Behaviour of Long-Term Yields in a Lévy Term Structure

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Abstract

We study the behaviour of the long-term yield in a HJM setting for forward rates driven by Lévy processes. The long-term rates are investigated by examining continuously compounded spot rate yields with maturity going to infinity. In this paper we generalise the model of El Karoui, Frachot and Geman [19] by using Lévy processes instead of Brownian motions as driving processes of the forward rate dynamics, and analyse the behaviour of the long-term yield under certain conditions which encompass the asymptotic behaviour of the interest rate model's volatility function as well as the variation of the paths of the Lévy process. One of the main results is that the long-term volatility has to vanish except in the case of a Lévy process with only negative jumps and paths of finite variation serving as random driver. Furthermore, we study the required asymptotic behaviour of the volatility function so that the long-term drift exists.

1 Introduction

Long-term interest rates are important for the pricing and hedging of a number of different financial instruments, including for example long-term fixed income securities, life insurance, accident insurance or long-term interest rate swaps. Further, we are seeing now a resurgence of very long-term debt issuance. For instance, the United Kingdom plans to issue bonds with time to maturity of 100 years and even perpetuals (see, for example, Focus Money [12]). Besides these financial instruments there are situations in which the time horizon of cash flows extends beyond the limit of the observable term structure of interest rates: valuation of required financial resources for public and private retirement systems, funding of long-term infrastructure projects and determining compensatory adjustments in the course of a divorce or an accident. From an economic and financial point of view it is then important to investigate the behaviour of interest rates over a long time horizon.

In the literature there are different understandings of the meaning of "long-term" interest rates. Yao [29] investigates yield curves with time to maturity beyond 30 years, Shiller [27] looks at bonds with over 20 years to maturity

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whereas the European Central Bank [9] designates its values of market yields of government bonds with maturities close to 10 years as long-term interest rates. A natural approach to the concept “long-term” is to consider interest rate systems in the limit as the maturity goes to infinity, as considered, for example, by Dybvig et al. [5] and by El Karoui et al. [19]. We adapt this approach here and define the long-term interest rate ℓ_t at time t as the yield of zero-coupon bonds prevailing at time t with time to maturity tending to infinity.

Though the modelling and the corresponding convergence behaviour of the long-term yield has been topic of various publications, further research is still required. Some researchers approached the topic from a macroeconomic point of view like Mankiw [23] or Gürkaynak et al. [15]. Other more mathematical works were published by Dybvig et al. [5], El Karoui et al. [19], Yao [29] and Zubchenko [31]. The main result of [5] is that the long-term rate is necessarily a non-decreasing process. This result has been clarified and extended in several other works, e.g. [24], [17], [14], and [18]. In [19] the behaviour of the long-term yield is studied in a HJM-type model, where the stochastic driver is a Brownian motion. However the resulting analysis in [19] is not completely satisfactory since no jumps are included in the underlying model. The aim of this paper is then to generalise this approach by considering interest rates dynamics driven by a multivariate Lévy process. In this more general setting we have to impose stronger requirements on the volatility process than in [19] in order to guarantee the convergence of the long-term yield. Nevertheless, we are able to provide explicit formulas for the long-term yield, long-term drift and long-term volatility, all depending only on the Lévy process and the volatility. One of the main results is that the long-term volatility has to vanish except in the case of a Lévy process with only negative jumps and paths of finite variation serving as random driver (see Theorem 3.6).

The results of our paper provide a further answer concerning the investigation of the behaviour of long-term interest rates. A main advantage of our approach is to consider a wider class of models that allow jumps to affect the dynamics of the long-term yields. With the use of Lévy processes as stochastic driver a more accurate fit of the resulting long-term yield curve can be obtained than in the Brownian motion setting, since jumps can be encompassed. Furthermore, we provide a characterisation of the volatility long-term structure that generates non-trivial and non-explosive long-term interest rates.

The paper is organised as follows. In Section 2 we introduce a Lévy HJM framework derived from [8] and [20] and state the requirements on the volatility needed for the convergence of the long-term yield in Assumption I. We are able to consider a quite general setting except for restrictions given by Assumptions (A4) and (A5), concerning the integrability of the volatility process. To maintain a sufficient flexibility of the setting, (A4) is only required for the long-term time horizon and may not be satisfied for shorter maturities. Assumption (A5), however, is needed for all maturities for the calculations of the yield curve. Then, in Section 3 the long-term yield is examined with regard to its convergence properties. Thereafter, in Section 4 we provide an example, where we compute explicitly the long-term yield using a jump-diffusion Lévy process as stochastic driver and a volatility process fulfilling the required assumptions.

2 Lévy HJM Framework

In 1992 Heath et al. proposed in [16] a new framework for modelling the entire forward curve directly, where the forward rate is described by a continuous Itô process driven by a possibly d-dimensional Brownian motion. Here we consider an extended version of the HJM-Framework where a Lévy process serves as random driver in order to include also jumps in the model for the bond dynamics as in [7], [8], [11], [20] and [30]. In particular we assume a frictionless market for bonds.

We denote the jump of a stochastic process $(Y_t)_{t\geq 0}$ at time s by $\Delta Y_s := Y_s - Y_{s-}$ whereas $Y_{s-} := \lim_{u\uparrow s} Y_u$. Let $X = \left(X^1, \ldots, X^d\right)$ be a d-dimensional Lévy process with Lévy measure ν and decomposition

$$X_t = \gamma t + W_t + \sum_{s\in[0,t]} \Delta X_s 1_{\{\|\Delta X_s\|>1\}} + \int_{\{\|x\|\leq 1\}} x\left(N_t(\cdot, dx) - t\,\nu(dx)\right) ,$$

where $W := (W_t)_{t\geq 0}$ is a Brownian motion on $\mathbb{R}^d$ with positive definite covariance matrix $A \in \mathbb{R}^{d\times d}$, $\gamma \in \mathbb{R}^d$, and for any set $B \in \mathbb{R}^d$, $0 \notin \bar{B}$, $N_t^B = \int_B N_t(\cdot, dx)$ is a Poisson process independent of W. The Lévy process is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the completed canonical filtration $(\mathcal{F}_t)_{t\geq 0}$ associated with X.

Following [2] and [10] we consider the following bond market setting. By a T-maturity zero-coupon bond we mean a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $t \leq T$ is denoted by $P(t,T)$. The bond market satisfies the following hypotheses: (1) there exists a frictionless market for T-bonds for every maturity $T \geq 0$; (2) $P(T,T) = 1$ for every $T \geq 0$; (3) for each fixed t, the zero-coupon bond price $P(t,T)$ is differentiable with respect to the maturity T.

We assume that for all $T \geq 0$ the bond price is given by

$$P(t,T) = P(0,T)\cdot \beta_t \cdot \frac{\exp\Big(\int_0^t \sigma(s,T)\,dX_s\Big)}{\mathbb{E}\Big[\exp\Big(\int_0^t \sigma(s,T)\,dX_s\Big)\Big]}\,,\ t\in[0,T]\,, \tag{2.1}$$

where $\beta_t := \exp\Big(\int_0^t r_s\,ds\Big)$ is the money market account with r_t denoting the short rate at time t. In particular this implies that $\frac{P(t,T)}{\beta_t}, t \in [0,T]$, is a martingale under $\mathbb{P}$, i.e. $\mathbb{P}$ is assumed to be the risk-neutral measure.

The volatility function σ is d-dimensional and deterministic in every dimension, i.e.

$$\sigma(t,T) = \left(\sigma^1(t,T), \ldots, \sigma^d(t,T)\right)\,, \tag{2.2}$$

where

$$\sigma^i : [0,\infty] \times [0,\infty] \to [0,\infty]\,,\quad (s,t) \mapsto \sigma^i(s,t) \tag{2.3}$$

denotes the volatility of the i-th random component of the yield. In particular $\int_0^t \sigma(s,T)\,dX_s = \sum_{i=1}^d \int_0^t \sigma^i(s,T)\,dX_s^i$. The partial derivatives of σ are denoted the following way:

$$\sigma_1(s,T) := \frac{\partial}{\partial s}\sigma(s,t)\,,\ \sigma_2(s,t) := \frac{\partial}{\partial t}\sigma(s,t)\ .$$

We denote the Euclidean scalar product on $\mathbb{R}^d$ as $x\cdot y := \sum_{i=1}^d x^i y^i$ for $x, y \in \mathbb{R}^d$, $x := \left(x^1, \ldots, x^d\right)$, $y := \left(y^1, \ldots, y^d\right)$ and the respective norm by $\|\cdot\|$.

