A stochastic maximum principle for processes driven by fractional Brownian motion

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Abstract

We prove a stochastic maximum principle for controlled processes $X(t) = X^{(u)}(t)$ of the form

$$dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB^{(H)}(t)$$

where $B^{(H)}(t)$ is $m$-dimensional fractional Brownian motion with Hurst parameter $H = (H_1, \cdots, H_m) \in (\frac{1}{2}, 1)^m$. As an application we solve a problem about minimal variance hedging in an incomplete market driven by fractional Brownian motion.

1 Introduction

Let $H = (H_1, \cdots, H_m)$ with $\frac{1}{2} < H_j < 1$, $j = 1, 2, \ldots, m$, and let $B^{(H)}(t) = (B_1^{(H)}(t), \ldots, B_m^{(H)}(t))$, $t \in \mathbb{R}$ be $m$-dimensional fractional Brownian motion, i.e. $B^{(H)}(t) = B^{(H)}(t, \omega)$, $(t, \omega) \in \mathbb{R} \times \Omega$ is a Gaussian process in $\mathbb{R}^m$ such that

$$\mathbb{E} [B^{(H)}(t)] = B^{(H)}(0) = 0$$

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and

\[
(1.2) \quad \mathbb{E} \left[ B_j^{(H)}(s) B_k^{(H)}(t) \right] = \frac{1}{2} \left\{ |s|^{2H_j} + |t|^{2H_k} - |t-s|^{2H_j} \right\} \delta_{jk}; 1 \leq j, k \leq n, \quad s, t \in \mathbb{R},
\]

where

\[
\delta_{jk} = \begin{cases} 
0 & \text{when } j \neq k \\
1 & \text{when } j = k 
\end{cases}
\]

Here \( \mathbb{E} = \mathbb{E}_\mu \) denotes the expectation with respect to the probability law \( \mu = \mu_H \) for \( B^{(H)}(\cdot) \). This means that the components \( B_1^{(H)}(\cdot), \ldots, B_m^{(H)}(\cdot) \) of \( B^{(H)}(\cdot) \) are \( m \) independent 1-dimensional fractional Brownian motions with Hurst parameters \( H_1, H_2, \ldots, H_m \), respectively. We refer to [MvN], [NVV] and [S] for more information about fractional Brownian motion. Because of its interesting properties (e.g. long range dependence and self-similarity of the components) \( B^{(H)}(t) \) has been suggested as a replacement of standard Brownian motion \( B(t) \) (corresponding to \( H_j = \frac{1}{2} \) for all \( j = 1, \ldots, m \)) in several stochastic models, including finance.

Unfortunately, \( B^{(H)}(\cdot) \) is neither a semimartingale nor a Markov process, so the powerful tools from the theories of such processes are not applicable when studying \( B^{(H)}(\cdot) \). Nevertheless, an efficient stochastic calculus of \( B^{(H)}(\cdot) \) can be developed. This calculus uses an Itô type of integration with respect to \( B^{(H)}(\cdot) \) and white noise theory. See [DHP] and [H2] for details. For applications to finance see [HO2], [HOS1] [HOS2]. In [Hu1], [Hu2], [HOZ] and [OZ] the theory is extended to multi-parameter fractional Brownian fields \( B^{(H)}(x); x \in \mathbb{R}^d \) and applied to stochastic partial differential equations driven by such fractional white noise.

The purpose of this paper is to establish a stochastic maximum principle for stochastic control of processes driven by \( B^{(H)}(\cdot) \). We illustrate the result by applying it to a problem about minimal variance hedging in finance.

## 2 Preliminaries

For the convenience of the reader we recall here some of the basic results of fractional Brownian motion calculus. Let \( B^{(H)}(t) \) be 1-dimensional in the following.

Define, for given \( H \in (\frac{1}{2}, 1) \),

\[
(2.1) \quad \phi(s,t) = \phi_H(s,t) = H(2H - 1)|s-t|^{2H-2}; \quad s, t \in \mathbb{R}.
\]

As in [HO2] we will assume that \( \Omega \) is the space \( S'(\mathbb{R}) \) of tempered distributions on \( \mathbb{R} \), which is the dual of the Schwartz space \( S(\mathbb{R}) \) of rapidly decreasing functions on \( \mathbb{R} \). If \( \omega \in S'(\mathbb{R}) \) and \( f \in S(\mathbb{R}) \) we let \( \langle \omega, f \rangle = \omega(g) \) denote the action of \( \omega \) applied to \( f \). It can be extended to all \( f : \mathbb{R} \to \mathbb{R} \) such that

\[
\|f\|_\phi^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} f(s)f(t)\phi(s,t)ds dt < \infty.
\]

The space of all such (deterministic) functions \( f \) is denoted by \( L^2_\phi(\mathbb{R}) \).

If \( F : \Omega \to \mathbb{R} \) is a given function we let

\[
(2.2) \quad D_t^\phi F = \int_{\mathbb{R}} D_r F \cdot \phi(r,t)dr
\]
denote the Malliavin $\phi$-derivative of $F$ at $t$ (if it exists) (see [DHP, Definition 3.4]). Define $\mathcal{L}_\phi^{1,2}$ to be the set of (measurable) processes $g(t, \omega): \mathbb{R} \times \Omega \to \mathbb{R}$ such that $D^\phi_s g(s)$ exists for a.a. $s \in \mathbb{R}$ and

$$
\|g\|_{L_\phi^{1,2}}^2 := E\left[\int_\mathbb{R} \int_\mathbb{R} g(s)g(t)\phi(s, t)ds \, dt + \left(\int_\mathbb{R} D^\phi_s g(s)ds\right)^2\right] < \infty
$$

We let $\int_\mathbb{R} \sigma(t, \omega)dB^{(H)}(t)$ denote the fractional Itô-integral of the process $\sigma(t, \omega)$ with respect to $B^{(H)}(t)$, as defined in [DHP]. In particular, this means that if $\sigma$ belongs to the family $\mathcal{S}$ of step functions of the form

$$
\sigma(t, \omega) = \sum_{i=1}^{N} \sigma_i(\omega)\chi_{(t_i, t_{i+1})}(t), \quad (t, \omega) \in \mathbb{R} \times \Omega,
$$

where $0 \leq t_1 < t_2 < \cdots < t_{N+1}$, then

$$
\int_\mathbb{R} \sigma(t, \omega)dB^{(H)}(t) = \sum_{i=1}^{N} \sigma_i(\omega) \diamond (B^{(H)}(t_{i+1}) - B^{(H)}(t_i))
$$

where $\diamond$ denotes the Wick product. For $\sigma(t) = \sigma(t, \omega) \in \mathcal{S} \cap \mathcal{L}_\phi^{1,2}$ we have the isometry

$$
E\left[\int_\mathbb{R} \sigma(t, \omega)dB^{(H)}(t)\right]^2 = E\left[\int_\mathbb{R} \sigma(s)\sigma(t)\phi(s, t)ds \, dt + \left(\int_\mathbb{R} D^\phi_s \sigma(s)ds\right)^2\right] = \|\sigma\|_{L_\phi^{1,2}}^2,
$$

where $E = E_{\mu_H}$. Using this we can extend the integral $\int_\mathbb{R} \sigma(t, \omega)dB^{(H)}(t)$ to $L_\phi^{1,2}$. Note that if $\sigma, \theta \in L_\phi^{1,2}$, we have, by polarization,

$$
E \left[\int_\mathbb{R} \sigma(t, \omega)dB^{(H)}(t) \int_\mathbb{R} \theta(t, \omega)dB^{(H)}(t)\right] = E \left[\int_\mathbb{R} \sigma(s)\theta(t)\phi(s, t)ds \, dt + \int_\mathbb{R} D^\phi_s \sigma(s)ds \int_\mathbb{R} D^\phi_t \theta(t)dt\right].
$$

Also note that we need not assume that the integrand $\sigma \in L_\phi^{1,2}$ is adapted to the filtration $\mathcal{F}_t^{(H)}$ generated by $B^{(H)}(s, \cdot); s \leq t$.

