an n-dimensional Brownian motion, see Theorem 3.4. Specifically, we show that under mild conditions, such a local martingale X has same law under P as under every $Q \in \mathcal{M}_{loc}(X)$. Furthermore, Theorem 3.11 gives a result about optional projections into a filtration $\mathbb{F}$ that is smaller than the natural filtration $\mathbb{F}^X$ of X. Here the optional projection is taken with respect to an ELMM Q such that X has same law under P as under Q. A class of local martingales X such that all ELMM for X have this characteristic is indeed provided by Theorem 3.4. Important applications of the setting of Theorem 3.11, i.e., when $\mathbb{F} \subseteq \mathbb{F}^X$, are given by delayed information and by the model of Cetin et al. [2004], where the market does not see the value of a firm but only knows when the firm has positive cash balances or when it has negative or zero cash balances.
The rest of the paper is organised as follows. In Section 2 we describe our setting and formulate the aims of our study as five mathematical problems about optional projections of strict local martingales which we study in the sequel. In particular we provide a synthetic overview of the main results of the paper in the light of these problems. In Section 3 we give some general results about optional projections of local martingales under equivalent local martingale measures, that will be used in Sections 4 and 5. More pre-

cisely, Section 4 is devoted to the inverse three-dimensional Bessel process, projected into different filtrations, whereas in Section 5 we focus on a class of two-dimensional stochastic volatility processes. The novelty here is that we provide such results not only under a reference measure P but more in general for a set of equivalent local martingale measures. We conclude the paper with Appendix A, where we characterize the local martingale property of the optional projection of a local martingale via optimal transport.

2 Mathematical setting

Consider a probability space $(\Omega, \mathcal{F}, P)$ equipped with two filtrations $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, satisfying the usual hypothesis of right-continuity and completeness, with $\mathbb{F} \subset \mathbb{G}$. Moreover, let X be a non-negative càdlàg $(P, \mathbb{G})$-local martingale. Unless differently specified, we suppose X to be a strict $(P, \mathbb{G})$-local martingale. Moreover, we always suppose that $\mathcal{F} = \mathcal{G}_\infty$.
For the rest of the paper, we adopt the following notation.

Notation 2.1. We denote by ${}^oX$ the optional projection of X into $\mathbb{F}$, i.e., the unique càdlàg process satisfying

$$\mathbb{1}_{\{\tau<\infty\}}\, {}^oX_\tau = \mathbb{E}[\mathbb{1}_{\{\tau<\infty\}} X_\tau | \mathcal{F}_\tau] \quad a.s.$$

for every $\mathbb{F}$-stopping time τ. We also define ${}^{Q,o}X$ to be the optional projection of X under $Q \sim P$ into $\mathbb{F}$, i.e., $\mathbb{1}_{\{\tau<\infty\}}\, {}^{Q,o}X_\tau = \mathbb{E}^Q[\mathbb{1}_{\{\tau<\infty\}} X_\tau | \mathcal{F}_\tau]$ a.s., for every $\mathbb{F}$-stopping time τ. We call ${}^{Q,o}X$ the Q-optional projection of X. If we don't specify the measure, the optional projection is with respect to P.
We call $\mathbb{F}^X$ the natural filtration of X. Moreover, if Q is a probability measure equivalent to P, we define $Z_\infty := \frac{dQ}{dP}$ and denote by ${}^{\mathcal{F}}Z$, ${}^{\mathcal{F}^X}Z$, ${}^{\mathcal{G}}Z$, the càdlàg processes characterised by

$${}^{\mathcal{F}}Z_t = \mathbb{E}[Z_\infty|\mathcal{F}_t], \quad {}^{\mathcal{F}^X}Z_t = \mathbb{E}[Z_\infty|\mathcal{F}^X_t], \quad {}^{\mathcal{G}}Z_t = \mathbb{E}[Z_\infty|\mathcal{G}_t], \quad t \geq 0 \qquad (2.1)$$

respectively. Moreover, for $\mathbb{H} = \mathbb{F}, \mathbb{F}^X, \mathbb{G}$, we denote

$$\begin{aligned}
\mathcal{M}_{loc}(X, \mathbb{H}) &= \{Q \sim P, \quad \text{X is a } (Q, \mathbb{H})\text{-local martingale}\},\\
\mathcal{M}_M(X, \mathbb{H}) &= \{Q \sim P, \quad \text{X is a } (Q, \mathbb{H})\text{-true martingale}\},\\
\mathcal{M}_L(X, \mathbb{H}) &= \{Q \sim P, \quad \text{X is a } (Q, \mathbb{H})\text{-strict local martingale}\}.
\end{aligned}$$

We also set

$$\mathcal{M}^o_{loc}(X,\mathbb{F}) := \left\{Q \sim P,\ {}^{Q,o}X \text{ is a } (Q,\mathbb{F})\text{-local martingale}\right\},$$
$$\mathcal{M}_{loc}(X,\mathbb{G},\mathbb{F}) := \left\{Q \sim P,\ X \text{ is a } (Q,\mathbb{G})\text{-local martingale, } {}^{\mathcal{G}}Z \text{ is } \mathbb{F}\text{-adapted, } \frac{dQ}{dP} \text{ is } \mathcal{F}_\infty\text{-measurable}\right\},$$
$$\mathcal{M}^o_{loc}(X,\mathbb{G},\mathbb{F}) := \left\{Q \sim P,\ {}^{Q,o}X \text{ is a } (Q,\mathbb{F})\text{-local martingale, } {}^{\mathcal{G}}Z \text{ is } \mathbb{F}\text{-adapted, } \frac{dQ}{dP} \text{ is } \mathcal{F}_\infty\text{-measurable}\right\}.$$

Moreover, for a given a $(Q,\mathbb{G})$-Brownian motion $B = (B_t)_{t\geq 0}$ and a suitably integrable, $\mathbb{G}$-adapted process $\alpha = (\alpha_t)_{t\geq 0}$, we denote by $\mathcal{E}_t\left(\int \alpha_s \cdot dB_s\right)$ the stochastic exponential at time t of the process $(\int_0^u \alpha_s \cdot dB_s)_{u\geq 0}$.

The facts from the following remark are used throughout the paper.

Remark 2.2. *All the processes considered in the paper are càdlàg and such that their optional projections under any considered Q are càdlàg. Càdlàg processes are indistinguishable if they are modifications of each other. Thus, given any such càdlàg processes Y^1 and Y^2 and such measures Q^1 and Q^2, we have that ${}^{Q^1,o}Y^1 = {}^{Q^2,o}Y^2$ if*

$$E^{Q^1}[Y^1|\mathcal{F}_t] = E^{Q^2}[Y^2|\mathcal{F}_t] \quad \textit{for all } t.$$

Remark 2.3. *Since we assume $\mathcal{F} = \mathcal{G}_\infty$, the density dQ/dP of any equivalent probability measure Q with respect to the original measure P must be $\mathcal{G}_\infty$-measurable. This implies that*

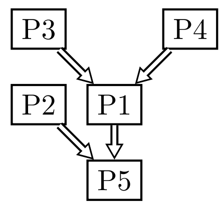
$$\mathcal{M}_{loc}(X,\mathbb{G},\mathbb{G}) = \mathcal{M}_{loc}(X,\mathbb{G}).$$

We investigate when the following properties hold:

$$\mathcal{M}_{loc}(X,\mathbb{G}) \cap \mathcal{M}^o_{loc}(X,\mathbb{F}) \neq \emptyset; \tag{P1}$$
$$\mathcal{M}_{loc}({}^oX,\mathbb{F}) \neq \emptyset; \tag{P2}$$
$$\mathcal{M}_L(X,\mathbb{G}) \cap \mathcal{M}^o_{loc}(X,\mathbb{F}) \neq \emptyset; \tag{P3}$$
$$\mathcal{M}_{loc}(X,\mathbb{G},\mathbb{F}) = \mathcal{M}^o_{loc}(X,\mathbb{G},\mathbb{F}); \tag{P4}$$
$$\bigcup_{Q\in\mathcal{M}_{loc}(X,\mathbb{G})} \mathcal{M}_{loc}({}^{Q,o}X,\mathbb{F}) \neq \emptyset. \tag{P5}$$

Note that $\mathcal{M}_L(X,\mathbb{G}), \mathcal{M}_{loc}(X,\mathbb{G}) \neq \emptyset$, as $P \in \mathcal{M}_L(X,\mathbb{G})$ by hypothesis. Properties (P1), (P2), (P3) and (P5) trivially hold if ${}^oX$ is an $\mathbb{F}$-local martingale, so the more interesting case is when ${}^oX$ is not a local martingale.

Under this hypothesis, properties (P1)-(P5) can hold or not depending on both the process X and the filtration $\mathbb{F}$, as illustrated in Sections 4 and 5. In particular, the properties are related as we discuss in the sequel.
Note that if one of (P3) or (P4) holds, then (P1) holds as well. Moreover, (P5) is the weakest one: anyone of (P1), (P2), (P3) or (P4) implies (P5). This can be summarized in the following scheme:



Note that problems (P2) and (P5) are related to the perception of arbitrages under the smaller filtration $\mathbb{F}$. In particular, when property (P5) holds, this means that there exist at least one equivalent probability measure Q that defines an arbitrage-free market under restricted information.
We finish the section by summarizing our main results. Note that properties (P1), (P2), (P3) and (P5) trivially hold for the three-dimensional Bessel process projected into the filtration generated by (B^1, B^2), as the optional projection is again a strict local martingale, see Section 4.1.

- **Property** (P1): in Section 5 we introduce a stochastic volatility process X, which is a strict local martingale under suitable conditions on the coefficients of its SDE, but whose optional projection into a specific sub-filtration is not a local martingale, see Theorem 5.10. Property (P1) holds because X admits a true martingale measure, see Proposition 5.2. On the contrary, (P1) is not true for the inverse three-dimensional Bessel process projected into a delayed filtration, i.e., $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, $\mathcal{F}_t = \mathcal{G}_{t-\epsilon}$, $\epsilon > 0$, see Section 4.2..

- **Property** (P2): a particular case of the stochastic volatility model introduced in Section 5 gives a strict local martingale N such that ${}^oN$ is not a local martingale but $\mathcal{M}_{loc}({}^oN, \mathbb{F}) \neq \emptyset$, see Example 5.7. In this setting (P2) holds. For the optional projection into the delayed filtration of the process introduced in Example 4.7, (P2) holds true as well. On the other hand, the property does not hold in the case of the inverse three-dimensional Bessel process projected into the delayed filtration,, see Theorem 4.5.

- **Property** (P3): in Example 5.8 we consider the sum of the stochastic volatility process of Section 5 and a suitable strict local martingale adapted to a Brownian filtration $\mathbb{F}$: this is a strict local martingale whose optional projection into $\mathbb{F}$ is not a local martingale, but such that (P3) is satisfied. On the contrary, this property is not true for the inverse three-dimensional Bessel process projected into the delayed filtration, see Section 4.2.

- **Property** (P4): it is satisfied for the inverse three-dimensional Bessel process projected into the filtration generated by (B^1, B^2), see Theorem 4.2. The property does not hold for any of the examples where (P1) does not hold, e.g., in the case of the inverse three-dimensional Bessel process projected into a delayed filtration.