Assumption I. *We assume that this volatility function σ satisfies the following properties:*

(A1) *For all $i \in \{1, \ldots, d\} : \sigma^i(s,t) > 0$ for all $t \in [0, \infty]$, $s \in [0, t[$.*

(A2) *For all $i \in \{1, \ldots, d\} : \sigma^i(s,t) = 0$ for all $s \geq t$, $t \in [0, \infty]$.*

(A3) *For all $i \in \{1, \ldots, d\} : \sigma^i, \sigma^i_1, \sigma^i_2$ are càglàd in both components.*

(A4) *There exists a càglàd function $\phi \in L^1(\mathbb{R}_+)$ such as for all $T \geq 1$*

$$\frac{\|\sigma(s,T)\|}{T} \leq \phi(s) \quad \text{for all } 0 \leq s \leq T \ .$$

(A5) *There exists a function $\psi \in L^1(\mathbb{R}_+)$ such as for all $T > 0$ and for an $\epsilon \in (0,1)$*

$$\frac{|\log \mathbb{E}[\exp((1+\epsilon)\,\sigma(r,T) \cdot X_1)]|}{T} \leq \psi(r) \quad \text{for all } 0 \leq r \leq T \ . \tag{2.4}$$

Note that (A5) also implies that $\exp(\sigma(s,T) \cdot X_1) \in L^1(\mathbb{P})$ for all $0 \leq s \leq T$. Otherwise we could find a T and $s \leq T$ such that $\mathbb{E}[\exp(\sigma(s,T) \cdot X_1)] = \infty$ and then ψ cannot dominate the left-hand side of (2.4) for all T.

Assumption (A4) is needed for the convergence of the volatility function of long-term interest rates, hence we impose it for sufficiently long times of maturity, say $T \geq 1$. In this way, we also guarantee complete flexibility in the choice of the model for the volatility of interest rates with short-term maturity. This allows to find realistic models by distinguishing between short-term and long-term interest rates modelling. On the contrary, condition (A5) is required to hold for any $T > 0$ in order to obtain the analytical results of Lemma 2.4.

In the subsequent calculations we will use the logarithm of the moment-generating function of X_1, defined as

$$\theta(u) := \log \mathbb{E}[\exp(u \cdot X_1)]\,,\ u \in \mathbb{R}^d. \tag{2.5}$$

Corollary 2.1. *Under* (A5)*, we get for all $t, T \geq 0$:*

$$\mathbb{E}\left[\exp\left(\int_0^t \sigma(s,T)\,dX_s\right)\right] = \exp\left(\int_0^t \theta(\sigma(s,T))\,ds\right). \tag{2.6}$$

Proof. The proof uses the idea of Lemma 3.1 of [8]. However, we do not assume here the boundedness of σ as in [8]. Hence, we have to follow a slightly different approach. We take a partition $0 = t_0 < t_1 < \cdots < t_{N+1} = t$ of $[0,t]$ and get, as described in [8]:

$$\mathbb{E}\left[\exp\left(\sum_{k=0}^{N} \sigma(t_k,T) \cdot \left(X_{t_{k+1}} - X_{t_k}\right)\right)\right] = \exp\left(\sum_{k=0}^{N} \theta(\sigma(t_k,T))\,(t_{k+1} - t_k)\right). \tag{2.7}$$

According to Theorem 53 of Chapter I, Section 7 of [25], the right-hand side of the (2.7) converges to $\exp\left(\int_0^t \theta(\sigma(s,T))\,ds\right)$ almost surely. Now, we take a look at the left-hand side. By Theorem 21 of Chapter II, Section 5 of [25] it follows that

$$\exp\left(\sum_{k=0}^{N} \sigma(t_k, T) \cdot \left(X_{t_{k+1}} - X_{t_k}\right)\right) \overset{N\to\infty}{\longrightarrow} \exp\left(\int_0^t \sigma(s,T)\, dX_s\right) \tag{2.8}$$

in probability. To show (2.6) we have to prove that the convergence in (2.8) holds in L^1. To this purpose we show that the approximating sequence in (2.8) is uniformly integrable by using Theorem II.22 of [4]. We define

$$K := \left\{ \exp\left(\sum_{k=0}^{N} \sigma(t_k, T) \cdot \left(X_{t_{k+1}} - X_{t_k}\right)\right) | N \geq 1 \right\}.$$

For all $N \geq 1$ we have

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\sum_{k=0}^{N} \sigma(t_k, T) \cdot \left(X_{t_{k+1}} - X_{t_k}\right)\right)\right] &\overset{(2.7)}{=} \exp\left(\sum_{k=0}^{N} \theta(\sigma(t_k, T))\,(t_{k+1} - t_k)\right) \\
&\overset{(2.4)}{\leq} \exp\left(T \sum_{k=0}^{N} \psi(t_k)\,(t_{k+1} - t_k)\right) \\
&< \infty\,,
\end{aligned} \tag{2.9}$$

since by (A5) $\psi \in L^1(\mathbb{R}_+)$. It follows from (2.9) that $K \subseteq L^1(\mathbb{P})$. Further, we have to show that there exists a positive function $G : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{x\to\infty} \frac{G(x)}{x} = \infty \quad \text{and} \quad \sup_{f\in K} \mathbb{E}[G(|f|)] < \infty. \tag{2.10}$$

Let us take $G(x) = x^{1+\epsilon}$ with $0 < \epsilon < 1$. Then we have

$$\lim_{x\to\infty} \frac{G(x)}{x} = \lim_{x\to\infty} x^{\epsilon} = \infty$$

and

$$\begin{aligned}
\sup_{f\in K} \mathbb{E}[G(|f|)] &= \sup_{N\geq 1} \mathbb{E}\left[\left(\exp\left(\sum_{k=0}^{N} \sigma(t_k, T) \cdot \left(X_{t_{k+1}} - X_{t_k}\right)\right)\right)^{1+\epsilon}\right] \\
&\overset{(2.7)}{=} \sup_{N\geq 1} \exp\left(\sum_{k=0}^{N} \theta((1+\epsilon)\,\sigma(t_k, T))\,(t_{k+1} - t_k)\right) \\
&\overset{(2.4)}{\leq} \sup_{N\geq 1} \exp\left(T \sum_{k=0}^{N} \psi(t_k)\,(t_{k+1} - t_k)\right) \\
&< \infty
\end{aligned}$$

because $\psi \in L^1(\mathbb{R}_+)$. Hence, since $K \subseteq L^1(\mathbb{P})$ and (2.10) holds, it follows by Theorem II.22 of [4] that K is uniformly integrable. $\square$

Note that from Assumption (A5) and Corollary 2.1 follows that for all $t, T \geq 0$

$$\mathbb{E}\left[\exp\left(\int_0^t \sigma(s,T)\,dX_s\right)\right] < \infty\,.$$

This holds because for all $t, T \geq 0$

$$\left|\int_0^t \theta(\sigma(s,T))\,ds\right| \;\leq\; \int_0^t |\theta(\sigma(s,T))|\;ds \overset{(2.4)}{\leq} T\int_0^t \psi(s)\;ds < \infty$$

since $\psi \in L^1(\mathbb{R}_+)$, and consequently

$$\exp\left(\int_0^t \theta(\sigma(s,T))\,ds\right) < \infty$$

for all $t, T \geq 0$.

Putting all this together, we have derived the following representation for the bond price process.

Corollary 2.2. *Under* Assumption I *the following equation holds for the bond price:*

$$P(t,T) = P(0,T)\cdot\exp\left(\int_0^t (r_s - \theta(\sigma(s,T)))\,ds + \int_0^t \sigma(s,T)\,dX_s\right)\,.$$

Proof. The result follows by Corollary 2.1 and the definition of the money-market account $(\beta_t)_{t\geq 0}$ since we have

$$\int_0^{t_1}\int_0^{t_2} \sigma_2(v,s)\,dX_v\,ds = \int_0^{t_2}\int_0^{t_1} \sigma_2(v,s)\,ds\,dX_v \quad \forall\, t_1, t_2 \geq 0, \tag{2.11}$$

due to the Fubini theorem for semimartingales (see Chapter IV, Section 6, Theorem 65 of [25]). □

We now consider the instantaneous forward rate with maturity T prevailing at time t, $f(t,T) := -\frac{\partial}{\partial T}\log P(t,T)$. In the Lévy HJM framework we obtain that $f(t,T)\,, t \in [0,T]$, has the following form.