An important property of this fractional Itô-integral is that

$$
E \left[\int_\mathbb{R} \sigma(t, \omega)dB^{(H)}(t)\right] = 0 \quad \text{for all } \sigma \in L_\phi^{1,2}.
$$

(see [DHP, Theorem 3.9]).

We give three versions of the fractional Itô formula, in increasing order of complexity.

Theorem 2.1 ([DHP], Theorem 4.1) Let $f \in C^2(\mathbb{R})$ with bounded second order derivatives. Then for $t \geq 0$

$$
f(B^{(H)}(t)) = f(B^{(H)}(0)) + \int_0^t f'(B^{(H)}(s))dB^{(H)}(s) + \int_0^t \int_0^s s^{2H-1} f''(B^{(H)}(s))ds \, ds.
$$
Theorem 2.2 ([DHP], Theorem 4.3) Let \( X(t) = \int_0^t \sigma(s, \omega)dB^H(s) \), where \( \sigma \in \mathcal{L}^{1,2}_\phi \) and assume \( f \in C^2(\mathbb{R}_+ \times \mathbb{R}) \) with bounded second order derivatives. Then for \( t \geq 0 \)

\[
f(t, X(t)) = f(0, 0) + \int_0^t \frac{\partial f}{\partial s}(s, X(s))ds + \int_0^t \frac{\partial f}{\partial x}(s, X(s))\sigma(s)dB^H(s) + \int_0^t \frac{\partial^2 f}{\partial x^2}(s, X(s))\sigma(s)D_s^\phi X(s)ds.
\]

Finally we give an \( m \)-dimensional version:

Let \( B^H(t) = (B_1^H(t), \ldots, B_m^H(t)) \) be an \( m \)-dimensional fractional Brownian motion with Hurst parameter \( H = (H_1, \ldots, H_m) \in (1/2, 1)^m \), as in Section 1. Since we are here dealing with \( m \) independent fractional Brownian motions we may regard \( \Omega \) as the product of \( m \) independent copies of \( \tilde{\Omega} \) and write \( \omega = (\omega_1, \ldots, \omega_m) \) for \( \omega \in \Omega \). Then in the following the notation \( D_{k,s}^\phi Y \) means the Malliavin \( \phi \)-derivative with respect to \( \omega_k \) and could also be written

\[
D_{k,s}^\phi Y = \int_\mathbb{R} \phi_{H_k}(s,t)D_{k,t}Y dt = \int_\mathbb{R} \phi_{H_k}(s,t)\frac{\partial Y}{\partial \omega_k}(t,\omega)dt.
\]

Similar to the 1-dimensional case discussed in Section 1, we can define the multi-dimensional fractional (Wick-Itô) integral

\[
\int_\mathbb{R} f(t, \omega)dB^H(t) = \sum_{j=1}^m \int_\mathbb{R} f_j(t, \omega)dB_j^H(t) \in L^2(\mu)
\]

for all processes \( f(t, \omega) = (f_1(t, \omega), \ldots, f_m(t, \omega)) \in \mathbb{R}^m \) such that, for all \( j = 1, 2, \ldots, m \),

\[
\|f_j\|_{\mathcal{L}^{1,2}_\phi}^2 := \mathbb{E}\left[ \int_\mathbb{R} \int_\mathbb{R} f_j(s)f_j(t)\phi_j(s,t)dsdt + \left( \int_\mathbb{R} D_{j,t}^\phi f_j(t)dt \right)^2 \right] < \infty
\]

where \( \phi_j = \phi_{H_j}, 1 \leq j \leq m \).

Denote the set of all such \( m \)-dimensional processes \( f \) by \( \mathcal{L}^{1,2}_\phi(m) \), where \( \phi = (\phi_1, \ldots, \phi_m) \).

It can be proved (see [BO]) that for \( f, g \in \mathcal{L}^{1,2}_\phi(m) \) we have the following fractional multi-dimensional Itô isometry

\[
\mathbb{E}\left[ \left( \int_\mathbb{R} dB^H(t) \right) \cdot \left( \int_\mathbb{R} gdB^H(t) \right) \right] = \mathbb{E}\left[ \sum_{i=1}^m \int_\mathbb{R} \int_\mathbb{R} f_i(s)g_i(t)\phi_i(s,t)dsdt \right] + \sum_{i,j=1}^m \left( \int_\mathbb{R} D_{j,t}^\phi f_i(t)dt \right) \cdot \left( \int_\mathbb{R} D_{i,t}^\phi g_j(t)dt \right).
\]

We put

\[
(f, g)_{\mathcal{L}^{1,2}_\phi(m)} = \mathbb{E}\left[ \sum_{i=1}^m \int_\mathbb{R} \int_\mathbb{R} f_i(s)g_i(t)\phi_i(s,t)dsdt \right] + \sum_{i,j=1}^m \left( \int_\mathbb{R} D_{j,t}^\phi f_i(t)dt \right) \cdot \left( \int_\mathbb{R} D_{i,t}^\phi g_j(t)dt \right).
\]
and define
\[ \mathbb{L}^{1,2}_\phi(m) = \{ f \in \mathcal{L}^{1,2}_\phi(m); \|f\|^{2,1,2}_{\phi}(m) := (f, f)_{\mathbb{L}^{1,2}_\phi(m)} < \infty \} . \]

Now suppose \( \sigma_i \in \mathcal{L}^{1,2}_\phi(m) \) for \( 1 \leq i \leq n \). Then we can define \( X(t) = (X_1(t), \ldots, X_n(t)) \)
where
\[ X_i(t, \omega) = \sum_{j=1}^{m} \int_0^t \sigma_{ij}(s, \omega) dB_j^{(H)}(s); 1 \leq i \leq n. \]

We have the following multi-dimensional fractional Itô formula:

**Theorem 2.3** Let \( f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n) \) with bounded second order derivatives. Then, for \( t \geq 0 \),
\[
 f(t, X(t)) = f(0, 0) + \int_0^t \frac{\partial f}{\partial s}(s, X(s))ds + \int_0^t \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(s, X(s))dX_i(s) \\
+ \int_0^t \left\{ \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X(s)) \sum_{k=1}^{m} \sigma_{ik}(s) D_{k,s}^{\phi}(X_j(s)) \right\} ds \\
= f(0, 0) + \int_0^t \frac{\partial f}{\partial s}(s, X(s))ds + \sum_{j=1}^{m} \int_0^t \left[ \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(s, X(s)) \sigma_{ij}(s, \omega) \right] dB_j^{(H)}(s) \\
+ \int_0^t \text{Tr} \left[ \Lambda^T(s) f_{xx}(s, X(s)) \right] ds .
\]

Here \( \Lambda = [\Lambda_{ij}] \in \mathbb{R}^{n \times m} \) with
\[
 \Lambda_{ij}(s) = \sum_{k=1}^{m} \sigma_{ik} D_{k,s}^{\phi}(X_j(s)); \quad 1 \leq i \leq n, \quad 1 \leq j \leq m,
\]
and \((\cdot)^T\) denotes matrix transposed and \(\text{Tr}[:\cdot:]\) denotes matrix trace.

