Local Risk-Minimization under the Benchmark Approach

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Abstract

We study the pricing and hedging of derivatives in incomplete financial markets by considering the local risk-minimization method in the context of the benchmark approach, which will be called benchmarked local risk-minimization. Given a benchmarked contingent claim we identify the minimal possible price for its benchmarked hedgeable part, and the benchmarked profit and loss with zero mean and minimal variance. Examples demonstrate that the proposed benchmarked local risk-minimization allows to handle under extremely weak assumptions a much richer modeling world than the classical methodology.

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1 Introduction

The valuation and hedging of derivatives in incomplete financial markets is a frequently studied problem in mathematical finance. The goal of this paper is to discuss the concept of local risk-minimization under the *benchmark approach* (see e.g. [8], [9], [15], [21] and [22]), a general modeling framework that only requires the existence of a benchmark, the numéraire portfolio. According to this approach, even under the absence of an equivalent local martingale measure (in short ELMM), contingent claims can be consistently evaluated by means of the so-called *real world pricing formula*, which generalizes standard valuation formulas, where the discounting factor is the numéraire portfolio and the pricing measure is the

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physical probability measure $\mathbb{P}$. Local risk-minimization under the benchmark approach has been also studied in [7] in the case of jump-diffusion markets. In this paper our approach is more general, since we do not assume any specific market model for the primitive assets. Our aim is to investigate at a general level the connection between (local) risk-minimization and real world pricing as well as to show some innovative features of *benchmarked local risk-minimization*, (i.e. local risk-minimization under the benchmark approach), as explained in the sequel.

First of all, we study the local risk-minimization method in the case when the benchmarked asset prices are $\mathbb{P}$-local martingales, which will correspond to *benchmarked risk-minimization*. This includes continuous market models (see Section 3.1) and a wide class of jump-diffusion models (see for example [22], Chapter 14, pages 513-549). This property implies several advantages since in market models, where the discounted asset prices are given by $\mathbb{P}$-local martingales, the local risk-minimization method coincides with risk-minimization, as introduced originally in [11]. In the local risk-minimization approach, the risk-minimizing strategy is often calculated by switching to a particular martingale measure $\widehat{\mathbb{P}}$ (the *minimal martingale measure*) and computing the Galtchouck-Kunita-Watanabe (in short GKW) decomposition of a benchmarked contingent claim $\hat{H}$ under $\widehat{\mathbb{P}}$. However, this method has two main disadvantages:

(i) the minimal measure $\widehat{\mathbb{P}}$ may not exist, as it is often the case in the presence of jumps affecting the asset price dynamics;

(ii) if $\widehat{\mathbb{P}}$ exists, the GKW decomposition of $\hat{H}$ under $\widehat{\mathbb{P}}$ must satisfy some particular integrability conditions under the real world probability measure $\mathbb{P}$ to give the Föllmer-Schweizer decomposition of $\hat{H}$.

On the contrary, the risk-minimization approach that we discuss in this paper for the case of benchmarked market models, does not face the same technical difficulties as the local risk-minimization one. It formalizes in a straightforward mathematical way the economic intuition of risk and delivers always an optimal strategy for a given benchmarked contingent claim $\hat{H} \in L^2(\mathcal{F}_T, \mathbb{P})$[1], obtained by computing the GKW decomposition of $\hat{H}$ under $\mathbb{P}$.

Furthermore, in this setting we establish a fundamental relation between real world pricing and benchmarked risk-minimization. In market models, where the asset prices are given by $\mathbb{P}$-local martingales, by Theorem 3.6 we will obtain the result that the benchmarked portfolio's value of the risk-minimizing strategy for $\hat{H} \in L^2(\mathcal{F}_T, \mathbb{P})$ coincides with the real world pricing formula for $\hat{H}$. The benchmarked contingent claim $\hat{H}$ can be written as

$$\hat{H} = \hat{H}_0 + \int_0^T \xi_u^{\hat{H}} \mathrm{d}\hat{S}_u + L_T^{\hat{H}} \quad \mathbb{P} - \text{a.s.}, \tag{1.1}$$

[1]The space $L^2(\mathcal{F}_T, \mathbb{P})$ denotes the set of all $\mathcal{F}_T$-measurable random variables H such that $\mathbb{E}\left[H^2\right] = \int H^2 \mathrm{d}\mathbb{P} < \infty$.

where $L^{\hat{H}}$ is a square-integrable $\mathbb{P}$-martingale with $L_0^{\hat{H}} = 0$ strongly orthogonal[2] to $\hat{S}$. Decomposition (1.1) allows us to decompose every square-integrable benchmarked contingent claim as the sum of its *hedgeable part* $\hat{H}^h$ and its *unhedgable part* $\hat{H}^u$ such that we can write

$$\hat{H} = \hat{H}^h + \hat{H}^u,$$

where

$$\hat{H}^h := \hat{H}_0 + \int_0^T \xi_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u$$

and

$$\hat{H}^u := L_T^{\hat{H}}.$$

Here the notation $\int \xi^{\hat{H}} \cdot \mathrm{d}\hat{S}$ characterizes the integral of the vector process $\xi^{\hat{H}}$ with respect to the vector process $\hat{S}$ (see e.g. [20]). Note that the benchmarked hedgeable part $\hat{H}^h$ can be replicated perfectly, i.e.

$$\hat{U}_{H^h}(t) = \mathbb{E}\left[\hat{H}^h \middle| \mathcal{F}_t\right] = \hat{H}_0 + \int_0^t \xi_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u\,,$$

and $\xi^{\hat{H}}$ yields the fair strategy for the self-financing replication of the hedgeable part of $\hat{H}$. The remaining benchmarked unhedgeable part can be diversified and will be covered through the profit and loss process $L^{\hat{H}}$. The connection between risk-minimization and real world pricing is then an important insight, which gives a clear reasoning for the pricing and hedging of contingent claims via real world pricing also in incomplete markets.

A natural question concerns then the invariance of the risk-minimizing strategy under a change of numéraire. By [3] this property always holds in the case of continuous assets prices. Here we show that this result is also true when only the orthogonal martingale structure is generated by continuous $\mathbb{P}$-(local) martingales.

Then we also study the case when the benchmarked processes are $\mathbb{P}$-supermartingales. In the general case, when benchmarked asset prices are given by strict $\mathbb{P}$-supermartingales, we are able to generalize a result of [10], where we show how to do local risk-minimization under incomplete information without assuming continuity of paths for the underlying assets. The proof we provide holds when the discounted asset prices are special semimartingales in $\mathcal{S}^2(\mathbb{P})$[3], hence in particular

[2]Two $\mathbb{P}$-local martingales M and N are called *strongly orthogonal* if their product MN is a $\mathbb{P}$-local martingale.

[3]Given the Doob-Meyer decomposition

$$X_t = X_0 + M_t + V_t, \quad t \in [0,T],$$

of a $\mathbb{P}$-semimartingale X into a $\mathbb{P}$-local martingale $M = \{M_t,\ t \in [0,T]\}$ and an $\mathbb{F}$-predictable process $V = \{V_t,\ t \in [0,T]\}$ of finite variation, we say that $X \in \mathcal{S}^2(\mathbb{P})$ if the following integrability condition is satisfied

$$\mathbb{E}\left[X_0^2 + [X]_T + |V|_T^2\right] < \infty.$$

Here $|V| = \{|V|_t,\ t \in [0,T]\}$ denotes the total variation of the process V.

for all benchmarked underlying assets in $\mathcal{S}^2(\mathbb{P})$ by the Doob decomposition.

Finally, we provide some examples to illustrate how to compute the Föllmer-Schweizer decomposition in the minimal market model, where there exists no ELMM, but the primitive assets are still local $\mathbb{P}$-martingales if benchmarked.

The local risk-minimization method under the benchmark approach has acquired new importance for pricing and hedging in hybrid markets and insurance markets (see [1] and [4]). Since hybrid markets are intrinsically incomplete, perfect replication of contingent claims is not always possible and one has to apply one of the several methods for pricing and hedging in incomplete markets. Local risk-minimization appears to be one of the most suitable methods when the market is affected by orthogonal sources of randomness, such as the ones represented by mortality risk and catastrophic risks. The results of this paper provide the new simplified framework for applying of benchmarked local risk-minimization.

2 Financial Market

To describe a financial market in continuous time, we introduce a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a time horizon $T \in (0, \infty)$ and a filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \le t \le T}$ that is assumed to satisfy $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \in [0, T]$, as well as the usual hypotheses of completeness and right-continuity and saturation by all $\mathbb{P}$-null sets of $\mathcal{F}$.
In our market model we can find d adapted, nonnegative primary security account processes represented by (càdlàg) $\mathbb{P}$-semimartingales $S^j = \{S_t^j,\ t \in [0, T]\}$, $j \in \{1, 2, \ldots, d\}$, $d \in \{1, 2, \ldots\}$. Additionally, the 0-th security account S_t^0 denotes the value of the adapted strictly positive savings account at time $t \in [0, T]$. The j-th *primary security account* holds units of the j-th primary security plus its accumulated dividends or interest payments, $j \in \{1, 2, \ldots, d\}$. In this setting, market participants can trade in order to reallocate their wealth.

Definition 2.1. *We call a* strategy *a $(d+1)$-dimensional process $\delta = \{\delta_t = (\delta_t^0, \delta_t^1, \ldots, \delta_t^d)^\top,\ t \in [0, T]\}$, where for each $j \in \{0, 1, \ldots, d\}$, the process $\delta^j = \{\delta_t^j,\ t \in [0, T]\}$ is $\mathbb{F}$-predictable and integrable with respect to $S^j = \{S_t^j,\ t \in [0, T]\}$.*

Here δ_t^j, $j \in \{0, 1, \ldots, d\}$, denotes the number of units of the j-th security account that are held at time $t \ge 0$ in the corresponding *portfolio* $S^\delta = \{S_t^\delta,\ t \in [0, T]\}$. Following [3], we define the value S^δ of this portfolio as given by a càdlàg optional process such that

$$S_{t-}^\delta := \delta_t \cdot S_t = \sum_{j=0}^{d} \delta_t^j S_t^j, \quad t \in [0, T],$$

where $S = \{S_t = (S_t^0, S_t^1, \ldots, S_t^d)^\top,\ t \in [0, T]\}$. A strategy δ and the corresponding portfolio $\hat{S}^\delta$ are said to be *self-financing* if

$$S_t^\delta = S_0^\delta + \int_0^t \delta_u \cdot \mathrm{d}S_u, \quad t \in [0, T], \tag{2.1}$$

where $\delta = \{\delta_t = (\delta_t^0, \delta_t^1, \dots, \delta_t^d)^\top,\ t \in [0,T]\}$. Note that the stochastic integral of the vector process δ with respect to S is well-defined because of our assumptions on δ. Furthermore, $a^\top$ denotes the transpose of a. In general, we do not request strategies to be self-financing. Denote by $\mathcal{V}_x^+$, $(\mathcal{V}_x)$, the set of all strictly positive, (nonnegative), finite, self-financing portfolios, with initial capital $x > 0$, $(x \geq 0)$.