- **Property** (P5): it holds for all the examples considered except for the inverse three-dimensional Bessel process projected into the delayed filtration, see Theorem 4.5.

3 General results

We start by providing some preliminary results which will be used throughout the rest of the paper. They are also of independent interest.

Lemma 3.1. *Let $X = (X_t)_{t\geq 0}$ be a non-negative $\mathbb{G}$-local martingale adapted to the filtration $\mathbb{F} \subseteq \mathbb{G}$. Then*

$$\mathcal{M}_{loc}(X, \mathbb{G}) \subseteq \mathcal{M}_{loc}(X, \mathbb{F}).$$

Proof. Consider the sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ defined by

$$\tau_n := \inf\{t \geq 0 : X_t > n\} \wedge n, \quad n \geq 1.$$

Note that $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of $\mathbb{F}$-stopping times since X is $\mathbb{F}$-adapted. We have

$$\sup_{0\leq t\leq \tau_n} |X_t| \leq n + |X_{\tau_n}|$$

for every $n \in \mathbb{N}$. Let now $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$. Then $|X_{\tau_n}| \in L^1(Q)$, since X is a $(Q, \mathbb{G})$-local martingale bounded from below and thus a $(Q, \mathbb{G})$-supermartingale. Then X is localized by $(\tau_n)_{n\in\mathbb{N}}$, and therefore $Q \in \mathcal{M}_{loc}(X, \mathbb{F})$. □ □

Lemma 3.2. *Let Q be a probability measure equivalent to P, such that ${}^{\mathcal{G}}Z$ in* (2.1) *is $\mathbb{F}$-adapted. Then ${}^{Q,o}X = {}^{o}X$.*

Proof. Recall that ${}^{\mathcal{F}}Z_t := \mathbb{E}[dQ/dP|\mathcal{F}_t]$. Since ${}^{\mathcal{G}}Z$ is $\mathbb{F}$-adapted, we have

$$ {}^{\mathcal{G}}Z_t = \mathbb{E}[{}^{\mathcal{G}}Z_t|\mathcal{F}_t] = \mathbb{E}[\mathbb{E}[dQ/dP|\mathcal{G}_t]|\mathcal{F}_t] = \mathbb{E}[dQ/dP|\mathcal{F}_t] = {}^{\mathcal{F}}Z_t, \quad t \geq 0 \tag{3.1} $$

and thus

$$ {}^{Q,o}X_t = \mathbb{E}^Q[X_t|\mathcal{F}_t] = ({}^{\mathcal{F}}Z_t)^{-1}\mathbb{E}[{}^{\mathcal{G}}Z_t X_t|\mathcal{F}_t] = \mathbb{E}[X_t|\mathcal{F}_t] = {}^{o}X_t, \quad t \geq 0. $$

□ □

We now give some general results about optional projections under changes of equivalent measures. Some of the examples in Sections 4 and 5 are based on these findings.
The following theorem provides a condition under which the Q-optional projection of X is an $\mathbb{F}$-local martingale under any ELMM Q.

Theorem 3.3. *Assume that X admits an $\mathbb{F}$-localizing sequence which makes it a bounded $(P, \mathbb{G})$-martingale. Then ${}^{Q,o}X$ is a $(Q, \mathbb{F})$-local martingale for every $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$.*

Proof. Let $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$, and $(\tau_n)_{n\in\mathbb{N}}$ be the assumed localizing sequence. Since X^{τ_n} is bounded for every $n \in \mathbb{N}$, $(\tau_n)_{n\in\mathbb{N}}$ localizes X under Q as well, and the result follows by Theorem 3.7 in Föllmer and Protter [2011]. □ □

We now give a theorem which provides a class of local martingales whose law under P is invariant under change of any equivalent local martingale measure. This result is of independent interest and also useful in our context, see Theorem 3.11.

Theorem 3.4. *Let $\mathbb{G}$ be the natural filtration of an n-dimensional Brownian motion $B = (B_t)_{t\geq 0}$, $n \in \mathbb{N}$. Moreover, let $X = (X_t)_{t\geq 0}$ be a $(P, \mathbb{G})$-local martingale, given by*

$$ dX_t = \sigma(t, X_t)dW_t, \quad t \geq 0, \tag{3.2} $$

where W is a one-dimensional $(P, \mathbb{G})$-Brownian motion, and the function $\sigma(\cdot, \cdot)$ is such that there exists a unique strong solution to (3.2). Suppose also that $\sigma(t, X_t) \neq 0$ a.s. for almost every $t \geq 0$.
Then X has the same law under P as under any $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$. In particular, if X is a $(P, \mathbb{G})$-strict local martingale, it is a $(Q, \mathbb{G})$-strict local martingale under any $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$, and if it is a $(P, \mathbb{G})$-true martingale, it is a $(Q, \mathbb{G})$-true martingale under any $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$.

Proof. The Martingale representation Theorem applied to the filtration $\mathbb{G}$ implies that there exists a unique $\mathbb{R}^n$-valued process $\sigma^W = (\sigma_t^W)_{t\geq 0}$, progressive and such that $\int_0^t (\sigma_s^W)^2 ds < \infty$ a.s. for all $t \geq 0$, such that

$$W_t = \int_0^t \sigma_s^W \cdot dB_s, \quad a.s., \quad t \geq 0. \tag{3.3}$$

Consider a probability measure $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$, defined by a density

$$\frac{dQ}{dP}|_{\mathcal{G}_t} = \mathcal{E}_t \left(\int \alpha_s \cdot dB_s \right), \quad t \geq 0, \tag{3.4}$$

where $\alpha = (\alpha_t)_{t\geq 0}$ is a suitably integrable, $\mathbb{G}$-adapted processes. Girsanov's Theorem implies that the dynamics of W under Q are given by

$$W_t = \int_0^t \sigma_s^W \cdot d\tilde{B}_s + \int_0^t (\sigma_s^W \cdot \alpha_s) ds, \quad t \geq 0, \tag{3.5}$$

where the process $\tilde{B} = (\tilde{B}_t)_{t\geq 0}$ defined by

$$\tilde{B}_t = B_t - \int_0^t \alpha_s ds, \quad t \geq 0 \tag{3.6}$$

is a $(Q, \mathbb{G})$-Brownian motion. But since $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$ and $\sigma(t, X_t) \neq 0$ a.s. for almost every $t \geq 0$, we obtain by (3.2) that

$$\sigma_t^W \cdot \alpha_t = 0 \quad P \otimes dt - \text{almost surely}, \quad t \geq 0,$$

and W is a $(Q, \mathbb{G})$-local martingale.
Since W is a $(P, \mathbb{G})$-Brownian motion, Lévy's Characterization Lemma of the one-dimensional Brownian motion implies that W is also a $(Q, \mathbb{G})$-Brownian motion, and the result follows. □ □

The next results regard the case when there are no ELMMs defined by a non trivial density adapted to $\mathbb{F}^X$.

Proposition 3.5. *Let X be a non-negative $(P, \mathbb{G})$-local martingale, and suppose that*

$$\mathcal{M}_{loc}(X, \mathbb{F}^X, \mathbb{F}^X) = \{P\}.$$

Let Q be a probability measure with $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$, and ${}^{\mathcal{G}}Z$ be the density process defined in (2.1)*. Then*

$$\mathbb{E}[{}^{\mathcal{G}}Z_t | \mathcal{F}_t^X] = 1, \quad a.s., \quad t \geq 0.$$

Proof. If $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$, the X is a $(Q, \mathbb{F}^X)$-local martingale by Lemma 3.1. This implies that $X \cdot {}^{\mathcal{F}^X}Z$ is a $(P, \mathbb{F}^X)$-local martingale, where ${}^{\mathcal{F}^X}Z$ is defined in (2.1). By the assumption $\mathcal{M}_{loc}(X, \mathbb{F}^X, \mathbb{F}^X) = \{P\}$,

$$1 = \mathbb{E}[Z_\infty | \mathcal{F}_t^X] = \mathbb{E}\left[\mathbb{E}[Z_\infty | \mathcal{G}_t] | \mathcal{F}_t^X\right] = \mathbb{E}[{}^{\mathcal{G}}Z_t | \mathcal{F}_t^X], \quad a.s., \quad t \geq 0.$$

□ □

Corollary 3.6. *Let X be a non-negative $(P, \mathbb{G})$-local martingale, and suppose that*

$$\mathcal{M}_{loc}(X, \mathbb{F}^X, \mathbb{F}^X) = \{P\}.$$

If X is a $(P, \mathbb{G})$-strict local martingale, then it is a $(Q, \mathbb{G})$-strict local martingale under any $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$, and if it is a $(P, \mathbb{G})$-true martingale, it is a $(Q, \mathbb{G})$-true martingale under any $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$.

Proof. Suppose that X is a $(P, \mathbb{G})$-strict local martingale. Then there exists $t \geq 0$ such that $\mathbb{E}^P[X_t] < X_0$. For the same t, we have

$$\mathbb{E}^Q[X_t] = \mathbb{E}^P[Z_\infty X_t] = \mathbb{E}^P[{}^{\mathcal{G}}Z_t X_t] \quad = \mathbb{E}^P\left[X_t \mathbb{E}^P[{}^{\mathcal{G}}Z_t | \mathcal{F}_t^X]\right] = \mathbb{E}^P\left[X_t\right] < X_0$$

a.s., where the last equality follows from Proposition 3.5. Therefore, X is a $(Q, \mathbb{G})$-strict local martingale. Analogously, it can be seen that if X is a $(P, \mathbb{G})$-true martingale, it is a $(Q, \mathbb{G})$-true martingale. □ □

Corollary 3.7. *Let X be a non-negative $(P, \mathbb{G})$-local martingale, and suppose that*

$$\mathcal{M}_{loc}(X, \mathbb{F}^X, \mathbb{F}^X) = \{P\}.$$

Then for every probability measure $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$, and every sub-filtration $\mathbb{F} \subseteq \mathbb{F}^X$, we have ${}^oX = {}^{Q,o}X$.