The following useful result is a multidimensional version of Theorem 4.2 in [DHP]:

**Theorem 2.4** Let
\[
 X(t) = \sum_{j=1}^{m} \int_0^t \sigma_j(r, \omega) dB_j^{(H)}(r); \quad \sigma = (\sigma_1, \ldots, \sigma_m) \in \mathcal{L}^{1,2}_\phi(m) .
\]

Then
\[
 D_{k,s}^{\phi} X(t) = \sum_{j=1}^{m} \int_0^t D_{k,s}^{\phi} \sigma_j(r) dB_j^{(H)}(r) + \int_0^t \sigma_k(r) \phi_{H_k}(s, r)dr, \quad 1 \leq k \leq m .
\]

In particular, if \( \sigma_j(r) \) is deterministic for all \( j \in \{1, 2, \ldots, m\} \) then
\[
 D_{k,s}^{\phi} X(t) = \int_0^t \sigma_k(r) \phi_{H_k}(s, r)dr .
\]
Now we have the following integration by parts formula.

**Corollary 2.5** Let $X(t)$ and $Y(t)$ be two processes of the form

$$dX(t) = \mu(t, \omega)dt + \sigma(t, \omega)dB^{(H)}(t), \quad X(0) = x \in \mathbb{R}^n$$

and

$$dY(t) = \nu(t, \omega)dt + \theta(t, \omega)dB^{(H)}(t), \quad Y(0) = y \in \mathbb{R}^n,$$

where $\mu : \mathbb{R} \times \Omega \to \mathbb{R}^n$, $\nu : \mathbb{R} \times \Omega \to \mathbb{R}^n$, $\sigma : \mathbb{R} \times \Omega \to \mathbb{R}^{n \times m}$ and $\theta : \mathbb{R} \times \Omega \to \mathbb{R}^{n \times m}$ are given processes with rows $\sigma_i, \theta_i \in L^{1,2}_\phi(m)$ for $1 \leq i \leq n$ and $B^{(H)}(\cdot)$ is an $m$-dimensional fractional Brownian motion.

a) Then, for $T > 0$,

$$E[X(T) \cdot Y(T)] = x \cdot y + E\left[ \int_0^T X(s)dY(s) \right] + E\left[ \int_0^T Y(s)dX(s) \right]$$

$$+ E\left[ \int_0^T \int_0^T \sum_{i=1}^n \sum_{k=1}^m \sigma_{ik}(s)\theta_{ik}(t)\phi_{H_k}(s,t)ds \right]$$

(2.23)

$$+ E\left[ \sum_{i=1}^n \sum_{j,k=1}^m \left( \int_\mathbb{R} D^\phi_{j,t}\sigma_{ik}(t)dt \right) \left( \int_\mathbb{R} D^\phi_{k,t}\theta_{ij}(t)dt \right) \right]$$

provided that the first two integrals exist.

b) In particular, if $\sigma(\cdot)$ or $\theta(\cdot)$ is deterministic then

$$E\left[ X(T) \cdot Y(T) \right] = x \cdot y + E\left[ \int_0^T X(s)dY(s) \right] + E\left[ \int_0^T Y(s)dX(s) \right]$$

(2.24)

$$+ E\left[ \int_0^T \int_0^T \sum_{i=1}^n \sum_{k=1}^m \sigma_{ik}(s)\theta_{ik}(t)\phi_{H_k}(s,t)dsdt \right].$$

**Proof** This follows from Theorem 2.3 applied to the function $f(t, x, y) = xy$, combined with (2.13). \qed

3 **Stochastic differential equations**

For given functions $b : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ and $\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ consider the stochastic differential equation

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB^{(H)}(t), \quad t \in [0, T],$$

where the initial value $X(0) \in L^2(\mu_\phi)$ or the terminal value $X(T) \in L^2(\mu_\phi)$ is given. The Itô isometry for the stochastic integral becomes

$$E\left( \int_0^T \sigma(t, X(t))dB^{(H)}(t) \right)^2 = E\left( \int_0^T \int_0^T \sigma(t, X(t))\sigma(s, X(s))\phi(s, t)dsdt \right)$$

(3.2)

$$+ E\left\{ \left( \int_0^T \sigma'_x(s, X(s))D^\phi_s X(s)ds \right)^2 \right\}. $$
Because of the appearance of the term $D^2_s X(s)$ on the right-hand-side of the above identity, we may not directly apply the Picard iteration to solve (3.1).

In this section, we will solve the following quasi-linear stochastic differential equations using the theory developed in [H01], [H02]:

\begin{equation}
X(t) = b(t, X(t))dt + \sigma_t X(t) dB(t),
\end{equation}

where $\sigma_t$ and $a_t$ are given deterministic functions, $b(t, x) = b(t, x, \omega)$ is (almost surely) continuous with respect to $t$ and $x$ and globally Lipschitz continuous on $x$, the initial condition $X(0)$ or the terminal condition $X(T)$ is given. For simplicity we will discuss the case when $a_t = 0$ for all $t \in [0, T]$. Namely, we shall consider

\begin{equation}
X(t) = b(t, X(t))dt + \sigma_t X(t) dB(t).
\end{equation}

We need the following result, which is a fractional version of Gjessing’s lemma (see e.g. Theorem 2.10.7 in [HOUZ]).

**Lemma 3.1** Let $G \in L^2(\mu_H)$ and

\[ F = \exp\left( \int_R f(t) dB(t) \right) = \exp\left( \int_R f(t) dB(t) - \frac{1}{2} \|f\|_\phi^2 \right), \]

where $f$ is deterministic and such that

\[ \|f\|_\phi^2 := \int_R f(s)f(t)\phi(s,t)dsdt < \infty. \]

Then

\begin{equation}
F \circ G = F \tau_f G,
\end{equation}

where $\circ$ is the Wick product defined in [H02], $\tau_f$ is given by

\begin{equation}
\int_R f(s)g(t)\phi(s,t)dsdt = \int_R \hat{f}(s)g(s)ds \quad \forall g \in C_0^\infty(R)
\end{equation}

and

\[ \tau_f G(\omega) = G(\omega - \int_0^\omega \hat{f}(s)ds). \]

**Proof** By [DHP, Theorem 3.1] it suffices to show the result in the case when

\[ G(\omega) = \exp\left( \int_R g(t) dB(t) \right) = \exp\langle \omega, g \rangle, \]

where $g$ is deterministic and $\|g\|_\phi < \infty$. In this case we have

\[ F \circ G = \exp\left( \int_R [f(t) + g(t)] dB(t) \right) = \exp\left( \int_R [f(t) + g(t)] dB(t) - \frac{1}{2} \|f\|_\phi^2 - \frac{1}{2} \|g\|_\phi^2 - \langle f, g \rangle \right), \]
where
\[(f, g)_{\phi} = \int_{\mathbb{R}^2} f(s)g(t)\phi(s, t)dsdt.\]

