A stochastic maximum principle for processes driven by fractional Brownian motion

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Abstract

We prove a stochastic maximum principle for controlled processes $X(t) = X^{(u)}(t)$ of the form

$$dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB^{(H)}(t)$$

where $B^{(H)}(t)$ is m-dimensional fractional Brownian motion with Hurst parameter $H = (H_1, \cdots, H_m) \in (\frac{1}{2}, 1)^m$. As an application we solve a problem about minimal variance hedging in an incomplete market driven by fractional Brownian motion.

1 Introduction

Let $H = (H_1, \cdots, H_m)$ with $\frac{1}{2} < H_j < 1$, $j = 1, 2, \ldots, m$, and let $B^{(H)}(t) = (B_1^{(H)}(t), \ldots, B_m^{(H)}(t))$, $t \in \mathbb{R}$ be m-dimensional fractional Brownian motion, *i.e.* $B^{(H)}(t) = B^{(H)}(t, \omega)$, $(t, \omega) \in \mathbb{R} \times \Omega$ is a Gaussian process in $\mathbb{R}^m$ such that

$$\mathbb{E}\left[B^{(H)}(t)\right] = B^{(H)}(0) = 0 \tag{1.1}$$

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and

$$\mathbb{E}\left[B_j^{(H)}(s)B_k^{(H)}(t)\right] = \tfrac{1}{2}\left\{|s|^{2H_j} + |t|^{2H_j} - |t-s|^{2H_j}\right\}\delta_{jk}\,;1 \le j,k \le n\,, \quad s\,,t \in \mathbb{R}\,, \tag{1.2}$$

where

$$\delta_{jk} = \begin{cases} 0 & \text{when } j \neq k \\ 1 & \text{when } j = k \end{cases}$$

Here $\mathbb{E} = \mathbb{E}_\mu$ denotes the expectation with respect to the probability law $\mu = \mu_H$ for $B^{(H)}(\cdot)$. This means that the components $B_1^{(H)}(\cdot)$, $\cdots$, $B_m^{(H)}(\cdot)$ of $B^{(H)}(\cdot)$ are m independent 1-dimensional fractional Brownian motions with Hurst parameters $H_1\,,H_2\,,\cdots,H_m$, respectively. We refer to [MvN], [NVV] and [S] for more information about fractional Brownian motion. Because of its interesting properties (e.g. long range dependence and self-similarity of the components) $B^{(H)}(t)$ has been suggested as a replacement of *standard Brownian motion* $B(t)$ (corresponding to $H_j = \frac{1}{2}$ for all $j = 1,\cdots,m$) in several stochastic models, including finance.

Unfortunately, $B^{(H)}(\cdot)$ is neither a semimartingale nor a Markov process, so the powerful tools from the theories of such processes are not applicable when studying $B^{(H)}(\cdot)$. Nevertheless, an efficient stochastic calculus of $B^{(H)}(\cdot)$ can be developed. This calculus uses an Itô type of integration with respect to $B^{(H)}(\cdot)$ and white noise theory. See [DHP] and [HØ2] for details. For applications to finance see [HØ2], [HØS1] [HØS2]. In [Hu1], [Hu2], [HØZ] and [ØZ] the theory is extended to multi-parameter fractional Brownian fields $B^{(H)}(x)\,;x \in \mathbb{R}^d$ and applied to stochastic partial differential equations driven by such fractional white noise.

The purpose of this paper is to establish a stochastic maximum principle for stochastic control of processes driven by $B^{(H)}(\cdot)$. We illustrate the result by applying it to a problem about minimal variance hedging in finance.

2 Preliminaries

For the convenience of the reader we recall here some of the basic results of fractional Brownian motion calculus. Let $B^{(H)}(t)$ be 1-dimensional in the following.

Define, for given $H \in (\frac{1}{2}\,,1)$,

$$\phi(s,t) = \phi_H(s,t) = H(2H-1)|s-t|^{2H-2}\,; \qquad s,t \in \mathbb{R}\,. \tag{2.1}$$

As in [HØ2] we will assume that Ω is the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions on $\mathbb{R}$, which is the dual of the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions on $\mathbb{R}$. If $\omega \in \mathcal{S}'(\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$ we let $\langle \omega, f\rangle = \omega(g)$ denote the action of ω applied to f. It can be extended to all $f : \mathbb{R} \to \mathbb{R}$ such that

$$\|f\|_\phi^2 := \int_{\mathbb{R}}\int_{\mathbb{R}} f(s)f(t)\phi(s,t)ds\,dt < \infty\,.$$

The space of all such (deterministic) functions f is denoted by $L_\phi^2(\mathbb{R})$.

If $F : \Omega \to \mathbb{R}$ is a given function we let

$$D_t^\phi F = \int_{\mathbb{R}} D_r F \cdot \phi(r,t)dr \tag{2.2}$$

denote the Malliavin ϕ-derivative of F at t (if it exists) (see [DHP, Definition 3.4]. Define $\mathcal{L}_\phi^{1,2}$ to be the set of (measurable) processes $g(t,\omega) : \mathbb{R}\times\Omega \to \mathbb{R}$ such that $D_s^\phi g(s)$ exists for a.a. $s \in \mathbb{R}$ and

$$\|g\|^2_{\mathcal{L}_\phi^{1,2}} := \mathbb{E}\Big[\int_{\mathbb{R}}\int_{\mathbb{R}} g(s)g(t)\phi(s,t)ds\,dt + \Big(\int_{\mathbb{R}} D_s^\phi g(s)ds\Big)^2\Big] < \infty \tag{2.3}$$

We let $\int_{\mathbb{R}} \sigma(t,\omega)dB^{(H)}(t)$ denote the *fractional Itô-integral* of the process $\sigma(t,\omega)$ with respect to $B^{(H)}(t)$, as defined in [DHP]. In particular, this means that if σ belongs to the family $\mathbb{S}$ of step functions of the form

$$\sigma(t,\omega) = \sum_{i=1}^N \sigma_i(\omega)\chi_{[t_i,t_{i+1})}(t)\,, \quad (t,\omega)\in\mathbb{R}\times\Omega\,,$$

where $0 \le t_1 < t_2 < \cdots < t_{N+1}$, then

$$\int_{\mathbb{R}} \sigma(t,\omega)dB^{(H)}(t) = \sum_{i=1}^N \sigma_i(\omega)\diamond\big(B^{(H)}(t_{i+1}) - B^{(H)}(t_i)\big)\,, \tag{2.4}$$

where $\diamond$ denotes the Wick product. For $\sigma(t) = \sigma(t,\omega) \in \mathbb{S}\cap\mathcal{L}_\phi^{1,2}$ we have the isometry

$$\mathbb{E}\Big[\int_{\mathbb{R}}\sigma(t,\omega)dB^{(H)}(t)\Big]^2 = \mathbb{E}\Big[\int_{\mathbb{R}^2}\sigma(s)\sigma(t)\phi(s,t)ds\,dt + \Big(\int_{\mathbb{R}} D_s^\phi\sigma(s)ds\Big)^2\Big] = \|\sigma\|^2_{\mathcal{L}_\phi^{1,2}}\,, \tag{2.5}$$

where $\mathbb{E} = \mathbb{E}_{\mu_H}$. Using this we can extend the integral $\int_{\mathbb{R}}\sigma(t,\omega)dB^{(H)}(t)$ to $\mathcal{L}_\phi^{1,2}$. Note that if σ, $\theta \in \mathcal{L}_\phi^{1,2}$, we have, by polarization,

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}}\sigma(t,\omega)dB^{(H)}(t)\int_{\mathbb{R}}\theta(t,\omega)dB^{(H)}(t)\right]\\
&\qquad = \mathbb{E}\left[\int_{\mathbb{R}^2}\sigma(s)\theta(t)\phi(s,t)dsdt + \int_{\mathbb{R}} D_s^\phi\sigma(s)ds\int_{\mathbb{R}} D_t^\phi\theta(t)dt\right]\,.
\end{aligned} \tag{2.6}$$

Also note that we need not assume that the integrand $\sigma \in \mathcal{L}_\phi^{1,2}$ is adapted to the filtration $\mathcal{F}_t^{(H)}$ generated by $B^{(H)}(s,\cdot)$; $s \le t$.