Definition 2.2. *A portfolio $S^{\delta_*} \in \mathcal{V}_1^+$ is called a* numéraire portfolio*, if any nonnegative portfolio $S^\delta \in \mathcal{V}_1^+$, when denominated in units of S^{δ_*}, forms a $\mathbb{P}$-supermartingale, that is,*

$$\frac{S_t^\delta}{S_t^{\delta_*}} \geq \mathbb{E}\left[\frac{S_s^\delta}{S_s^{\delta_*}}\middle|\mathcal{F}_t\right], \tag{2.2}$$

for all $0 \leq t \leq s \leq T$.

To establish the modeling framework, we make the following (extremely weak) key assumption, which is satisfied for almost all models of practical interest, see e.g. [22] and [15].

Assumption 2.3. *There exists a numéraire portfolio $S^{\delta_*} \in \mathcal{V}_1^+$.*

From now on, let us choose the numéraire portfolio as *benchmark*. We call any security, when expressed in units of the numéraire portfolio, a benchmarked security and refer to this procedure as *benchmarking*. The benchmarked value of a portfolio S^δ is of particular interest and is given by the ratio

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}}$$

for all $t \in [0,T]$. If a benchmarked price process is a $\mathbb{P}$-martingale, then we call it *fair*. In this case we would have equality in relation (2.2) of Definition 2.2.
The benchmark approach developed in [15], [18] and [22] uses the numéraire portfolio for derivative pricing without using equivalent martingale measures. In portfolio optimization the numéraire portfolio, which is also the growth optimal portfolio, is in many other ways the best performing self-financing portfolio, see [17] and [19]. As shown in [22], jump-diffusion and Itô process driven market models have a numéraire portfolio under very general assumptions, where benchmarked nonnegative portfolios turn out to be $\mathbb{P}$-local martingales and, thus, $\mathbb{P}$-supermartingales. In [15] the question on the existence of a numéraire portfolio in a general semimartingale market is studied.
In order to guarantee the economic viability of our framework, we check whether obvious arbitrage opportunities are excluded. A strong form of arbitrage would arise when a market participant could generate strictly positive wealth from zero initial capital via his or her nonnegative portfolio of total wealth.

Definition 2.4. *A benchmarked nonnegative self-financing portfolio $\hat{S}^\delta$ is a* strong arbitrage *if it starts with zero initial capital, that is $\hat{S}_0^\delta = 0$, and generates some strictly positive wealth with strictly positive probability at a later time $t \in (0,T]$, that is $\mathbb{P}(\hat{S}_t^\delta > 0) > 0$.*

Thanks to the supermartingale property (2.2), the existence of the numéraire portfolio guarantees that strong arbitrage is automatically excluded in the given general setting, see [22]. However, some weaker forms of arbitrage may still exist. These would require to allow for negative portfolios of total wealth of those market participants who fully focus on exploiting such weaker forms of arbitrage, which is not possible in reality due to bankruptcy laws. This emphasizes the fact that an economically motivated notion of arbitrage should rely on nonnegative portfolios.
Within this paper, we consider a discounted European style contingent claim. Such a benchmarked claim $\hat{H}$ (expressed in units of the benchmark) is given by the $\mathcal{F}_T$-measurable, nonnegative random payoff $\hat{H}$ that is delivered at time T. We will here always assume that a benchmarked contingent claim $\hat{H}$ belongs to $L^2(\mathcal{F}_T, \mathbb{P})$.
Given a benchmarked contingent claim $\hat{H}$, there are at least two tasks that a potential seller of $\hat{H}$ may want to accomplish: the *pricing* by assigning a value to $\hat{H}$ at times $t < T$; and the *hedging* by covering as much as possible against potential losses arising from the uncertainty of $\hat{H}$. If the market is complete, then there exists a self-financing strategy δ whose terminal value $\hat{S}^{\delta}_T$ equals $\hat{H}$ with probability one, see [22]. More precisely, the *real world pricing formula*

$$\hat{S}^{\delta_H}_t = \mathbb{E}\left[\hat{H}\middle|\mathcal{F}_t\right] \tag{2.3}$$

provides the description for the benchmarked fair portfolio at time $t \in [0, T]$, which is the least expensive $\mathbb{P}$-supermartingale that replicates the benchmarked payoff $\hat{H}$ if it admits a replicating self-financing strategy δ^H with $\hat{S}^{\delta^H}_T = \hat{H}$. Here $\hat{S}^{\delta_H}$ forms by definition a $\mathbb{P}$-martingale. The benchmark approach allows other self-financing hedge portfolios to exist for $\hat{H}$, see [22]. However, these nonnegative portfolios are not $\mathbb{P}$-martingales and, as $\mathbb{P}$-supermartingales, more expensive than the $\mathbb{P}$-martingale $\hat{S}^{\delta_H}$ given in (2.3), see [22].
Completeness is a rather delicate property that does not cover a large class of realistic market models. Here we choose the (local) risk-minimization approach (see e.g. [10], [11] and [24]) to price non-hedgeable contingent claims.
In this paper, we first investigate the case of benchmarked securities that represent $\mathbb{P}$-local martingales and study risk-minimization as originally introduced in [11]. We will see that this covers many cases in the context of the benchmark approach including all continuous financial market models, a wide range of jump-diffusion driven market models and cases like the minimal market model that do not have an equivalent risk neutral probability measure. Then we will study the general case when benchmarked securities are $\mathbb{P}$-supermartingales that are not necessarily $\mathbb{P}$-local martingales. As indicated earlier, we will refer to local risk-minimization under the benchmark approach as *benchmarked local risk-minimization*.

3 Local Risk-Minimization with Benchmarked Assets

Our aim is to investigate a concept of local risk-minimization similar to the one in [13] and [24], which used the savings account as reference unit. Here we use the numéraire portfolio as discounting factor and benchmark. The main feature of a local risk-minimization concept is the fact that one insists on the replication requirement $\hat{S}^\delta_T = \hat{H}$. If $\hat{H}$ is not hedgeable, then this forces one to work with strategies that are not self-financing and the aim becomes to minimize the resulting intrinsic risk or cost under a suitable criterion. As we will see, rather natural and tractable are quadratic hedging criteria, where we refer to [24] and [13] for extensive surveys.

Important is the fact that there are realistic situations that we will cover, which would be excluded because a minimal martingale measure may not exist for the respective models. For example, in the case of local risk-minimization of financial derivatives based on insurance products, the minimal martingale measure may often not exist because of the presence of jumps in the underlying.

We recall that under Assumption 2.3, the benchmarked value of any nonnegative, self-financing portfolio forms a $\mathbb{P}$-supermartingale, see (2.2). In particular, the vector of the $d+1$ benchmarked primary security accounts $\hat{S} = (\hat{S}^0, \hat{S}^1, \ldots, \hat{S}^d)^\top$ forms with each of its components a nonnegative $\mathbb{P}$-*supermartingale*. By Theorem VII.12 of [6], we know that the vector process $\hat{S}$ has a unique decomposition of the form

$$\hat{S}_t = \hat{S}_0 + M_t + V_t, \quad t \in [0,T], \tag{3.1}$$

where M is a vector $\mathbb{P}$-local martingale and V is a right-continuous $\mathbb{F}$-predictable finite variation vector process with $M_0 = V_0 = \mathbf{0}$, with $\mathbf{0}$ denoting the $(d+1)$-dimensional null vector. This expresses the fact that every right-continuous $\mathbb{P}$-supermartingale is a special $\mathbb{P}$-semimartingale.

3.1 Benchmarked Local Martingales

We now discuss the case when benchmarked securities are $\mathbb{P}$-local martingales. Let us assume that the vector of the $d+1$ discounted primary security accounts $\frac{S}{S^0} =: X = \{X_t = (1, X^1_t, \ldots, X^d_t)^\top,\ t \in [0,T]\}$ is a *continuous* $\mathbb{P}$-semimartingale with canonical decomposition $X = X_0 + M^X + A^X$. The processes $M^X = \{M^X_t : t \in [0,T]\}$ and $A^X = \{A^X_t : t \in [0,T]\}$ are both $\mathbb{R}^{d+1}$-valued, continuous and null at 0. Moreover, M^X is a vector $\mathbb{P}$-local martingale and A^X is an adapted, finite variation vector process. The bracket process $\langle M^X \rangle$ of M^X is the adapted, continuous $(d+1) \times (d+1)$-matrix-valued process with components $\langle M^X \rangle^{i,j}_t = \langle (M^X)^i, (M^X)^j \rangle_t$ denoting covariation for $i,j = 0,1,\ldots,d$ and $t \in [0,T]$.

Since Assumption 2.3 is in force, Theorem 2.4 of [14] ensures that the *structure condition*[4] is satisfied and the discounted numéraire portfolio $\bar{S}^{\delta_*}_t = \frac{S^{\delta_*}_t}{S^0_t}$ at any

[4] We say that X satisfies the *structure condition* if A^X is absolutely continuous with respect to $\langle M^X \rangle$, in the sense that there exists an $\mathbb{F}$-predictable process $\hat{\lambda} = \{\hat{\lambda}_t, t \in [0,T]\}$ such that $A^X = \int \mathrm{d}\langle M^X \rangle \hat{\lambda}$, i.e. $(A^X_t)^i = \sum_{j=0}^d \int_0^t \hat{\lambda}^j_u \mathrm{d}\langle M^X \rangle^{ij}_u$, for $i \in \{0,\ldots,d\}$ and $t \in [0,T]$, and the *mean-variance tradeoff process* $\hat{K}_t = \int_0^t \hat{\lambda}^\top_u \mathrm{d}\langle M^X \rangle_u \hat{\lambda}_u$ is finite $\mathbb{P}$-a.s. for each $t \in [0,T]$.

time t is given by

$$\bar{S}_t^{\delta_*} = \frac{1}{\hat{Z}_t}, \quad t \in [0,T],$$

where the process $\hat{Z}$ corresponds to the stochastic exponential

$$\hat{Z}_t = \mathcal{E}\left(-\hat{\lambda} \cdot M^X\right)_t = \exp\left(-\hat{\lambda} \cdot M_t^X - \frac{1}{2}\hat{K}_t\right), \quad t \in [0,T],$$

which is then well-defined and a strictly positive $\mathbb{P}$-local martingale. Via Itô's product rule, it is easy to check that the vector process $\hat{S}$ of benchmarked primary security accounts is a $\mathbb{P}$-local martingale, and thus, a $\mathbb{P}$-supermartingale. Indeed, since $X_t = X_0 + M_t^X + \int_0^t \hat{\lambda}_s \mathrm{d}\langle M^X \rangle_s$, we have

$$\begin{aligned} \mathrm{d}\hat{S}_t &= \mathrm{d}(X_t \hat{Z}_t) = \hat{Z}_t \mathrm{d}X_t + X_t \mathrm{d}\hat{Z}_t + \mathrm{d}\langle X, \hat{Z}\rangle_t \\ &= \hat{Z}_t (1 - X_t \hat{\lambda}_t) \mathrm{d}M_t^X, \quad t \in [0,T]. \end{aligned}$$

This implies that whenever we consider continuous primary security account processes, they are $\mathbb{P}$-local martingales when expressed in units of the numéraire portfolio.