But
\[
\tau_f G = \exp^2 \left( \int_{\mathbb{R}} g(t)dB^{(H)}(t) - \int_{\mathbb{R}} f(t)g(t)dt \right)
\]
\[= \exp^2 \left( \int_{\mathbb{R}} g(t)dB^{(H)}(t) - (f, g)_{\phi} \right).\]

Hence
\[
F\tau_f G = \exp \left( \int_{\mathbb{R}} f(t)dB^{(H)}(t) - \frac{1}{2}\|f\|_{\phi}^2 + \int_{\mathbb{R}} g(t)dB^{(H)}(t) - \frac{1}{2}\|g\|_{\phi}^2 - (f, g)_{\phi} \right) = F \circ G.
\]

We now return to Equation (3.3). First let us solve the equation when \(b = 0\) and with initial value \(X(0)\) given. Namely, let us consider
\[
(3.7) \quad dX(t) = -\sigma_t X(t)dB^{(H)}(t), \quad X(0) \text{ given}.
\]

With the notion of Wick product, this equation can be written (see [HO2, Def 3.11])
\[
(3.8) \quad \dot{X}(t) = -\sigma_t X(t) \circ W^{(H)}(t),
\]
where \(W^{(H)} = \dot{B}^{(H)}\) is the fractional white noise. Using the Wick calculus, we obtain
\[
X(t) = X(0) \circ J_{\sigma}(t)
\]
\[
:= X(0) \circ \exp^2 \left( -\int_0^t \sigma_s W^{(H)}(s)ds \right)
\]
\[
= X(0) \circ \exp \left( -\int_0^t \sigma_s dB^{(H)}(s) - \frac{1}{2}\|\sigma\|_{\phi, t}^2 \right), \quad (3.9)
\]
where
\[
(3.10) \quad \|\sigma\|_{\phi, t}^2 := \int_0^t \int_0^t \sigma_u \sigma_v \phi(u, v)du dv.
\]

To solve Equation (3.4) we let
\[
(3.11) \quad Y_t := X(t) \circ J_{\sigma}(t).
\]

This means
\[
(3.12) \quad X(t) = Y_t \circ \dot{J}_{\sigma}(t),
\]
where
\[
(3.13) \quad \dot{J}_{\sigma}(t) = J_{-\sigma}(t) = \exp \left( \int_0^t \sigma_s dB^{(H)}(s) - \frac{1}{2}\|\sigma\|_{\phi, t}^2 \right).
\]
Thus we have
\[
\frac{dY_t}{dt} = \frac{dX(t)}{dt} \circ J_\sigma(t) + X(t) \circ \frac{dJ_\sigma(t)}{dt}
\]
\[
= \frac{dX(t)}{dt} \circ J_\sigma(t) - \sigma_t J_\sigma(t) \circ X(t) \circ W^{(H)}(t)
\]
\[
= J_\sigma(t) \circ b(t, X(t), \omega)
\]
\[
= J_\sigma(t) b(t, \tau_\phi X(t), \omega + \int_0^\infty \hat{\sigma}(s) ds)
\]
where
\[
\int_{\mathbb{R}^2} \sigma_s g(t) \phi(s, t) ds dt = \int_{\mathbb{R}} \hat{\sigma}_s g(s) ds \quad \forall g \in C_0^\infty(\mathbb{R})
\]

We are going to relate \(\tau_\phi X(t)\) to \(Y_t\).
\[
\tau_\phi X_t(t, \omega) = \tau_\phi [J_\phi(t) \sigma \circ Y_t(t, \omega)]
\]
\[
= \tau_\phi [J_\phi(t) \tau_\phi Y_t]
\]
\[
= \tau_\phi [J_\phi(t) J_\phi(t)] Y_t.
\]
Since \(\tau_\phi J_\phi(t) = [J_\phi(t)]^{-1}\), we obtain an equation equivalent to (3.4) for \(Y_t\):
\[
\frac{dY_t}{dt} = J_\phi(t) b(t, [J_\phi(t)]^{-1} Y_t, \omega + \int_0^\infty \hat{\sigma}(s) ds).
\]

This is a deterministic equation. The initial value \(X(0)\) is equivalent to initial value \(Y_0 = X(0) \circ J_\phi(0) = X(0)\). Thus we can solve the quasilinear equation with given initial value.

The terminal value \(X(T)\) can also be transformed into the terminal value on \(Y(T) = X(T) \circ J_\phi(T)\). Thus the equation with given terminal value can be solved in a similar way. Note, however, that in this case the solution need not be \(F^{(H)}\)-adapted (see the next section).

**Example 3.2** In the equation (3.4) let us consider the case \(b(t, x) = b_t x\) for some deterministic locally bounded function \(b_t\) of \(t\). This means that we are considering the linear stochastic differential equation:
\[
\frac{dX(t)}{dt} = b_t X(t) dt + \sigma_t X(t) dB^{(H)}(t).
\]

In this case it is easy to see that the equation (3.15) satisfied by \(Y\) is
\[
\dot{Y}_t = b(t) Y_t.
\]

When the initial value is \(Y(0) = x\) (constant), \(x \in \mathbb{R}\), then
\[
Y_t = x e^{\int_0^t b(s) ds}.
\]

Thus the solution of (3.16) with \(X(0) = x\) can be expressed as
\[
X(t) = Y(t) \circ J_\phi(t)
\]
\[
= x \exp \left\{ \int_0^t b(s) ds + \int_0^t \sigma_s dB^{(H)}(s) - \frac{1}{2} \|\sigma\|^2_{\phi, t} \right\}.
\]
If we assume the terminal value $X(T)$ given, then
\[ Y(t) = Y(T) e^{\int_t^T b(s) ds} = X(T) \circ J_\sigma(T) e^{\int_t^T b(s) ds}. \]

Hence
\[ X(t) = Y(t) \circ J_{-\sigma}(t) = X(T) \circ \exp \left\{ \int_t^T b(s) ds \right\} - \int_t^T \sigma_s dB^{(H)}(s) - \frac{1}{2} \int_t^T \int_t^T \sigma(u)\sigma(v)\phi(u,v) du dv \right\}. \]

(3.18)

4 Fractional backward stochastic differential equations

Let $b : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ be a given function and let $F : \Omega \rightarrow \mathbb{R}$ be a given $\mathcal{F}^{(H)}_T$-measurable random variable, where $T > 0$ is a constant. Consider the problem of finding $\mathcal{F}^{(H)}_T$-adapted processes $p(t), q(t)$ such that
\[ dp(t) = b(t, p(t), q(t)) dt + q(t) dB^{(H)}(t); \quad t \in [0, T], \]

(4.1)
\[ P(T) = F \quad \text{a.s.} \]

(4.2)

This is a fractional backward stochastic differential equation (FBSDE) in the two unknown processes $p(t)$ and $q(t)$. We will not discuss general theory for such equations here, but settle with a solution in a linear variant of (4.1)-(4.2), namely
\[ dp(t) = [\alpha(t) + b_p(t) + c_q(t)] dt + q(t) dB^{(H)}(t); \quad t \in [0, T], \]

(4.3)
\[ P(T) = F \quad \text{a.s.}, \]

(4.4)

where $b_t$ and $c_t$ are given continuous deterministic functions and $\alpha(t) = \alpha(t, \omega)$ is a given $\mathcal{F}^{(H)}_T$-adapted process s.t. $\int_0^T |\alpha(t, \omega)| dt < \infty$ a.s.