Lemma 2.3. *For all $T \geq 0$ the forward rate process $f(\cdot,T)$ exists under* Assumption I *and for all $t \in [0,T]$ has the form*

$$f(t,T) = f(0,T) + \int_0^t \theta'(\sigma(s,T))\cdot\sigma_2(s,T)\,ds - \int_0^t \sigma_2(s,T)\,dX_s, \tag{2.12}$$

where $f(0,T)$ is determined by the initial forward rate structure.

Proof. The proof follows starting with equation (2.1) and using Corollary 2.1, Theorem "Differentiationssatz" in Chapter 2.6 of [22], and the integration by parts formula. □

Note that the integral $\int_0^t \theta'(\sigma(s,T)) \cdot \sigma_2(s,T)\, ds$ is well-defined because of Assumption (A3).

We now introduce the continuously compounded spot rate for $[t,T]$ as

$$Y(t,T) := -\frac{\log P(t,T)}{T-t} . \tag{2.13}$$

From now on we will indicate the function $T \mapsto Y(t,T)$ as yield curve. We recall that the term "yield curve" is used differently in the literature. For example, in [2] it is a combination of simply compounded spot rates for maturities up to one year and annually compounded spot rates for maturities greater than one year. In this paper we will use (2.13) as the yield in t for $[t,T]$ which is equivalent to the definition in Section 2.4.4 of [10].

Now, we compute the yield in our Lévy setting.

Lemma 2.4. *Let* $0 \le t < T$. *Under* Assumption I *the* yield $Y(t,T)\,, t \in [0,T]$, *is:*

$$Y(t,T) = Y(0;t,T) + \int_0^t \frac{\theta(\sigma(s,T)) - \theta(\sigma(s,t))}{T-t}\, ds - \int_0^t \left(\frac{\sigma(s,T) - \sigma(s,t)}{T-t} \right) dX_s\,, \tag{2.14}$$

where

$$Y(0;t,T) := \frac{1}{T-t} \left(\int_t^T f(0,u)\, du \right) . \tag{2.15}$$

Proof. Since $P(t,T) = \exp\Big(- \int_t^T f(t,s)\ ds \Big)$ we obtain

$$\begin{aligned}
Y(t,T) &= \frac{1}{T-t} \left(\int_t^T f(t,u)\, du \right) \\
&\overset{(2.12)}{=} \int_t^T \frac{f(0,u)}{T-t}\, du + \int_t^T \int_0^t \frac{\theta'(\sigma(s,u)) \cdot \sigma_2(s,u)}{T-t}\, ds\, du - \int_t^T \int_0^t \frac{\sigma_2(s,u)}{T-t}\, dX_s\, du \\
&\underset{(2.15)}{\overset{(*)}{=}} Y(0;t,T) + \frac{1}{T-t} \left(\int_0^t \int_t^T \theta'(\sigma(s,u)) \cdot \sigma_2(s,u)\, du\, ds - \int_t^T \int_0^t \sigma_2(s,u)\, dX_s\, du \right) \\
&\overset{(2.11)}{=} Y(0;t,T) + \frac{1}{T-t} \left(\int_0^t \int_t^T \theta'(\sigma(s,u)) \cdot \sigma_2(s,u)\, du\, ds - \int_0^t \int_t^T \sigma_2(s,u)\, du\, dX_s \right) \\
&= Y(0;t,T) + \int_0^t \frac{\theta(\sigma(s,T)) - \theta(\sigma(s,t))}{T-t}\, ds - \int_0^t \left(\frac{\sigma(s,T) - \sigma(s,t)}{T-t} \right) dX_s\,.
\end{aligned}$$

At $(*)$ we used the standard Fubini theorem for deterministic functions. $\square$

3 Long-Term Yield in a Lévy Setting

3.1 Vanishing Long-Term Volatility

In this section we introduce the definition of the long-term yield and analyse its behaviour in the Lévy HJM framework outlined in Section 2.

Definition 3.1. *The* long-term yield $(\ell_t)_{t\geq 0}$ *is the process denoted by*

$$\ell_t = \lim_{T\to\infty} Y(t,T)\,, \tag{3.1}$$

where $(Y(t,T))_{t\geq 0}$ *is the yield process for* $T \geq 0$ *defined by equation* (2.14)*.*

Definition 3.2. *Let* $0 \leq t < T$*. If the bond price is defined as in* (2.1)*, the* long-term drift $(\mu_\infty(t))_{t\geq 0}$ *is the deterministic process denoted by*

$$\mu_\infty(t) = \lim_{T\to\infty} \frac{\theta(\sigma(t,T))}{T-t}, \tag{3.2}$$

where $\theta(\cdot)$ *is the logarithm of the moment-generating function of* X_1 *defined by equation* (2.5) *and* $(\sigma(t,T))_{t\geq 0}$ *is the deterministic volatility process defined by equations* (2.2) *and* (2.3)*.*
Furthermore, the long-term volatility $(\sigma_\infty(t))_{t\geq 0}$ *is the d-dimensional, deterministic process denoted by*

$$\sigma_\infty(t) = \lim_{T\to\infty} \frac{\sigma(t,T)}{T-t}, \tag{3.3}$$

where $(\sigma(t,T))_{t\geq 0}$ *is the deterministic volatility process defined by equations* (2.2) *and* (2.3)*.*

Here we are supposing that the limits (3.1), (3.2) and (3.3) are well-defined. By using the following results, we characterise the long-term yield as an integral of μ_∞ and σ_∞.

Proposition 3.3. *Let* $0 \leq t < T$*. The* long-term yield *at* 0 *is*

$$\lim_{T\to\infty} Y(0;t,T) = \lim_{T\to\infty} Y(0,T) = \ell_0 \quad a.s. \tag{3.4}$$

Proof. We have that

$$\begin{aligned}
\lim_{T\to\infty} Y(0;t,T) &\overset{(2.15)}{=} \lim_{T\to\infty} \frac{1}{T-t}\int_t^T f(0,u)\,du \overset{(*)}{=} \lim_{T\to\infty} \frac{1}{T-t}\int_0^T f(0,u)\,du \\
&= -\lim_{T\to\infty} \frac{1}{T}\log P(0,T) = \lim_{T\to\infty} Y(0,T) \overset{(3.1)}{=} \ell_0\,.
\end{aligned}$$

In $(*)$ we used the fact that by our assumptions on the bond market, there exists for all $t \geq 0$ a bond price $P(0,t) = \exp\left(-\int_0^t f(0,u)\,du\right)$, hence $\int_0^t f(0,u)\,du < \infty$ and $\lim_{T\to\infty} \frac{1}{T-t}\int_0^t f(0,u)\,du = 0$ for all $t \geq 0$. $\square$

Proposition 3.4. *Under the setting outlined in* Section 2 *and* Assumption I*, it holds for all* $t \geq 0$*:*

$$\lim_{T\to\infty} \int_0^t \left(\frac{\sigma(s,T) - \sigma(s,t)}{T-t} \right) dX_s = \int_0^t \sigma_\infty(s)\ dX_s,$$

where $(\sigma_\infty(s))_{s\geq 0}$ *is the long-term volatility process defined by equation* (3.3) *and the convergence is uniformly on compacts in probability (ucp).*

Proof. Let $t \leq T$ and $T \geq 1$. We note that

$$\lim_{T\to\infty} \int_0^t \frac{\sigma(s,T) - \sigma(s,t)}{T-t}\ dX_s = \lim_{T\to\infty} \int_0^t \frac{\sigma(s,T) - \sigma(s,t)}{T}\ dX_s\ . \tag{3.5}$$

Hence we study only the limit on the right-hand side.
First, we note that for all compact intervals $[a,b]$ with $a, b \geq 0$:

$$\frac{1}{T} \sup_{t\in[a,b]} \left| \int_0^t \sigma(s,t)\ dX_s \right| < \infty \text{ a.s.}$$

because X is a semimartingale and σ is simple predictable as a deterministic process (see Theorem 11 of Chapter II, Section 4 of [25]) and converges to 0 a.s., hence in probability. Therefore

$$\frac{1}{T} \int_0^t \sigma(s,t)\ dX_s \overset{T\to\infty}{\longrightarrow} 0 \text{ in ucp.} \tag{3.6}$$