An important property of this fractional Itô-integral is that

$$\mathbb{E}\left[\int_{\mathbb{R}}\sigma(t,\omega)dB^{(H)}(t)\right] = 0 \quad \text{for all} \quad \sigma\in\mathcal{L}_\phi^{1,2}\,. \tag{2.7}$$

(see [DHP, Theorem 3.9]).

We give three versions of the fractional Itô formula, in increasing order of complexity.

Theorem 2.1 ([DHP], Theorem 4.1) *Let $f \in C^2(\mathbb{R})$ with bounded second order derivatives. Then for $t \ge 0$*

$$f(B^{(H)}(t)) = f(B^{(H)}(0)) + \int_0^t f'(B^{(H)}(s))dB^{(H)}(s) + H\int_0^t s^{2H-1}f''(B^{(H)}(s))ds\,. \tag{2.8}$$

Theorem 2.2 ([DHP], Theorem 4.3) *Let $X(t) = \int_0^t \sigma(s,\omega)dB^{(H)}(s)$, where $\sigma \in \mathcal{L}_\phi^{1,2}$ and assume $f \in C^2(\mathbb{R}_+ \times \mathbb{R})$ with bounded second order derivatives. Then for $t \geq 0$*

$$
\begin{aligned}
f(t,X(t)) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s,X(s))ds \\
&\quad + \int_0^t \frac{\partial f}{\partial x}(s,X(s))\sigma(s)dB^{(H)}(s) + \int_0^t \frac{\partial^2 f}{\partial x^2}(s,X(s))\sigma(s)D_s^\phi X(s)ds\,. \qquad (2.9)
\end{aligned}
$$

Finally we give an m-dimensional version:

Let $B^{(H)}(t) = \left(B_1^{(H)}(t)\,, \cdots, B_m^{(H)}(t)\right)$ be an m-dimensional fractional Brownian motion with Hurst parameter $H = (H_1, \cdots, H_m) \in (1/2,1)^m$, as in Section 1. Since we are here dealing with m independent fractional Brownian motions we may regard Ω as the product of m independent copies of $\bar{\Omega}$ and write $\omega = (\omega_1, \ldots, \omega_m)$ for $\omega \in \Omega$. Then in the following the notation $D_{k,s}^\phi Y$ means the Malliavin ϕ-derivative with respect to ω_k and could also be written

$$
D_{k,s}^\phi Y = \int_{\mathbb{R}} \phi_{H_k}(s,t)D_{k,t}Y dt = \int_{\mathbb{R}} \phi_{H_k}(s,t)\frac{\partial Y}{\partial \omega_k}(t,\omega)dt\,. \qquad (2.10)
$$

Similar to the 1-dimensional case discussed in Section 1, we can define the multi-dimensional fractional (Wick-Itô) integral

$$
\int_{\mathbb{R}} f(t,\omega)dB^{(H)}(t) = \sum_{j=1}^m \int_{\mathbb{R}} f_j(t,\omega)dB_j^{(H)}(t) \in L^2(\mu) \qquad (2.11)
$$

for all processes $f(t,\omega) = (f_1(t,\omega), \ldots, f_m(t,\omega)) \in \mathbb{R}^m$ such that, for all $j = 1, 2, \ldots, m$,

$$
\|f_j\|^2_{\mathcal{L}^{1,2}_{\phi_j}} := \mathbb{E}\Big[\int_{\mathbb{R}}\int_{\mathbb{R}} f_j(s)f_j(t)\phi_j(s,t)ds\,dt + \Big(\int_{\mathbb{R}} D_{j,t}^{\phi_j} f_j(t)dt\Big)^2\Big] < \infty \qquad (2.12)
$$

where $\phi_j = \phi_{H_j}$; $1 \leq j \leq m$.

Denote the set of all such m-dimensional processes f by $\mathcal{L}_\phi^{1,2}(m)$, where $\phi = (\phi_1, \ldots, \phi_m)$.

It can be proved (see [BØ]) that for $f, g \in \mathcal{L}_\phi^{1,2}(m)$ we have the following *fractional multi-dimensional Itô isometry*

$$
\begin{aligned}
\mathbb{E}\Big[\Big(\int_{\mathbb{R}} f dB^{(H)}\Big)\cdot\Big(\int_{\mathbb{R}} g dB^{(H)}\Big)\Big] &= \mathbb{E}\Big[\sum_{i=1}^m \int_{\mathbb{R}}\int_{\mathbb{R}} f_i(s)g_i(t)\phi_i(s,t)ds\,dt \\
&\quad + \sum_{i,j=1}^m \Big(\int_{\mathbb{R}} D_{j,t}^\phi f_i(t)dt\Big)\cdot\Big(\int_{\mathbb{R}} D_{i,t}^\phi g_j(t)dt\Big)\Big]. \qquad (2.13)
\end{aligned}
$$

We put

$$
\begin{aligned}
(f,g)_{\mathbb{L}_\phi^{1,2}(m)} &= \mathbb{E}\Big[\sum_{i=1}^m \int_{\mathbb{R}}\int_{\mathbb{R}} f_i(s)g_i(t)\phi_i(s,t)ds\,dt \\
&\quad + \sum_{i,j=1}^m \Big(\int_{\mathbb{R}} D_{j,t}^\phi f_i(t)dt\Big)\cdot\Big(\int_{\mathbb{R}} D_{i,t}^\phi g_j(t)dt\Big)\Big] \qquad (2.14)
\end{aligned}
$$

and define

$$\mathbb{L}_\phi^{1,2}(m) = \left\{ f \in \mathcal{L}_\phi^{1,2}(m); \left\|f\right\|^2_{\mathbb{L}_\phi^{1,2}(m)} := (f,f)_{\mathbb{L}_\phi^{1,2}(m)} < \infty \right\} .$$

Now suppose $\sigma_i \in \mathcal{L}_\phi^{1,2}(m)$ for $1 \le i \le n$. Then we can define $X(t) = (X_1(t), \cdots, X_n(t))$ where

$$X_i(t,\omega) = \sum_{j=1}^m \int_0^t \sigma_{ij}(s,\omega) dB_j^{(H)}(s)\,; 1 \le i \le n\,. \tag{2.15}$$

We have the following multi-dimensional fractional Itô formula:

Theorem 2.3 *Let $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^n)$ with bounded second order derivatives. Then, for $t \ge 0$,*

$$\begin{aligned} f(t,X(t)) &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s,X(s))ds + \int_0^t \sum_{i=1}^n \frac{\partial f}{\partial x_i}(s,X(s))dX_i(s) \\ &\quad + \int_0^t \left\{ \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(s,X(s)) \sum_{k=1}^m \sigma_{ik}(s) D_{k,s}^\phi (X_j(s)) \right\} ds \qquad (2.16) \\ &= f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s,X(s))ds + \sum_{j=1}^m \int_0^t \left[\sum_{i=1}^n \frac{\partial f}{\partial x_i}(s,X(s)) \sigma_{ij}(s,\omega) \right] dB_j^{(H)}(s) \\ &\quad + \int_0^t \; Tr\left[\Lambda^T(s) f_{xx}(s,X(s))\right] ds\,. \qquad (2.17) \end{aligned}$$