To solve (4.3)-(4.4) we proceed as follows: By the fractional Girsanov theorem (see e.g. [HÖ2, Theorem 3.18]) we can rewrite (4.3) as
\[ dp(t) = [\alpha(t) + b_p(t)] dt + q(t) d\hat{B}^{(H)}(t); \quad t \in [0, T], \]

(4.5)

where
\[ \hat{B}^{(H)}(t) = B^{(H)}(t) + \int_0^t c_s ds \]

(4.6)
is a fractional Brownian motion (with Hurst parameter $H$) under the new probability measure $\hat{\mu}$ on $\mathcal{F}^{(H)}_T$ defined by
\[ \frac{d\hat{\mu}(\omega)}{d\mu(\omega)} = \exp \{ -\langle \omega, \hat{c} \rangle \} = \exp \left\{ -\int_0^T \hat{c}(s) dB^{(H)}(s) - \frac{1}{2} \| \hat{c} \|^2_\phi \right\}. \]

(4.7)
where \( \dot{c} = \dot{c}_t \) is the continuous function with supp (\( \dot{c} \)) \( \subset [0, T] \) satisfying

\[
\int_0^T \dot{c}_s(s, t) ds = c_t ; \quad 0 \leq t \leq T ,
\]

and

\[
\| \dot{c} \|^2_\alpha = \int_0^T \int_0^T \dot{c}(s) \dot{c}(t) \phi(s, t) ds dt .
\]

If we multiply (4.5) with the integrating factor

\[
\beta_t := \exp(- \int_0^t \Delta_s ds) ,
\]

we get

\[
d(\beta_s p(s)) = \beta_s \alpha(s) ds + \beta_s q(s) d\dot{B}^{(H)}(s) ,
\]

or, by integrating (4.9) from \( s = t \) to \( s = T \),

\[
\beta_T F = \beta_t p(t) + \int_t^T \beta_s \alpha(s) ds + \int_t^T \beta_s q(s) d\dot{B}^{(H)}(s) .
\]

Assume from now on that

\[
\| \alpha \|^2_{\mathcal{L}^2_{\phi}[0,T]} := \mathbb{E}_\mu \left[ \int_0^{[0,T] \times [0,T]} \alpha(s) \alpha(t) \phi(s, t) ds dt + \left( \int_0^T \dot{D}_s \alpha(s) ds \right)^2 \right] < \infty .
\]

By the fractional Itô isometry (see [DHP, Theorem 3.7] or [HOS2, (1.10)]) applied to \( \dot{B} \), \( \dot{\mu} \)
we then have

\[
\mathbb{E}_\dot{\mu} \left[ \left( \int_0^T \alpha(s) d\dot{B}^{(H)}(s) \right)^2 \right] = \| \alpha \|^2_{\mathcal{L}^2_{\phi}[0,T]} .
\]

From now on let us also assume that

\[
\mathbb{E}_\dot{\mu} \left[ F^2 \right] < \infty .
\]

We now apply the quasi-conditional expectation operator (see [HÖ2, Definition 4.9a]])

\[
\mathbb{E}_\dot{\mu} \left[ \cdot | \mathcal{F}^{(H)}_t \right]
\]

to both sides of (4.10) and get

\[
\beta_T \mathbb{E}_\dot{\mu} \left[ F | \mathcal{F}^{(H)}_t \right] = \beta_t p(t) + \int_t^T \beta_s \mathbb{E}_\dot{\mu} \left[ \alpha(s) | \mathcal{F}^{(H)}_t \right] ds .
\]

Here we have used that \( p(t) \) is \( \mathcal{F}^{(H)}_t \)-measurable, that the filtration \( \mathcal{F}^{(H)}_t \) generated by
\( \dot{B}^{(H)}(s) ; s \leq t \) is the same as \( \mathcal{F}^{(H)}_t \), and that

\[
\mathbb{E}_\dot{\mu} \left[ \int_t^T f(s, \omega) d\dot{B}^{(H)}(s) | \mathcal{F}^{(H)}_t \right] = 0 , \quad \text{for all} \quad t \leq T
\]
for all $f \in \mathcal{L}^{1,2}_\phi[0,T]$. See [HØ2, Def 4.9] and [HOS2, Lemma 1.1].

From (4.14) we get the solution

$$p(t) = \exp\left(-\int_t^T b_s ds\right) \hat{E}_{\hat{\mu}} \left[F|\mathcal{F}^{(H)}_t\right] + \int_t^T \exp\left(-\int_s^t b_r dr\right) \hat{E}_{\hat{\mu}} \left[\alpha(s)|\mathcal{F}^{(H)}_t\right] ds; \quad t \leq T.$$  

(4.16)

In particular, choosing $t = 0$ we get

$$p(0) = \exp\left(-\int_0^T b_s ds\right) \hat{E}_{\hat{\mu}} \left[F\right] + \int_0^T \exp\left(-\int_0^s b_r dr\right) \hat{E}_{\hat{\mu}} \left[\alpha(s)\right] ds.$$  

(4.17)

Note that $p(0)$ is $\mathcal{F}^{(H)}_0$-measurable and hence a constant. Choosing $t = 0$ in (4.10) we get

$$G = \int_0^T \beta_s q(s) d\hat{B}^{(H)}(s),$$  

where

$$G = G(\omega) = \beta_T F(\omega) - \int_0^T \beta_s \alpha(s,\omega) ds - p(0),$$  

(4.19)

with $p(0)$ given by (4.17).

By the fractional Clark-Ocone theorem [HØ1, Theorem 4.15 b)] applied to $(\hat{B}^{(H)}, \hat{\mu})$ we have

$$G = \hat{E}_{\hat{\mu}}[G] + \int_0^T \hat{E}_{\hat{\mu}} \left[\hat{D}_s G|\mathcal{F}^{(H)}_s\right] d\hat{B}^{(H)}(s),$$  

(4.20)

where $\hat{D}$ denotes the Malliavin derivative at $s$ with respect to $\hat{B}^{(H)}(\cdot)$. Comparing (4.18) and (4.20) we see that we can choose

$$q(t) = \exp\left(\int_0^t b_r dr\right) \hat{E}_{\hat{\mu}} \left[\hat{D}_t G|\mathcal{F}^{(H)}_t\right].$$  

(4.21)

We have proved the first part of the following result:

**Theorem 4.1** Assume that (4.11) and (4.13) hold. Then a solution $(p(t), q(t))$ of (4.3)-(4.4) is given by (4.16) and (4.21). The solution is unique among all $\mathcal{F}^{(H)}$-adapted processes $p(\cdot), q(\cdot) \in \mathcal{L}^{1,2}_\phi[0,T]$.