We define $H^T := \left(H_s^T\right)_{s\geq 0}$ with

$$H_s^T := \frac{\sigma(s,T)}{T}\ .$$

Then for $T \to \infty:\ H_s^T \to \sigma_\infty(s)$ a.s. for all $s \geq 0$.
Due to (A4) there exists $\phi := (\phi(s))_{s\geq 0}$ with

$$\frac{\|\sigma(s,T)\|}{T} \leq \phi(s)\ \ ,\ 0 \leq s \leq T\,,$$

for $T \geq 1$.
Therefore, we get for all $0 \leq s \leq T$:

$$\left\|H_s^T\right\| \ \leq\ \phi(s)\,,$$

where ϕ is a càglàd deterministic function, hence ϕ is locally bounded. Then $\phi \in L(X)$ because of Theorem 15 of Chapter IV, Section 2 of [25].
Now, by the dominated convergence theorem for semimartingales (see Chapter IV, Section 2, Theorem 32 of [25]) it follows that

$$\int_0^t \frac{\sigma(s,T)}{T}\ dX_s \overset{T\to\infty}{\longrightarrow} \int_0^t \sigma_\infty(s)\ dX_s \tag{3.7}$$

in ucp. By Lemma 5.8 of [13] and (3.5), (3.6) and (3.7) we get:

$$\int_0^t \left(\frac{\sigma(s,T) - \sigma(s,t)}{T-t} \right) dX_s \overset{T\to\infty}{\longrightarrow} \int_0^t \sigma_\infty(s)\ dX_s \text{ in ucp.}$$

□

Proposition 3.5. *Under the setting outlined in* Section 2 *and* Assumption I *it is for all $t \geq 0$:*

$$\lim_{T\to\infty} \int_0^t \frac{\theta(\sigma(s,T)) - \theta(\sigma(s,t))}{T-t}\ ds = \int_0^t \mu_\infty(s)\ ds\ ,$$

where $(\mu_\infty(s))_{s\geq 0}$ is the long-term drift process defined by equation (3.2) *and the convergence is almost surely.*

Proof. Let $T > 0$. First, we note that for all $t \geq 0$:

$$\lim_{T\to\infty} \int_0^t \frac{\theta(\sigma(s,T)) - \theta(\sigma(s,t))}{T-t}\ ds = \lim_{T\to\infty} \int_0^t \frac{\theta(\sigma(s,T)) - \theta(\sigma(s,t))}{T}\ ds. \tag{3.8}$$

Next, notice that for all fixed $t \geq 0$:

$$\frac{1}{T} \int_0^t \theta(\sigma(s,t))\ ds \overset{T\to\infty}{\longrightarrow} 0 \ \text{ a.s.} \tag{3.9}$$

because $\theta(\sigma(s,t))$ is a càglàd function for all $s, t \geq 0$.
Since (A5) holds, by the dominated convergence theorem for deterministic functions it follows a.s. that for all $t \geq 0$:

$$\begin{aligned}
&\lim_{T\to\infty} \int_0^t \frac{\theta(\sigma(s,T)) - \theta(\sigma(s,t))}{T-t}\ ds \\
\overset{(3.8)}{=}\ &\lim_{T\to\infty} \frac{1}{T} \int_0^t \theta(\sigma(s,T))\ ds - \lim_{T\to\infty} \frac{1}{T} \int_0^t \theta(\sigma(s,t))\ ds \\
\overset{(3.9)}{=}\ &\lim_{T\to\infty} \frac{1}{T} \int_0^t \theta(\sigma(s,T))\ ds = \int_0^t \lim_{T\to\infty} \frac{\theta(\sigma(s,T))}{T}\ ds \\
\overset{(3.2)}{=}\ &\int_0^t \mu_\infty(s)\ ds.
\end{aligned}$$

□

By Lemma 2.4, Proposition 3.3, Proposition 3.4, and Proposition 3.5 the long-term yield can be written in the following way, whereas the convergence is in ucp:

$$\ell_t = \ell_0 + \int_0^t \mu_\infty(s)\ ds - \int_0^t \sigma_\infty(s)\, dX_s,\ t \geq 0\ . \tag{3.10}$$

By the Lévy-Khintchine formula (see Theorem 43 of Chapter I, Section 4 in [25]) we get a representation of the moment-generating function given by

$$M_{X_t}(u) := \mathbb{E}\big[e^{u\cdot X_t}\big] = e^{-t\psi(-iu)},\ u \in \mathbb{R}^d\,, \tag{3.11}$$

where

$$\psi(u) := \frac{1}{2}u\cdot Au - i\gamma\cdot u + \int_{\mathbb{R}^d}\big(1 - e^{iu\cdot x} + iu\cdot x1_{\{\|x\|\le 1\}}\big)\,\nu(dx)$$

with (A,ν,γ) being the characteristic triplet of X, i.e. A is the covariance matrix of the d-dimensional Brownian motion W, ν is the Lévy measure on $\mathbb{R}^d$, and γ is a vector on $\mathbb{R}^d$. Then, (A5) ensures that the moment-generating function of X_1 with parameter σ is well-defined, hence we get:

$$\begin{aligned} M_{X_1}(\sigma(t,T)) \quad &= \quad \exp\Bigg(\gamma\cdot\sigma(t,T) + \frac{1}{2}\sigma(t,T)\cdot A\sigma(t,T) \\ &\quad + \int_{\{\|x\|>1\}}\Big(e^{\sigma(t,T)\cdot x} - 1\Big)\ \nu(dx) \\ &\quad + \int_{\{\|x\|\le 1\}}\Big(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x\Big)\ \nu(dx)\Bigg). \end{aligned} \tag{3.12}$$

This leads to the following formula for the long-term drift:

$$\begin{aligned} \theta(\sigma(t,T)) \quad &\overset{(2.5)}{=} \quad \log\mathbb{E}[\exp(\sigma(t,T)\cdot X_1)] \\ &\overset{(3.11)}{=} \quad \log M_{X_1}(\sigma(t,T)) \\ &\overset{(3.12)}{=} \quad \gamma\cdot\sigma(t,T) + \frac{1}{2}\sigma(t,T)\cdot A\sigma(t,T) \\ &\quad + \int_{\{\|x\|>1\}}\Big(e^{\sigma(t,T)\cdot x} - 1\Big)\ \nu(dx) \\ &\quad + \int_{\{\|x\|\le 1\}}\Big(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x\Big)\ \nu(dx)\,. \end{aligned} \tag{3.13}$$

Therefore the long-term drift is for all $t \ge 0$:

$$\begin{aligned} \mu_\infty(t) \quad &\overset{(3.2)}{=} \quad \lim_{T\to\infty}\frac{\theta(\sigma(t,T))}{T-t} \\ &\overset{(3.13)}{=} \quad \lim_{T\to\infty}\frac{1}{T-t}\Bigg(\gamma\cdot\sigma\,(t,T) + \frac{1}{2}\sigma(t,T)\cdot A\sigma(t,T) \\ &\quad + \int_{\{\|x\|>1\}}\Big(e^{\sigma(t,T)\cdot x} - 1\Big)\ \nu(dx) \\ &\quad + \int_{\{\|x\|\le 1\}}\Big(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x\Big)\ \nu(dx)\Bigg)\,. \end{aligned} \tag{3.14}$$

Now, we want to know what happens to the long-term drift if the long-term volatility exists. For this purpose we examine different cases of Lévy processes regarding the jump sizes as well as the variation of the paths.

Theorem 3.6. *Let* $X := (X_t)_{t\ge 0}$ *be a Lévy process satisfying equation* (2.1) *and* $0 < \|\sigma_\infty(t)\| < \infty$*. Under* Assumption I*:*

(i) If X has only positive jumps or both positive and negative jumps, then $\mu_\infty(t) \equiv \infty$.

(ii) If X has only negative jumps and paths of infinite variation, then $\mu_\infty(t) \equiv \infty$.

(iii) If X has only negative jumps and paths of finite variation, then $\mu_\infty(t) \in \mathbb{R}$ for all $t \geq 0$.