Here $\Lambda = [\Lambda_{ij}] \in \mathbb{R}^{n\times m}$ with

$$\Lambda_{ij}(s) = \sum_{k=1}^m \sigma_{ik} D_{k,s}^\phi \left(X_j(s)\right)\,; \quad 1 \le i \le n\,, \quad 1 \le j \le m\,, \tag{2.18}$$

$$f_{xx} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{1 \le i,j \le n}\,, \tag{2.19}$$

and $(\cdot)^T$ denotes matrix transposed and $\mathrm{Tr}[\cdot]$ *denotes matrix trace.*

The following useful result is a multidimensional version of Theorem 4.2 in [DHP]:

Theorem 2.4 *Let*

$$X(t) = \sum_{j=1}^m \int_0^t \sigma_j(r,\omega) dB_j^{(H)}(r)\,; \qquad \sigma = (\sigma_1, \ldots, \sigma_m) \in \mathcal{L}_\phi^{1,2}(m)\,. \tag{2.20}$$

Then

$$D_{k,s}^\phi X(t) = \sum_{j=1}^m \int_0^t D_{k,s}^\phi \sigma_j(r) dB_j^{(H)}(r) + \int_0^t \sigma_k(r) \phi_{H_k}(s,r) dr\,, \quad 1 \le k \le m\,. \tag{2.21}$$

In particular, if $\sigma_j(r)$ is deterministic for all $j \in \{1, 2, \cdots, m\}$ then

$$D_{k,s}^\phi X(t) = \int_0^t \sigma_k(r) \phi_{H_k}(s,r) dr\,. \tag{2.22}$$

Now we have the following integration by parts formula.

Corollary 2.5 *Let $X(t)$ and $Y(t)$ be two processes of the form*

$$dX(t) = \mu(t,\omega)dt + \sigma(t,\omega)dB^{(H)}(t)\,, \quad X(0) = x \in \mathbb{R}^n$$

and

$$dY(t) = \nu(t,\omega)dt + \theta(t,\omega)dB^{(H)}(t)\,, \quad Y(0) = y \in \mathbb{R}^n\,,$$

where $\mu : \mathbb{R} \times \Omega \to \mathbb{R}^n$, $\nu : \mathbb{R} \times \Omega \to \mathbb{R}^n$, $\sigma : \mathbb{R} \times \Omega \to \mathbb{R}^{n\times m}$ and $\theta : \mathbb{R} \times \Omega \to \mathbb{R}^{n\times m}$ are given processes with rows σ_i, $\theta_i \in \mathcal{L}_\phi^{1,2}(m)$ for $1 \le i \le n$ and $B^H(\cdot)$ is an m-dimensional fractional Brownian motion.

a) *Then, for $T > 0$,*

$$\begin{aligned}\mathbb{E}[X(T)\cdot Y(T)] &= x\cdot y + \mathbb{E}\Big[\int_0^T X(s)dY(s)\Big] + \mathbb{E}\Big[\int_0^T Y(s)dX(s)\Big] \\ &\quad + \mathbb{E}\Big[\int_0^T\int_0^T \sum_{i=1}^n\sum_{k=1}^m \sigma_{ik}(s)\theta_{ik}(t)\phi_{H_k}(s,t)ds\,dt\Big] \\ &\quad + \mathbb{E}\Big[\sum_{i=1}^n\sum_{j,k=1}^m \Big(\int_{\mathbb{R}} D^\phi_{j,t}\sigma_{ik}(t)dt\Big)\Big(\int_{\mathbb{R}} D^\phi_{k,t}\theta_{ij}(t)dt\Big)\Big] \end{aligned} \tag{2.23}$$

provided that the first two integrals exist.

b) *In particular, if $\sigma(\cdot)$ or $\theta(\cdot)$ is deterministic then*

$$\begin{aligned}\mathbb{E}\left[X(T)\cdot Y(T)\right] &= x\cdot y + \mathbb{E}\left[\int_0^T X(s)dY(s)\right] + \mathbb{E}\left[\int_0^T Y(s)dX(s)\right] \\ &\quad + \mathbb{E}\left[\int_0^T\int_0^T \sum_{i=1}^n\sum_{k=1}^m \sigma_{ik}(s)\theta_{ik}(t)\phi_{H_k}(s,t)dsdt\right]. \end{aligned} \tag{2.24}$$

Proof This follows from Theorem 2.3 applied to the function $f(t,x,y) = xy$, combined with (2.13). □

3 Stochastic differential equations

For given functions $b : \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ and $\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ consider the stochastic differential equation

$$dX(t) = b(t,X(t))dt + \sigma(t,X(t))dB^{(H)}(t)\,, \quad t \in [0,T]\,, \tag{3.1}$$

where the initial value $X(0) \in L^2(\mu_\phi)$ or the terminal value $X(T) \in L^2(\mu_\phi)$ is given. The Itô isometry for the stochastic integral becomes

$$\begin{aligned}\mathbb{E}\left(\int_0^T \sigma(t,X(t))dB^{(H)}(t)\right)^2 &= \mathbb{E}\left(\int_0^T\int_0^T \sigma(t,X(t))\sigma(s,X(s))\phi(s,t)dsdt\right) \\ &\quad + \mathbb{E}\left\{\left(\int_0^T \sigma'_x(s,X(s))D^\phi_s X(s)ds\right)^2\right\}. \end{aligned} \tag{3.2}$$

Because of the appearance of the term $D_s^\phi X(s)$ on the right-hand-side of the above identity, we may not directly apply the Picard iteration to solve (3.1).

In this section, we will solve the following quasi-linear stochastic differential equations using the theory developed in [HØ1], [HØ2]:

$$dX(t) = b(t, X(t))dt + (\sigma_t X(t) + a_t)\, dB^{(H)}(t)\,, \tag{3.3}$$

where σ_t and a_t are given deterministic functions, $b(t,x) = b(t,x,\omega)$ is (almost surely) continuous with respect to t and x and globally Lipschitz continuous on x, the initial condition $X(0)$ or the terminal condition $X(T)$ is given. For simplicity we will discuss the case when $a_t = 0$ for all $t \in [0,T]$. Namely, we shall consider

$$dX(t) = b(t, X(t))dt + \sigma_t X(t) dB^{(H)}(t)\,. \tag{3.4}$$

We need the following result, which is a fractional version of Gjessing's lemma (see e.g. Theorem 2.10.7 in [HØUZ]).

Lemma 3.1 *Let $G \in L^2(\mu_H)$ and*

$$F = \exp^\diamond\left(\int_{\mathbb{R}} f(t) dB^{(H)}(t)\right) = \exp\left(\int_{\mathbb{R}} f(t) dB^{(H)}(t) - \tfrac{1}{2}\|f\|_\phi^2\right)\,,$$

where f is deterministic and such that

$$\|f\|_\phi^2 := \int_{\mathbb{R}^2} f(s)f(t)\phi(s,t)dsdt < \infty\,.$$

Then

$$F \diamond G = F\tau_{\hat f} G\,, \tag{3.5}$$

where $\diamond$ is the Wick product defined in [HØ2], $\hat f$ is given by

$$\int_{\mathbb{R}^2} f(s)g(t)\phi(s,t)dsdt = \int_{\mathbb{R}} \hat f(s)g(s)ds \quad \forall g \in C_0^\infty(\mathbb{R}) \tag{3.6}$$

and

$$\tau_{\hat f} G(\omega) = G(\omega - \int_0^\cdot \hat f(s)ds)\,.$$

Proof By [DHP, Theorem 3.1] it suffices to show the result in the case when

$$G(\omega) = \exp^\diamond\left(\int_{\mathbb{R}} g(t) dB^{(H)}(t)\right) = \exp^\diamond\langle \omega, g\rangle\,,$$

where g is deterministic and $\|g\|_\phi < \infty$. In this case we have

$$\begin{aligned} F \diamond G &= \exp^\diamond\left(\int_{\mathbb{R}} [f(t) + g(t)]\, dB^{(H)}(t)\right) \\ &= \exp\left(\int_{\mathbb{R}} [f(t) + g(t)]\, dB^{(H)}(t) - \tfrac{1}{2}\|f\|_\phi^2 - \tfrac{1}{2}\|g\|_\phi^2 - (f,g)_\phi\right)\,, \end{aligned}$$

where

$$(f,g)_\phi = \int_{\mathbb{R}^2} f(s)g(t)\phi(s,t)dsdt\,.$$

But

$$\begin{aligned}\tau_{\hat f}G &= \exp^\diamond\left(\int_{\mathbb{R}} g(t)dB^{(H)}(t) - \int_{\mathbb{R}} \hat f(t)g(t)dt\right)\\ &= \exp^\diamond\left(\int_{\mathbb{R}} g(t)dB^{(H)}(t) - (f,g)_\phi\right)\,.\end{aligned}$$