Proof. Let $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$. Then

$$\begin{aligned}
\mathbb{E}^Q[X_t | \mathcal{F}_t] &= \left(\mathbb{E}^P[Z_\infty | \mathcal{F}_t]\right)^{-1} \mathbb{E}^P[{}^{\mathcal{G}}Z_t X_t | \mathcal{F}_t] \\
&= \left(\mathbb{E}^P\left[\mathbb{E}^P[Z_\infty | \mathcal{F}_t^X] | \mathcal{F}_t\right]\right)^{-1} \mathbb{E}^P\left[\mathbb{E}^P[{}^{\mathcal{G}}Z_t X_t | \mathcal{F}_t^X] | \mathcal{F}_t\right] \\
&= \mathbb{E}^P\left[X_t \mathbb{E}^P[{}^{\mathcal{G}}Z_t | \mathcal{F}_t^X] | \mathcal{F}_t\right] \\
&= \mathbb{E}^P\left[X_t | \mathcal{F}_t\right], \quad a.s., \quad t \geq 0,
\end{aligned}$$

where the second equality follows from $\mathbb{F} \subseteq \mathbb{F}^X$ and the last from Proposition 3.5. □ □

The rest of the section is concerned with the equivalent measure extension problem of Larsson [2014], together with the most important results relating this problem to the optional projection of strict local martingales.
In the setting introduced in Section 2, define first the stopping times

$$\tau_n := \inf\{t \geq 0 : X_t \geq n\} \wedge n, \qquad \tau := \lim_{n\to\infty} \tau_n,$$

and note that $\mathcal{G}_{\tau-} = \vee_{n\geq 1}\mathcal{G}_{\tau_n}$.
The Föllmer measure Q_0 is defined on $\mathcal{G}_{\tau-}$ as the probability measure that coincides with Q_n on $\mathcal{G}_{\tau_n}$ for each $n \geq 1$, where $Q_n \sim P$ is defined on $\mathcal{G}_{\tau_n}$ by $dQ_n = X_{\tau_n} dP$. For more details see [Larsson, 2014, Section 2].
The measure Q_0 is then only defined on $\mathcal{G}_{\tau-}$. It is then a natural question whether Q_0 can be extended to $\mathcal{G}_\infty$, i.e., whether it is possible to find a measure $\tilde{Q}$ on $(\Omega, \mathcal{G}_\infty)$ such that $\tilde{Q} = Q_0$ on $\mathcal{G}_{\tau-}$. There are several ways in which Q_0 can be extended to a measure $\tilde{Q}$ on $\mathcal{G}_\infty$, see Larsson [2014]. A further problem is whether Q_0 admits an extension to $\mathcal{G}_\infty$ as specified below.

Problem 3.8 (Equivalent measure extension problem, Problem 1 of Larsson [2014])**.** Consider the probability space $(\Omega, \mathcal{F}, P)$ equipped with two filtrations $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ with $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \geq 0$, and let X be a $(P, \mathbb{G})$-local martingale. Given the probability measure Q_0 introduced above, find a probability measure Q on $(\Omega, \mathcal{G}_\infty)$ such that:

1. $Q = Q_0$ on $\mathcal{G}_{\tau-}$;
2. The restrictions of P and Q to $\mathcal{F}_t$ are equivalent for each $t \geq 0$.

The existence of a solution to the equivalent measure extension problem is connected with the behaviour of the optional projection of X into $\mathbb{F}$ by the following theorem.

Theorem 3.9 (Corollary 1 of Larsson [2014])**.** *Let X be a strict $\mathbb{G}$-local martingale. If ${}^oX$ is an $\mathbb{F}$-local martingale, then the equivalent measure extension problem has no solution.*

We now provide a result about optional projections under equivalent local martingale measures into a filtration $\mathbb{F} \subseteq \mathbb{F}^X$. We start with a lemma.

Lemma 3.10. *Let X be a non-negative $\mathbb{G}$-local martingale. Suppose that the equivalent measure extension problem admits a solution for P and the two filtrations $\mathbb{F} \subseteq \mathbb{G}$, with $\mathbb{F} \subseteq \mathbb{F}^X$. Then it also admits a solution for P, $\mathbb{F}$ and $\mathbb{F}^X$.*

Proof. Note first that X is an $\mathbb{F}^X$-local martingale by Lemma 3.1. Let Q be a solution of the equivalent measure extension problem for P, $\mathbb{F}$ and $\mathbb{G}$. Also let Q_0 and Q_0^X be the Föllmer measures on $\mathcal{G}_{\tau-}$ and $\mathcal{F}^X_{\tau-}$, respectively. By construction, we have that Q_0^X coincides with Q_0 on $\mathcal{F}^X_{\tau-}$. This implies that Q is also an extension of Q_0^X, equivalent to P on $\mathcal{F}_t$ for every $t \geq 0$. Then Q gives a solution for the equivalent measure extension problem for P, $\mathbb{F}$ and $\mathbb{F}^X$. □ □

Theorem 3.11. *Consider a probability measure $\tilde{P} \in \mathcal{M}_L(X, \mathbb{G})$ and suppose that X has the same law under P as under $\tilde{P}$. Also assume that the equivalent measure extension problem admits a solution for P, and that $\mathbb{F} \subseteq \mathbb{F}^X$. Then the $\tilde{P}$-optional projection ${}^{\tilde{P},o}X$ of X into $\mathbb{F}$ is not a $(\tilde{P}, \mathbb{F})$-local martingale.*

Proof. By Lemma 3.10, we have that the equivalent measure extension problem admits a solution for P, $\mathbb{F}$ and $\mathbb{F}^X$. Consider now the construction of the Föllmer measure illustrated above. The stopping times

$$\tau_n := \inf\{t \geq 0 : X_t \geq n\} \wedge n, \qquad \tau := \lim_{n \to \infty} \tau_n$$

have the same law under P as under $\tilde{P}$. Moreover, as by assumption X has the same law under P as under $\tilde{P}$ and since $dQ_n = X_{\tau_n} dP$ and $d\tilde{Q}_n = X_{\tau_n} d\tilde{P}$, the measures Q_n and $\tilde{Q}_n$ coincide on $\mathcal{F}^X_{\tau_n}$, so the equivalent measure extension problem also admits a solution for $\tilde{P}$, $\mathbb{F}$ and $\mathbb{F}^X$. By Theorem 3.9, it follows that the $\tilde{P}$-optional projection of X into $\mathbb{F}$ is not a $(\tilde{P}, \mathbb{F})$-local martingale. □ □

Note that Theorem 3.4 implies that Theorem 3.11 can be applied to all processes with dynamics as in (3.2). An important application when $\mathbb{F} \subseteq \mathbb{F}^X$ is the study of delayed information.

4 The inverse three-dimensional Bessel process

In this section we study problems (P1)-(P5) in the case of the inverse three-dimensional Bessel process. Let $B^1 = (B^1_t)_{t \geq 0}, B^2 = (B^2_t)_{t \geq 0}, B^3 = (B^3_t)_{t \geq 0}$ be standard, independent Brownian motions, starting at $(B^1_0, B^2_0, B^3_0) = (1, 0, 0)$, on $(\Omega, \mathcal{F}, P)$. We specify the filtration later. In the notation of Section 2, we now assume that the non-negative local martingale X is given by the inverse three-dimensional Bessel process

$$X_t := \left((B^1_t)^2 + (B^2_t)^2 + (B^3_t)^2\right)^{-1/2}, \quad t \geq 0. \tag{4.1}$$

Itô's formula implies that under the original probability measure P, X has dynamics

$$dX_t = -X_t^3 \left(B_t^1 dB_t^1 + B_t^2 dB_t^2 + B_t^3 dB_t^3 \right), \quad t \geq 0, \tag{4.2}$$

with $(B_0^1, B_0^2, B_0^3) = (1, 0, 0)$. We note that X also solves the SDE

$$dX_t = -X_t^2 dW_t, \quad t \geq 0, \tag{4.3}$$

where the process W with

$$W_t = \int_0^t X_s \left(B_s^1 dB_s^1 + B_s^2 dB_s^2 + B_s^3 dB_s^3 \right), \quad t \geq 0 \tag{4.4}$$

is a one-dimensional Brownian motion as it is a continuous local martingale with $[W, W]_t = t$.
In the notation of Section 2 we now consider two different choices for the filtration $\mathbb{G}$: in Section 4.1 we let $\mathbb{G}$ be the filtration generated by (B^1, B^2, B^3), whereas in Section 4.2 it is generated by the Brownian motion W in (4.4). In both cases, X is a strict $\mathbb{G}$-local martingale, and it is therefore interesting to investigate properties (P1)-(P5) when X is projected into a smaller filtration $\mathbb{F}$. It is well known that the projection of X is not a local martingale when projected into the natural filtration of B^1, see also Kardaras and Ruf [2019b] for a more general study of the projection of functions of the sum of two independent Bessel processes of dimension $n \geq 2$ and $m - n$, with $0 \leq m < n$. In our work, we consider the case when $\mathbb{F}$ is generated by B^1 and B^2 in Section 4.1. On the other hand, in Section 4.2 we study an example of delayed information, which describes in fact a situation which often happens in practice: here $\mathbb{F} = (\mathcal{F}_t)$, with $\mathcal{F}_t = \mathcal{G}_{t-\epsilon}$, $\epsilon > 0$, meaning that investors have access to the information of the process with a strictly positive time delay ϵ.

Remark 4.1. *In order to study property* (P1)*, it is of course important to have some knowledge about the set $\mathcal{M}_{loc}(X, \mathbb{G})$. In particular, one can ask whether the market is complete, i.e., $\mathcal{M}_{loc}(X, \mathbb{G}) = \{P\}$, or incomplete, that means that there exists infinitely many measures $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$.*
In the case of the inverse three-dimensional Bessel process, this depends on the choice of $\mathbb{G}$: if $\mathbb{G}$ is generated by one Brownian motion, as it happens in Section 4.2, it is well known that the market is complete, so that the probability P is the only measure under which X is a local martingale, see also Delbaen and Schachermayer [1994].

On the other hand, let now $\mathbb{G}$ be the natural filtration of (B^1, B^2, B^3), as it is the case in Section 4.1. In this case, we have $\mathcal{M}_{loc}(X, \mathbb{G}) \neq \{P\}$. Namely, consider for example the $\mathbb{G}$-adapted processes

$$\alpha_t^1 = -\frac{B_t^2}{(B_t^1)^2 + (B_t^2)^2 + 1}, \qquad \alpha_t^2 = \frac{B_t^1}{(B_t^1)^2 + (B_t^2)^2 + 1}, \quad t \geq 0 \tag{4.5}$$

and $Z = (Z_t)_{t \geq 0}$ defined by

$$Z_t = \mathcal{E}_t(L), \quad t \geq 0, \tag{4.6}$$

with

$$L_t := \int_0^t e^{-\delta s} \alpha_s^1 dB_s^1 + \int_0^t e^{-\delta s} \alpha_s^2 dB_s^2, \quad t \geq 0,$$

for a given $\delta > 0$. With this choice of α^i, $i = 1, 2$, the stochastic exponential Z in (4.6) is a $\mathbb{G}$-adapted process such that $[X, Z] = 0$ a.s.. Moreover, the Novikov's condition for Z is fulfilled since

$$\exp\left(\frac{1}{2}[Z, Z]_\infty\right) = \exp\left(\frac{1}{2}\int_0^\infty e^{-2\delta s}\|\alpha_s\|^2 ds\right) \leq \exp\left(\frac{1}{4\delta}\right),$$

and Z is then a uniformly integrable martingale. For this reason, defining Q by

$$\frac{dQ}{dP}|_{\mathcal{G}_t} = Z_t, \quad t \geq 0,$$

we have that $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$, $Q \neq P$.
However, Theorem 3.4 implies that $\mathcal{M}_M(X, \mathbb{H}) = \emptyset$ for every filtration $\mathbb{H}$ to which X is adapted, i.e., there does not exist any measure $Q \sim P$ such that X is a true martingale under Q. This also means that, for the examples of Sections 4.1 and 4.2, property (P1) *holds if and only if* (P3) *holds.*

4.1 Optional projection into the filtration generated by (B^1, B^2)

In this section, we provide an example for which property (P4) is satisfied, i.e., we introduce two filtrations $\mathbb{F} \subseteq \mathbb{G}$ such that for the inverse Bessel process X we have

$$\mathcal{M}_{loc}(X, \mathbb{G}, \mathbb{F}) = \mathcal{M}_{loc}^o(X, \mathbb{G}, \mathbb{F}).$$

In particular, we let $\mathbb{G}$ be the natural filtration of (B^1, B^2, B^3), and $\mathbb{F}$ be generated by (B^1, B^2). As usual, we denote the optional projection of the inverse three-dimensional Bessel process X into $\mathbb{F}$ by ${}^oX$. Theorem 5.2 of

Föllmer and Protter [2011] states that ${}^oX$ is an $\mathbb{F}$-local martingale and has the form

$$ {}^oX_t = u(B^1_t, B^2_t, t), \quad t \geq 0 $$

with

$$ u(x,y,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{x^2+y^2}{4t}\right) K_0\left(\frac{x^2+y^2}{4t}\right), $$

where we denote by K_n, $n \geq 0$, the modified Bessel functions of the second kind. In particular, it holds

$$ \partial_x u(x,y,t) = x\psi(x,y,t), \qquad \partial_y u(x,y,t) = y\psi(x,y,t), \tag{4.7} $$

where

$$ \psi(x,y,t) = \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{x^2+y^2}{4t}\right)\left(K_0\left(\frac{x^2+y^2}{4t}\right) - K_1\left(\frac{x^2+y^2}{4t}\right)\right). \tag{4.8} $$

Since ${}^oX$ is an $\mathbb{F}$-local martingale, we focus here on property (P4).
The following theorem provides a positive answer to property (P4) in this example.