In the general case when S_t can have jumps, it is not possible to provide an analogous explicit description of the numéraire portfolio $S_t^{\delta_*}$ or, more precisely, its generating strategy δ_*. An implicit description can be found in [15], Theorem 3.15, or more generally in [12], Theorem 3.2 and Corollary 3.2. In both cases, δ_* can be obtained by pointwise maximization of a function that is given explicitly in terms of semimartingale characteristics. If S is discontinuous, such a pointwise maximizer is only defined implicitly and neither of the above descriptions provides explicit expressions for δ_*.

However, a wide class of jump-diffusion market models is driven by primary security account processes that turn out to be, when expressed in units of the numéraire portfolio, $\mathbb{P}$-local martingales, see e.g. [22], Chapter 14. For example, this is the case in jump-diffusion markets, that is, when security price processes exhibit intensity based jumps due to event risk, see [22], Chapter 14, page 513. These results allow us to consider below risk-minimization in the case when the benchmarked assets are given by $\mathbb{P}$-local martingales.

3.1.1 Risk-Minimization with Benchmarked Assets

Since at this stage we refer to the case where benchmarked securities represent $\mathbb{P}$-local martingales (i.e. we assume $V \equiv \mathbf{0}$ for all $t \in [0,T]$ in (3.1)), we study risk-minimization as originally introduced in [11] under the benchmark approach, that is, benchmarked risk-minimization. In particular, since we are considering a (general) discounting factor (different from the usual money market account), we follow the approach of [3] for local risk-minimization under a given numéraire.

Definition 3.1. *An* L^2-admissible strategy *is any* $\mathbb{R}^{d+1}$*-valued* $\mathbb{F}$*-predictable vector process* $\delta = \{\delta_t = (\delta_t^0, \delta_t^1, \ldots, \delta_t^d)^\top,\ t \in [0,T]\}$ *such that* *(i) the associated portfolio $\hat{S}^\delta$ is a square-integrable stochastic process whose left-limit is equal to $\hat{S}^\delta_{t-} = \delta_t \cdot \hat{S}_t$,*

(ii) the stochastic integral $\int \delta \cdot \mathrm{d}\hat{S}$ is such that

$$\mathbb{E}\left[\int_0^T \delta_u^\top \mathrm{d}[\hat{S}]_u \delta_u\right] < \infty. \tag{3.2}$$

Here $[\hat{S}] = ([\hat{S}^i, \hat{S}^j])_{i,j=1,\dots,d}$ denotes the matrix-valued optional covariance process of $\hat{S}$.

Recall that the market may be not complete. We also admit strategies that are not self-financing and may generate benchmarked profits or losses over time as defined below.

Definition 3.2. *For any L^2-admissible strategy δ, the* benchmarked profit & loss *process $\hat{C}^\delta$ is defined by*

$$\hat{C}^\delta_t := \hat{S}^\delta_t - \int_0^t \delta_u \cdot \mathrm{d}\hat{S}_u - \hat{S}^\delta_0, \quad t \in [0,T]. \tag{3.3}$$

Here $\hat{C}^\delta_t$ describes the total costs incurred by δ over the interval $[0,t]$.

Definition 3.3. *For an L^2-admissible strategy δ, the corresponding* risk *at time t is defined by*

$$\hat{R}^\delta_t := \mathbb{E}\left[\left(\hat{C}^\delta_T - \hat{C}^\delta_t\right)^2 \middle| \mathcal{F}_t\right], \quad t \in [0,T],$$

where the benchmarked profit & loss process $\hat{C}^\delta$, given in (3.3)*, is assumed to be square-integrable.*

If $\hat{C}^\delta$ is constant, then it equals zero and the strategy is self-financing. Our goal is to find an L^2-admissible strategy δ, which minimizes the associated risk measured by the fluctuations of its benchmarked profit & loss process in a suitable sense.

Definition 3.4. *Given a benchmarked contingent claim $\hat{H} \in L^2(\mathcal{F}_T, \mathbb{P})$, an L^2-admissible strategy δ is said to be* benchmarked risk-minimizing *if the following conditions hold:*

(i) $\hat{S}^\delta_T = \hat{H}$, $\mathbb{P}$-a.s.;

(ii) for any L^2-admissible strategy $\tilde{\delta}$ such that $\hat{S}^{\tilde{\delta}}_T = \hat{S}^\delta_T$ $\mathbb{P}$-a.s., we have

$$\hat{R}^\delta_t \leq \hat{R}^{\tilde{\delta}}_t \quad \mathbb{P}-\text{a.s. for every } \mathrm{t} \in [0,\mathrm{T}].$$

Lemma 3.5. *The benchmarked profit & loss process $\hat{C}^\delta$ defined in* (3.3) *associated to a benchmarked risk-minimizing strategy δ is a $\mathbb{P}$-martingale for all $t \in [0,T]$.*

For the proof of Lemma 3.5, we refer to Section A in the Appendix. Hence benchmarked risk-minimizing strategies are "self-financing on average". We will see, to find a benchmarked risk-minimizing strategy corresponds to finding a suitable decomposition of the benchmarked claim adapted to this setting. Let $\mathcal{M}_0^2(\mathbb{P})$ be the space of all square-integrable $\mathbb{P}$-martingales starting at null at the initial time.

Theorem 3.6. *Every benchmarked contingent claim $\hat{H} \in L^2(\mathcal{F}_T, \mathbb{P})$ admits a unique benchmarked risk-minimizing strategy δ with portfolio value $\hat{S}^\delta$ and benchmarked profit & loss process $\hat{C}^\delta$, given by*

$$\begin{aligned}
\delta &= \delta^{\hat{H}}, \\
\hat{S}_t^\delta &= \hat{H}_t = \mathbb{E}\left[\hat{H}\,\middle|\,\mathcal{F}_t\right], \quad t \in [0,T], \\
\hat{C}^\delta &= \hat{H}_0 - \hat{S}_0^\delta + L^{\hat{H}} = L^{\hat{H}},
\end{aligned}$$

where $\delta^{\hat{H}}$ and $L^{\hat{H}}$ are provided by the Galtchouk-Kunita-Watanabe decomposition of $\hat{H}$, i.e.

$$\hat{H} = \hat{H}_0 + \int_0^T \delta_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u + L_T^{\hat{H}}, \quad \mathbb{P} - \text{a.s.} \tag{3.4}$$

with $\hat{H}_0 \in \mathbb{R}$, where $\delta^{\hat{H}}$ is an $\mathbb{F}$-predictable vector process satisfying the integrability condition (3.2) *and $L^{\hat{H}} \in \mathcal{M}_0^2(\mathbb{P})$ is strongly orthogonal to each component of $\hat{S}$.*

Proof. The proof follows from Theorem 2.4 of [24] and Lemma 3.5. □

Note that since $\hat{S}_0^\delta = \hat{H}_0$, the benchmarked profit & loss $\hat{C}_t^\delta$ equals $L_t^{\hat{H}}$ for all $t \in [0,T]$. Thus, the problem of minimizing risk is reduced to finding the representation (3.4). A natural question is whether the benchmarked risk-minimizing strategy is invariant under a change of numéraire. We address this issue in Section 3.3.1.

3.2 Relationship to Real World Pricing

Definition 3.7. *We say that a nonnegative benchmarked contingent claim $\hat{H} \in L^2(\mathcal{F}_T, \mathbb{P})$ is* hedgeable *if there exists an L^2-admissible self-financing strategy $\xi^{\hat{H}} = \{\xi_t^{\hat{H}} = (\xi_t^{\hat{H},1}, \xi_t^{\hat{H},2}, \ldots, \xi_t^{\hat{H},d})^\top,\ t \in [0,T]\}$ such that*

$$\hat{H} = \hat{H}_0 + \int_0^T \xi_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u.$$

Decomposition (3.4) and Definition 3.7 allow us to decompose every nonnegative, square-integrable benchmarked contingent claim as the sum of its *hedgeable part* $\hat{H}^h$ and its *unhedgeable part* $\hat{H}^u$ such that we can write

$$\hat{H} = \hat{H}^h + \hat{H}^u, \tag{3.5}$$

where

$$\hat{H}^h := \hat{H}_0 + \int_0^T \xi_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u$$

and

$$\hat{H}^u := L_T^{\hat{H}}.$$

Recall that $L^{\hat{H}} = \{L_t^{\hat{H}}, t \in [0,T]\}$ is a $\mathbb{P}$-martingale in $\mathcal{M}_0^2(\mathbb{P})$, strongly orthogonal to each component of $\hat{S}$. There is a close relationship between benchmarked risk-minimization and real world pricing, as we will see now. Let us apply the real world pricing formula (2.3) to the benchmarked contingent claim $\hat{H}$ in order to get its benchmarked fair price $\hat{U}_H(t)$ at time t. Recall, by its martingale property that the benchmarked fair price is the best forecast of its future benchmarked prices. Due to the supermartingale property (2.2), it follows that we characterize, when using the real world pricing formula (2.3) for obtaining the fair price of the hedgeable part, the least expensive replicating portfolio for $\hat{H}^h$ by taking the conditional expectation $\mathbb{E}\left[\hat{H}^h \middle| \mathcal{F}_t\right]$ under the real world probability measure $\mathbb{P}$. Then by (3.5) we have

$$\hat{U}_H(t) = \mathbb{E}\left[\hat{H} \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\hat{H}^h \middle| \mathcal{F}_t\right] + \mathbb{E}\left[\hat{H}^u \middle| \mathcal{F}_t\right] = \hat{U}_{H^h}(t) + \hat{U}_{H^u}(t),$$

for every $t \in [0,T]$. Note that the benchmarked hedgeable part $\hat{H}^h$ can be replicated perfectly, i.e.