Proof. Let $0 < \|\sigma_\infty(t)\| < \infty$. Then there exists at least one $i \in \{1, \dots, d\}$ such that

$$\sigma^i(t,T) \in \mathrm{O}(T-t)\,, \text{ i.e. } \lim_{T\to\infty} \sigma^i(t,T) = \infty. \tag{3.15}$$

Further, this means that

$$\lim_{T\to\infty} e^{\sigma(t,T)\cdot x} = +\infty \quad \text{if } x \in \mathbb{R}^d_+ \tag{3.16}$$

and

$$\lim_{T\to\infty} e^{\sigma(t,T)\cdot x} = 0 \quad \text{if } x \in \mathbb{R}^d_- \;. \tag{3.17}$$

The long-term drift is given by equation (3.14). Therefore we will investigate the different summands in (3.14). First, there is

$$\gamma \cdot \left(\lim_{T\to\infty} \frac{\sigma(t,T)}{T-t} \right) \overset{(3.3)}{=} \gamma \cdot \sigma_\infty(t) \in \mathbb{R}\;. \tag{3.18}$$

because of (3.15). Again by (3.15) and the fact that A is a positive definite matrix the second summand is

$$\frac{1}{2} \lim_{T\to\infty} \frac{\sigma(t,T)\cdot A\sigma(t,T)}{T-t} = \begin{cases} \infty & \text{if } A \neq 0, \\ 0 & \text{if } A = 0. \end{cases} \tag{3.19}$$

Now, we examine the third and fourth summand according to the different cases and define the following sets:

$$A^{t,T} := \left\{ x \in \mathbb{R}^d \mid \|x\| > 1, \sigma(t,T)\cdot x < 0 \right\},$$
$$B^{t,T} := \left\{ x \in \mathbb{R}^d \mid \|x\| > 1, \sigma(t,T)\cdot x \geq 0 \right\}.$$

Then, the third summand of the representation of the long-term drift (3.14) can be written as follows

$$\begin{aligned} & \lim_{T\to\infty} \frac{1}{T-t} \int_{\{\|x\|>1\}} \left(e^{\sigma(t,T)\cdot x} - 1 \right) \nu(dx) \\ = \; & \lim_{T\to\infty} \frac{1}{T-t} \left(\int_{A^{t,T}} \left(e^{\sigma(t,T)\cdot x} - 1 \right) \nu(dx) + \int_{B^{t,T}} \left(e^{\sigma(t,T)\cdot x} - 1 \right) \nu(dx) \right). \end{aligned}$$

For all $x \in A^{t,T}$

$$-1 \leq e^{\sigma(t,T)\cdot x} - 1 \leq 0$$

and hence for all $x \in A^{t,T}$

$$-\frac{\nu(A^{t,T})}{T-t} \leq \frac{1}{T-t} \int_{A^{t,T}} \left(e^{\sigma(t,T)\cdot x} - 1 \right) \nu(dx) \leq 0\,.$$

Since $A^{t,T} \subseteq \{\|x\| > 1\}$ and ν is a Lévy measure we have that

$$\nu(\{\|x\| > 1\}) < \infty,$$

hence

$$\lim_{T\to\infty} \frac{1}{T-t} \int\limits_{A^{t,T}} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu(dx) = 0\,. \tag{3.20}$$

Then, we can apply Fatou's Lemma for $\left\{x \in \mathbb{R}^d_+ \mid \|x\| > 1\right\}$ due to the fact that for $x \in \mathbb{R}^d_+$

$$\frac{1}{T-t}\left(e^{\sigma(t,T)\cdot x} - 1\right) \geq \frac{1}{T-t}\left(\sigma(t,T)\cdot x\right) \geq 0\,.$$

Since $\left\{x \in \mathbb{R}^d_+ \mid \|x\| > 1\right\} \subseteq B^{t,T}$, we get:

$$\begin{aligned}
& \lim_{T\to\infty} \frac{1}{T-t} \int_{B^{t,T}} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu(dx) \\
\geq \quad & \lim_{T\to\infty} \frac{1}{T-t} \int_{\left\{x\in\mathbb{R}^d_+ \mid \|x\|>1\right\}} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu(dx) \\
\overset{\text{Fatou}}{\geq} \quad & \int_{\left\{x\in\mathbb{R}^d_+ \mid \|x\|>1\right\}} \liminf_{T\to\infty} \frac{1}{T-t} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu(dx) \\
= \quad & \infty
\end{aligned} \tag{3.21}$$

because of (3.16) and the fact that $\nu\left(\left\{x \in \mathbb{R}^d_+ \mid \|x\| > 1\right\}\right) > 0$. For the last summand it is sufficient to note that:

$$\lim_{T\to\infty} \frac{1}{T-t} \int_{\{\|x\|\leq 1\}} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x\right) \nu(dx) \geq 0\,. \tag{3.22}$$

To (i): Let X be a Lévy process with only positive or both positive and negative jumps. Then we get $\mu_\infty(t) \equiv \infty$ by (3.18), (3.19), (3.20), (3.21), and (3.22).
To (ii): Let X be a Lévy process with only negative jumps and paths of infinite variation, i.e $A \neq 0$ or $\int_{\{\|x\|\leq 1\}} \|x\| \; \nu(dx) = \infty$ (see Proposition 3 of [6]). If $A \neq 0$, then $\mu_\infty(t) \equiv \infty$ because of (3.19) and since all other terms are non-negative. If $A = 0$, then $\int_{\{\|x\|\leq 1\}} \|x\|\, \nu(dx) = \infty$, since X has paths of infinite variation. In this case, we notice that inequality (3.22) still holds in the case of a Lévy process with only negative jumps and we can apply Fatou's Lemma for the fourth summand of equation (3.14) to get:

$$\begin{aligned}
& \lim_{T\to\infty} \int_{\left\{\|x\|\leq 1, x\in\mathbb{R}^d_-\right\}} \frac{1}{T-t} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x\right) \nu(dx) \\
\overset{\text{Fatou}}{\geq} \quad & \int_{\left\{\|x\|\leq 1, x\in\mathbb{R}^d_-\right\}} \liminf_{T\to\infty} \frac{1}{T-t} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x\right) \nu(dx) \\
= \quad & -\sigma_\infty(t) \cdot \int_{\left\{\|x\|\leq 1, x\in\mathbb{R}^d_-\right\}} x\, \nu(dx) = \infty
\end{aligned}$$

because of (3.17). Since all other terms are non-negative, this implies $\mu_\infty(t) \equiv \infty$.

To (iii): Let X be a Lévy process with only negative jumps and paths of finite variation, i.e. $A = 0$ and $\int_{\{\|x\|\leq 1\}} \|x\| \; \nu(dx) < \infty$ (see Proposition 3 of [6]). Due to the finite variation paths, the process X can be written as follows:

$$X_t = \gamma^* t + \int x \, N_t(\cdot, dx)$$

with $\gamma^* := \gamma - \int_{\{\|x\|\leq 1\}} x \, \nu(dx)$.
Using Corollary 3.1 of [3] and the fact that we consider only negative jumps, i.e. $\nu\big(\mathbb{R}^d \backslash \mathbb{R}^d_-\big) = 0$, we get:

$$\mu_\infty(t) = \gamma^* \cdot \sigma_\infty(t) + \lim_{T\to\infty} \frac{1}{T-t} \int_{\mathbb{R}^d_-} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu(dx)\,,$$

whereas $\gamma^* \cdot \sigma_\infty(t) \in \mathbb{R}$ because of (3.15) and the fact that X has paths of finite variation. We know that for all $x \in \mathbb{R}^d_-$ and $t, T \geq 0$

$$-1 \leq e^{\sigma(t,T)\cdot x} - 1 \leq 0 \tag{3.23}$$

and therefore

$$-\frac{\nu\big(\big\{x \in \mathbb{R}^d_- \mid \|x\| > 1\big\}\big)}{T-t} \leq \int_{\left\{x\in\mathbb{R}^d_- \,\mid\, \|x\|>1\right\}} \frac{e^{\sigma(t,T)\cdot x} - 1}{T-t} \nu(dx) \leq 0\,.$$

Since $\nu\big(\big\{x \in \mathbb{R}^d_- \mid \|x\| > 1\big\}\big) < \infty$, we get

$$\lim_{T\to\infty} \frac{1}{T-t} \int_{\left\{x\in\mathbb{R}^d_- \,\mid\, \|x\|>1\right\}} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu(dx) = 0\,. \tag{3.24}$$

Next, according to (3.23), we conclude that

$$\lim_{T\to\infty} \frac{1}{T-t} \int_{\left\{x\in\mathbb{R}^d_- \,\mid\, \|x\|\leq 1\right\}} \left(e^{\sigma(t,T)\cdot x} - 1\right) \nu(dx) \leq 0\,. \tag{3.25}$$

Since $\exp(y) \geq 1 + y$ for all $y \in \mathbb{R}^d$, it follows for all $t, T \geq 0$ that

$$\frac{\sigma\,(t,T)}{T-t} \cdot \int_{\left\{x\in\mathbb{R}^d_- \,\mid\, \|x\|\leq 1\right\}} x \, \nu(dx) \leq \int_{\left\{x\in\mathbb{R}^d_- \,\mid\, \|x\|\leq 1\right\}} \frac{e^{\sigma(t,T)\cdot x} - 1}{T-t} \, \nu(dx)\;. \tag{3.26}$$