Theorem 4.2. *Let $\mathbb{F}$ be the natural filtration of (B^1, B^2). Then*

$$ \mathcal{M}_{loc}(X, \mathbb{G}, \mathbb{F}) = \mathcal{M}^o_{loc}(X, \mathbb{G}, \mathbb{F}). $$

Proof. We first prove that $\mathcal{M}_{loc}(X, \mathbb{G}, \mathbb{F}) \subseteq \mathcal{M}^o_{loc}(X, \mathbb{G}, \mathbb{F})$.
Introduce the sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ with

$$ \tau_n = \inf\left\{(B^1_t)^2 + (B^2_t)^2 \leq \frac{1}{n}\right\}, \quad n \geq 1. $$

We have $\lim_{n\to\infty} \tau_n = \infty$ because the origin $(0,0)$ is polar for a two-dimensional Brownian motion, so this is a localizing sequence of $\mathbb{F}$-stopping times that makes X a bounded martingale. Consider now $Q \in \mathcal{M}_{loc}(X, \mathbb{G}, \mathbb{F})$. Theorem 3.3 implies that ${}^{Q,o}X$ is a $(Q, \mathbb{F})$-local martingale, i.e., $Q \in \mathcal{M}^o_{loc}(X, \mathbb{G}, \mathbb{F})$.
We now prove that $\mathcal{M}^o_{loc}(X, \mathbb{G}, \mathbb{F}) \subseteq \mathcal{M}_{loc}(X, \mathbb{G}, \mathbb{F})$. Take $Q \in \mathcal{M}^o_{loc}(X, \mathbb{G}, \mathbb{F})$, i.e., suppose that ${}^{Q,o}X$ is a $(Q, \mathbb{F})$-local martingale.
Since ${}^oX$ is a $(P, \mathbb{F})$-local martingale and ${}^{Q,o}X = {}^oX$ by Lemma 3.2, from $Q \in \mathcal{M}^o_{loc}(X, \mathbb{G}, \mathbb{F})$ it follows that $[{}^{\mathcal{G}}Z, {}^oX]$ is a local martingale, because ${}^{\mathcal{G}}Z = {}^{\mathcal{F}}Z$ as seen in (3.1).
Note now that since the density of Q with respect to P is $\mathbb{F}$-adapted, it holds

$$ {}^{\mathcal{G}}Z_t = \frac{dQ}{dP}|_{\mathcal{G}_t} = \mathcal{E}_t\left(\int \alpha^1_s dB^1_s + \int \alpha^2_s dB^2_s\right), \quad t \geq 0, \tag{4.9} $$

where α^1 and α^2 are $\mathbb{F}$-adapted processes such that the Doléans exponential in (4.9) is well defined and a uniformly integrable martingale.
By (4.7), (4.8) and (4.9), we have

$$[{}^{\mathcal{G}}Z, {}^{o}X]_t = \int_0^t {}^{\mathcal{G}}Z_s \psi(B_s^1, B_s^2, s)(\alpha_s^1 B_s^1 + \alpha_s^2 B_s^2) ds, \quad t \geq 0. \tag{4.10}$$

Since $\psi(x, y, t) < 0$ for $x, y < \infty$ and $t > 0$ (see for example Yang and Chu [2017]), equation (4.10) together with the fact that $[{}^{\mathcal{G}}Z, {}^{o}X]$ is a local martingale imply that

$$\alpha_t^1 B_t^1 + \alpha_t^2 B_t^2 = 0 \quad P\text{-a.s.}, \quad t \geq 0 \tag{4.11}$$

as P is equivalent to Q. Moreover, from (4.2) and (4.9) it follows that

$$[{}^{\mathcal{G}}Z, X]_t = -\int_0^t {}^{\mathcal{G}}Z_s X_s^3 (\alpha_s^1 B_s^1 + \alpha_s^2 B_s^2) ds, \quad t \geq 0$$

and this is zero P-a.s. by (4.11). Since X is $(P, \mathbb{G})$-local martingale, this implies that X is also a $(Q, \mathbb{G})$-local martingale. Hence $Q \in \mathcal{M}_{loc}(X, \mathbb{G}, \mathbb{F})$.
□ □

4.2 Delayed information

We now consider a market model with delayed information: here $\mathbb{G}$ is the filtration generated by the Brownian motion W in (4.4), whereas $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is given by $\mathcal{F}_t = \mathcal{G}_{t-\epsilon}$, $\epsilon > 0$. As explained above, this means that investors have access to the information about W, with respect to which X is adapted by (4.3), only with a positive delay ϵ.
In this setting, we show that none of the properties properties (P1)-(P5) is satisfied for X. However, in Example 4.7 we introduce a modification M of the inverse Bessel process X such that $\mathcal{M}_{loc}({}^{o}M, \mathbb{F}) \neq \emptyset$, so that properties (P2) and (P5) are satisfied. Still, (P1), (P3) and (P4) are not fulfilled in this case either.
We start our analysis with the following

Lemma 4.3. *For every $\epsilon > 0$, we have*

$$\mathbb{E}[X_{t+\epsilon} | \sigma(B_t^1, B_t^2, B_t^3)] = X_t \, erf\left(\frac{1}{X_t \sqrt{2\epsilon}}\right),$$

where $erf(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

Proof. We have

$$\mathbb{E}[X_{t+\epsilon}|\sigma(B_t^1, B_t^2, B_t^3)] = u(\epsilon, B_t^1, B_t^2, B_t^3),$$

with

$$u(t,a,b,c) = (2\pi t)^{-3/2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2t}\left((x-a)^2+(y-b)^2+(z-c)^2\right)}}{\sqrt{x^2+y^2+z^2}} dzdydx =: I.$$

We set $R = \sqrt{a^2+b^2+c^2}$, $r = \sqrt{x^2+y^2+z^2}$. The above integral can be written in spherical coordinates as

$$\begin{aligned} I &= (2\pi t)^{-3/2} \int_0^{2\pi}\int_0^{\infty} r^2 \frac{1}{r} \int_0^{\pi} \sin(\theta) e^{-\frac{1}{2t}(r^2-2rR\cos(\theta)+R^2)} d\theta dr d\phi \\ &= \frac{2}{R\sqrt{\pi}} \int_0^{\frac{R}{\sqrt{2t}}} e^{-r^2} dr = \frac{1}{\sqrt{a^2+b^2+c^2}} \mathrm{erf}\left(\sqrt{\frac{a^2+b^2+c^2}{2t}}\right). \end{aligned}$$

Thus

$$\begin{aligned} \mathbb{E}[X_{t+\epsilon}|\sigma(B_t^1, B_t^2, B_t^3)] &= \frac{1}{\sqrt{(B_t^1)^2+(B_t^2)^2+(B_t^3)^2}} \mathrm{erf}\left(\sqrt{\frac{(B_t^1)^2+(B_t^2)^2+(B_t^3)^2}{2\epsilon}}\right) \\ &= X_t \mathrm{erf}\left(\frac{1}{X_t\sqrt{2\epsilon}}\right). \end{aligned}$$

□ □

Proposition 4.4. *Let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be the filtration generated by the Brownian motion W in (4.4), and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be given by $\mathcal{F}_t := \mathcal{G}_{t-\epsilon}$, $t \geq 0$, $\epsilon > 0$. Then*

$$^o X_{t+\epsilon} = \mathbb{E}[X_{t+\epsilon}|\mathcal{F}_{t+\epsilon}] = X_t \mathit{erf}\left(\frac{1}{X_t\sqrt{2\epsilon}}\right), \quad t \geq 0.$$

Proof. Due to the Markov property of W and since $\sigma(W_t) \subset \sigma(B_t^1, B_t^2, B_t^3)$ by (4.3) and (4.4), from Lemma 4.3 it follows

$$\begin{aligned} \mathbb{E}[X_{t+\epsilon}|\mathcal{F}_{t+\epsilon}] &= \mathbb{E}[X_{t+\epsilon}|\sigma(W_t)] = \mathbb{E}\left[\mathbb{E}\left[X_{t+\epsilon}|\sigma(B_t^1, B_t^2, B_t^3)\right]|\sigma(W_t)\right] \\ &= \mathbb{E}\left[X_t \mathrm{erf}\left(\frac{1}{X_t\sqrt{2\epsilon}}\right)\middle|\sigma(W_t)\right] = X_t \mathrm{erf}\left(\frac{1}{X_t\sqrt{2\epsilon}}\right), \quad t \geq 0 \end{aligned}$$

as X_t is $\sigma(W_t)$-measurable. □ □

By Proposition 4.4, we have

$$^{o}X_{t+\epsilon} = f(X_t), \quad t \geq 0,$$

where $f(x) = x \cdot \mathrm{erf}\left(\frac{1}{x\sqrt{2\epsilon}}\right)$. Since

$$f'(x) = -\frac{\sqrt{2}e^{-\frac{1}{2\epsilon x^2}}}{x\sqrt{\pi\epsilon}} + \mathrm{erf}\left(\frac{1}{x\sqrt{2\epsilon}}\right), \qquad f''(x) = -\frac{\sqrt{2}\epsilon^{-\frac{3}{2}}e^{-\frac{1}{2\epsilon x^2}}}{x^4\sqrt{\pi}},$$

Itô's formula gives

$$\begin{aligned} d\,^{o}X_{t+\epsilon} &= \left(-\frac{\sqrt{2}e^{-\frac{1}{2\epsilon X_t^2}}}{X_t\sqrt{\pi\epsilon}} + \mathrm{erf}\left(\frac{1}{X_t\sqrt{2\epsilon}}\right)\right) dX_t - \frac{\sqrt{2}\epsilon^{-\frac{3}{2}}e^{-\frac{1}{2\epsilon X_t^2}}}{X_t^4\sqrt{\pi}} d[X,X]_t \\ &= \left(\frac{\sqrt{2}e^{-\frac{1}{2\epsilon X_t^2}}}{\sqrt{\pi\epsilon}} X_t - \mathrm{erf}\left(\frac{1}{X_t\sqrt{2\epsilon}}\right) X_t^2\right) dW_t - \sqrt{\frac{2}{\pi}}\epsilon^{-\frac{3}{2}} e^{-\frac{1}{2\epsilon X_t^2}}\, dt. \end{aligned} \tag{4.12}$$

By the above expression, we note that the optional projection is a strict $\mathbb{F}$-supermartingale, as the drift is strictly negative. Since by Remark 4.1 we have $\mathcal{M}_{loc}(X, \mathbb{G}) = \{P\}$, this implies that

$$\mathcal{M}_{loc}(X, \mathbb{G}) \cap \mathcal{M}^{o}_{loc}(X, \mathbb{F}) = \emptyset, \tag{4.13}$$

i.e., properties (P1), (P3) and (P4) do not hold.
Moreover, we give the following theorem, which implies that properties (P2) and (P5) are not satisfied.