Hence

$$F\tau_{\hat f}G = \exp\left(\int_{\mathbb{R}} f(t)dB^{(H)}(t) - \tfrac12\|f\|_\phi^2 + \int_{\mathbb{R}} g(t)dB^{(H)}(t) - \tfrac12\|g\|_\phi^2 - (f,g)_\phi\right) = F\diamond G\,.$$

□

We now return to Equation (3.3). First let us solve the equation when $b=0$ and with initial value $X(0)$ given. Namely, let us consider

$$dX(t) = -\sigma_t X(t)dB^{(H)}(t)\,,\quad X(0)\quad \text{given}\,. \tag{3.7}$$

With the notion of Wick product, this equation can be written (see [HØ2, Def 3.11])

$$\dot X(t) = -\sigma_t X(t)\diamond W^{(H)}(t)\,, \tag{3.8}$$

where $W^{(H)} = \dot B^{(H)}$ is the fractional white noise. Using the Wick calculus, we obtain

$$\begin{aligned}X(t) &= X(0)\diamond J_\sigma(t)\\ &:= X(0)\diamond\exp^\diamond\left(-\int_0^t \sigma_s W^{(H)}(s)ds\right)\\ &= X(0)\diamond\exp\left(-\int_0^t \sigma_s dB^{(H)}(s) - \tfrac12\|\sigma\|_{\phi,t}^2\right)\,,\end{aligned} \tag{3.9}$$

where

$$\|\sigma\|_{\phi,t}^2 := \int_0^t\int_0^t \sigma_u\sigma_v\phi(u,v)dudv\,. \tag{3.10}$$

To solve Equation (3.4) we let

$$Y_t := X(t)\diamond J_\sigma(t)\,. \tag{3.11}$$

This means

$$X(t) = Y_t\diamond \hat J_\sigma(t)\,, \tag{3.12}$$

where

$$\hat J_\sigma(t) = J_{-\sigma}(t) = \exp\left(\int_0^t \sigma_s dB^{(H)}(s) - \tfrac12\|\sigma\|_{\phi,t}^2\right)\,. \tag{3.13}$$

Thus we have

$$\begin{aligned}
\frac{dY_t}{dt} &= \frac{dX(t)}{dt} \diamond J_\sigma(t) + X(t) \diamond \frac{dJ_\sigma(t)}{dt} \\
&= \frac{dX(t)}{dt} \diamond J_\sigma(t) - \sigma_t J_\sigma(t) \diamond X(t) \diamond W^{(H)}(t) \\
&= J_\sigma(t) \diamond b(t, X(t), \omega) \\
&= J_\sigma(t) b(t, \tau_{-\hat{\sigma}} X(t), \omega + \int_0^{\cdot} \hat{\sigma}(s) ds) \,,
\end{aligned}$$

where

$$\int_{\mathbb{R}^2} \sigma_s g(t) \phi(s,t) ds dt = \int_{\mathbb{R}} \hat{\sigma}_s g(s) ds \quad \forall g \in C_0^\infty(\mathbb{R}) \,. \tag{3.14}$$

We are going to relate $\tau_{\hat{\sigma}} X(t)$ to Y_t.

$$\begin{aligned}
\tau_{-\hat{\sigma}} X_t(t,\omega) &= \tau_{-\hat{\sigma}} [J_{-\sigma}(t) \sigma \diamond Y_t(t,\omega)] \\
&= \tau_{-\hat{\sigma}} [J_{-\sigma}(t) \tau_{\hat{\sigma}} Y_t] \\
&= \tau_{-\hat{\sigma}} J_{-\sigma}(t) Y_t \,.
\end{aligned}$$

Since $\tau_{-\hat{\sigma}} J_{-\sigma}(t) = [J_{-\hat{\sigma}}(t)]^{-1}$, we obtain an equation equivalent to (3.4) for Y_t:

$$\frac{dY_t}{dt} = J_{-\sigma}(t) b(t, [J_{-\sigma}(t)]^{-1} Y_t, \omega + \int_0^{\cdot} \hat{\sigma}(s) ds). \tag{3.15}$$

This is a deterministic equation. The initial value $X(0)$ is equivalent to initial value $Y_0 = X(0) \diamond J_{-\sigma}(0) = X(0)$. Thus we can solve the quasilinear equation with given initial value.

The terminal value $X(T)$ can also be transformed into the terminal value on $Y(T) = X(T) \diamond J_{-\sigma}(T)$. Thus the equation with given terminal value can be solved in a similar way. Note, however, that in this case the solution need not be $\mathcal{F}_\cdot^{(H)}$-adapted (see the next section).

Example 3.2 In the equation (3.4) let us consider the case $b(t,x) = b_t x$ for some deterministic locally bounded function b_t of t. This means that we are considering the linear stochastic differential equation:

$$dX(t) = b_t X(t) dt + \sigma_t X(t) dB^{(H)}(t) \,. \tag{3.16}$$

In this case it is easy to see that the equation (3.15) satisfied by Y is

$$\dot{Y}_t = b(t) Y_t \,.$$

When the initial value is $Y(0) = x$ (constant), $x \in \mathbb{R}$, then

$$Y_t = x e^{\int_0^t b(s) ds} \,.$$

Thus the solution of (3.16) with $X(0) = x$ can be expressed as

$$\begin{aligned}
X(t) &= Y(t) \diamond J_{-\sigma}(t) \\
&= x \exp\left\{ \int_0^t b(s) ds + \int_0^t \sigma_s dB^{(H)}(s) - \tfrac{1}{2} \|\sigma\|_{\phi,t}^2 \right\} \,.
\end{aligned} \tag{3.17}$$

If we assume the terminal value $X(T)$ given, then

$$\begin{aligned} Y(t) &= Y(T)e^{\int_t^T b(s)ds} \\ &= X(T) \diamond J_\sigma(T)e^{\int_t^T b(s)ds} . \end{aligned}$$

Hence

$$\begin{aligned} X(t) =& Y(t) \diamond J_{-\sigma}(t) = X(T) \diamond \exp\Big\{ \int_t^T b(s)ds \\ & - \int_t^T \sigma_s dB^{(H)}(s) - \tfrac{1}{2} \int_t^T \int_t^T \sigma(u)\sigma(v)\phi(u,v)dudv \Big\} . \end{aligned} \tag{3.18}$$

4 Fractional backward stochastic differential equations

Let $b : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a given function and let $F : \Omega \to \mathbb{R}$ be a given $\mathcal{F}_T^{(H)}$-measurable random variable, where $T > 0$ is a constant. Consider the problem of finding $\mathcal{F}^{(H)}$*-adapted processes* $p(t)$, $q(t)$ such that

$$dp(t) = b(t, p(t), q(t))dt + q(t)dB^{(H)}(t) \,; \quad t \in [0,T] \,, \tag{4.1}$$

$$P(T) = F \quad \text{a.s.} \tag{4.2}$$

This is a *fractional backward stochastic differential equation* (FBSDE) in the two unknown processes $p(t)$ and $q(t)$. We will not discuss general theory for such equations here, but settle with a solution in a linear variant of (4.1)-(4.2), namely

$$dp(t) = [\alpha(t) + b_t p(t) + c_t q(t)]\, dt + q(t)dB^{(H)}(t) \,; \quad t \in [0,T] \,, \tag{4.3}$$

$$P(T) = F \quad \text{a.s.} \,, \tag{4.4}$$

where b_t and c_t are given continuous deterministic functions and $\alpha(t) = \alpha(t,\omega)$ is a given $\mathcal{F}^{(H)}$-adapted process s.t. $\int_0^T |\alpha(t,\omega)|dt < \infty$ a.s.