Since X has paths of finite variation, we have

$$-\infty < \int_{\left\{x\in\mathbb{R}^d_- \,\mid\, \|x\|\leq 1\right\}} x \, \nu(dx) \leq 0\,,$$

and then because of (3.15) it follows for all $t \geq 0$

$$-\infty < \sigma_\infty(t) \cdot \int_{\left\{x\in\mathbb{R}^d_- \,\mid\, \|x\|\leq 1\right\}} x \, \nu(dx) \leq 0\,. \tag{3.27}$$

Putting together (3.25), (3.26), and (3.27), we conclude

$$\lim_{T\to\infty} \frac{1}{T-t} \int_{\left\{x\in\mathbb{R}^d_- \,\mid\, \|x\|\leq 1\right\}} \left(e^{\sigma(t,T)x} - 1\right) \nu(dx) \in \mathbb{R}_-\,. \tag{3.28}$$

In the end, by using (3.24) and (3.28) we get

$$\lim_{T\to\infty}\frac{1}{T-t}\int_{\mathbb{R}_-^d}\left(e^{\sigma(t,T)x}-1\right)\nu(dx)\in\mathbb{R}_-\,. \tag{3.29}$$

Considering the special case of a finite activity Lévy process, i.e. $\nu\left(\mathbb{R}^d\right)<\infty$ (see Proposition 2 of [6]), with finite variation , we even get

$$\lim_{T\to\infty}\frac{1}{T-t}\int_{\mathbb{R}_-^d}\left(e^{\sigma(t,T)x}-1\right)\nu(dx)=0 \tag{3.30}$$

because of $\nu\left(\mathbb{R}_-^d\right)<\infty$ and

$$-\frac{\nu\left(\mathbb{R}_-^d\right)}{T-t}\le\frac{1}{T-t}\int_{\mathbb{R}_-^d}\left(e^{\sigma(t,T)x}-1\right)\nu\left(dx\right)\le 0$$

due to the fact that $\exp(y)\ge 1+y$ for all $y\in\mathbb{R}^d$. Using (3.29), (3.30), and $\gamma^*\cdot\sigma_\infty\left(t\right)\in\mathbb{R}$ we conclude that $\mu_\infty\left(t\right)\in\mathbb{R}$ for all $t\ge 0$. □

To summarise, we have showed in this section that the long-term volatility process always vanishes except in the case of a finite variation Lévy process with only negative jumps.

3.2 Long-Term Yield as Non-Decreasing Process

A well-known fact of interest rate modelling is that if the market is arbitrage-free, the long-term yields, as well as the long-term forward rates, can never fall. This statement was first shown in 1996 by Dybvig et al. [5] and therefore is commonly referred to as Dybvig-Ingersoll-Ross Theorem (DIR-Theorem). Following [5], several researchers took a deeper look into this topic and further results were achieved. First, [24] clarified some aspects of the original proof, then a generalisation of the proof of the DIR-Theorem in an elegant mathematical way, where no additional assumptions to an arbitrage-free market have to be stated, was provided in [17]. Recently, further generalisations on the DIR-Theorem have been shown in [14] and [18]. Furthermore, in [18] the maximal discrepancy between $Y(s,T)$ and $Y(t,T)$ for a long-term, but finite, maturity T is discussed.

We remark that the result that the long-term yield is a non-decreasing process does not depend on the assumptions of a frictionless bond market where all bonds have final payoff $P(T,T)=1$. In reality, the two conditions are not always satisfied: zero-coupon bonds are not traded for all maturities, and $P(T,T)$ might be less than one if the issuer of the T-bond defaults. Yet, an alteration of these conditions would not have any influence on the validity of the DIR-Theorem.

We now see that a non-decreasing long-term yield does not contradict the realistic behaviour of the bond price process. It follows from the assumption of $(\ell_t)_{t\ge 0}$ being a non-decreasing process that for all $t\le s$ we have $\ell_s\ge\ell_t$. This implies that there exists $M>0$ such that for all $T>M$

$$Y(s,T)\ge Y(t,T)\,,$$

i.e.

$$P(s,T) \le P(t,T)^{\frac{T-s}{T-t}} .$$

For $s > t$, we have that $0 \le \frac{T-s}{T-t} < 1$ and this leads to the statement that there exists $M > 0$ such that for all $T > M$

$$P(s,T) \le P(t,T)^{a(t,s,T)}$$

with $a(t,s,T) \in [0,1)$ for all $0 \le t < s \le T$. This is economically realistic because the fluctuations of the bond price will decrease if the time to maturity decreases. Furthermore for a maturity that is far away from the time of observation, it is comprehensible that the bond price $P(s,T)$ is always lower or equal than $P(t,T)^{a(t,s,T)}$ for $s > t$ because any incident that could occur between the times t and s only has minor effects on long-term observations and can be captured in $a(t,s,T)$.

3.3 Asymptotic Behaviour of the Long-Term Rate

In Section 3.1 we have seen that if $0 < \|\sigma_\infty(t)\| < \infty$, then $\mu_\infty(t)$ is infinite, except in the case when X is a finite variation Lévy process with only negative jumps. Here, we investigate the asymptotic behaviour of the long-term rate μ_∞ if $\sigma_\infty(t) = 0$ for all $t \in \mathbb{R}_+$.

Proposition 3.7. *Let $\sigma(t,T) \in \mathrm{O}(1)$, i.e. $\sigma^i(t,T) \in \mathrm{O}(1)$ for all $i \in \{1,\dots,d\}$, for every $t \ge 0$. Under* Assumption I *we get*

$$\mu_\infty(t) = 0$$

and therefore $(\ell_t)_{t\ge 0}$ is constant.

Proof. Let $t \ge 0$ and $\sigma(t,T) \in \mathrm{O}(1)$, then $\sigma^i(t,T) \le c$ for T big enough for all $i \in \{1,\dots,d\}$. Therefore $\|\sigma_\infty(t)\| = 0$. In (3.14) we obtain

$$\gamma \cdot \sigma_\infty(t) = 0 \tag{3.31}$$

and

$$0 \le \frac{1}{2} \lim_{T\to\infty} \frac{\sigma(t,T) \cdot A\sigma(t,T)}{T-t} \le \underbrace{\frac{1}{2}\vec{c} \cdot A\vec{c}}_{<\infty} \lim_{T\to\infty} \frac{1}{T-t} = 0 \; . \tag{3.32}$$

Due to Assumption (A5) the moment-generating function of X_1 with parameter σ is well-defined, hence for all $t,T \in \mathbb{R}_+$:

$$\int_{\mathbb{R}^d} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T) \cdot x 1_{\{\|x\|\le 1\}} \right) \nu(dx) < \infty. \tag{3.33}$$

This inequality even holds for $t,T \in \bar{\mathbb{R}}_+$ because of $\sigma(t,T) \in \mathrm{O}(1)$. Then, we put together the third and fourth summand of equation (3.14) and get with (3.33):

$$\lim_{T\to\infty} \frac{1}{T-t} \int_{\mathbb{R}^d} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma\,(t,T) \cdot x\, 1_{\{\|x\|\le 1\}} \right) \nu(dx) = 0 \; . \tag{3.34}$$

By (3.31), (3.32) and (3.34) it follows $\mu_\infty(t) = 0$ for all $t \ge 0$.
This yields $\ell_t = \ell_0$ for all $t \ge 0$ following (3.1) and (3.4), i.e. $(\ell_t)_{t\ge 0}$ is constant. □

Proposition 3.8. *Let $t \geq 0$. If $\mu_\infty(t) < \infty$, then $\sigma(t,T) \in \mathrm{O}(\log(T-t))$ under* Assumption I. *In this case $\mu_\infty(t) \geq 0$.*

Proof. Let $t \geq 0$. First, we want to check which convergence behaviour of the volatility function is necessary to guarantee that $\mu_\infty(t) \neq \infty$ holds. In (3.14) we need to have

$$\gamma \cdot \sigma_\infty(t) < \infty,$$

i.e. $\sigma^i_\infty(t) = \lim\limits_{T\to\infty} \frac{\sigma^i(t,T)}{T-t} < \infty$ for all $i \in \{1,\ldots,d\}$ and therefore:

$$\forall\, i \in \{1,\ldots,d\} : \sigma^i(t,T) \in \mathrm{O}(T-t)\,. \tag{3.35}$$

Next, the second summand needs to be

$$\frac{1}{2}\lim_{T\to\infty}\frac{\sigma(t,T)\cdot A\sigma(t,T)}{T-t} < \infty,$$

hence

$$\forall\, i \in \{1,\ldots,d\} : \sigma^i(t,T) \in \mathrm{O}\left(\sqrt{T-t}\right). \tag{3.36}$$