Theorem 4.5. *Let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be the filtration generated by the Brownian motion W in* (4.4) *and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, with $\mathcal{F}_t := \mathcal{G}_{t-\epsilon}$, $t \geq 0$, $\epsilon > 0$. Then*

$$\mathcal{M}_{loc}(^{o}X, \mathbb{F}) = \emptyset.$$

To prove Theorem 4.5, we rely on some results provided by Mijatovic and Urusov [2012], which we now recall. Consider the state space $J = (l, r)$, $-\infty \leq l < r \leq \infty$ and a J-valued diffusion $Y = (Y_t)_{t\geq 0}$ on some filtered probability space, governed by the SDE

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dB_t, \quad t \geq 0, \tag{4.14}$$

where $Y_0 = y_0 \in J$ and B is a one-dimensional Brownian motion. Moreover, $\mu_Y(\cdot)$, $\sigma_Y(\cdot)$ are deterministic functions, that from now on we simply denote by μ_Y and σ_Y, such that

$$\sigma_Y(x) \neq 0 \quad \forall x \in J \tag{4.15}$$

and

$$\frac{1}{\sigma_Y^2},\ \frac{\mu_Y}{\sigma_Y^2} \in L^1_{loc}(J). \tag{4.16}$$

Here $L^1_{loc}(J)$ denotes the class of locally integrable functions ψ on J, i.e., the measurable functions $\psi : (J, \mathcal{B}(J)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that are Lebesgue integrable on compact subsets of J.
Consider the stochastic exponential

$$\mathcal{E}_t\left(\int g(Y_u)dB_u\right), \quad t \geq 0 \tag{4.17}$$

with $g(\cdot)$ such that

$$\frac{g^2}{\sigma_Y^2} \in L^1_{loc}(J). \tag{4.18}$$

Put $\bar{J} = [l, r]$ and, fixing an arbitrary $c \in J$, define

$$\rho(x) := \exp\left\{-\int_c^x \frac{2\mu_Y}{\sigma_Y^2}(y)dy\right\}, \quad x \in J, \tag{4.19}$$

$$\tilde{\rho}(x) := \rho(x)\exp\left\{-\int_c^x \frac{2g}{\sigma_Y}(y)dy\right\}, \quad x \in J, \tag{4.20}$$

$$s(x) := \int_c^x \rho(y)dy, \quad x \in \bar{J}, \tag{4.21}$$

$$\tilde{s}(x) := \int_c^x \tilde{\rho}(y)dy, \quad x \in \bar{J}. \tag{4.22}$$

Denote $\rho = \rho(\cdot)$, $s = s(\cdot)$, $s(r) = \lim_{x\to r^-} s(x)$, $s(l) = \lim_{x\to l^+} s(x)$, and analogously for $\tilde{s}(\cdot)$ and $\tilde{\rho}(\cdot)$.
Define

$$L^1_{loc}(r-) := \left\{\psi : (J, \mathcal{B}(J)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))\Big| \int_x^r |\psi(y)|dy < \infty \text{ for some } x \in J\right\},$$

and $L^1_{loc}(l+)$ analogously. We report here Theorem 2.1 in Mijatovic and Urusov [2012].

Theorem 4.6. *Let the functions μ_Y, σ_Y, and g satisfy conditions* (4.15), (4.16) *and* (4.18), *and let Y be a solution of the SDE* (4.14).
Then the Doléans exponential given by (4.17) *is a true martingale if and only if both of the following requirements are satisfied:*

1. *It does not hold*

$$\tilde{s}(r) < \infty \quad \text{and} \quad \frac{\tilde{s}(r) - \tilde{s}}{\tilde{\rho}\sigma_Y^2} \in L^1_{loc}(r-), \tag{4.23}$$

or it holds

$$s(r) < \infty \quad \text{and} \quad \frac{(s(r) - s)g^2}{\rho\sigma_Y^2} \in L^1_{loc}(r-). \tag{4.24}$$

2. *It does not hold*

$$\tilde{s}(l) > -\infty \quad \text{and} \quad \frac{\tilde{s} - \tilde{s}(l)}{\tilde{\rho}\sigma_Y^2} \in L^1_{loc}(l+),$$

or it holds

$$s(l) > -\infty \quad \text{and} \quad \frac{(s - s(l))g^2}{\rho\sigma_Y^2} \in L^1_{loc}(l+).$$

We now use Theorem 4.6 in order to prove Theorem 4.5.

Proof of Theorem 4.5. By (4.12) we have

$$d\,{}^oX_{t+\epsilon} = \mu(X_t)dt + \sigma(X_t)dW_t, \quad t \geq 0$$

with

$$\mu(x) = -\sqrt{\frac{2}{\pi}}\epsilon^{-\frac{3}{2}}e^{-\frac{1}{2\epsilon x^2}}, \qquad \sigma(x) = x\sqrt{\frac{2}{\pi\epsilon}}e^{-\frac{1}{2\epsilon x^2}} - x^2\text{erf}\left(\frac{1}{x\sqrt{2\epsilon}}\right). \tag{4.25}$$

By Girsanov's Theorem there exists a probability measure $Q \in \mathcal{M}_{loc}({}^oX, \mathbb{F})$ only if the Doléans exponential

$$\frac{dQ}{dP}|_{\mathcal{G}_t} = Z_t = \mathcal{E}_t\left(\int \alpha_s dW_s\right), \quad t \geq 0 \tag{4.26}$$

with

$$\alpha_t = -\frac{\mu(X_t)}{\sigma(X_t)}, \quad t \geq 0 \tag{4.27}$$

is a true martingale.
In order to prove that this is not the case, we apply Theorem 4.6. In our case, by equations (4.3), (4.25) and (4.27), we have $Y = X$, $J = (0, \infty)$, $\mu_Y \equiv 0$, $\sigma_Y(x) = -x^2$ and

$$g(x) = \frac{\sqrt{2/\pi}\epsilon^{-\frac{3}{2}}e^{-\frac{1}{2\epsilon x^2}}}{x\left(\sqrt{\frac{2}{\pi\epsilon}}e^{-\frac{1}{2\epsilon x^2}} - x \cdot \text{erf}\left(\frac{1}{x\sqrt{2\epsilon}}\right)\right)}.$$

Note that conditions (4.15) and (4.16) are satisfied. In order to prove that (4.18) also holds, it is enough to check that

$$x \cdot \operatorname{erf}\left(\frac{1}{x\sqrt{2\epsilon}}\right) - \sqrt{\frac{2}{\pi\epsilon}}e^{-\frac{1}{2\epsilon x^2}} > 0 \quad \text{for every } x \in (0,\infty).$$

This holds if and only if

$$\operatorname{erf}(y)\frac{1}{y\sqrt{2\epsilon}} - \sqrt{\frac{2}{\pi\epsilon}}e^{-y^2} > 0 \quad \text{for every } y \in (0,\infty),$$

i.e., if and only if

$$F(y) := \operatorname{erf}(y) - \frac{2}{\sqrt{\pi}}ye^{-y^2} > 0 \quad \text{for every } y \in (0,\infty).$$

The last condition if fulfilled since $F(0) = 0$ and $F'(y) = \frac{4}{\sqrt{\pi}}y^2e^{-y^2} > 0$ for every $y > 0$, then we have

$$x \cdot \operatorname{erf}\left(\frac{1}{x\sqrt{2\epsilon}}\right) - \sqrt{\frac{2}{\pi\epsilon}}e^{-\frac{1}{2\epsilon x^2}} > 0 \quad \text{for every } x \in (0,\infty) \tag{4.28}$$

and the assumptions of Theorem 4.6 are thus satisfied.
We now show that condition (4.24) fails whereas (4.23) is satisfied, implying that the density Z introduced in (4.26) is not a martingale.
Consider first ρ and s defined in (4.19) and (4.21), respectively. We have $\rho \equiv 1$, so that $s(x) = x - c$, for any $c > 0$. This implies that $s(\infty) = +\infty$, so that condition (4.24) fails.
We now check condition (4.23). We have

$$\lim_{x\to\infty} \frac{e^{-\frac{1}{2\epsilon x^2}}}{x^2\left(\sqrt{\frac{2}{\pi\epsilon}}e^{-\frac{1}{2\epsilon x^2}} - x \cdot \operatorname{erf}\left(\frac{1}{x\sqrt{2\epsilon}}\right)\right)} = -3\sqrt{\frac{\pi}{2}}\epsilon^{3/2},$$

so

$$\lim_{x\to\infty} -x\frac{2g(x)}{\sigma_Y(x)} = \lim_{x\to\infty} 2\sqrt{2/\pi}\epsilon^{-\frac{3}{2}}\frac{e^{-\frac{1}{2\epsilon x^2}}}{x^2\left(\sqrt{\frac{2}{\pi\epsilon}}e^{-\frac{1}{2\epsilon x^2}} - x \cdot \operatorname{erf}\left(\frac{1}{x\sqrt{2\epsilon}}\right)\right)} = -6.$$

Hence we have that, for every $\delta > 0$, there exists $\bar{x} > 0$ such that

$$\left| -x\frac{2g(x)}{\sigma_Y(x)} + 6\right| \leq \delta \quad \text{for every } x \geq \bar{x}. \tag{4.29}$$

We fix $\delta < 1$ and choose $\bar{x} > 0$ such that (4.29) holds. For every $x > \bar{x}$ we get the estimate

$$\left| -\int_{\bar{x}}^{x} \frac{2g(y)}{\sigma_Y(y)} dy + \int_{\bar{x}}^{x} \frac{6}{y} dy \right| \leq \int_{\bar{x}}^{x} \left| -\frac{2g(y)}{\sigma_Y(y)} + \frac{6}{y} \right| dy \leq \int_{\bar{x}}^{x} \frac{1}{y} \left| -y\frac{2g(y)}{\sigma_Y(y)} + 6 \right| dy$$
$$\leq \delta \left(\log(x) - \log(\bar{x}) \right).$$

Thus, for every $x > \bar{x}$, we have

$$(-6-\delta)\left(\log(x) - \log(\bar{x})\right) \leq -\int_{\bar{x}}^{x} \frac{2g(y)}{\sigma_Y(y)} dy \leq (-6+\delta)\left(\log(x) - \log(\bar{x})\right).$$