To solve (4.3)-(4.4) we proceed as follows: By the fractional Girsanov theorem (see e.g. [HØ2, Theorem 3.18]) we can rewrite (4.3) as

$$dp(t) = [\alpha(t) + b_t p(t)]\, dt + q(t)d\hat{B}^{(H)}(t) \,; \quad t \in [0,T] \,, \tag{4.5}$$

where

$$\hat{B}^{(H)}(t) = B^{(H)}(t) + \int_0^t c_s ds \tag{4.6}$$

is a fractional Brownian motion (with Hurst parameter H) under the new probability measure $\hat{\mu}$ on $\mathcal{F}_T^{(H)}$ defined by

$$\frac{d\hat{\mu}(\omega)}{d\mu(\omega)} = \exp^\diamond \{-\langle\omega, \hat{c}\rangle\} = \exp\Big\{ - \int_0^T \hat{c}(s)dB^{(H)}(s) - \tfrac{1}{2}\|\hat{c}\|_\phi^2 \Big\}, \tag{4.7}$$

where $\hat{c} = \hat{c}_t$ is the continuous function with supp $(\hat{c}) \subset [0,T]$ satisfying

$$(4.8) \qquad \int_0^T \hat{c}_s \phi(s,t) ds = c_t\,; \quad 0 \le t \le T\,,$$

and

$$\|\hat{c}\|_\phi^2 = \int_0^T \int_0^T \hat{c}(s)\hat{c}(t)\phi(s,t) ds\, dt\,.$$

If we multiply (4.5) with the integrating factor

$$\beta_t := \exp(-\int_0^t b_s ds)\,,$$

we get

$$(4.9) \qquad d(\beta_s p(s)) = \beta_s \alpha(s) ds + \beta_s q(s) d\hat{B}^{(H)}(s)\,,$$

or, by integrating (4.9) from $s = t$ to $s = T$,

$$(4.10) \qquad \beta_T F = \beta_t p(t) + \int_t^T \beta_s \alpha(s) ds + \int_t^T \beta_s q(s) d\hat{B}^{(H)}(s)\,.$$

Assume from now on that

$$(4.11) \qquad \|\alpha\|^2_{\hat{\mathcal{L}}_\phi^{1,2}[0,T]} := \mathbb{E}_{\hat{\mu}}\left[\int_{[0,T]\times[0,T]} \alpha(s)\alpha(t)\phi(s,t) ds dt + \left(\int_0^T \hat{D}_s^\phi \alpha(s) ds\right)^2\right] < \infty\,.$$

By the fractional Itô isometry (see [DHP, Theorem 3.7] or [HØS2, (1.10)]) applied to $\hat{B}$, $\hat{\mu}$ we then have

$$(4.12) \qquad \mathbb{E}_{\hat{\mu}}\left[\left(\int_0^T \alpha(s) d\hat{B}^{(H)}(s)\right)^2\right] = \|\alpha\|^2_{\hat{\mathcal{L}}_\phi^{1,2}[0,T]}\,.$$

From now on let us also assume that

$$(4.13) \qquad \mathbb{E}_{\hat{\mu}}\left[F^2\right] < \infty\,.$$

We now apply the quasi-conditional expectation operator (see [HØ2, Definition 4.9a)])

$$\tilde{\mathbb{E}}_{\hat{\mu}}\left[\cdot|\mathcal{F}_t^{(H)}\right]$$

to both sides of (4.10) and get

$$(4.14) \qquad \beta_T \tilde{\mathbb{E}}_{\hat{\mu}}\left[F|\mathcal{F}_t^{(H)}\right] = \beta_t p(t) + \int_t^T \beta_s \tilde{\mathbb{E}}_{\hat{\mu}}\left[\alpha(s)|\mathcal{F}_t^{(H)}\right] ds\,.$$

Here we have used that $p(t)$ is $\mathcal{F}_t^{(H)}$-measurable, that the filtration $\hat{\mathcal{F}}_t^{(H)}$ generated by $\hat{B}^{(H)}(s)\,; s \le t$ is the same as $\mathcal{F}_t^{(H)}$, and that

$$(4.15) \qquad \tilde{\mathbb{E}}_{\hat{\mu}}\left[\int_t^T f(s,\omega) d\hat{B}^{(H)}(s)|\hat{\mathcal{F}}_t^{(H)}\right] = 0\,, \quad \text{for all} \quad t \le T$$

for all $f \in \hat{\mathcal{L}}_{\phi}^{1,2}[0,T]$. See [HØ2, Def 4.9] and [HØS2, Lemma 1.1].

From (4.14) we get the solution

$$
\begin{aligned}
p(t) \quad = \quad & \exp\left(-\int_t^T b_s ds\right) \tilde{\mathbb{E}}_{\hat{\mu}}\left[F|\mathcal{F}_t^{(H)}\right] \\
& + \int_t^T \exp\left(-\int_t^s b_r dr\right) \tilde{\mathbb{E}}_{\hat{\mu}}\left[\alpha(s)|\mathcal{F}_t^{(H)}\right] ds\,; \quad t \leq T\,.
\end{aligned} \tag{4.16}
$$

In particular, choosing $t = 0$ we get

$$
p(0) = \exp\left(-\int_0^T b_s ds\right) \tilde{\mathbb{E}}_{\hat{\mu}}\left[F\right] + \int_0^T \exp\left(-\int_0^s b_r dr\right) \tilde{\mathbb{E}}_{\hat{\mu}}\left[\alpha(s)\right] ds\,. \tag{4.17}
$$

Note that $p(0)$ is $\mathcal{F}_0^{(H)}$-measurable and hence a constant. Choosing $t = 0$ in (4.10) we get

$$
G = \int_0^T \beta_s q(s) d\hat{B}^{(H)}(s)\,, \tag{4.18}
$$

where

$$
G = G(\omega) = \beta_T F(\omega) - \int_0^T \beta_s \alpha(s,\omega) ds - p(0)\,, \tag{4.19}
$$

with $p(0)$ given by (4.17).