Furthermore the following inequality has to be satisfied:

$$\lim_{T\to\infty}\frac{1}{T-t}\int_{\mathbb{R}^d}\left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x 1_{\{\|x\|\leq 1\}}\right)\,\nu(dx) < \infty. \tag{3.37}$$

For inequality (3.37) to hold, it is sufficient that

$$\lim_{T\to\infty}\frac{1}{T-t}\left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T)\cdot x 1_{\{\|x\|\leq 1\}}\right) < \infty \tag{3.38}$$

by Fatou's Lemma. Let $C^{t,T}$ and $D^{t,T}$ defined as

$$C^{t,T} := \left\{x \in \mathbb{R}^d \,|\, \sigma(t,T)\cdot x < 0\right\}$$

and

$$D^{t,T} := \left\{x \in \mathbb{R}^d \,|\, \sigma(t,T)\cdot x \geq 0\right\}\,,$$

i.e.

$$C^{t,T} \cup D^{t,T} = \mathbb{R}^d.$$

We see that for $x \in C^{t,T}$ inequality (3.38) is equal to

$$\lim_{T\to\infty}\frac{1}{T-t}\left(-\sigma(t,T)\cdot x 1_{\{\|x\|\leq 1\}}\right) = -\sigma_\infty(t)\cdot x 1_{\{\|x\|\leq 1\}} < \infty\,,$$

and it follows again condition (3.35).
Furthermore, for $x \in D^{t,T}$ inequality (3.38) leads to

$$\lim_{T\to\infty}\frac{1}{T-t}e^{\sigma(t,T)\cdot x} < \infty$$

or equivalently for all $i \in \{1,\ldots,d\}$ that $\lim\limits_{T\to\infty}\frac{1}{T-t}e^{\sigma^i(t,T)\,x^i} < \infty$. Hence we must have

$$\forall\, i \in \{1,\ldots,d\} : \sigma^i(t,T) \in \mathrm{O}(\log(T-t))\,. \tag{3.39}$$

By regarding (3.35), (3.36), and (3.39) it follows that the condition $\mu_\infty(t) < \infty$ implies $\sigma(t,T) \in \mathrm{O}(\log(T-t))$.
In particular if $\sigma(t,T) \in \mathrm{O}(\log(T-t))$, then $\mu_\infty(t) \geq 0$ since $\|\sigma_\infty(t)\| = 0$ and

$$\lim_{T\to\infty} \frac{1}{T-t} \int_{\mathbb{R}^d} \left(e^{\sigma(t,T)\cdot x} - 1 - \sigma(t,T) \cdot x 1_{\{\|x\|\leq 1\}} \right) \nu(dx) \geq 0 \,.$$

□

Corollary 3.9. *Let $t \geq 0$. If $0 < \mu_\infty(t) < \infty$, then under* Assumption I, *$\sigma(t,T)$ is asymptotically lower bounded and belongs to* $\mathrm{O}(\log(T-t))$.

Proof. This follows by Proposition 3.7 and Proposition 3.8. □

We summarise the results in the following table.

Long-term drift	Long-term volatility	Long-term yield	Volatility curve	Lévy process
$\mu_\infty(\cdot) = \infty$	$0 < \|\sigma_\infty(\cdot)\| < \infty$	infinite	$\sigma(t,T) \sim \mathrm{O}(T-t)$	Only positive jumps
$\mu_\infty(\cdot) = \infty$	$0 < \|\sigma_\infty(\cdot)\| < \infty$	infinite	$\sigma(t,T) \sim \mathrm{O}(T-t)$	Positive and negative jumps
$\mu_\infty(\cdot) \in \mathbb{R}$	$0 < \|\sigma_\infty(\cdot)\| < \infty$	non-decreasing	$\sigma(t,T) \sim \mathrm{O}(T-t)$	Finite variation and only negative jumps
$\mu_\infty(\cdot) = \infty$	$0 < \|\sigma_\infty(\cdot)\| < \infty$	infinite	$\sigma(t,T) \sim \mathrm{O}(T-t)$	Infinite variation and only negative jumps
$\mu_\infty(\cdot) = \infty$	$\|\sigma_\infty(\cdot)\| = 0$	infinite	$\sigma(t,T) \sim \mathrm{O}(\sqrt{T-t})$	Positive and negative jumps
$\mu_\infty(\cdot) = 0$	$\|\sigma_\infty(\cdot)\| = 0$	constant	$\sigma(t,T) \sim \mathrm{O}(1)$	Positive and negative jumps
$\mu_\infty(\cdot) \in \mathbb{R}_+$	$\|\sigma_\infty(\cdot)\| = 0$	non-decreasing	$\sigma(t,T) \sim \mathrm{O}(\log(T-t))$	Positive and negative jumps

4 Jump-diffusion Processes as Random Driver

4.1 Jump Distributions

In this section we want to find out some possible jump size distributions fitting to our model using a jump-diffusion Lévy process, such that the long-term yield is a non-decreasing process which is not equal to infinity. We consider a 1-dimensional Lévy process $X := (X_t)_{t\geq 0}$ of jump-diffusion type with the following form:

$$X_t = \gamma t + aW_t + Z_t \,, \tag{4.1}$$

$\gamma \in \mathbb{R}$, $a \in \mathbb{R}_+$. Here $W := (W_t)_{t\geq 0}$ is a 1-dimensional Brownian motion and $Z := (Z_t)_{t\geq 0}$ is a compound Poisson process given by

$$Z_t = \sum_{i=1}^{N_t} Y_i \,,$$

where $N := (N_t)_{t\geq 0}$ is a Poisson process and $Y_i, i \geq 1$, are i.i.d. random variables. Then the Lévy measure ν is finite (see Chapter I, Section 4, Example 2 of

[25]), i.e. $\nu(\mathbb{R}) < \infty$ and in particular $\nu(\mathbb{R}) = \lambda > 0$, where λ denotes the jump arrival intensity of the Poisson process N (see Chapter I, Section 3, Theorem 23 of [25]). To define the parametric model completely, the distribution of the jump sizes has to be specified and we have to find out if (A5) of Assumption I can hold for the respective jump size distribution. All other statements of Assumption I are only dependent on the volatility function.

To fulfil (A5), according to Proposition 3.4 of [3] equation (3.2.13) of [28] and (4.1), we have to show that there exists a function $\psi \in L^1(\mathbb{R}_+)$ such that

$$\frac{m\,\sigma(t,T)\,|\gamma|}{T} + \frac{m^2\,\sigma(t,T)^2\,a^2}{2T} + \frac{\lambda}{T}\left|\int_{\mathbb{R}}\left(e^{m\,\sigma(t,T)\,x} - 1\right) f(dx)\right| \leq \psi(t) \tag{4.2}$$

for all $T > 0$, $0 \leq t \leq T$ and $m := 1 + \epsilon$ with $0 < \epsilon < 1$. The density function of the jump size distribution is denoted by f.

Remark. *Note that not every distribution is eligible as jump distribution. For example the normal distribution, the gamma distribution and the continuous uniform distribution are possible jump distributions because with the right choice of volatility function they can fulfil inequality* (4.2)*. At the same time, the exponential distribution, the Laplace distribution and the generalised hyperbolic distribution are examples of distributions that cannot fulfil inequality* (4.2) *unless the chosen volatility function is bounded.*

4.2 Example

Let us consider a compound Poisson process where the jumps are Gaussian distributed with $Y \sim \mathcal{N}(0, \eta^2)$ and $\eta > 0$. In this case the probability of a positive jump is equal to the probability of a negative jump, i.e. $\mathbb{P}(Y \geq 0) = \mathbb{P}(Y \leq 0) = \frac{1}{2}$. The density of Y is

$$f(x) = \frac{1}{\sqrt{2\pi}\eta}\exp\left(-\frac{x^2}{2\eta^2}\right). \tag{4.3}$$

We know from Proposition 3.4 of [3] that the characteristic function of the compound Poisson process Z is

$$\mathbb{E}\left[e^{iuZ_t}\right] = e^{t\phi(u)}, u \in \mathbb{R},$$

with

$$\phi(u) = \int_{\mathbb{R}}\left(e^{iux} - 1\right)\nu(dx), u \in \mathbb{R}.$$