**Proof** It remains to prove uniqueness. The uniqueness of $p(\cdot)$ follows from the way we deduced formula (4.16) from (4.3)-(4.4). The uniqueness of $q$ is deduced from (4.18) and (4.20) by the following argument: Substituting (4.20) from (4.18) and using that $\hat{E}_{\hat{\mu}}(G) = 0$ we get

$$0 = \int_0^T \left(\beta_s q(s) - \hat{E}_{\hat{\mu}} \left[\hat{D}_s G|\mathcal{F}^{(H)}_s\right]\right) d\hat{B}^{(H)}(s).$$

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Hence by the fractional Itô isometry (4.12)
\[
0 = \mathbb{E}_\mu \left[ \left\{ \int_0^T \left( \beta_s q(s) - \hat{\mathbb{E}}_\mu \left[ \hat{D}_s G[H] \right] \right) dB^{(H)}(s) \right\}^2 \right]
\]
\[
= \| \beta_s q(s) - \hat{\mathbb{E}}_\mu \left[ \hat{D}_s G[H] \right] \|^2 \mathbb{L}^2_{\mu^*}[0,T],
\]
from which it follows that
\[
\beta_s q(s) - \hat{\mathbb{E}}_\mu \left[ \hat{D}_s G[H] \right] = 0 \quad \text{for a.a.} (s, \omega) \in [0,T] \times \Omega.
\]

\[\square\]

## 5 A stochastic maximum principle

We now apply the theory in the previous section to prove a maximum principle for systems driven by fractional Brownian motion. See e.g. [H], [P] and [YZ] and the references therein for more information about the maximum principle in the classical Brownian motion case.

Suppose \( X(t) = X^{(u)}(t) \) is a controlled system of the form

\[
dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB^{(H)}(t) \; ; \quad X(0) = x \in \mathbb{R}^n,
\]

where \( b : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n \) and \( \sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n \times m} \) are given \( C^1 \) functions. The control process \( u(\cdot) : [0,T] \times \Omega \to U \subset \mathbb{R}^k \) is assumed to be \( \mathcal{F}^{(H)} \)-adapted. \( U \) is a given closed convex set in \( \mathbb{R}^k \).

Let \( f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}, \ g : \mathbb{R}^n \to \mathbb{R} \) and \( G : \mathbb{R}^n \to \mathbb{R}^N \) be given \( C^1 \) functions and consider a performance functional \( J(u) \) of the form

\[
J(u) = \mathbb{E} \left[ \int_0^T f(t, X(t), u(t))dt + g(X(T)) \right]
\]

and a terminal condition given by

\[
\mathbb{E} [G(X(T))] = 0.
\]

Let \( \mathcal{A} \) denote the set of all \( \mathcal{F}^{(H)}_t \)-adapted processes \( u : [0,T] \times \Omega \to U \) such that \( X^{(u)}(t) \) exists and does not explode in \([0,T]\) and

\[
\mathbb{E} \left[ \int_0^T \| f(t, X(t), u(t)) \|dt + g^-(X(T)) + G^-(X(T)) \right] < \infty
\]

where \( y^- = \max(0, y) \) for \( y \in \mathbb{R} \), and such that (5.3) holds. If \( u \in \mathcal{A} \) and \( X^{(u)}(t) \) is the corresponding state process we call \((u, X^{(u)})\) an admissible pair. Consider the problem to find \( J^* \) and \( u^* \in \mathcal{A} \) such that

\[
J^* = \sup \{ J(u) ; u \in \mathcal{A} \} = J(u^*)
\]

If such \( u^* \in \mathcal{A} \) exists, then \( u^* \) is called an optimal control and \((u^*, X^*)\), where \( X^* = X^{u^*} \), is called an optimal pair.
Let $\mathcal{R}^{n \times m}$ be the set of continuous function from $[0, T]$ into $\mathbb{R}^{n \times m}$. Define the Hamiltonian $H: [0, T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathcal{R}^{n \times m} \to \mathbb{R}$ by

$$
H(t, x, u, p, q(\cdot)) = f(t, x, u) + b(t, x, u)^T p + \sum_{i=1}^n \sum_{k=1}^m \sigma_{ik}(t, x, u) \int_0^T q_k(s) \phi_{H_k}(s, t) ds.
$$

Consider the following fractional stochastic backward differential equation in the pair of unknown $\mathcal{F}_t^{H(t)}$-adapted processes $p(t) \in \mathbb{R}^n$, $q(t) \in \mathcal{R}^{n \times m}$, called the adjoint processes:

$$
\begin{align*}
(dp(t) &= -H_x(t, X(t), u(t), p(t), q(\cdot)) dt + q(t) dB^{(H(t))}(t) ; \quad t \in [0, T] \\
p(T) &= g_x(X(T)) + \lambda^T G_x(X(T)).
\end{align*}
$$

where $H_x = \nabla_x H = \left( \frac{\partial H}{\partial x_1}, \cdots, \frac{\partial H}{\partial x_n} \right)^T$ is the gradient of $H$ with respect to $x$ and similarly with $g_x$ and $G_x$. $X(t) = X(u)(t)$ is the process obtained by using the control $u \in \mathcal{A}$ and $\lambda \in \mathbb{R}^n_+$ is a constant. The equation (5.6) is called the adjoint equation and $p(t)$ is sometimes interpreted as the shadow price (of a resource).

**Theorem 5.1 (The fractional stochastic maximum principle)** Suppose $\hat{u} \in \mathcal{A}$ and put $\hat{X} = X^{(\hat{u})}$. Suppose there exists a solution $\hat{p}(t), \hat{q}(t)$ of the corresponding adjoint equation (5.7) for some $\lambda \in \mathbb{R}^n_+$ and such that the following, (5.8)--(5.11), hold:

$$
\begin{align*}
X^{(u)}(t) \hat{q}(t) &\in L^1_{\phi} & \text{and} & \hat{p}(t) \sigma(t, X^{(u)}(t), u(t)) \in L^1_{\phi} \quad \text{for all} \ u \in \mathcal{A} \\
H(t, \cdot, \cdot, \hat{p}(t), \hat{q}(t)) &\quad \text{and} \quad G(\cdot) \quad \text{are concave, for all} \ t \in [0, T], \\
H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{q}(\cdot)) &= \max_{v \in \mathcal{U}} H(t, \hat{X}(t), v, \hat{p}(t), \hat{q}(\cdot)), \\
\Delta_4 &:= E \left[ \sum_{i=1}^n \sum_{j,k=1}^m \left( \int_0^T D_{jk}^{\hat{p}}(\sigma_{ik}(t, X(t), u(t)) \right)
\right. \\
&- \sigma_{ik}(t, \hat{X}(t), \hat{u}(t)) \right] \left( \int_0^T D_{kj}^{\hat{p}} \hat{q}_{ij}(t) dt \right) \leq 0 \quad \text{for all} \ u \in \mathcal{A}.
\end{align*}
$$

Then if $\lambda \in \mathbb{R}^n_+$ is such that $(\hat{u}, \hat{X})$ is admissible (in particular, (5.3) holds), the pair $(\hat{u}, \hat{X})$ is an optimal pair for problem (5.5).