This implies that, taking $\tilde{\rho}$ as in (4.20) and choosing $c = \bar{x}$, for every $x > \bar{x}$,

$$\left(\frac{x}{\bar{x}}\right)^{-6-\delta} \leq \tilde{\rho}(x) \leq \left(\frac{x}{\bar{x}}\right)^{-6+\delta}. \tag{4.30}$$

Hence, taking $\tilde{s}$ as in (4.22) and choosing again $c = \bar{x}$, for every $x > \bar{x}$ we have

$$\int_{\bar{x}}^{x} \left(\frac{y}{\bar{x}}\right)^{-6-\delta} dy \leq \tilde{s}(x) = \int_{\bar{x}}^{x} \tilde{\rho}(y) dy \leq \int_{\bar{x}}^{x} \left(\frac{y}{\bar{x}}\right)^{-6+\delta} dy$$

so that $\tilde{s}(\infty) < \infty$ and in particular

$$\bar{x}^{6+\delta} \frac{x^{-5-\delta}}{5+\delta} \leq \tilde{s}(\infty) - \tilde{s}(x) = \int_{x}^{\infty} \tilde{\rho}(y) dy \leq \bar{x}^{6-\delta} \frac{x^{-5+\delta}}{5-\delta}.$$

Together with (4.30), this implies that

$$\frac{\bar{x}^{2\delta}}{5+\delta} x^{1-2\delta} \leq \frac{\tilde{s}(\infty) - \tilde{s}(x)}{\tilde{\rho}(x)} \leq \frac{\bar{x}^{-2\delta}}{5-\delta} x^{1+2\delta}$$

for every $x > \bar{x}$, so

$$\frac{\bar{x}^{2\delta}}{5+\delta} x^{-3-2\delta} \leq \frac{\tilde{s}(\infty) - \tilde{s}(x)}{\sigma_Y^2(x)\tilde{\rho}(x)} \leq \frac{\bar{x}^{-2\delta}}{5-\delta} x^{-3+2\delta}.$$

Therefore, as $\delta < 1$ by the choice of $\bar{x}$, we have that $\frac{\tilde{s}(\infty)-\tilde{s}}{\tilde{\rho}\sigma_Y^2} \in L^1_{loc}(\infty-)$ and condition (4.23) is satisfied. By Theorem 4.6, it follows that Z defined in (4.26) is not a martingale. □ □

We give now an example of a process whose optional projection into the delayed filtration is not a local martingale but admits an equivalent local martingale measure.

Example 4.7. Consider again the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ generated by the Brownian motion W in (4.4), and define $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, with $\mathcal{F}_t := \mathcal{G}_{t-\epsilon}$, $t \geq 0$, $\epsilon > 0$. Introduce the process $M = (M_t)_{t\geq 0}$ with $M_t = X_t - \int_0^t (1+s)dW_s$, where X is the inverse three-dimensional Bessel process. Thus

$$\begin{aligned}
{}^oM_{t+\epsilon} = \mathbb{E}[M_{t+\epsilon}|\mathcal{G}_t] &= \mathbb{E}\left[X_{t+\epsilon} - \int_0^{t+\epsilon}(1+s)dW_s \Big| \mathcal{G}_t\right] \\
&= X_t \mathrm{erf}\left(\frac{1}{X_t\sqrt{2\epsilon}}\right) - \int_0^t (1+s)dW_s, \quad t \geq 0,
\end{aligned}$$

where the last equality comes from Proposition 4.4 and from the martingale property of $\int (1+s)dW_s$. From (4.12) we have therefore

$$d\,{}^oM_{t+\epsilon} = \left(\sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{1}{2\epsilon X_t^2}} X_t - \mathrm{erf}\left(\frac{1}{X_t\sqrt{2\epsilon}}\right) X_t^2 - (1+t)\right) dW_t - \sqrt{\frac{2}{\pi}} \epsilon^{-\frac{3}{2}} e^{-\frac{1}{2\epsilon X_t^2}} dt, \quad t \geq 0.$$

It is then clear that ${}^oM$ is not an $\mathbb{F}$-local martingale. This implies that

$$\mathcal{M}_{loc}(M, \mathbb{G}) \cap \mathcal{M}^o_{loc}(M, \mathbb{F}) = \emptyset,$$

since $\mathcal{M}_{loc}(M, \mathbb{G}) = \{P\}$ by Remark 4.1, so that properties (P1), (P3) and (P4) have a negative answer.
We now introduce the Doléans exponential

$$\bar{Z}_t = \mathcal{E}_t\left(\int \bar{\alpha}_s dW_s\right), \quad t \geq 0,$$

with

$$\bar{\alpha}_t = \frac{\sqrt{\frac{2}{\pi}}\epsilon^{-\frac{3}{2}} e^{-\frac{1}{2\epsilon X_t^2}}}{\sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{1}{2\epsilon X_t^2}} X_t - \mathrm{erf}\left(\frac{1}{X_t\sqrt{2\epsilon}}\right) X_t^2 - (1+t)}, \quad t \geq 0,$$

and define the measure $\bar{Q}$ by $\frac{d\bar{Q}}{dP}|_{\mathcal{G}_t} = \bar{Z}_t$, $t \geq 0$. By (4.28) we have that $|\bar{\alpha}_t| \leq \sqrt{\frac{2}{\pi}}\epsilon^{-\frac{3}{2}}(1+t)^{-1}$ for all $t \geq 0$. Thus

$$\exp\left(\frac{1}{2}\int_0^\infty |\bar{\alpha}_s|^2 ds\right) \leq \exp\left(\frac{1}{\pi}\epsilon^{-3}\int_0^\infty (1+s)^{-2} ds\right) < \infty,$$

so Novikov's condition is satisfied and $\bar{Z}$ is a uniformly integrable martingale. By Girsanov's Theorem, $\bar{Q} \in \mathcal{M}_{loc}({}^oM, \mathbb{G})$. Hence, properties (P2) and (P5) are satisfied.

Remark 4.8. *In the above analysis* $\mathbb{G}$ *represents the natural filtration of* X*. By* (4.13) *we obtain that* (P1) *is not satisfied in this setting. However,* (P1) *still does not hold if* $\mathbb{G}$ *is given by the filtration generated by* (B^1, B^2, B^3)*, and* $\mathbb{F}$ *is the delayed filtration of* $\mathbb{F}^X$*, i.e.,* $\mathcal{F}_t = \mathcal{F}^X_{t-\epsilon}$*,* $t \geq 0$*,* $\epsilon > 0$*. In this case there exist infinitely many measures in* $\mathcal{M}_{loc}(X, \mathbb{G})$*, but* $\mathcal{M}_{loc}(X, \mathbb{F}^X, \mathbb{F}^X) = \{P\}$*. Corollary 3.7 implies that for every* $Q \in \mathcal{M}_{loc}(X, \mathbb{G})$ *we have*

$$\mathbb{E}^Q[X_t|\mathcal{F}_t] = \mathbb{E}^P[X_t|\mathcal{F}_t], \quad a.s., \quad t \geq 0.$$

Then ${}^{Q,o}X$ *is a* $(Q, \mathbb{F})$*-local martingale if and only if* Q *is an equivalent local martingale measure for* ${}^oX$*, which has dynamics given in* (4.12)*. However, this cannot be the case because such a measure* Q *would be defined by a density which is not a true martingale, by the same arguments as in the proof of Theorem 4.5.*

5 A stochastic volatility example

In this section we assume that X is given by a stochastic volatility process which is a local martingale with respect to a filtration $\mathbb{G}$. We then consider a sub-filtration $\hat{\mathbb{F}}$ of $\mathbb{G}$ such that property (P1) is satisfied even if the optional projection of X into $\hat{\mathbb{F}}$ is not a local martingale. We provide Examples 5.7 and 5.8 to show when property (P2) or (P3) hold, respectively.
We introduce a three-dimensional Brownian motion $B = (B^1, B^2, B^3)$ on a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{G} = (\mathcal{G}_t)_{t\geq 0})$, and consider a stochastic volatility model of the form

$$dX_t = \sigma_1 v_t^\alpha X_t dB_t^1 + \sigma_2 v_t^\alpha X_t dB_t^2, \quad t \geq 0, \quad X_0 = x > 0, \tag{5.1}$$

$$dv_t = a_1 v_t dB_t^1 + a_2 v_t dB_t^2 + a_3 v_t dB_t^3 + \rho(L - v_t)dt, \quad t \geq 0, \quad v_0 = 1, \tag{5.2}$$

where $\alpha, \rho, L \in \mathbb{R}^+$ and $\sigma_1, \sigma_2, a_1, a_2, a_3 \in \mathbb{R}$.

Remark 5.1. *The class of stochastic volatility processes* (5.1)-(5.2) *reduces to the class considered in Sin [1998] when* $a_3 = 0$ *and to the class presented in Biagini et al. [2014] when* $\rho = 0$ *and* $\alpha = 1$*. Therefore, all the results of this section can be applied to these particular cases.*

The next proposition states that, under a given condition on the coefficients of (5.1)-(5.2), X is a strict $\mathbb{G}$-local martingale under P but $\mathcal{M}_M(X, \mathbb{G}) \neq \emptyset$.

Proposition 5.2. *Consider the unique strong solution*[1] (X, v) *to the system of SDEs* (5.1)-(5.2)*. Then:*

1. X *is a local martingale, and is a true martingale if and only if*

$$a_1\sigma_1 + a_2\sigma_2 \leq 0.$$

2. *For every* $T > 0$ *there exists a probability measure* Q *equivalent to* P *on* $\mathcal{G}_T$ *such that* X *is a true* Q*-martingale on* $[0, T]$*.*

Proof. The proofs of the two claims are easy extensions of the proofs of Theorem 3.2 of Sin [1998] and of Theorem 5.1 of Biagini et al. [2014], respectively. □ □

We now give two results that provide a relation between the expectation of X and the explosion time of a process associated to the volatility v.

Lemma 5.3. *Let* (X, v) *satisfy the system of SDEs* (5.1)-(5.2)*. Then*

$$\mathbb{E}[X_t] = X_0 P(\{\hat{v} \text{ does not explode on } [0, t]\}), \quad t \geq 0, \tag{5.3}$$

where $\hat{v} = (\hat{v}_t)_{t \geq 0}$ *is given by*

$$\begin{aligned} d\hat{v}_t =& a_1\hat{v}_t dB_t^1 + a_2\hat{v}_t dB_t^2 + a_3\hat{v}_t dB_t^3 + \rho(L - \hat{v}_t)dt \\ &+ (a_1\sigma_1 + a_2\sigma_2)\hat{v}_t^{\alpha+1}dt, \quad t \geq 0, \end{aligned} \tag{5.4}$$

$\hat{v}_0 = 1.$

Proof. This result is a particular case of Proposition 5.9, which we give below. □ □

Lemma 5.4. *The (unique) solution to equation* (5.4) *explodes to* $+\infty$ *in finite time with positive probability if and only if* $a_1\sigma_1 + a_2\sigma_2 > 0$*. Moreover, if* $a_1\sigma_1 + a_2\sigma_2 > 0$*, it does not reach zero in finite time.*

Proof. The result is given in Lemma 4.3 of Sin [1998] when $a_3 = 0$, and proved by using Feller's test of explosions. In this case, the test is applicable because $\hat{v}$ is a one-dimensional Itô diffusion with respect to the Brownian motion $1/|a|(a \cdot B)$, with $a = (a_1, a_2)$ and $B = (B^1, B^2)$. The author introduces $\sigma = (\sigma_1, \sigma_2)$ and proves that $\hat{v}$ explodes with positive probability in finite time and does not reach the origin in finite time if $a \cdot \sigma > 0$. In our case, the proof comes as an easy extension by considering now $a = (a_1, a_2, a_3)$ and $\sigma = (\sigma_1, \sigma_2, 0)$. □ □

[1] Existence and uniqueness of a strong solution to (5.1)-(5.2) can be proved as an extension of [Sin, 1998, Remark 2.2].