$$\hat{U}_{H^h}(t) = \mathbb{E}\left[\hat{H}^h \middle| \mathcal{F}_t\right] = \hat{H}_0 + \int_0^t \xi_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u.$$

In particular, for $t = 0$ one has for the benchmarked hedgeable part

$$\hat{U}_{H^h}(0) = \mathbb{E}\left[\hat{H}^h \middle| \mathcal{F}_0\right] = \hat{H}_0.$$

On the other hand, we have for the benchmarked unhedgeable part

$$\hat{U}_{H^u}(t) = \mathbb{E}\left[\hat{H}^u \middle| \mathcal{F}_t\right] = L_t^{\hat{H}}$$

with

$$\hat{U}_{H^u}(0) = 0.$$

Consequently, for the nonnegative benchmarked payoff $\hat{H}$, its benchmarked fair price $\hat{U}_H(0)$ at time $t = 0$, is given by

$$\hat{U}_H(0) = \hat{U}_{H^h}(0) + \hat{U}_{H^u}(0) = \mathbb{E}\left[\hat{H}^h \middle| \mathcal{F}_0\right] + \mathbb{E}\left[\hat{H}^u \middle| \mathcal{F}_0\right] = \hat{H}_0.$$

The real world pricing formula (2.3) appears in the form of a conditional expectation and, thus, as a projection in a least squares sense. More precisely, the benchmarked fair price $\hat{U}_H(0)$ can be interpreted as the least squares projection

of $\hat{H}$ into the space of $\mathcal{F}_0$-measurable benchmarked values. Note that the benchmarked fair price $\hat{U}_{H^u}(0)$ of the unhedgeable part $\hat{H}^u = L_T^{\hat{H}}$ is zero at time $t = 0$. Recall that the benchmarked hedgeable part is priced at time $t = 0$ such that the minimal possible price, the fair price, results. Viewed from time $t = 0$ the benchmarked profit & loss $\hat{C}_T^{\delta} = \hat{L}_T^{\hat{H}}$, see Theorem 3.6, has then zero mean and minimal variance $\mathrm{Var}\left(\mathrm{L}_{\mathrm{T}}^{\hat{\mathrm{H}}}\right)$. This means that the application of the real world pricing formula to a benchmarked payoff at time $t = 0$ leaves its benchmarked unhedgeable part *totally untouched*. This is reasonable because any extra trading could only create unnecessary uncertainty and potential additional benchmarked profits or losses. Of course, once the benchmarked fair price is used to establish a hedge portfolio, a benchmarked profit & loss emerges according to Theorem 3.6 if there was an unhedgeable part in the benchmarked contingent claim.
The following practically important insight is worth mentioning:

Remark 3.8. *From a large financial institution's point of view, the benchmarked profits & losses in its derivative book have zero mean and minimal variance when evaluated under real world pricing and viewed at time $t = 0$. If they are large in number and* independent, *then the Law of Large Numbers reduces asymptotically the variance of the benchmarked pooled profit & loss to zero and, thus, its value to zero.*
Obviously, requesting from clients higher prices than fair prices would make the bank less competitive. On the other hand, charging lower prices than fair prices would make it unsustainable in the long run because it would suffer on average a loss. In this sense fair pricing of unhedgeable claims is most natural and yields economically correct prices. Accordingly, benchmarked risk-minimization is a very natural risk management strategy for pricing and hedging. Moreover, it is mathematically convenient and for many models rather tractable when using the GKW-decomposition.

With the above notation, we obtain by Theorem 3.6 and (3.4) for the benchmarked payoff $\hat{H} \in L^2(\mathcal{F}_T, \mathbb{P})$ the following decomposition:

$$\hat{H} = \hat{U}_{H^h}(0) + \int_0^T \xi_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u + \hat{U}_{H^u}(T).$$

Since $\hat{U}_{H^u}(T) = \hat{H}^u = L_T^{\hat{H}}$, it follows

$$\hat{H} = \hat{U}_{H^h}(0) + \int_0^T \xi_u^{\hat{H}} \cdot \mathrm{d}\hat{S}_u + L_T^{\hat{H}}.$$

This allows us to summarize the relationship between benchmarked risk-minimization and real world pricing. In our setting $\hat{H} \in L^2(\mathcal{F}_T, \mathbb{P})$ admits a benchmarked risk-minimizing strategy and the decomposition for $\hat{H}$, provided by the real world pricing formula, coincides with the decomposition (3.4), where $\xi^{\hat{H}}$ yields the fair strategy for the self-financing replication of the hedgeable part of $\hat{H}$. The remaining benchmarked unhedgeable part given by the profit & loss process $L^{\hat{H}}$ can be diversified. Note that diversification takes place under the real world probability measure and *not* under some putative risk neutral measure. This is an important insight, which gives a clear reasoning for the pricing and hedging of contingent claims via real world pricing in incomplete markets.

3.3 Local Risk-Minimization with Benchmarked Assets

We now consider the general situation, where the vector of the $d+1$ benchmarked primary security accounts $\hat{S} = (\hat{S}^0, \hat{S}^1, \dots, \hat{S}^d)^\top$ forms with each of its components a nonnegative strict locally square-integrable $\mathbb{P}$-supermartingale with decomposition (3.1), and hence a special $\mathbb{P}$-semimartingale. In view of Proposition 3.1 in [24], Definition 3.3 does not hold in this non-martingale case due to a compatibility problem. Indeed as observed in [24], at any time t we minimize $\hat{R}^\delta_t$ over all admissible continuations from t on and obtain a continuation which is optimal when viewed in t only. But for $s < t$, the s-optimal continuation from s onward highlights what to do on the whole interval $(s,T] \supset (t,T]$ and this may be different from what the t-optimal continuation from t on prescribes. However, it is possible to characterize benchmarked pseudo-locally risk-minimizing strategies[5] through the following well-known result, see [24].

Proposition 3.9. *A benchmarked contingent claim $\hat{H} \in L^2(\mathcal{F}_T, \mathbb{P})$ admits a benchmarked pseudo-locally risk-minimizing strategy δ with $\hat{S}^\delta_T = \hat{H}$ $\mathbb{P}$-a.s. if and only if $\hat{H}$ can be written as*

$$\hat{H} = \hat{H}_0 + \int_0^T \xi^{\hat{H}}_u \cdot \mathrm{d}\hat{S}_u + L^{\hat{H}}_T, \quad \mathbb{P} - \text{a.s.} \tag{3.6}$$

with $\hat{H}_0 \in L^2(\mathcal{F}_0, \mathbb{P})$, $\xi^{\hat{H}}$ is an $\mathbb{F}$-predictable vector process satisfying the following integrability condition

$$\mathbb{E}\left[\int_0^T (\xi^{\hat{H}}_s)^\top \mathrm{d}[M]_s \xi^{\hat{H}}_s + \left(\int_0^T |(\xi^{\hat{H}}_s)^\top||\mathrm{d}V_s|\right)^2\right] < \infty,$$

where for $\omega \in \Omega$, $\mathrm{d}V_s(\omega)$ denotes the (signed) Lebesgue-Stieltjes measure corresponding to the finite variation function $s \mapsto V_s(\omega)$ and $|\mathrm{d}V_s|(\omega)$ the associated total variation measure, and $L^{\hat{H}} \in \mathcal{M}^2_0(\mathbb{P})$ is strongly orthogonal to M. The strategy δ is then given by

$$\delta_t = \xi^{\hat{H}}_t, \quad t \in [0,T],$$

[5]The original definition of a *locally risk-minimizing* strategy is given in [24] and formalizes the intuitive idea that changing an optimal strategy over a small time interval increases the risk, at least asymptotically. Since it is a rather technical definition, it has been introduced the concept of a *pseudo-locally risk-minimizing* strategy that is both easier to find and to characterize, as Proposition 3.9 will show in the following. Moreover, in the one-dimensional case and if $\hat{S}$ is sufficiently well-behaved, pseudo-optimal and locally risk-minimizing strategies are the same.

its benchmarked value process is

$$\hat{S}_t^\delta = \hat{S}_0^\delta + \int_0^t \delta_s \cdot \mathrm{d}\hat{S}_s + \hat{C}_t^\delta, \quad t \in [0,T],$$

and the benchmarked profit & loss process equals

$$\hat{C}_t^\delta = \hat{H}_0 - \hat{S}_0^\delta + \hat{L}_t^{\hat{H}}, \quad t \in [0,T].$$

Decompositions (3.4) and (3.6) for $\hat{H} \in L^2(\mathcal{F}_T, \mathbb{P})$ are also known in the literature as the Föllmer-Schweizer decompositions for $\hat{H}$.

3.3.1 Invariance under a change of numéraire

A natural question to clarify is when risk-minimizing strategies are invariant under a change of numéraire. Indeed, if the primary security accounts S^j, $j \in \{0,1,\dots,d\}$, are continuous, then Theorem 3.1 of [3] ensures that the strategy is invariant under a change of numéraire. In this case the process $\xi^{\hat{H}}$ appearing in decomposition (3.6) also provides the classical locally risk-minimizing strategy (if it exists) for the discounted contingent claim $\bar{H} := \dfrac{H}{S_T^0}$. However, if the primary security accounts S^j are only right-continuous, it is still possible to extend some results of [3] as follows:
Consider two discounting factors S^0 and S^{δ_*}. Given an L^2-admissible strategy δ, we now assume that the two stochastic integrals $\int_0^\cdot \delta_s \cdot \mathrm{d}\bar{S}_s := \int_0^\cdot \delta_s \cdot \mathrm{d}\left(\frac{S_s}{S_s^0}\right)$ and $\int_0^\cdot \delta_s \cdot \mathrm{d}\hat{S}_s$ exist. Denote by $\bar{C}^\delta$ and $\hat{C}^\delta$ the profit & loss processes associated to the strategy δ denominated in units of S^0 and S^{δ_*}, respectively.