By the fractional Clark-Ocone theorem [HØ1, Theorem 4.15 b)] applied to $(\hat{B}^{(H)}, \hat{\mu})$ we have

$$
G = \mathbb{E}_{\hat{\mu}}[G] + \int_0^T \tilde{\mathbb{E}}_{\hat{\mu}}\left[\hat{D}_s G|\hat{\mathcal{F}}_s^{(H)}\right] d\hat{B}^{(H)}(s)\,, \tag{4.20}
$$

where $\hat{D}$ denotes the Malliavin derivative at s with respect to $\hat{B}^{(H)}(\cdot)$. Comparing (4.18) and (4.20) we see that we can choose

$$
q(t) = \exp\left(\int_0^t b_r dr\right) \tilde{\mathbb{E}}_{\hat{\mu}}\left[\hat{D}_t G|\mathcal{F}_t^{(H)}\right]\,. \tag{4.21}
$$

We have proved the first part of the following result:

Theorem 4.1 *Assume that (4.11) and (4.13) hold. Then a solution $(p(t), q(t))$ of (4.3)–(4.4) is given by (4.16) and (4.21). The solution is unique among all $\mathcal{F}_{\cdot}^{(H)}$-adapted processes $p(\cdot)$, $q(\cdot) \in \hat{\mathcal{L}}_{\phi}^{1,2}[0,T]$.*

Proof It remains to prove uniqueness. The uniqueness of $p(\cdot)$ follows from the way we deduced formula (4.16) from (4.3)-(4.4). The uniqueness of q is deduced from (4.18) and (4.20) by the following argument: Substituting (4.20) from (4.18) and using that $\mathbb{E}_{\hat{\mu}}(G) = 0$ we get

$$
0 = \int_0^T \left(\beta_s q(s) - \tilde{\mathbb{E}}_{\hat{\mu}}\left[\hat{D}_s G|\hat{\mathcal{F}}_s^{(H)}\right]\right) d\hat{B}^{(H)}(s)\,.
$$

Hence by the fractional Itô isometry (4.12)

$$
\begin{aligned}
0 &= \mathbb{E}_{\hat{\mu}}\left[\left\{\int_0^T \left(\beta_s q(s) - \tilde{\mathbb{E}}_{\hat{\mu}}\left[\hat{D}_s G|\hat{\mathcal{F}}_s^{(H)}\right]\right) d\hat{B}^{(H)}(s)\right\}^2\right] \\
&= \|\beta_s q(s) - \tilde{\mathbb{E}}_{\hat{\mu}}\left[\hat{D}_s G|\hat{\mathcal{F}}_s^{(H)}\right]\|^2_{\hat{\mathcal{L}}_\phi^{1,2}[0,T]},
\end{aligned}
$$

from which it follows that

$$
\beta_s q(s) - \tilde{\mathbb{E}}_{\hat{\mu}}\left[\hat{D}_s G|\hat{\mathcal{F}}_s^{(H)}\right] = 0 \quad \text{for} \quad a.a.(s,\omega) \in [0,T] \times \Omega\,.
$$

□

5 A stochastic maximum principle

We now apply the theory in the previous section to prove a maximum principle for systems driven by fractional Brownian motion. See e.g. [H], [P] and [YZ] and the references therein for more information about the maximum principle in the classical Brownian motion case.

Suppose $X(t) = X^{(u)}(t)$ is a controlled system of the form

$$
dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB^{(H)}(t)\,; \quad X(0) = x \in \mathbb{R}^n\,, \tag{5.1}
$$

where $b : [0.T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}$ are given C^1 functions. The control process $u(\cdot) : [0,T] \times \Omega \to U \subset \mathbb{R}^k$ is assumed to be $\mathcal{F}^{(H)}$-adapted. U is a given closed convex set in $\mathbb{R}^k$.

Let $f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R}^n \to \mathbb{R}^N$ be given C^1 functions and consider a *performance functional* $J(u)$ of the form

$$
J(u) = \mathbb{E}\left[\int_0^T f(t, X(t), u(t))dt + g(X(T))\right] \tag{5.2}
$$

and a *terminal condition* given by

$$
\mathbb{E}\left[G(X(T))\right] = 0\,. \tag{5.3}
$$

Let $\mathcal{A}$ denote the set of all $\mathcal{F}_t^{(H)}$-adapted processes $u : [0,T] \times \Omega \to U$ such that $X^{(u)}(t)$ exists and does not explode in $[0,T]$ and

$$
E\Big[\int_0^T |f(t, X(t), u(t))|dt + g^-(X(T)) + G^-(X(T))\Big] < \infty \tag{5.4}
$$

where $y^- = \max(0, y)$ for $y \in \mathbb{R}$, and such that (5.3) holds. If $u \in \mathcal{A}$ and $X^{(u)}(t)$ is the corresponding state process we call $(u, X^{(u)})$ an *admissible pair*. Consider the problem to find J^* and $u^* \in \mathcal{A}$ such that

$$
J^* = \sup\{J(u)\,; u \in \mathcal{A}\} = J(u^*)\,. \tag{5.5}
$$

If such $u^* \in \mathcal{A}$ exists, then u^* is called an *optimal control* and (u^*, X^*), where $X^* = X^{u^*}$, is called an *optimal pair*.

Let $\mathcal{R}^{n\times m}$ be the set of continuous function from $[0,T]$ into $\mathbb{R}^{n\times m}$. Define the *Hamiltonian* $H:[0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^n\times\mathcal{R}^{n\times m}\to\mathbb{R}$ by

$$H(t,x,u,p,q(\cdot))=f(t,x,u)+b(t,x,u)^Tp+\sum_{i=1}^{n}\sum_{k=1}^{m}\sigma_{ik}(t,x,u)\int_0^T q_{ik}(s)\phi_{H_k}(s,t)ds\,. \tag{5.6}$$

Consider the following *fractional stochastic backward differential equation* in the pair of unknown $\mathcal{F}_t^{(H)}$-adapted processes $p(t)\in\mathbb{R}^n$, $q(t)\in\mathbb{R}^{n\times m}$, called the *adjoint processes*:

$$\begin{cases} dp(t)=-H_x(t,X(t),u(t),p(t),q(\cdot))dt+q(t)dB^{(H)}(t)\,; \quad t\in[0,T] \\ p(T)=g_x(X(T))+\lambda^TG_x(X(T))\,. \end{cases} \tag{5.7}$$

where $H_x=\nabla_xH=\left(\frac{\partial H}{\partial x_1},\cdots,\frac{\partial H}{\partial x_n}\right)^T$ is the gradient of H with respect to x and similarly with g_x and G_x. $X(t)=X^{(u)}(t)$ is the process obtained by using the control $u\in\mathcal{A}$ and $\lambda\in\mathbb{R}_+^n$ is a constant. The equation (5.6) is called the adjoint equation and $p(t)$ is sometimes interpreted as the *shadow price* (of a resource).

Theorem 5.1 (The fractional stochastic maximum principle) *Suppose $\hat{u}\in\mathcal{A}$ and put $\hat{X}=X^{(\hat{u})}$. Suppose there exists a solution $\hat{p}(t),\hat{q}(t)$ of the corresponding adjoint equation (5.7) for some $\lambda\in\mathbb{R}_+^n$ and such that the following, (5.8)–(5.11), hold:*

$$X^{(u)}(t)\hat{q}(t)\in\mathcal{L}_\phi^{1,2} \quad \textit{and} \quad \hat{p}^T(t)\sigma(t,X^{(u)}(t),u(t))\in\mathcal{L}_\phi^{1,2} \quad \textit{for all } u\in\mathcal{A} \tag{5.8}$$

$$H(t,\cdot,\cdot,\hat{p}(t),\hat{q}(t))\,,\quad g(\cdot)\ \text{ and }\ G(\cdot)\ \textit{are concave, for all } t\in[0,T]\,, \tag{5.9}$$

$$H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(\cdot))=\max_{v\in U}H(t,\hat{X}(t),v,\hat{p}(t),\hat{q}(\cdot))\,, \tag{5.10}$$

$$\begin{aligned}\Delta_4:=\mathbb{E}\Big[\sum_{i=1}^{n}\sum_{j,k=1}^{m}\Big(\int_0^T D_{j,t}^{\phi_j}\{\sigma_{ik}(t,X(t),u(t))\\ -\sigma_{ik}(t,\hat{X}(t),\hat{u}(t))\}dt\Big)\Big(\int_0^T D_{k,t}^{\phi_k}\hat{q}_{ij}(t)dt\Big)\Big]\le 0 \qquad \textit{for all } u\in\mathcal{A}\,.\end{aligned} \tag{5.11}$$

Then if $\lambda\in\mathbb{R}_+^n$ is such that $(\hat{u},\hat{X})$ is admissible (in particular, (5.3) holds), the pair $(\hat{u},\hat{X})$ is an optimal pair for problem (5.5).