We now give an example for which property (P2) is satisfied. We start with the following lemma, which is Proposition 5.2 of Karatzas and Ruf [2016].

Lemma 5.5. *Fix an open interval $I = (\ell, r)$ with $-\infty \leq \ell < r \leq \infty$ and consider the stochastic differential equation*

$$dY_t = s(Y_t)\left(dW_t + b(Y_t)dt\right), \quad t \geq 0, \quad Y_0 = \xi, \tag{5.5}$$

where $\xi \in I$ and W denotes a Brownian motion. Suppose that the functions $b : (I, \mathcal{B}(I)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $s : (I, \mathcal{B}(I)) \to (\mathbb{R} \setminus \{0\}, \mathcal{B}(\mathbb{R} \setminus \{0\}))$ are measurable and satisfy

$$\int_K \left(\frac{1}{s^2(y)} + \left| \frac{b(y)}{s(y)} \right| \right) dy < \infty \qquad \text{for every compact set } K \subset I. \tag{5.6}$$

Call τ^ξ the first time when the weak solution Y to (5.5), unique in the sense of probability distribution, exits the open interval I. Introduce the function $U : (0, \infty) \times I \to \mathbb{R}^+$ defined by

$$U(t, \xi) := P(\tau^\xi > t).$$

If the functions $s(\cdot)$ and $b(\cdot)$ are locally Hölder continuous on I, the function $U(\cdot, \cdot)$ is of class $C\left([0, \infty) \times I\right) \cap C^{1,2}\left((0, \infty) \times I\right)$.

Applying Lemma 5.5 to our setting, we get the following result.

Lemma 5.6. *Consider the solution $\hat{v}$ to equation (5.4), supposing $\rho = 0$ and $a_1\sigma_1 + a_2\sigma_2 > 0$. Define the function $m : (0, \infty) \to \mathbb{R}^+$ by*

$$m_t = P(\{\hat{v} \text{ does not explode on } [0, t]\}). \tag{5.7}$$

Then $m \in C^1((0, \infty))$.

Proof. Note that $\hat{v}$ is a one-dimensional Itô diffusion with respect to the Brownian motion $W = 1/|a|(a \cdot B)$, with $a = (a_1, a_2, a_3)$ and $B = (B^1, B^2, B^3)$. In particular, we have

$$d\hat{v}_t = |a|\hat{v}_t dW_t + (a_1\sigma_1 + a_2\sigma_2)\hat{v}^{\alpha+1}dt, \quad t \geq 0.$$

We are thus in the setting of Lemma 5.5 with $I = (0, \infty)$ and

$$s(x) = |a|x, \qquad b(x) = \frac{a_1\sigma_1 + a_2\sigma_2}{|a|}x^\alpha.$$

Condition (5.6) is satisfied because, for every K compact interval of $(0,\infty)$, we have

$$\int_K \left(\frac{1}{s^2(y)} + \left| \frac{b(y)}{s(y)} \right| \right) dy = \int_K \left(\frac{1}{|a|y^2} + \left| \frac{a_1\sigma_1 + a_2\sigma_2}{|a|^2} y^{\alpha-1} \right| \right) dy < \infty.$$

Moreover, $s(\cdot)$ and $b(\cdot)$ are locally Hölder continuous on $(0,\infty)$. The result follows from Lemma 5.5, since $\hat{v}$ does not reach zero in finite time by Lemma 5.4. □ □

We are now ready to state our first result in this setting.

Example 5.7. Consider the solution (X, v) to the system of SDEs (5.1)-(5.2), supposing $\rho = 0$ and $a_1\sigma_1 + a_2\sigma_2 > 0$. Let $\mathbb{H}$ be the filtration generated by (B^1, B^2, B^3), and $\mathbb{F}$ the filtration generated by a fourth Brownian motion B^4, independent of (B^1, B^2, B^3). Introduce the $\mathbb{F}$-local martingale $N = (N_t)_{t\geq 0}$ whose dynamics are given by

$$dN_t = (1 + m'_t)(1 + t)dB^4_t, \quad t \geq 0,$$

where m' is the first derivative of the function m we introduce in (5.7). Then the process $R := N + X$ is an $\mathbb{H} \vee \mathbb{F}$-local martingale, and its optional projection into $\mathbb{F}$ is ${}^oR = N + m$ by Lemma 5.3. Thus ${}^oR$ is not an $\mathbb{F}$-local martingale, because m is not constant by Lemma 5.4.
We introduce the stochastic exponential

$$Z_t = \mathcal{E}_t\left(\int \alpha_s dB^4_s \right), \quad t \geq 0$$

with

$$\alpha_t = \frac{m'_t}{(1 + m'_t)(1 + t)}, \quad t \geq 0.$$

Note that $m' \in C((0,\infty))$ by Lemma 5.6. It follows that Novikov's condition is satisfied, so Z is an $\mathbb{F}$-uniformly integrable martingale and we can introduce the probability measure $Q \sim P$ defined by Z. Then ${}^oR = N + m$ is a $(Q, \mathbb{F})$-local martingale. Hence, property (P2) is satisfied.

Example 5.8. Again in the setting and with the notations of Example 5.7, introduce a $(P, \mathbb{F})$-strict local martingale Y, independent of $\mathbb{H}$. Let Q be the probability measure from Proposition 5.2 under which X is a true martingale. Since the density of Q with respect to P only depends on (B^1, B^2, B^3), Y is a strict local martingale also with respect to Q, and $U := Y + X$ as well. The Q-optional projection of U into $\mathbb{F}$ is given by

$$^{Q,o}U_t = Y_t + \mathbb{E}[X_t] = Y_t + X_0, \quad t \geq 0,$$

which is a $(Q, \mathbb{F})$-local martingale.
Thus $P \notin \mathcal{M}^o_{loc}(U, \mathbb{F})$ but

$$\mathcal{M}_L(U, \mathbb{H} \vee \mathbb{F}) \cap \mathcal{M}^o_{loc}(U, \mathbb{F}) \neq \emptyset,$$

i.e., property (P3) holds for U.

We now find an example of a sub-filtration $\hat{\mathbb{F}} \subset \mathbb{G}$ such that the optional projection of X into $\hat{\mathbb{F}}$ is not an $\hat{\mathbb{F}}$-local martingale. The next proposition is a generalization of Lemma 4.2 of Sin [1998].

Proposition 5.9. *Suppose that the two-dimensional process (X, v) satisfies the system of SDEs* (5.1)-(5.2), *and call $\mathbb{F}$ the natural filtration of B^1. Introduce the process $(\hat{X}_t)_{t\geq 0}$ defined by*

$$\hat{X}_t = B^1_t - \sigma_1 \int_0^t v^\alpha_s ds, \quad t \geq 0 \tag{5.8}$$

and call $\hat{\mathbb{F}}$ the natural filtration of $\hat{X}$. Then for every $\hat{\mathbb{F}}$- stopping time $\hat{\tau}$ there exists an $\mathbb{F}$-stopping time τ such that

$$\mathbb{E}[X_{T\wedge\hat{\tau}}] = X_0 P(\{\hat{v} \textit{ does not explode on } [0, T \wedge \tau]\}), \tag{5.9}$$

where $\hat{v}$ is defined in (5.4).

Proof. By (5.1), X is a positive local martingale. Define a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ by

$$\tau_n = \inf\left\{ t \in \mathbb{R}^+ : |\sigma_1 + \sigma_2|^2 \int_0^t v^{2\alpha}_s ds \geq n \right\} \wedge T,$$

with $v = (v_t)_{t\geq 0}$ in (5.2). Then the process X^n defined by

$$X^n_t = X_{t\wedge\tau_n}, \quad t \geq 0 \tag{5.10}$$

is a local martingale for $n \in \mathbb{N}$. Define Z^n by

$$Z^n_t = \sigma_1 \int_0^{t\wedge\tau_n} v^\alpha_s dB^1_s + \sigma_2 \int_0^{t\wedge\tau_n} v^\alpha_s dB^2_s, \quad t \geq 0.$$

Then X^n is the stochastic exponential of Z^n, and since $[Z^n, Z^n]_t \leq n$ for all $t \geq 0$, X^n is a $(P, \mathbb{G})$-martingale for every $n \in \mathbb{N}$ by Novikov's condition and $(\tau_n)_{n\in\mathbb{N}}$ reduces X with respect to $(P, \mathbb{G})$.
Since X^n stopped at $\hat{\tau}$ is also a martingale, we can define a new probability measure Q_n on $(\Omega, \mathcal{G}_T)$ as

$$Q_n(A) = \frac{1}{X_0}\mathbb{E}[X_{T\wedge\tau_n\wedge\hat{\tau}}\mathbb{1}_A] \qquad \text{for all } A \in \mathcal{G}_T.$$

By the Lebesgue dominated convergence theorem,

$$\mathbb{E}[X_{T\wedge\hat{\tau}}] = \lim_{n\to\infty}\mathbb{E}[X_{T\wedge\tau_n\wedge\hat{\tau}}\mathbb{1}_{\{\tau_n\geq T\wedge\hat{\tau}\}}] = X_0 \lim_{n\to\infty} Q_n\left(\tau_n \geq T\wedge\hat{\tau}\right), \quad (5.11)$$

by definition of Q_n. Moreover, Girsanov's Theorem implies that the processes $B^{(n,1)}$, $B^{(n,2)}$ defined by

$$B_t^{(n,1)} = B_t^1 - \sigma_1\int_0^t \mathbb{1}_{\{s\leq\tau_n\wedge\hat{\tau}\}}v_s^\alpha ds, \quad t\geq 0 \qquad (5.12)$$

$$B_t^{(n,2)} = B_t^2 - \sigma_2\int_0^t \mathbb{1}_{\{s\leq\tau_n\wedge\hat{\tau}\}}v_s^\alpha ds, \quad t\geq 0 \qquad (5.13)$$

are Brownian motions under Q_n, $n \geq 0$. Therefore under Q_n, the process v has the dynamics

$$\begin{aligned} dv_t = {} & a_1 v_t dB_t^{(n,1)} + a_2 v_2 dB_t^{(n,2)} + a_3 v_t dB_t^3 + \rho(L - v_t)dt \\ & + \mathbb{1}_{\{t\leq\tau_n\wedge\hat{\tau}\}}(a_1\sigma_1 + a_2\sigma_2)v_t^{\alpha+1}dt, \quad t\geq 0, \quad v_0 = 1. \end{aligned} \qquad (5.14)$$

Consider now the process $\hat{X}$ introduced in (5.8) and define $\hat{v}$ as the unique, strong solution[2] of the SDE

$$d\hat{v}_t = a_1\hat{v}_t dB_t^1 + a_2\hat{v}_t dB_t^2 + a_3 v_t dB_t^3 + \rho(L-\hat{v}_t)dt + (a_1\sigma_1 + a_2\sigma_2)\hat{v}_t^{\alpha+1}dt, \qquad (5.15)$$

$t \geq 0$. Note that on $[0, \tau_n \wedge \hat{\tau}]$, $(\hat{X}, v)$ has the same distribution under Q_n as $(B^1, \hat{v})$ under P.
By the Doob measurability theorem (see, e.g., [Kallenberg, 2006, Lemma 1.13]), there exists a measurable function $h : \mathcal{C}[0,\infty) \to \mathbb{R}^+$ such that $\hat{\tau} = h(\hat{X}_\cdot)$. Set $\tau = h(B_\cdot^1)$. As $T \wedge \hat{\tau}$ is a $\sigma(\hat{X})$-stopping time there exists, by the Doob measurability theorem again, a $\mathcal{B}(\mathcal{C}[0,t])$-measurable function Ψ_t such that $\mathbb{1}_{\{t\geq T\wedge\hat{\tau}\}} = \Psi_t(\hat{X}_\cdot^t)$. Thus

$$\mathbb{1}_{\{\tau_n\geq T\wedge\hat{\tau}\}} = \Psi_{\tau_n}(\hat{X}_\cdot^{\tau_n}), \quad n \in \mathbb{N}.$$

[2]It can be seen that admits an unique, strong solution following the same arguments as in the proof of Lemma 4.2 of Sin [1998].