Lemma 3.10. *If $\bar{C}^\delta$ and $\hat{C}^\delta$ are the profit & loss processes of the strategy δ, then*

$$\mathrm{d}\hat{C}_t^\delta = \hat{S}_{t-}^0 \mathrm{d}\bar{C}_t^\delta + \mathrm{d}[\bar{C}^\delta, \hat{S}^0]_t. \tag{3.7}$$

Proof. This result extends Lemma 3.1 in [3]. For the reader's convenience we provide here briefly the proof of (3.7). It is formally analogous to the one of Lemma 3.1 in [3]. By Itô's formula, we have

$$\begin{aligned}
\mathrm{d}\hat{S}_t^\delta &= \mathrm{d}\left(\frac{S_t^\delta}{S_t^{\delta_*}}\right) = \mathrm{d}\left(\frac{S_t^\delta}{S_t^0} \cdot \frac{S_t^0}{S_t^{\delta_*}}\right) = \frac{S_{t-}^\delta}{S_{t-}^0} \mathrm{d}\left(\frac{S_t^0}{S_t^{\delta_*}}\right) + \frac{S_{t-}^0}{S_{t-}^{\delta_*}} \mathrm{d}\left(\frac{S_t^\delta}{S_t^0}\right) + \mathrm{d}\left[\frac{S^\delta}{S^0}, \frac{S^0}{S^{\delta_*}}\right]_t \\
&= \delta_{t-} \frac{S_{t-}}{S_{t-}^0} \mathrm{d}\left(\frac{S_t^0}{S_t^{\delta_*}}\right) + \frac{S_{t-}^0}{S_{t-}^{\delta_*}} \mathrm{d}\left(\frac{S_t^\delta}{S_t^0}\right) + \mathrm{d}\left[\frac{S^\delta}{S^0}, \frac{S^0}{S^{\delta_*}}\right]_t.
\end{aligned}$$

Since

$$\mathrm{d}\left(\frac{S_t^\delta}{S_t^0}\right) = \delta_{t-} \mathrm{d}\left(\frac{S_t}{S_t^0}\right) + \mathrm{d}\bar{C}_t^\delta,$$

then

$$\mathrm{d}\left[\frac{S^\delta}{S^0}, \frac{S^0}{S^{\delta_*}}\right]_t = \delta_{t-}\mathrm{d}\left[\frac{S}{S^0}, \frac{S^0}{S^{\delta_*}}\right]_t + \mathrm{d}\left[\bar{C}^\delta, \frac{S^0}{S^{\delta_*}}\right]_t.$$

Finally

$$\begin{aligned}
\mathrm{d}\hat{S}^\delta_t &= \delta_{t-}\left\{\frac{S_{t-}}{S^0_{t-}}\mathrm{d}\left(\frac{S^0_t}{S^{\delta_*}_t}\right) + \frac{S^0_{t-}}{S^{\delta_*}_{t-}}\mathrm{d}\left(\frac{S_t}{S^0_t}\right) + \mathrm{d}\left[\frac{S}{S^0}, \frac{S^0}{S^{\delta_*}}\right]_t\right\} \\
&\quad + \frac{S^0_{t-}}{S^{\delta_*}_{t-}}\mathrm{d}\bar{C}^\delta_t + \mathrm{d}\left[\bar{C}^\delta, \frac{S^0}{S^{\delta_*}}\right]_t \\
&= \delta_{t-}\mathrm{d}\left(\frac{S_t}{S^{\delta_*}_t}\right) + \frac{S^0_{t-}}{S^{\delta_*}_{t-}}\mathrm{d}\bar{C}^\delta_t + \mathrm{d}\left[\bar{C}^\delta, \frac{S^0}{S^{\delta_*}}\right]_t \\
&= \delta_{t-}\mathrm{d}\hat{S}^\delta_t + \hat{S}^0_{t-}\mathrm{d}\bar{C}^\delta_t + \mathrm{d}\left[\bar{C}^\delta, \hat{S}^0\right]_t.
\end{aligned}$$

□

By Lemma 3.10, the profit & loss process of a risk-minimizing strategy (with respect to a given discounting factor) is given by a $\mathbb{P}$-martingale. This property provides a fundamental characterization of (local) risk-minimizing strategies (with respect to a given discounting factor). Here we show that they are invariant under a change of numéraire.

Proposition 3.11. *Under the same hypotheses of the previous lemma, if the process $\bar{C}^\delta$ is a continuous $\mathbb{P}$-local martingale strongly orthogonal to the martingale part of $\bar{S}$, then $\hat{C}^\delta$ is also a (continuous) $\mathbb{P}$-local martingale strongly orthogonal to the martingale part of $\hat{S}$.*

Proof. This result generalizes Proposition 3.1 of [3]. The proof essentially follows from Itô's formula and Lemma 3.10. From integration by parts formula, we have that

$$\mathrm{d}\left(\frac{S_t}{S^{\delta_*}_t}\right) = \mathrm{d}\left(\frac{S_t}{S^0_t}\cdot\frac{S^0_t}{S^{\delta_*}_t}\right) = \frac{S_{t-}}{S^0_{t-}}\mathrm{d}\left(\frac{S^0_t}{S^{\delta_*}_t}\right) + \frac{S^0_{t-}}{S^{\delta_*}_{t-}}\mathrm{d}\left(\frac{S_t}{S^0_t}\right) + \mathrm{d}\left[\frac{S}{S^0}, \frac{S^0}{S^{\delta_*}}\right]_t$$

where by Itô's formula

$$\begin{aligned}
\mathrm{d}\left(\frac{S^0_t}{S^{\delta_*}_t}\right) = \mathrm{d}\left[\left(\frac{S^{\delta_*}_t}{S^0_t}\right)^{-1}\right] &= -\left(\frac{S^0_{t-}}{S^{\delta_*}_{t-}}\right)^2\mathrm{d}\left(\frac{S^{\delta_*}_t}{S^0_t}\right) + \left(\frac{S^0_{t-}}{S^{\delta_*}_{t-}}\right)^3\mathrm{d}\left[\frac{S^{\delta_*}}{S^0}, \frac{S^{\delta_*}}{S^0}\right]_t \\
&\quad + \underbrace{\Delta\left(\frac{S^{\delta_*}_t}{S^0_t}\right)^{-1} + \left(\frac{S^0_{t-}}{S^{\delta_*}_{t-}}\right)^2\Delta\left(\frac{S^{\delta_*}_t}{S^0_t}\right) - \left(\frac{S^0_{t-}}{S^{\delta_*}_{t-}}\right)^3\left[\Delta\left(\frac{S^{\delta_*}_t}{S^0_t}\right)\right]^2}_{:=\mathrm{d}\Sigma_t}.
\end{aligned}$$

From Lemma 3.10, we have

$$
\begin{aligned}
\mathrm{d}\left[\hat{C}^{\delta}, \frac{S}{S^{\delta_*}}\right]_t &= \frac{S^0_{t-}}{S^{\delta_*}_{t-}}\mathrm{d}\left[\bar{C}^{\delta}, \frac{S}{S^{\delta_*}}\right]_t + \underbrace{\mathrm{d}\left[\left[\bar{C}^{\delta}, \frac{S^0}{S^{\delta_*}}\right], \frac{S}{S^{\delta_*}}\right]_t}_{0} \\
&= \frac{S_{t-}}{S^{\delta_*}_{t-}}\mathrm{d}\left[\bar{C}^{\delta}, \frac{S^0}{S^{\delta_*}}\right]_t + \underbrace{\left(\frac{S^0_{t-}}{S^{\delta_*}_{t-}}\right)^2 \mathrm{d}\left[\bar{C}^{\delta}, \frac{S}{S^0}\right]_t}_{0} + \underbrace{\frac{S^0_{t-}}{S^{\delta_*}_{t-}}\mathrm{d}\left[\bar{C}^{\delta}, \left[\frac{S}{S^0}, \frac{S^0}{S^{\delta_*}}\right]\right]_t}_{0} \\
&\quad + \underbrace{\mathrm{d}\left[\left[\bar{C}^{\delta}, \frac{S^0}{S^{\delta_*}}\right], \frac{S}{S^{\delta_*}}\right]_t}_{0} \\
&= \frac{S_{t-}}{S^{\delta_*}_{t-}}\mathrm{d}\left[\bar{C}^{\delta}, \frac{S^0}{S^{\delta_*}}\right]_t, \qquad (3.8)
\end{aligned}
$$

where we have used the fact that $\bar{C}^{\delta}$ is a continuous $\mathbb{P}$-local martingale strongly orthogonal to the martingale part of $\bar{S}$. Furthermore,

$$
\begin{aligned}
\mathrm{d}\left[\bar{C}^{\delta}, \frac{S^0}{S^{\delta_*}}\right]_t &= -\left(\frac{S^0_{t-}}{S^{\delta_*}_{t-}}\right)^2 \mathrm{d}\left[\bar{C}^{\delta}, \frac{S^{\delta_*}}{S^0}\right]_t + \left(\frac{S^0_{t-}}{S^{\delta_*}_{t-}}\right)^3 \mathrm{d}\left[\bar{C}^{\delta}, \left[\frac{S^{\delta_*}}{S^0}, \frac{S^{\delta_*}}{S^0}\right]\right]_t + \mathrm{d}\left[\bar{C}^{\delta}, \Sigma\right]_t \\
&= -\left(\frac{S^0_{t-}}{S^{\delta_*}_{t-}}\right)^2 \mathrm{d}\left[\bar{C}^{\delta}, \frac{S^{\delta_*}}{S^0}\right]_t = -\left(\frac{S^0_{t-}}{S^{\delta_*}_{t-}}\right)^2 \mathrm{d}\left\langle \bar{C}^{\delta}, \frac{S^{\delta_*}}{S^0}\right\rangle_t .
\end{aligned}
\qquad (3.9)
$$

If now $\bar{C}^{\delta}$ is a continuous $\mathbb{P}$-local martingale strongly orthogonal to $\bar{S}$, then

$$
\mathrm{d}\left\langle \bar{C}^{\delta}, \frac{S^{\delta_*}}{S^0}\right\rangle_t = 0, \quad t \in [0,T].
$$

By (3.8) and (3.9), we have that also $\mathrm{d}\left[\hat{C}^{\delta}, \frac{S}{S^{\delta_*}}\right]_t = \mathrm{d}\left\langle \bar{C}^{\delta}, \frac{S}{S^{\delta_*}}\right\rangle_t = 0$, hence $\hat{C}^{\delta}$ is strongly orthogonal to the martingale part of S. This concludes the proof. □

Now it is possible to state the main result that guarantees that invariance under change of numéraire is kept in the case of right-continuous asset price processes if we assume that the profit & loss process $\bar{C}^{\delta}$ is continuous.