Proof We first give a proof in the case when $G(x)=0$, *i.e.* when there is no terminal condition.

With $(\hat{u},\hat{X})$ as above consider

$$\begin{aligned}\Delta &:=\mathbb{E}\left[\int_0^T f(t,\hat{X}(t),\hat{u}(t))dt-\int_0^T f(t,X(t),u(t))dt\right]\\ &=\mathbb{E}\left[\int_0^T H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(\cdot))dt-\int_0^T H(t,X(t),u(t),\hat{p}(t),\hat{q}(\cdot))dt\right]\\ &\quad-\mathbb{E}\left[\int_0^T\left\{b(t,\hat{X}(t),\hat{u}(t))\right\}^T\hat{p}(t)dt-\int_0^T b(t,X(t),u(t))^T\hat{p}(t)dt\right]\\ &\quad-\mathbb{E}\left[\int_0^T\int_0^T\sum_{i=1}^{n}\sum_{k=1}^{m}\left\{\sigma_{ik}(s,\hat{X}(s),\hat{u}(s))-\sigma_{ik}(s,X(s),u(s))\right\}\hat{q}_{ik}(t)\phi_{H_k}(s,t)dsdt\right]\\ &=:\Delta_1+\Delta_2+\Delta_3\,.\end{aligned} \tag{5.12}$$

Since $(x,u) \to H(x,u) = H(t,x,u,p,q(\cdot))$ is concave we have

$$H(x,u) - H(\hat{x},\hat{u}) \leq H_x(\hat{x},\hat{u}) \cdot (x-\hat{x}) + H_u(\hat{x},\hat{u}) \cdot (u-\hat{u})$$

for all (x,u), $(\hat{x},\hat{u})$. Since $v \to H(\hat{X}(t),v)$ is maximal at $v = \hat{u}(t)$ we have

$$H_u(\hat{x},\hat{u}) \cdot (u(t)-\hat{u}(t)) \leq 0 \quad \forall t\,.$$

Therefore

$$\begin{aligned} \Delta_1 &\geq \mathbb{E}\left[\int_0^T -H_x(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(\cdot)) \cdot (X(t)-\hat{X}(t))dt\right] \\ &= \mathbb{E}\left[\int_0^T (X(t)-\hat{X}(t))^T d\hat{p}(t) - \int_0^T (X(t)-\hat{X}(t))^T \hat{q}(t) dB^{(H)}(t)\right] \end{aligned}$$

Since $\mathbb{E}\left[\int_0^T (X(t)-\hat{X}(t))^T \hat{q}(t) dB^{(H)}(t)\right] = 0$ by (2.7), this gives

$$\Delta_1 \geq \mathbb{E}\left[\int_0^T (X(t)-\hat{X}(t))^T d\hat{p}(t)\right]\,. \tag{5.13}$$

By (5.1) we have

$$\begin{aligned} \Delta_2 &= -\mathbb{E}\left[\int_0^T \Big\{b(t,\hat{X}(t),\hat{u}(t)) - b(t,X(t),u(t))\Big\} \cdot \hat{p}(t)dt\right] \\ &= -\mathbb{E}\left[\int_0^T \hat{p}(t)\Big(d\hat{X}(t) - dX(t)\Big)\right] \\ &\quad - \mathbb{E}\left[\int_0^T \hat{p}(t)^T \Big\{\sigma(t,\hat{X}(t),\hat{u}(t)) - \sigma(t,X(t),u(t))\Big\} dB^{(H)}(t)\right] \\ &= \mathbb{E}\left[\int_0^T \hat{p}(t)\Big(dX(t) - d\hat{X}(t)\Big)\right]\,. \end{aligned} \tag{5.14}$$

Finally, since g is concave we have

$$g(X(T)) - g(\hat{X}(T)) \leq g_x(\hat{X}(T)) \cdot (X(T) - \hat{X}(T)) \tag{5.15}$$

Combining (5.12)–(5.15) with Corollary 2.5 we get, using (5.2), (5.7) and (5.11),

$$\begin{aligned} J(\hat{u}) - J(u) &= \Delta + \mathbb{E}\Big[g(\hat{X}(T)) - g(X(T))\Big] \\ &\geq \Delta + \mathbb{E}\Big[g_x(\hat{X}(T)) \cdot (\hat{X}(T) - X(T))\Big] \\ &\geq \Delta - \mathbb{E}\Big[\hat{p}(T) \cdot \Big(X(T) - \hat{X}(T)\Big)\Big] \\ &= \Delta - \Bigg\{\mathbb{E}\left[\int_0^T \Big(X(t)-\hat{X}(t)\Big) \cdot d\hat{p}(t)\right] + \mathbb{E}\left[\int_0^T \hat{p}(t) \cdot \Big(dX(t) - d\hat{X}(t)\Big)\right] \\ &\quad + \mathbb{E}\left[\int_0^T\int_0^T \sum_{i=1}^n \sum_{k=1}^m \Big\{\sigma_{ik}(s,X(s),u(s)) - \sigma_{ik}(s,\hat{X}(s),\hat{u}(s))\Big\} \hat{q}_{ik}(t)\phi_{H_k}(s,t) ds\, dt\right. \\ &\quad + \mathbb{E}\left[\sum_{i=1}^n \sum_{j,k=1}^m \Big(\int_0^T D_{j,t}^{\phi_j}\{\sigma_{ik}(t,X(t),u(t)) - \sigma_{ik}(t,\hat{X}(t),\hat{u}(t))\}dt\Big)\Big(\int_0^T D_{k,t}^{\phi_k}\hat{q}_{ij}(t)\Big)\right]\Bigg\} \\ &\geq \Delta - (\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4) \geq 0\,. \end{aligned}$$

This shows that $J(\hat{u})$ is maximal among all admissible pairs $(u(\cdot), X(\cdot))$.

This completes the proof in the case with no terminal conditions ($G = 0$). Finally consider the general case with $G \neq 0$. Suppose that for some $\lambda_0 \in \mathbb{R}^n_+$ there exists $\hat{u}_{\lambda_0}$ satisfying (5.8)–(5.11). Then by the above argument we know that if we put

$$J_{\lambda_0}(u) = \mathbb{E}\left[\int_0^T f(t, X(t), u(t))dt + g(X(T)) + \lambda_0^T G(X(T))\right]$$

then $J_{\lambda_0}(\hat{u}_0) \geq J_{\lambda_0}(u)$ for all controls u (without terminal condition). If λ_0 is such that $\hat{u}_{\lambda_0}$ satisfies the terminal condition (*i.e.* $\hat{u}_{\lambda_0} \in \mathcal{A}$) and u is another control in $\mathcal{A}$ then

$$J(\hat{u}_{\lambda_0}) = J_{\lambda_0}(\hat{u}_{\lambda_0}) \geq J_{\lambda_0}(u) = J(u)$$

and hence $\hat{u}_{\lambda_0} \in \mathcal{A}$ maximizes $J(u)$ over all $u \in \mathcal{A}$. $\square$

Corollary 5.2 *Let $\hat{u} \in \mathcal{A}$, $\hat{X} = X^{(\hat{u})}$ and $(\hat{p}(t), \hat{q}(t))$ be as in Theorem 5.1. Assume that (5.8), (5.9) and (5.10) hold, and that condition (5.11) is replaced by the condition*

$$\hat{q}(\cdot) \ \text{ or } \ \sigma(\cdot, \hat{X}(\cdot), \hat{u}(\cdot)) \ \text{ is deterministic}\,. \tag{5.16}$$

Then if $\lambda \in \mathbb{R}^n_+$ is such that $(\hat{u}, \hat{X})$ is admissible, the pair $(\hat{u}, \hat{X})$ is an optimal pair for problem (5.5).