**Proof** We first give a proof in the case when $G(x) = 0$, i.e. when there is no terminal condition.

With $(\hat{u}, \hat{X})$ as above consider

$$
\begin{align*}
\Delta &:= E \left[ \int_0^T f(t, \hat{X}(t), \hat{u}(t)) dt - \int_0^T f(t, X(t), u(t)) dt \right] \\
&= E \left[ \int_0^T H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(\cdot)) dt - \int_0^T H(t, X(t), u(t), \hat{p}(t), \hat{q}(\cdot)) dt \right] \\
&- E \left[ \int_0^T \left\{ b(t, \hat{X}(t), \hat{u}(t)) \right\}^T \hat{p}(t) dt - \int_0^T b(t, X(t), u(t))^T \hat{p}(t) dt \right] \\
&- E \left[ \int_0^T \int_0^T \sum_{i=1}^n \sum_{k=1}^m \left\{ \sigma_{ik}(s, \hat{X}(s), \hat{u}(s)) - \sigma_{ik}(s, X(s), u(s)) \right\} \hat{q}_{ik}(t) \phi_{H_k}(s, t) ds dt \right] \\
&= \Delta_1 + \Delta_2 + \Delta_4.
\end{align*}
$$

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Since \((x, u) \mapsto H(x, u) = H(t, x, u, p, q(\cdot))\) is concave we have
\[
H(x, u) - H(\hat{x}, \hat{u}) \leq H_x(\hat{x}, \hat{u}) \cdot (x - \hat{x}) + H_u(\hat{x}, \hat{u}) \cdot (u - \hat{u})
\]
for all \((x, u), (\hat{x}, \hat{u})\). Since \(v \mapsto H(\hat{X}(t), v)\) is maximal at \(v = \hat{u}(t)\) we have
\[
H_u(\hat{x}, \hat{u}) \cdot (u(t) - \hat{u}(t)) \leq 0 \quad \forall t.
\]
Therefore
\[
\Delta_1 \geq \mathbb{E} \left[ \int_0^T -H_u(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(\cdot)) \cdot (X(t) - \hat{X}(t)) dt \right]
\]
\[
= \mathbb{E} \left[ \int_0^T (X(t) - \hat{X}(t))^T \hat{p}(t) - \int_0^T (X(t) - \hat{X}(t))^T \hat{q}(t) dB^{(H)}(t) \right]
\]
Since \(\mathbb{E} \left[ \int_0^T (X(t) - \hat{X}(t))^T \hat{q}(t) dB^{(H)}(t) \right] = 0\) by (2.7), this gives
\[
(5.13) \quad \Delta_1 \geq \mathbb{E} \left[ \int_0^T (X(t) - \hat{X}(t))^T \hat{p}(t) \right].
\]
By (5.1) we have
\[
\Delta_2 = -\mathbb{E} \left[ \int_0^T \left\{ b(t, \hat{X}(t), \hat{u}(t)) - b(t, X(t), u(t)) \right\} \cdot \hat{p}(t) dt \right]
\]
\[
= -\mathbb{E} \left[ \int_0^T \hat{p}(t) \left( d\hat{X}(t) - dX(t) \right) \right]
\]
\[
- \mathbb{E} \left[ \int_0^T \hat{p}(t)^T \left\{ \sigma(t, \hat{X}(t), \hat{u}(t)) - \sigma(t, X(t), u(t)) \right\} dB^{(H)}(t) \right]
\]
\[
(5.14) \quad = \mathbb{E} \left[ \int_0^T \hat{p}(t) \left( dX(t) - d\hat{X}(t) \right) \right].
\]
Finally, since \(g\) is concave we have
\[
(5.15) \quad g(X(T)) - g(\hat{X}(T)) \leq g_x(\hat{X}(T)) \cdot (X(T) - \hat{X}(T))
\]
Combining (5.12)–(5.15) with Corollary 2.5 we get, using (5.2), (5.7) and (5.11),
\[
J(\hat{u}) - J(u) = \Delta + \mathbb{E} \left[ g(\hat{X}(T)) - g(X(T)) \right]
\]
\[
\geq \Delta + \mathbb{E} \left[ g_x(\hat{X}(T)) \cdot (\hat{X}(T) - X(T)) \right]
\]
\[
\geq \Delta - \mathbb{E} \left[ \hat{p}(T) \cdot (X(T) - \hat{X}(T)) \right]
\]
\[
= \Delta - \left\{ \mathbb{E} \left[ \int_0^T (X(t) - \hat{X}(t)) \cdot \hat{p}(t) \right] + \mathbb{E} \left[ \int_0^T \hat{p}(t) \cdot \left( dX(t) - d\hat{X}(t) \right) \right] \right. 
\]
\[
+ \mathbb{E} \left[ \int_0^T \int_0^T \sum_{i=1}^n \sum_{k=1}^m \{ \sigma_i(s, X(s), u(s)) - \sigma_i(s, \hat{X}(s), \hat{u}(s)) \} \hat{q}_{ik}(t) \phi_{H_k}(s, t) ds dt \right] 
\]
\[
+ \mathbb{E} \left[ \sum_{i=1}^n \sum_{j,k=1}^m \left( \int_0^T D_{ij}^{(\cdot)} \{ \sigma_i(t, X(t), u(t)) - \sigma_i(t, \hat{X}(t), \hat{u}(t)) \} dt \right) \left( \int_0^T D_{jk}^{(\cdot)} \hat{q}_{ij}(t) \right) \right] \}
\]
\[
\geq \Delta - (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) \geq 0.
\]
This shows that \( J(\hat{u}) \) is maximal among all admissible pairs \((u(\cdot), X(\cdot))\).

This completes the proof in the case with no terminal conditions \((G = 0)\). Finally consider the general case with \( G \neq 0 \). Suppose that for some \( \lambda_0 \in \mathbb{R}_+^n \) there exists \( \hat{u}_{\lambda_0} \) satisfying (5.8)–(5.11). Then by the above argument we know that if we put
\[
J_{\lambda_0}(u) = \mathbb{E}\left[ \int_0^T f(t, X(t), u(t))dt + g(X(T)) + \lambda_0^T G(X(T)) \right]
\]
then \( J_{\lambda_0}(\hat{u}_0) \geq J_{\lambda_0}(u) \) for all controls \( u \) (without terminal condition). If \( \lambda_0 \) is such that \( \hat{u}_{\lambda_0} \) satisfies the terminal condition (i.e. \( u_{\lambda_0} \in \mathcal{A} \)) and \( u \) is another control in \( \mathcal{A} \) then
\[
J(\hat{u}_{\lambda_0}) = J_{\lambda_0}(\hat{u}_{\lambda_0}) \geq J_{\lambda_0}(u) = J(u)
\]
and hence \( \hat{u}_{\lambda_0} \in \mathcal{A} \) maximizes \( J(u) \) over all \( u \in \mathcal{A} \). \( \square \)

**Corollary 5.2** Let \( \hat{u} \in \mathcal{A} \), \( \hat{X} = X^{(\hat{u})} \) and \((\hat{p}(t), \hat{q}(t))\) be as in Theorem 5.1. Assume that (5.8), (5.9) and (5.10) hold, and that condition (5.11) is replaced by the condition
\[
(5.16) \quad \hat{q}(\cdot) \text{ or } \sigma(\cdot, \hat{X}(\cdot), \hat{u}(\cdot)) \text{ is deterministic}.
\]
Then if \( \lambda \in \mathbb{R}_+^n \) is such that \((\hat{u}, \hat{X})\) is admissible, the pair \((\hat{u}, \hat{X})\) is an optimal pair for problem (5.5).