Theorem 3.12. *Let δ be an L^2-admissible strategy with respect to the numéraires S^0 and S^{δ_*} and assume that $\bar{C}^{\delta}$ is continuous. If δ is locally risk-minimizing under the numéraire S^0, then δ is locally risk-minimizing also with respect to the numéraire S^{δ_*}, i.e. it is benchmarked locally risk-minimizing.*

Proof. The proof is an immediate consequence of Proposition 3.11: if δ is a locally risk-minimizing strategy under S^0, the cost $\hat{C}^{\delta}$ is a $\mathbb{P}$-local martingale strongly orthogonal to the martingale part of $\hat{S}$. But since the strategy is L^2-admissible with respect to S^{δ_*}, the profit & loss process $\hat{C}^{\delta}$ is actually a square integrable $\mathbb{P}$-martingale. □

3.4 Benchmarked local risk-minimization under incomplete information

Here we show an example of benchmarked local risk-minimization that works under general assumptions on $\hat{S}$. Similarly to [10], we consider a situation where the financial market would be complete if we had more information. The available information is described by the filtration $\mathbb{F}$. We suppose that the benchmarked claim $\hat{H}$ is attainable with respect to some larger filtration. Only at the terminal time T, but not at times $t < T$, all the information relevant for a perfect hedging of a claim will be available to us. So let $\tilde{\mathbb{F}} := (\tilde{\mathcal{F}}_t)_{0 \leq t \leq T}$ be a right-continuous filtration such that

$$\mathcal{F}_t \subseteq \tilde{\mathcal{F}}_t \subseteq \mathcal{F}, \quad t \in [0,T].$$

We now show how the results of [10] hold without assuming that the underlying asset price processes are continuous, if we consider a general benchmarked market. Note furthermore that we are not going to assume that the benchmarked assets are $\mathbb{P}$-local martingales.
Consider now the benchmarked asset price process $\hat{S}$. Then $\hat{S}$ is a $\mathbb{P}$-supermartingale and admits the Doob-Meyer's decomposition

$$\hat{S} = M - A,$$

where A is an $\mathbb{F}$-predictable increasing finite variation process and M a $\mathbb{P}$-local martingale.

Assumption 3.13. *Suppose now that the vector process $\hat{S}$ belongs to $\mathcal{S}^2(\mathbb{P})$, that is, the space of $\mathbb{P}$-semimartingales satisfying the integrability condition*

$$\mathbb{E}\left[\hat{S}_0^2 + [M]_T + |A|_T^2\right] < \infty,$$

where $|A| = \{|A|_t : \ t \in [0,T]\}$ is the total variation of A. In addition, the decomposition (3.1) of $\hat{S}$ with respect to $\mathbb{F}$ is still valid with respect to $\tilde{\mathbb{F}}$. In other words we assume that M is a $\mathbb{P}$-martingale with respect to $\tilde{\mathbb{F}}$, although it is adapted to the smaller filtration $\mathbb{F}$.

Suppose now that $\hat{H} \in L^2(\mathcal{F}_T, \mathbb{P})$ is attainable with respect to the larger filtration $\tilde{\mathbb{F}}$, i.e.

$$\hat{H} = \tilde{H}_0 + \int_0^T \tilde{\xi}_s^{\hat{H}} \mathrm{d}\hat{S}_s, \tag{3.10}$$

where $\tilde{H}_0$ is $\tilde{\mathcal{F}}_0$-measurable and the process $\tilde{\xi}^{\hat{H}} = \{\tilde{\xi}_t^{\hat{H}} = (\tilde{\xi}_t^{\hat{H},0}, \tilde{\xi}_t^{\hat{H},1}, \dots, \tilde{\xi}_t^{\hat{H},d})^\top, \ t \in [0,T]\}$ is predictable with respect to $\tilde{\mathbb{F}}$. We now need to specify suitable integrability conditions.

Assumption 3.14. *We suppose that the $(\tilde{\mathbb{F}}, \mathbb{P})$-semimartingale*

$$\tilde{H}_0 + \int_0^t \tilde{\xi}_s^{\hat{H}} \mathrm{d}\hat{S}_s, \quad t \in [0,T],$$

associated to $\hat{H}$ belongs to the space $\mathcal{S}^2(\mathbb{P})$, i.e.

$$\mathbb{E}\left[\tilde{H}_0^2 + \int_0^T (\tilde{\xi}_s^{\hat{H}})^\top \mathrm{d}[M]_s \tilde{\xi}_s^{\hat{H}} + \left(\int_0^T |(\tilde{\xi}_s^{\hat{H}})^\top||\mathrm{d}A_s|\right)^2\right] < \infty. \tag{3.11}$$

Theorem 3.15. *Suppose that $\hat{H}$ satisfies* (3.10) *and* (3.11). *Then $\hat{H}$ admits the representation*

$$\hat{H} = \tilde{H}_0 + \int_0^T \xi_s^{\hat{H}} \mathrm{d}\hat{S}_s + L_T^{\hat{H}}, \tag{3.12}$$

with $\hat{H}_0 = E[\tilde{H}_0|\mathcal{F}_0]$, where

$$\xi^{\hat{H}} := {}^p(\tilde{\xi}^{\hat{H}})$$

is the $\mathbb{F}$-predictable projection of the $\tilde{\mathbb{F}}$-predictable vector process $\tilde{\xi}^{\hat{H}}$, and where $L^{\hat{H}} = \{L_t^{\hat{H}} = (L_t^{\hat{H},0}, L_t^{\hat{H},1}, \dots, L_t^{\hat{H},d})^\top,\ t \in [0,T]\}$ is the square-integrable $(\mathbb{F},\mathbb{P})$-martingale, orthogonal to M, associated to

$$L_T^{\hat{H}} := \tilde{H}_0 - \hat{H}_0 + \int_0^T (\tilde{\xi}_s^{\hat{H}} - \xi_s^{\hat{H}}) \mathrm{d}\hat{S}_s \in L^2(\mathcal{F}_T, \mathbb{P}).$$

Proof. **Step 1.** First we need to check that all components in (3.12) are square-integrable. Denote by ${}^pX$ the (dual) $\mathbb{F}$-predictable projection of a process X. By (3.11) and by the properties of the $\mathbb{F}$-predictable projection (see [6], VI.57),

$$\infty > \mathbb{E}\left[\left(\int_0^T \tilde{\xi}_s^{\hat{H}} \mathrm{d}M_s\right)^2\right] = \mathbb{E}\left[\int_0^T (\tilde{\xi}_s^{\hat{H}})^2 \mathrm{d}[M]_s\right] = \mathbb{E}\left[\int_0^T {}^p\left((\tilde{\xi}_s^{\hat{H}})^2\right) \mathrm{d}\langle M\rangle_s\right],$$

where the last equality holds since $\langle M\rangle$ is the $\mathbb{F}$-predictable dual projection (see e.g. [6], VI.73) of $[M]$. By Jensen's inequality we have

$$\left({}^p(\tilde{\xi}_\tau^{\hat{H}})\right)^2 = (\xi_\tau^{\hat{H}})^2 = \left(E[\tilde{\xi}_\tau^{\hat{H}}|\mathcal{F}_{\tau-}]\right)^2 \le E[(\tilde{\xi}_\tau^{\hat{H}})^2|\mathcal{F}_{\tau-}] = {}^p\left((\tilde{\xi}_\tau^{\hat{H}})^2\right),$$

for every predictable $\mathbb{F}$-stopping time τ, on the set $\{\tau < \infty\}$. Hence if we consider $\tau = s$, again by the properties of the $\mathbb{F}$-predictable dual projection, it follows that

$$\mathbb{E}\left[\int_0^T {}^p\left((\tilde{\xi}_s^{\hat{H}})^2\right) \mathrm{d}\langle M\rangle_s\right] \ge \mathbb{E}\left[\int_0^T \left({}^p(\tilde{\xi}_s^{\hat{H}})\right)^2 \mathrm{d}\langle M\rangle_s\right] = \mathbb{E}\left[\int_0^T (\xi_s^{\hat{H}})^2 \mathrm{d}\langle M\rangle_s\right]$$

and finally

$$\int_0^T \xi_s^{\hat{H}} \mathrm{d}M_s \in L^2(\mathcal{F}_T, \mathbb{P}).$$

In order to show that

$$\int_0^T \xi_s^{\hat{H}} \mathrm{d}A_s \in L^2(\mathcal{F}_T, \mathbb{P}),$$

we prove that

$$\left|\mathbb{E}\left[Z_T\int_0^T \xi_s^{\hat{H}}\mathrm{d}A_s\right]\right| \leq c\cdot\|Z_T\|_2, \quad c\in\mathbb{R}$$

for any bounded $\mathcal{F}_T$-measurable random variable Z_T with L^2-norm $\|Z_T\|_2$. Let $Z = \{Z_t,\ t\in[0,T]\}$ denote a right-continuous version with left-limits of the $(\mathbb{F},\mathbb{P})$-martingale $E[Z_T|\mathcal{F}_t]$, $t\in[0,T]$, and put $Z^* = \sup_{0\leq t\leq T}|Z_t|$. Since A is $\mathbb{F}$-predictable, we can use some properties of the $\mathbb{F}$-predictable projection (see VI.45 and VI.57 in [6]), and obtain

$$\begin{aligned}\left|\mathbb{E}\left[Z_T\int_0^T \xi_s^{\hat{H}}\mathrm{d}A_s\right]\right| &= \left|\mathbb{E}\left[\int_0^T Z_{s-}\xi_s^{\hat{H}}\mathrm{d}A_s\right]\right|\\ &= \left|\mathbb{E}\left[\int_0^T Z_{s-}\tilde{\xi}_s^{\hat{H}}\mathrm{d}A_s\right]\right|\\ &\leq \left|Z^*\mathbb{E}\left[\int_0^T \tilde{\xi}_s^{\hat{H}}\mathrm{d}|A|_s\right]\right|\\ &\leq \|Z^*\|_2\left\|\int_0^T \tilde{\xi}_s^{\hat{H}}\mathrm{d}|A|_s\right\|_2\\ &\leq c\cdot\|Z^*\|_2, \quad c\in\mathbb{R},\end{aligned}$$

where in the last inequality we have used (3.11) and Doob's inequality for the supremum of a square-integrable $\mathbb{P}$-martingale.