6 A minimal variance hedging problem

To illustrate our main result, we use it to solve the following problem from mathematical finance:

Consider a financial market driven by two independent fractional Brownian motions $B_1(t) = B_1^{(H_1)}(t)$ and $B_2(t) = B^{(H_2)}(t)$, with $\frac{1}{2} < H_i < 1$, $i = 1, 2$, as follows:

(6.1)	(Bond price)	$dS_0(t) = 0\,; \quad S_0(0) = 1$
(6.2)	(Price of stock 1)	$dS_1(t) = dB_1(t)\,; \quad S_1(0) = s_1$
(6.3)	(Price of stock 2)	$dS_2(t) = dB_1(t) + dB_2(t)\,; \quad S_2(0) = s_2\,.$

If $\theta(t) = (\theta_0(t), \theta_1(t), \theta_2(t)) \in \mathbb{R}^3$ is a *portfolio* (giving the number of units of the bond, stock 1 and stock 2, respectively, held at time t) then the corresponding *value process* is

$$V^\theta(t) = \theta(t) \cdot S(t) = \sum_{i=0}^{2} \theta_i(t) S_i(t)\,. \tag{6.4}$$

The portfolio is called *self-financing* if

$$dV^\theta(t) = \theta(t) \cdot dS(t) = \theta_1(t)dB_1(t) + \theta_2(t)(dB_1(t) + dB_2(t))\,. \tag{6.5}$$

This market is called *complete* if any bounded $\mathcal{F}_T^{(H)}$-measurable random variable F can be *hedged* (or *replicated*), in the sense that there exists a (self-financing) portfolio $\theta(t)$ and an initial value $z \in \mathbb{R}$ such that

$$F(\omega) = z + \int_0^T \theta(t)dS(t) \qquad \text{for a.a. } \omega\,. \tag{6.6}$$

(See [HØ2] and [W] for a general discussion about this.)

Let us now assume that we are not allowed to trade in stock 1, i.e. we must have $\theta_1(t) \equiv 0$. How close to, say, $F(\omega) = B_1(T,\omega)$ can we get if we must hedge under this constraint?

If we put $\theta_2(t) = u(t)$ and interpret "close" as having a small $L^2(\mu)$ distance to F, then the problem can be stated as follows:

Find $z \in \mathbb{R}$ and admissible $u(t,\omega)$ such that

$$\begin{aligned} J(z,u) :&= \mathbb{E}\Big[\Big\{B_1(T) - \Big(z + \int_0^T u(t)(dB_1(t) + dB_2(t))\Big)\Big\}^2\Big] \\ &= z^2 + \mathbb{E}\Big[\Big\{\int_0^T (u(t)-1)dB_1(t) + \int_0^T u(t)dB_2(t)\Big\}^2\Big] \end{aligned} \tag{6.7}$$

is minimal. We see immediately that it is optimal to choose $z = 0$, so it remains to minimize over $u(t) = u(t,\omega)$ the functional

$$J(u) := \mathbb{E}\Big[\Big\{\int_0^T (u(t)-1)dB_1(t) + \int_0^T u(t)dB_2(t)\Big\}^2\Big]. \tag{6.8}$$

If we apply the fractional Itô isometry (2.13) we get, after some simplifications,

$$\begin{aligned} J(u) = \mathbb{E}\Big[&\int_0^T\int_0^T \big\{(u(s)-1)(u(t)-1)\phi_1(s,t) + u(s)u(t)\phi_2(s,t)\big\}ds\,dt \\ &+ \Big(\int_0^T \big\{D^\phi_{1,t}u(t) - D^\phi_{2,t}u(t)\big\}dt\Big)^2\Big]. \end{aligned} \tag{6.9}$$

However, it is difficult to see from this what the minimizing $u(t)$ is.

To approach this problem by using the fractional maximum principle, we define the state process $X(t)$ by

$$dX(t) = (u(t)-1)dB_1(t) + u(t)dB_2(t)\ . \tag{6.10}$$

Then the problem is equivalent to maximizing

$$J_1(u) := \mathbb{E}\big[-\tfrac{1}{2}X^2(T)\big]\ . \tag{6.11}$$

The Hamiltonian for this problem is

$$\begin{aligned} H(t,x,u,p,q(\cdot)) &= (u-1)\int_0^T q_1(s)\phi_1(s,t)ds + u\int_0^T q_2(s)\phi_2(s,t)ds \\ &= (u-1)\int_0^T q_1(s)\phi_1(s,t)ds + u\int_0^T q_2(s)\phi_2(s,t)ds \\ &= u\Big[\int_0^T q_1(s)\phi_1(s,t)ds + \int_0^T q_2(s)\phi_2(s,t)ds\Big] - \int_0^T q_1(s)\phi_1(s,t)ds\ . \end{aligned} \tag{6.12}$$

The adjoint equation is

$$dp(t) = q_1(t)dB_1(t) + q_2(t)dB_2(t)\ ; \qquad t < T \tag{6.13}$$

$$p(T) = -X(T)\ . \tag{6.14}$$

Comparing with (6.10) we see that this equation has the solution

$$q_1(t) = 1 - u(t), \quad q_2 = -u_2(t), \quad p(t) = -X(t)\ ; \qquad t \leq T\ . \tag{6.15}$$

Let $\hat{u}(t)$ be an optimal control candidate. Then by (6.12)

$$H(t,\hat{X}(t), v, \hat{p}(t), \hat{q}(\cdot)) = v\Big[\int_0^T \hat{q}_1(s)\phi_1(s,t)ds + \int_0^T \hat{q}_2(s)\phi_2(s,t)ds\Big] - \int_0^T \hat{q}_1(s)\phi_1(s,t)ds$$

$$= v\Big[\int_0^T (1-\hat{u}(t))\phi_1(s,t)ds - \int_0^T \hat{u}(s)\phi_2(s,t)ds\Big] - \int_0^T \hat{q}_1(s)\phi_1(s,t)ds\ . \tag{6.16}$$

The maximum principle requires that the maximum of this expression is attained at $v = \hat{u}(t)$. However, this is an affine function of v, so it is natural to guess that the coefficient of v must be 0, i.e.

$$\int_0^T (1-\hat{u}(s))\phi_1(s,t)ds - \int_0^T \hat{u}(s)\phi_2(s,t)ds = 0\ ,$$

which gives

$$\int_0^T \hat{u}(s)(\phi_1(s,t) + \phi_2(s,t))ds = \int_0^T \phi_1(s,t)ds\ . \tag{6.17}$$

This is a symmetric Fredholm integral equation of the first kind and it is known that it has a unique solution $\hat{u}(t) \in L^2[0,T]$. See e.g. [T, Section 3.15].

This choice of $\hat{u}(t)$ satisfies all the requirements of Theorem 5.1 (in fact, even those of Corollary 5.2) and we can conclude that this $\hat{u}(t)$ is optimal. Thus we have proved:

Theorem 6.1 (Solution of the minimal variance hedging problem)
The minimal value of

$$J(z,u) = \mathbb{E}\Big[\Big\{B_1(T) - \Big(z + \int_0^T u(t)(dB_1(t) + dB_2(t))\Big)\Big\}^2\Big]$$

is attained when $z = 0$ and $u = \hat{u}(t)$ satisfies (6.17). The corresponding minimal value is

$$\inf_{z,u} J(z,u) = \int_0^T \int_0^T \big\{(\hat{u}(s)-1)(\hat{u}(t)-1)\phi_1(s,t) + \hat{u}(s)\hat{u}(t)\phi_2(s,t)\big\}ds\,dt\ .$$

Remark Note that if $\phi_1 = \phi_2$ then $\hat{u}(t) \equiv \frac{1}{2}$, which is the same as the optimal value in the classical Brownian motion case $(H_1 = H_2 = \frac{1}{2})$.

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