## 6 A minimal variance hedging problem

To illustrate our main result, we use it to solve the following problem from mathematical finance:

Consider a financial market driven by two independent fractional Brownian motions \( B_1(t) = B_1^{(H_1)}(t) \) and \( B_2(t) = B_1^{(H_2)}(t) \), with \( \frac{1}{2} < H_i < 1, i = 1, 2 \), as follows:

\[
\begin{align*}
(6.1) & \quad \text{(Bond price)} \quad dS_0(t) = 0; \quad S_0(0) = 1 \\
(6.2) & \quad \text{(Price of stock 1)} \quad dS_1(t) = dB_1(t); \quad S_1(0) = s_1 \\
(6.3) & \quad \text{(Price of stock 2)} \quad dS_2(t) = dB_1(t) + dB_2(t); \quad S_2(0) = s_2.
\end{align*}
\]

If \( \theta(t) = (\theta_0(t), \theta_1(t), \theta_2(t)) \in \mathbb{R}^3 \) is a portfolio (giving the number of units of the bond, stock 1 and stock 2, respectively, held at time \( t \)) then the corresponding value process is
\[
(6.4) \quad V^\theta(t) = \theta(t) \cdot S(t) = \sum_{i=0}^2 \theta_i(t)S_i(t).
\]

The portfolio is called self-financing if
\[
(6.5) \quad dV^\theta(t) = \theta(t) \cdot dS(t) = \theta_1(t)dB_1(t) + \theta_2(t)(dB_1(t) + dB_2(t)).
\]

This market is called complete if any bounded \( \mathcal{F}_T^{(H)} \)-measurable random variable \( F \) can be hedged (or replicated), in the sense that there exists a (self-financing) portfolio \( \theta(t) \) and an initial value \( z \in \mathbb{R} \) such that
\[
(6.6) \quad F(\omega) = z + \int_0^T \theta(t)dS(t) \quad \text{for a.a. } \omega.
\]
(See [HO2] and [W] for a general discussion about this.)

Let us now assume that we are not allowed to trade in stock 1, i.e. we must have $\theta_1(t) \equiv 0$. How close to, say, $F(\omega) = B_1(T, \omega)$ can we get if we must hedge under this constraint?

If we put $\theta_2(t) = u(t)$ and interpret “close” as having a small $L^2(\mu)$ distance to $F$, then the problem can be stated as follows:

Find $z \in \mathbb{R}$ and admissible $u(t, \omega)$ such that

$$J(z, u) := \mathbb{E} \left[ \left\{ B_1(T) - \left( z + \int_0^T u(t)(dB_1(t) + dB_2(t)) \right) \right\}^2 \right]$$

$$(6.7)$$

is minimal. We see immediately that it is optimal to choose $z = 0$, so it remains to minimize over $u(t) = u(t, \omega)$ the functional

$$J(u) := \mathbb{E} \left[ \left\{ \int_0^T (u(t) - 1)dB_1(t) + \int_0^T u(t)dB_2(t) \right\}^2 \right].$$

If we apply the fractional Itô isometry (2.13) we get, after some simplifications,

$$J(u) = \mathbb{E} \left[ \int_0^T \int_0^T \{ (u(s) - 1)(u(t) - 1)\phi_1(s, t) + u(s)u(t)\phi_2(s, t) \} ds dt 
+ \left( \int_0^T \{ D_{1, u}^\phi - D_{2, u}^\phi \} dt \right)^2 \right].$$

$$(6.9)$$

However, it is difficult to see from this what the minimizing $u(t)$ is.

To approach this problem by using the fractional maximum principle, we define the state process $X(t)$ by

$$dX(t) = (u(t) - 1)dB_1(t) + u(t)dB_2(t).$$

$$(6.10)$$

Then the problem is equivalent to maximizing

$$J_1(u) := \mathbb{E} \left[ - \frac{1}{2} X^2(T) \right].$$

$$(6.11)$$

The Hamiltonian for this problem is

$$H(t, x, u, p, q(\cdot)) = (u - 1) \int_0^T q_1(s)\phi_1(s, t)ds + u \int_0^T q_2(s)\phi_2(s, t)ds$$

$$= (u - 1) \int_0^T q_1(s)\phi_1(s, t)ds + u \int_0^T q_2(s)\phi_2(s, t)ds$$

$$= u \left[ \int_0^T q_1(s)\phi_1(s, t)ds + \int_0^T q_2(s)\phi_2(s, t)ds \right] - \int_0^T q_1(s)\phi_1(s, t)ds .$$

$$(6.12)$$

The adjoint equation is

$$dp(t) = q_1(t)dB_1(t) + q_2(t)dB_2(t) ; \quad t < T$$

$$(6.13)$$

and

$$p(T) = -X(T).$$

$$(6.14)$$
Comparing with (6.10) we see that this equation has the solution
\[
q_1(t) = 1 - u(t), \quad q_2 = -u_2(t), \quad p(t) = -X(t); \quad t \leq T.
\]
Let \( \hat{u}(t) \) be an optimal control candidate. Then by (6.12)
\[
H(t, \hat{X}(t), v, \hat{p}(t), \hat{q}(\cdot)) = v \left[ \int_0^T \hat{q}_1(s) \phi_1(s, t) ds + \int_0^T \hat{q}_2(s) \phi_2(s, t) ds \right] - \int_0^T \hat{q}_1(s) \phi_1(s, t) ds
\]
(6.16)
\[
= \int_0^T (1 - \hat{u}(t)) \phi_1(s, t) ds - \int_0^T \hat{u}(s) \phi_2(s, t) ds - \int_0^T \hat{q}_1(s) \phi_1(s, t) ds.
\]
The maximum principle requires that the maximum of this expression is attained at \( v = \hat{u}(t) \). However, this is an affine function of \( v \), so it is natural to guess that the coefficient of \( v \) must be 0, i.e.
\[
\int_0^T (1 - \hat{u}(s)) \phi_1(s, t) ds - \int_0^T \hat{u}(s) \phi_2(s, t) ds = 0,
\]
which gives
\[
(6.17) \quad \int_0^T \hat{u}(s) (\phi_1(s, t) + \phi_2(s, t)) ds = \int_0^T \phi_1(s, t) ds.
\]
This is a symmetric Fredholm integral equation of the first kind and it is known that it has a unique solution \( \hat{u}(t) \in L^2[0, T] \). See e.g. [T, Section 3.15].

This choice of \( \hat{u}(t) \) satisfies all the requirements of Theorem 5.1 (in fact, even those of Corollary 5.2) and we can conclude that this \( \hat{u}(t) \) is optimal. Thus we have proved:

**Theorem 6.1 (Solution of the minimal variance hedging problem)**

The minimal value of
\[
J(z, u) = \mathbb{E} \left[ \left\{ B_1(T) - \left( z + \int_0^T u(t) (dB_1(t) + dB_2(t)) \right) \right\}^2 \right]
\]
is attained when \( z = 0 \) and \( u = \hat{u}(t) \) satisfies (6.17). The corresponding minimal value is
\[
\inf_{z, u} J(z, u) = \int_0^T \int_0^T \{(\hat{u}(s) - 1)(\hat{u}(t) - 1) \phi_1(s, t) + \hat{u}(s) \hat{u}(t) \phi_2(s, t)\} ds dt.
\]

**Remark** Note that if \( \phi_1 = \phi_2 \) then \( \hat{u}(t) \equiv \frac{1}{2} \), which is the same as the optimal value in the classical Brownian motion case \( H_1 = H_2 = \frac{1}{2} \).

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