Step 2. Clearly, $\tilde{H}_0 - \hat{H}_0 \in L^2(\Omega,\tilde{\mathcal{F}}_0,\mathbb{P})$ is orthogonal to all square-integrable stochastic integrals of M with respect to the filtration $\tilde{\mathbb{F}}$, hence in particular with respect to the filtration $\mathbb{F}$. Then, it only remains to show that

$$\mathbb{E}\left[\left(\int_0^T(\tilde{\xi}_s^{\hat{H}}-\xi_s^{\hat{H}})\mathrm{d}\hat{S}_s\right)\left(\int_0^T\mu_s\mathrm{d}M_s\right)\right]=0 \tag{3.13}$$

for all bounded $\mathbb{F}$-predictable processes $\mu = \{\mu_t,\ t\in[0,T]\}$. This will imply that the $(\mathbb{F},\mathbb{P})$-martingale $L^{\hat{H}}$ is orthogonal to M. First we note that (3.13) is equivalent to the following

$$\mathbb{E}\left[\left(\int_0^T\tilde{\xi}_s^{\hat{H}}\mathrm{d}\hat{S}_s\right)\left(\int_0^T\mu_s\mathrm{d}M_s\right)\right]=\mathbb{E}\left[\left(\int_0^T\xi_s^{\hat{H}}\mathrm{d}\hat{S}_s\right)\left(\int_0^T\mu_s\mathrm{d}M_s\right)\right]. \tag{3.14}$$

Then we decompose the left-side of (3.14) into

$$\mathbb{E}\left[\left(\int_0^T\tilde{\xi}_s^{\hat{H}}\mathrm{d}M_s\right)\left(\int_0^T\mu_s\mathrm{d}M_s\right)\right]+\mathbb{E}\left[\left(\int_0^T\tilde{\xi}_s^{\hat{H}}\mathrm{d}A_s\right)\left(\int_0^T\mu_s\mathrm{d}M_s\right)\right].$$

We have

$$\mathbb{E}\left[\left(\int_0^T\tilde{\xi}_s^{\hat{H}}\mathrm{d}M_s\right)\left(\int_0^T\mu_s\mathrm{d}M_s\right)\right]=\mathbb{E}\left[\int_0^T\tilde{\xi}_s^{\hat{H}}\cdot\mu_s\mathrm{d}[M]_s\right] \tag{3.15}$$

and

$$\mathbb{E}\left[\left(\int_0^T \tilde{\xi}_s^{\hat{H}} \mathrm{d}A_s\right)\left(\int_0^T \mu_s \mathrm{d}M_s\right)\right] = \mathbb{E}\left[\int_0^T \tilde{\xi}_s^{\hat{H}} \left(\int_{[0,s)} \mu_u \mathrm{d}M_u\right) \mathrm{d}A_s\right], \quad (3.16)$$

by the property of the $\mathbb{F}$-predictable projection (see VI.45 in [6]). Since $\langle M \rangle$ is the $\mathbb{F}$-predictable dual projection of $[M]$, we can rewrite (3.15) as follows

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \tilde{\xi}_s^{\hat{H}} \cdot \mu_s \mathrm{d}[M]_s\right] &= \mathbb{E}\left[\int_0^T {}^p(\tilde{\xi}_s^{\hat{H}} \cdot \mu_s) \mathrm{d}\langle M \rangle_s\right] \\
&= \mathbb{E}\left[\int_0^T {}^p(\tilde{\xi}_s^{\hat{H}}) \cdot \mu_s \mathrm{d}\langle M \rangle_s\right] \qquad (3.17) \\
&= \mathbb{E}\left[\int_0^T \xi_s^{\hat{H}} \cdot \mu_s \mathrm{d}\langle M \rangle_s\right],
\end{aligned}$$

where the second equality (3.17) follows from Remark 44(e) in [6]. Now, from the properties of the $\mathbb{F}$-predictable projection it is clear that $\tilde{\xi}^{\hat{H}}$ can be replaced by $\xi^{\hat{H}}$ in (3.16), and this yields (3.13). □

4 Applications

In the remaining part of the paper we discuss some examples that illustrate how classical local risk-minimization is generalized to benchmarked local risk-minimization in a market when there is no equivalent risk-neutral probability measure. Finally, we will demonstrate that jumps in asset price dynamics do not create a major problem, which is not easily resolved under classical local risk-minimization.

4.1 Risk-Minimization for a Defaultable Bond in the Minimal Market Model

The notion of a minimal market model has been introduced in a series of papers; see Chapter 13 of [22] for a recent textbook account. We should stress that existence of the numéraire portfolio ensures existence of the growth-optimal portfolio (in short GOP) in our setting and in addition they coincide (see e.g. Proposition 2.1 of [14]).

We begin by considering a continuous financial market model almost similarly as in Chapter 10 of [22]. More precisely, in this framework uncertainty is modeled by d independent standard Wiener processes $W^k = \{W_t^k,\ t \in [0,T]\}$, $k \in \{1, 2, \dots, d\}$. We assume that the value at time t of the savings account S^0 is given by

$$S_t^0 = \exp\left\{\int_0^t r_s \mathrm{d}s\right\} < \infty \quad (4.1)$$

for $t \in [0,T]$, where $r = \{r_t,\ t \in [0,T]\}$ denotes the adapted short term interest rate and that the dynamics of the primary security account processes $S^j = \{S^j_t,\ t \in [0,T]\}$, $j \in 1,2,\dots,d$, are given by the SDE

$$\mathrm{d}S^j_t = S^j_t \left(a^j_t \mathrm{d}t + \sum_{k=1}^{d} b^{j,k}_t \mathrm{d}W^k_t \right) \tag{4.2}$$

for $t \in [0,T]$ with $S^j_0 > 0$. The j-th appreciation rate $a^j = \{a^j_t,\ t \in [0,T]\}$ and the (j,k)-th volatility $b^{j,k} = \{b^{j,k}_t,\ t \in [0,T]\}$ are $\mathbb{F}$-predictable processes for $j,k \in \{1,2,\dots,d\}$ satisfying suitable integrability conditions. Furthermore the volatility matrix $\mathbf{b}_t = [b^{j,k}_t]^d_{j,k=1}$ is for Lebesgue almost-every $t \in [0,T]$ assumed to be invertible. This assumption avoids redundant primary security accounts and also ensures the existence of the GOP. By introducing the appreciation rate vector $a_t = (a^1_t, a^2_t, \dots, a^d_t)^\top$ and the unit vector $\mathbf{1} = (1,1,\dots,1)^\top$, we obtain the *market price of risk* vector

$$\theta_t = (\theta^1_t, \theta^2_t, \dots, \theta^d_t) = \mathbf{b}^{-1}_t [a_t - r_t \mathbf{1}] \tag{4.3}$$

for $t \in [0,T]$. The notion (4.3) allows us to rewrite the SDE (4.2) in the form

$$\mathrm{d}S^j_t = S^j_t \left\{ r_t \mathrm{d}t + \sum_{k=1}^{d} (\theta^k_t - \sigma^{j,k}_t)[\theta^k_t \mathrm{d}t + \mathrm{d}W^k_t] \right\} \tag{4.4}$$

with (j,k)-th volatility

$$\sigma^{j,k}_t = \theta^k_t - b^{j,k}_t$$

for $t \in [0,T]$ and $j,k \in \{1,2,\dots,d\}$. Then, for a given self-financing strategy δ, the corresponding portfolio value S^δ satisfies according to (2.1) and (4.4) the SDE

$$\mathrm{d}S^\delta_t = S^\delta_t r_t \mathrm{d}t + \sum_{k=1}^{d} \left(S^\delta_t \theta^k_t - \sum_{j=0}^{d} \delta^j_t S^j_t \sigma^{j,k}_t \right) \left(\theta^k_t \mathrm{d}t + \mathrm{d}W^k_t \right) \tag{4.5}$$

for $t \in [0,T]$. Now we can derive the GOP dynamics in this setting. The GOP is the portfolio that maximizes the expected log utility $\mathbb{E}\left[\log(S^\delta_\tau) \middle| \mathcal{F}_t\right]$ from terminal wealth for all $t \in [0,\tau]$ and $\tau \in [0,T]$. The optimal strategy $\delta_* = \{\delta_{*,t} = (\delta^0_{*,t}, \delta^1_{*,t}, \dots, \delta^d_{*,t})^\top,\ t \in [0,T]\}$ follows in a straightforward manner from solving the first order conditions for the log-utility maximization problem, see [16], where

$$\sum_{j=0}^{d} \delta^j_{*,t} S^j_t \sigma^{j,k}_t = 0 \tag{4.6}$$

for $k \in \{1,2,\dots,d\}$ and $t \in [0,T]$. By (4.5) and (4.6) the GOP satisfies the SDE

$$\mathrm{d}S^{\delta_*}_t = S^{\delta_*}_t \left[r_t \mathrm{d}t + \sum_{k=1}^{d} \theta^k_t \left(\theta^k_t \mathrm{d}t + \mathrm{d}W^k_t \right) \right] \tag{4.7}$$

for $t \in [0,T]$, where we set $S_0^{\delta_*} = 1$. Note that by (4.3) and (4.7) the volatilities θ_t^k, $k \in \{1,2,\ldots,d\}$ of the GOP are the market prices for risk. The SDE (4.7) reveals a close link between its drift and diffusion coefficient. More precisely, the risk premium of the GOP equals the square of its volatility, i.e. at time $t \in [0,T]$ we have

$$|\theta_t|^2 = \theta_t^\top \theta_t,$$

which amounts to the square of the total market price for risk $|\theta_t|$. To see this clearly, let us discount the GOP value $S_t^{\delta_*}$ at time t by the savings account value S_t^0, see (4.1). The discounted GOP

$$\bar{S}_t^{\delta_*} = \frac{S_t^{\delta_*}}{S_t^0}$$

satisfies by application of Itô's formula, (4.1) and (4.7), the SDE

$$\mathrm{d}\bar{S}_t^{\delta_*} = \mathrm{d}(\hat{S}_t^0)^{-1} = (\hat{S}_t^0)^{-1}|\theta_t|(|\theta_t|\mathrm{d}t + \mathrm{d}W_t) = \bar{S}_t^{\delta_*}|\theta_t|(|\theta_t|\mathrm{d}t + \mathrm{d}W_t), \tag{4.8}$$

where

$$\mathrm{d}W_t = \frac{1}{|\theta_t|}\sum_{k=1}^{d} \theta_t^k \mathrm{d}W_t^k$$

is the stochastic differential of a standard Wiener process W and the total market price of risk $|\theta_t|$ is given by the expression

$$|\theta_t| = \sqrt{\sum_{k=1}^{d} (\theta_t^k)^2}$$

for $t \in [0,T]$. Now, let us parameterize the discounted numéraire portfolio dynamics by its trend, that is, by the drift of the SDE (4.8). More precisely, according to Chapter 13, page 485 in [22] the discounted GOP drift at time $t \in [0,T]$ is of the form

$$\alpha_t = \bar{S}_t^{\delta_*}|\theta_t|^2 = (\hat{S}_t^0)^{-1}|\theta_t|^2, \tag{4.9}$$

which is assumed to be a continuous, strictly positive, $\mathbb{F}$-predictable parameter process. The discounted GOP drift, which models the long term trend of the economy, is chosen as the key parameter process. We call α_t also the *market trend* at time t. The parametrization given in (4.9) leads to a GOP volatility, or total market price of risk, of the form

$$|\theta_t| = \sqrt{\frac{\alpha_t}{\bar{S}_t^{\delta_*}}} = \sqrt{\alpha_t \hat{S}_t^0}$$