Therefore the long-term drift (3.14) for $t \geq 0$ has the following form:

$$\mu_\infty(t) = \lim_{T\to\infty}\left(\gamma\,\frac{\sigma(t,T)}{T-t} + \frac{a^2\sigma(t,T)^2}{2\,(T-t)} + \int_{\mathbb{R}}\frac{e^{\sigma(t,T)x} - 1}{T-t}\,\nu(dx)\right).$$

Let us take a closer look at the third summand:

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T-t}\int_{\mathbb{R}}\left(e^{\sigma(t,T)x}-1\right)\nu(dx) &= \lim_{T\to\infty}\frac{\lambda}{\sqrt{2\pi}\eta\,(T-t)}\int_{\mathbb{R}}\left(e^{\sigma(t,T)x}-1\right)e^{-\frac{x^2}{2\eta^2}}dx \\ &= \lambda\lim_{T\to\infty}\frac{e^{\frac{\sigma(t,T)^2\eta^2}{2}}}{T-t}.\end{aligned} \tag{4.4}$$

We consider the following volatility function:

$$\sigma(t,T) = \begin{cases} c\sqrt{\log(T-t+1)} & \text{if } t \le T-\delta \\ 0 & \text{if } t > T-\delta \end{cases} \tag{4.5}$$

with $c \in \mathbb{R}_+$ and $\delta > 0$ small enough. Then $\sigma(t,T) \in \mathrm{O}(\log(T-t))$ and $\sigma(t,T)$ is asymptotically lower bounded according to Corollary 3.9, for $t \le T - \delta$. We get

$$\sigma_\infty(t) \overset{(3.3)}{=} c \lim_{T\to\infty} \frac{\sqrt{\log(T-t+1)}}{T-t} = 0,\ t \in \mathbb{R}_+ .$$

Hence

$$\mu_\infty(t) \overset{(4.4)}{=} \lambda \lim_{T\to\infty} \frac{(T-t+1)^{1/2\, c^2\eta^2}}{T-t} = \begin{cases} \infty & \text{if } 1/2\, c^2\delta^2 > 1, \\ \lambda & \text{if } 1/2\, c^2\eta^2 = 1, \\ 0 & \text{if } 1/2\, c^2\eta^2 < 1. \end{cases} \tag{4.6}$$

In the case of $\frac{1}{2}\, c^2\eta^2 = 1$ the long-term yield has the following representation:

$$\ell_t \overset{(3.10)}{=} \ell_0 + \lambda t\,, \lambda > 0.$$

Remark. *Result (4.6) shows that the condition*

$$\sigma(t,T) \in \mathrm{O}(\log(T-t))$$

is necessary, but it may be not sufficient to have $\mu_\infty(t) < \infty$ for all $t \ge 0$.

We now set $\frac{1}{2}\, c^2\delta^2 = 1$ and check that the chosen volatility function (4.5) fulfils all prerequisites due to Assumption I.
Assumptions (A1) and (A2) are obviously fulfilled.
Next, σ is a continuous function in both variables, therefore càglàd. Further, the partial derivatives

$$\sigma_1(t,T) = \begin{cases} -\frac{c}{2\sqrt{\log(T-t+1)}(T-t+1)} & \text{if } t \le T-\delta\,, \\ 0 & \text{if } t > T-\delta\,, \end{cases}$$

and

$$\sigma_2(t,T) = \begin{cases} \frac{c}{2\sqrt{\log(T-t+1)}(T-t+1)} & \text{if } t \le T-\delta\,, \\ 0 & \text{if } t > T-\delta\,, \end{cases}$$

with $\delta > 0$ very small, are càglàd in both variables, hence (A3) holds.
To show (A4) we define

$$G(t,T) := \frac{\sigma(t,T)}{T}\,.$$

Then

$$\frac{\partial}{\partial t} G(t,T) = -\frac{c}{2T\sqrt{\log(T-t+1)}\,(T-t+1)} \quad \text{for } t \in [0, T-\delta]\,,$$

i.e. $\frac{\partial}{\partial t}G(t,T) < 0$, for all $0 \le t \le T-\delta$ and $T > 0$, hence G is a decreasing function in t with maximum for $t = 0$. One can easily see that $\lim_{T\to\infty} G(0,T) = 0$ and $G(0,T)$ is bounded for $T \ge 1$. Therefore for all $T \ge 1$ there exists $\phi \in L^1(\mathbb{R}_+)$ such that $G(t,T) \le \phi(t)$ for all $0 \le t \le T$.

It remains to verify (A5), i.e. we have to find a function $\psi \in L^1(\mathbb{R}_+)$ such that (4.2) holds. With density function (4.3), $0 < \epsilon < 1$, $m = 1+\epsilon$, $0 \le t \le T-\delta$, and $T > 0$, we get:

$$\begin{aligned}
&\frac{m\,\sigma(t,T)\,|\gamma|}{T} + \frac{m^2\,\sigma(t,T)^2\,a^2}{2T} + \frac{\lambda}{T}\left|\int_{\mathbb{R}}\left(e^{m\sigma(t,T)x}-1\right)f(dx)\right| \\
&= \frac{m\,\sigma(t,T)\,|\gamma|}{T} + \frac{m^2\,\sigma(t,T)^2\,a^2}{2T} + \frac{\lambda}{T}\left|\exp\left(\frac{1}{2}\,\sigma(t,T)^2\,\eta^2\right)-1\right| \\
&\le \frac{m\,\sigma(t,T)\,|\gamma|}{T} + \frac{m^2\,\sigma(t,T)^2\,a^2}{2T} + \frac{\lambda}{T}\exp\left(\frac{1}{2}\,\sigma(t,T)^2\,\eta^2\right) \\
&= \frac{m\,c\,|\gamma|\,\sqrt{\log(T-t+1)}}{T} + \frac{m^2\,c^2\,a^2\,\log(T-t+1)}{2T} + \frac{\lambda\,(T-t+1)}{T} =: H(t,T)\;.
\end{aligned}$$

Then

$$\frac{\partial}{\partial t}H(t,T) = -\frac{mc\,|\gamma|}{2T\sqrt{\log(T-t+1)}\,(T-t+1)} - \frac{m^2c^2a^2}{2T\,(T-t+1)} - \frac{\lambda}{T}\,,$$

i.e. $\frac{\partial}{\partial t}H(t,T) < 0$ for all $0 \le t \le T-\delta$ and $T > 0$, hence H is a decreasing function in t with maximum for $t = 0$. For $T \ge 1$ it is easy to find a dominating function $\psi \in L^1(\mathbb{R}_+)$ of H since

$$\begin{aligned}
&\frac{m\,c\,|\gamma|\,\sqrt{\log(T-t+1)}}{T} + \frac{m^2\,c^2\,a^2\,\log(T-t+1)}{2T} + \frac{\lambda\,(T-t+1)}{T} \\
&\le 3\left(m\,c\,|\gamma| + \frac{1}{2}\,m^2\,c^2\,a^2 + \lambda\right)\,.
\end{aligned} \tag{4.7}$$

Furthermore, we can find $\psi \in L^1(\mathbb{R}_+)$ satisfying (A5) for all $T > 0$, by considering two intervals, for instance $[0,1)$ and $[1,\infty)$. For $t \in [1,\infty)$ it follows $T \ge 1$ and we just have seen in (4.7) that a dominating $h \in L^1(\mathbb{R}_+)$ exists. If $t \in [0,1)$ we can show with lengthy computations that

$$\begin{aligned}
&\frac{m\,c\,|\gamma|\,\sqrt{\log(T-t+1)}}{T} + \frac{m^2\,c^2\,a^2\,\log(T-t+1)}{2T} + \frac{\lambda\,(T-t+1)}{T} \\
&\le g(t)\left(m\,c\,|\gamma| + \frac{1}{2}\,m^2\,c^2\,a^2 + \lambda\right)\,,
\end{aligned}$$

where for all $0 \le t < 1$

$$g(t) := \frac{6}{(1+t+\delta)\log(1+t+\delta)}$$

with $\delta > 0$ very small. Further, $g \in L^1([0,1])$ because

$$\int_0^1 \frac{1}{(1+t+\delta)\log(1+t+\delta)}\,dt = \log(\log(2+\delta)) - \log(\log(1+\delta)) < \infty\,.$$

In the end we have $\psi \in L^1(\mathbb{R}_+)$ satisfying (2.4) with

$$\psi(t) := k\left(g(t)\,1_{[0,1)}\,(t) + G(t)\,1_{[1,\infty)(t)}\right)\,,$$

where

$$k := m\,c\,|\gamma| + \frac{1}{2}\,m^2\,c^2\,a^2 + \lambda\,.$$

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