Analogously, by the construction of τ, we have

$$\mathbb{1}_{\{\hat{\tau}_n \geq T\wedge\tau\}} = \Psi_{\hat{\tau}_n}(B_\cdot^{1,\hat{\tau}_n}), \quad n \in \mathbb{N},$$

where $(\hat{\tau}_n)_{n\in\mathbb{N}}$ are stopping times for the natural filtration of $\hat{v}$, defined by

$$\hat{\tau}_n = \inf\left\{t \in \mathbb{R}^+ : |\sigma_1 + \sigma_2|^2 \int_0^s \hat{v}_u^{2\alpha} du \geq n\right\}, \quad n \geq 1.$$

Since on $[0, \tau_n \wedge \hat{\tau}]$, $(\hat{X}, v)$ has the same law under Q_n as $(B^1, \hat{v})$ under P, we have that $\Psi_{\tau_n}(\hat{X}_\cdot^{\tau_n})$ has the same law under Q_n as $\Psi_{\hat{\tau}_n}(B_\cdot^{1,\hat{\tau}_n})$ under P. Thus, from (5.11) we get

$$\begin{aligned}
\mathbb{E}[X_{T\wedge\hat{\tau}}] &= X_0 \lim_{n\to\infty} Q_n\left(\tau_n \geq T \wedge \hat{\tau}\right) \\
&= X_0 \lim_{n\to\infty} \mathbb{E}^{Q_n}\left[\Psi_{\tau_n}(\hat{X}_\cdot^{\tau_n})\right] \\
&= X_0 \lim_{n\to\infty} \mathbb{E}^{P}\left[\Psi_{\hat{\tau}_n}(B_\cdot^{1,\hat{\tau}_n})\right] \\
&= X_0 \lim_{n\to\infty} P\left(\hat{\tau}_n \geq T \wedge \tau\right) \\
&= X_0 P\left(\hat{\tau}_n \geq T \wedge \tau \text{ for some } n\right) \\
&= X_0 P\left(\hat{v} \text{ does not explode before time } T \wedge \tau\right),
\end{aligned}$$

and the proof is complete. □ □

We are now ready to give the following

Theorem 5.10. *Consider the stochastic volatility process X defined by*

$$dX_t = \sigma_1 v_t^\alpha X_t dB_t^1 + \sigma_2 v_t^\alpha X_t dB_t^2, \quad t \geq 0, \quad X_0 = x > 0, \tag{5.16}$$
$$dv_t = a_2 v_t dB_t^2 + \rho(L - v_t)dt, \quad t \geq 0, \quad v_0 = 1, \tag{5.17}$$

i.e., the model introduced in (5.1)-(5.2) *with $a_1 = a_3 = 0$, and suppose that $a_2\sigma_2 > 0$. Consider the filtration $\hat{\mathbb{F}} \subset \mathbb{G}$, generated by the process $\hat{X}$ defined in* (5.8)*. Then the P-optional projection of X into $\hat{\mathbb{F}}$ is not an $\hat{\mathbb{F}}$-local martingale.*

Proof. The process X in (5.16) is a strict local martingale by Proposition 5.2. By Proposition 5.9, for every $\hat{\mathbb{F}}$-stopping time $\hat{\tau}$ there exists a $\sigma(B^1)$-stopping time τ such that

$$\mathbb{E}[X_{T\wedge\hat{\tau}}] = X_0 P(\{\hat{v} \text{ does not explode on } [0, T \wedge \tau]\}), \tag{5.18}$$

where $\hat{v}$ is now given by

$$d\hat{v}_t = a_2 \hat{v}_t dB_t^2 + \rho(L - \hat{v}_t)dt + a_2\sigma_2 \hat{v}_t^{\alpha+1} dt, \quad t \geq 0, \quad \hat{v}_0 = 1.$$

Since $a_1\sigma_1 + a_2\sigma_2 = a_2\sigma_2 > 0$, Lemma 5.4 implies that

$$P(\{\hat{v} \text{ does not explode on } [0,t]\}) < 1 \quad \text{for all } t > 0.$$

In particular,

$$P(\{\hat{v} \text{ does not explode on } [0, T \wedge \eta]\}) < 1$$

for every $\sigma(B^1)$-stopping time η with $P(\eta = \infty) < 1$, because $\hat{v}$ is independent of B^1. Together with (5.18), this implies that X cannot be localized by any sequence of $\hat{\mathbb{F}}$-stopping times. Consequently, the optional projection of X into $\hat{\mathbb{F}}$ cannot be an $\hat{\mathbb{F}}$-local martingale by Theorem 3.7 of Föllmer and Protter [2011]. □ □

Proposition 5.2 and Theorem 5.10 provide a further example of two probability measures P and Q, of a P-local martingale X and of a non trivial filtration $\hat{\mathbb{F}} \subset \mathbb{G}$, such that the optional projection of X into $\hat{\mathbb{F}}$ under P is not a P-local martingale but the optional projection of X into $\hat{\mathbb{F}}$ under Q is a Q-martingale.

A Optional projections and optimal transport

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with filtrations $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, $\mathbb{F} \subset \mathbb{G}$, both satisfying the usual hypothesis of right-continuity and completeness.
For $\mathbb{G}$-adapted càdlàg processes X and Z, we denote $X \ll_{\mathbb{G}} Z$ if $Z - X$ is a nonnegative $\mathbb{G}$-supermartingale.

Proposition A.1. *Let X be a nonnegative, càdlàg $\mathbb{G}$-supermartingale. Then X is a $\mathbb{G}$-local martingale if and only if $X \ll_{\mathbb{G}} Z$ for all $\mathbb{G}$- supermartingales $Z \geq X$.*

Proof. Assume that X is a local martingale, and consider a supermartingale $Z \geq X$. Let $(\tau_n)_{n\geq 0}$ be a localizing sequence for X. By Fatou's lemma,

$$\mathbb{E}[Z_t - X_t|\mathcal{F}_s] \leq \liminf_{n\to\infty} \mathbb{E}[Z_{t\wedge\tau_n} - X_{t\wedge\tau_n}|\mathcal{F}_s] \leq Z_s - \liminf_{n\to\infty} \mathbb{E}[X_{t\wedge\tau_n}|\mathcal{F}_s] = Z_s - X_s$$

for every $0 \leq s \leq t$, so $Z - X$ is a supermartingale.

Suppose now that $Z - X$ is a nonnegative supermartingale for every supermartingale $Z \geq X$. Assume also that X is not a local martingale, i.e., that it is a strict supermartingale. Then X has the Doob-Meyer decomposition

$$X_t = M_t - Y_t, \quad t \geq 0,$$

where $M \geq X$ is a nonnegative local martingale and $Y \neq 0$ is a nondecreasing process. Thus $M \geq X$ is a supermartingale for which $M - X = Y$ is not a supermartingale, which is a contradiction. □ □

Proposition A.1 can be used to characterize when the optional projection of a local martingale remains a local martingale, and to provide a sufficient and necessary condition for this property via optimal transport.

Theorem A.2. *For a nonnegative $\mathbb{G}$-local martingale X, ${}^oX$ is a $\mathbb{F}$-local martingale if and only if, for every $\mathbb{F}$-supermartingale $Y \geq {}^oX$, there is a $\mathbb{G}$-supermartingale $Z \geq X$ with ${}^oZ \ll_{\mathbb{F}} Y$.*

Proof. If ${}^oX$ is a local martingale, we may choose $Z = X$. To prove the converse, $Z - X$ is a nonnegative $\mathbb{G}$-supermartingale, ${}^oZ - {}^oX$ is a nonnegative $\mathbb{F}$-supermartingale, so $Y - {}^oX = Y - {}^oZ + {}^oZ - {}^oX$ is a nonnegative $\mathbb{F}$-supermartingale. □ □

Given laws ν_s, ν_t on $\mathbb{R}_+$, we denote $\nu_s \ll_{cdo} \nu_t$ if $\int f d\nu_s \leq \int f d\nu_t$ for all $f \in C_d$, where C_d is the set of real-valued convex and decreasing functions on $\mathbb{R}_+$. Let now X be a $\mathbb{G}$-adapted nonnegative process with law ν and ν_s be the law of X_s, $s \geq 0$. If $\nu_s \ll_{cdo} \nu_t$ for all $s \leq t$, we say that the law ν is convex decreasing. In this case, there exists a Markov process with law ν which is a $\mathbb{G}$-supermartingale; see Theorem 3 of Kellerer [1972]. Also note that if a process is a $\mathbb{G}$-supermartingale, its law is convex decreasing.
We denote by $S(\nu)$ the set of joint laws of (Z, X), where Z ranges over all supermartingales dominating X. For $\pi \in S(\nu)$, π_t denotes the joint law of (Z_t, X_t). An application of Proposition A.1 leads to the following result.

Proposition A.3. *Given a nonnegative $\mathbb{G}$-adapted process X with the law ν, X is a $\mathbb{G}$-local martingale if and only if for every $t \geq 0$,*

$$\sup_{\pi \in S(\nu)} \sup_{s<t} \sup_{f \in C_d} \left[\int f(z-x) d\pi_t(z,x) - \int f(z-x) d\pi_s(z,x) \right] \leq 0. \quad \text{(A.1)}$$

Proof. If (A.1) does not hold, there exists a supermartingale $Z \geq X$ such that the law of $Z - X$ is not convex decreasing. Then $Z - X$ is not a supermartingale, so X is not a local martingale by Proposition A.1. Suppose now that X is not a local martingale. By Proposition A.1, there exists a supermartingale $Z \geq X$ such that $Z - X$ is not a supermartingale. Then the law of $Z - X$ is not convex decreasing, and (A.1) fails. □ □

Conflict of interest

The authors declare that they have no conflict of interest.

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