and then by (4.8) for the discounted GOP to the SDE

$$\mathrm{d}\bar{S}_t^{\delta_*} = \mathrm{d}(\hat{S}_t^0)^{-1} = \alpha_t \mathrm{d}t + \sqrt{(\hat{S}_t^0)^{-1}\alpha_t}\,\mathrm{d}W_t = \alpha_t \mathrm{d}t + \sqrt{\bar{S}_t^{\delta_*}\alpha_t}\,\mathrm{d}W_t, \tag{4.10}$$

for $t \in [0,T]$. We emphasize that α is, in principle, an observable financial quantity which appears directly in the quadratic variation of $(\hat{S}_t^0)^{-\frac{1}{2}}$. We also note that the parameter process α can be freely specified as an $\mathbb{F}$-predictable process such that the SDE (4.10) has a unique strong solution.
We fix in the following example the dynamics of the discounted drift process by assuming that it is given by the exponential function

$$\alpha_t = \alpha_0 \exp\{\eta t\}, \quad t \in [0,T],$$

where $\alpha_0 > 0$ is a scaling parameter and $\eta > 0$ denotes the long term net growth rate of the market. For simplicity, we set the interest rate to zero. With this choice $\hat{S}^0$ follows the stylized version of the *minimal market model* (in short MMM), see [22]. Note that the benchmarked savings account is the inverse of a squared Bessel process of dimension four, which is a square-integrable strict $\mathbb{P}$-local martingale and thus, a strict $\mathbb{P}$-supermartingale, see [23]. Assuming that the filtration is generated by the above Wiener process W, the candidate for the Radon-Nikodym derivative process for the putative risk neutral measure is a strict $\mathbb{P}$-supermartingale and not a $\mathbb{P}$-martingale. We stress the fact that in the stylized MMM there exists no equivalent risk neutral probability measure, see Chapter 13, page 499 in [22].

4.1.1 Defaultable Zero Coupon Bond

In the stylized MMM we now compute the price of a defaultable zero-coupon bond. Thanks to the real world pricing formula (2.3), $\mathbb{E}\left[\hat{S}_T^0 \middle| \mathcal{F}_t\right]$ provides the benchmarked fair price $\hat{P}(t,T)$ at time t of a zero coupon bond, which pays one unit of the domestic currency at maturity T. According to [22], the benchmarked price $\hat{P}(t,T)$ of a fair zero coupon bond under the stylized MMM is given by the explicit formula

$$\hat{P}(t,T) = \left(1 - \exp\left\{-(\hat{S}_t^0)^{-1} f(t)\right\}\right) \hat{S}_t^0, \tag{4.11}$$

where $f(t) = \frac{2\eta}{\alpha_0(\exp\{\eta T\} - \exp\{\eta t\})}$, $t \in [0,T]$. Since $\mathbb{E}\left[\hat{S}_T^0 \middle| \mathcal{F}_t\right]$ must be of the form

$$\mathbb{E}\left[\hat{S}_T^0 \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\hat{S}_T^0\right] + \int_0^t \zeta_u \mathrm{d}W_u,$$

for a suitable process ζ, by applying Itô's formula we obtain

$$\mathrm{d}\hat{P}(t,T) = -\hat{P}(t,T)\left(\hat{S}_t^0 - f(t)\frac{e^{-\frac{f(t)}{\hat{S}_t^0}}}{1 - e^{-\frac{f(t)}{\hat{S}_t^0}}}\right)\sqrt{(\hat{S}_t^0)^{-1}\alpha_t}\mathrm{d}W_t.$$

Finally,

$$\mathbb{E}\left[\hat{S}^0_T \middle| \mathcal{F}_t\right] = \hat{P}(0,T) - \int_0^t \hat{P}(u,T)\left(\hat{S}^0_u - f(u)\frac{e^{-\frac{f(u)}{\hat{S}^0_u}}}{1-e^{-\frac{f(u)}{\hat{S}^0_u}}}\right)\sqrt{(\hat{S}^0_u)^{-1}\alpha_u}\mathrm{d}W_u. \tag{4.12}$$

The benchmarked savings bond is a strict $\mathbb{P}$-supermartingale, whereas the benchmarked fair zero coupon bond is a $\mathbb{P}$-martingale. As a consequence, the fair price

$$P(t,T) = \hat{P}(t,T)S^{\delta_*}_t = 1 - \exp\{-(\hat{S}^0_t)^{-1}f(t)\}$$

at time t of a zero coupon bond that matures at time T is lower than the savings bond price $P^*_T(t) = 1$, for $t \in [0,T]$. Here $P^*_T(t)$ is simply the constant zero interest savings account.
Now, in the financial market described above, we apply benchmarked local risk-minimization to a defaultable zero-coupon bond with maturity T. Beyond the traded uncertainty given by the standard $\mathbb{F}$-Wiener process W, there is also an additional source of randomness due to the presence of a possible default that, according to intensity based modeling, shall be modeled via a compensated jump process. More precisely, we assume that the random time of default τ is represented by a stopping time in the given filtration $\mathbb{F}$. Let D be the default process, defined as $D_t = \mathbf{1}_{\{\tau \le t\}}$, for $t \in [0,T]$. We assume that τ admits an $\mathbb{F}$-intensity, that is, there exists an $\mathbb{F}$-adapted, nonnegative, (integrable) process λ such that the process

$$Q_t = D_t - \int_0^t \tilde{\lambda}_s \mathrm{d}s = D_t - \int_0^{\tau\wedge t} \lambda_s \mathrm{d}s, \quad t \in [0,T]$$

is a $\mathbb{P}$-martingale. Notice that for the sake of brevity we have written $\tilde{\lambda}_t = \lambda_t \mathbf{1}_{\{\tau \ge t\}}$. In particular, we obtain that the existence of the intensity implies that τ is a totally inaccessible $\mathbb{F}$-stopping time, see [6], so that $\mathbb{P}(\tau = \tilde{\tau}) = 0$ for any $\mathbb{F}$-predictable stopping time $\tilde{\tau}$. Furthermore, we suppose that the default time τ and the underlying Wiener process W, are independent. When λ is constant, τ is the moment of the first jump of a Poisson process.
In our setting the benchmarked payoff of a defaultable zero-coupon bond can be represented as follows:

$$\begin{aligned}\hat{H} &= \left(\mathbf{1}_{\{\tau > T\}} + \bar{\delta}\mathbf{1}_{\{\tau \le T\}}\right)\hat{S}^0_T \\ &= \left(1 + (\bar{\delta} - 1)D_T\right)\hat{S}^0_T,\end{aligned}$$

where $\bar{\delta}$ is supposed to be the random recovery rate. In particular, we assume that $\bar{\delta}$ is a random variable in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ depending only on T and τ, i.e.

$$\bar{\delta} = h(\tau \wedge T), \tag{4.13}$$

for some Borel function $h : (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $0 \leq h \leq 1$. Here we focus on the case when an agent recovers a random part of the promised claim at maturity. Moreover, we obtain that $\hat{H} \in L^2(\mathcal{F}_T, \mathbb{P})$. Thus, we can apply the results of Section 3 to compute the decomposition (3.4) for $\hat{H}$. Since the benchmarked primary security accounts are nonnegative $\mathbb{P}$-local martingales, decomposition (3.4) represents the GKW decomposition of $\hat{H}$ with respect to the square-integrable $\mathbb{P}$-local martingale $\hat{S}^0$. By applying the real world pricing formula, we obtain

$$\begin{aligned} \hat{U}_H(t) &= \mathbb{E}\left[\hat{H} \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[\hat{S}^0_T \left(1 + (h(\tau \wedge T) - 1) D_T\right) \middle| \mathcal{F}_t\right] \\ &= \underbrace{\mathbb{E}\left[\hat{S}^0_T \middle| \mathcal{F}_t\right]}_{:=\Phi_t} \underbrace{\mathbb{E}\left[1 + (h(\tau \wedge T) - 1) D_T \middle| \mathcal{F}_t\right]}_{:=\Psi_t}, \end{aligned} \tag{4.14}$$

for all $t \in [0, T]$. Note that the last equality in (4.14) follows from the independence of the numéraire portfolio and the default process, as is clear from (4.10). By substituting (4.12) and (4.11) into (4.14), we obtain

$$\hat{U}_H(t) = \Phi_t \cdot \Psi_t,$$

where

$$\begin{aligned} \Phi_t &= \hat{P}(0, T) \\ &\quad - \int_0^t \hat{P}(u, T) \left(\hat{S}^0_u - f(u) \cdot \frac{e^{-\frac{f(u)}{\hat{S}^0_u}}}{1 - e^{-\frac{f(u)}{\hat{S}^0_u}}}\right) \sqrt{(\hat{S}^0_u)^{-1} \alpha_u} \mathrm{d}W_u \end{aligned}$$

and

$$\Psi_t = \mathbb{E}\left[1 + (h(\tau \wedge T) - 1) D_T \middle| \mathcal{F}_t\right],$$

for each $t \in [0, T]$. Now it only remains to compute Ψ_t. First we note that

$$\begin{aligned} \Psi_t &= 1 + \mathbb{E}\left[h(\tau \wedge T) D_T \middle| \mathcal{F}_t\right] - \mathbb{E}\left[D_T \middle| \mathcal{F}_t\right] \\ &= 1 + \mathbb{E}\left[h(\tau \wedge T) D_T \middle| \mathcal{F}_t\right] - \left(1 - (1 - F_T)\right) \tilde{Q}_t \\ &= \mathbb{E}\left[h(\tau \wedge T) D_T \middle| \mathcal{F}_t\right] + (1 - F_T) \tilde{Q}_t, \end{aligned}$$

with

$$\tilde{Q}_t = \frac{1 - D_t}{1 - F_t}, \quad t \in [0, T],$$

where F stands for the cumulative distribution function of τ. We assume that $F_t < 1$, for every $t \in [0, T]$, so that $\tilde{Q}$ is well-defined. We note that the second equality in the above derivation follows from Corollary 4.1.2 of [5]. By using the same arguments as in [2], we obtain for every $t \in [0, T]$ the equation

$$\Psi_t = \mathbb{E}\left[g(\tau)\right] + \int_{]0,t]} \left(\tilde{g}(s) - \frac{1 - F_T}{1 - F_s}\right) \mathrm{d}Q_s, \tag{4.15}$$

where the function $\tilde{g} : \mathbb{R}^+ \to \mathbb{R}$ is given by the formula

$$\tilde{g}(t) = g(t) - e^{\int_0^t \lambda_s \mathrm{d}s} \mathbb{E}\left[\mathbf{1}_{\{\tau > t\}} g(\tau)\right],$$

with

$$g(x) = h(x \wedge T)\mathbf{1}_{\{x<T\}}.$$

Here h is the function introduced in (4.13). Moreover, we have used the relation

$$\mathrm{d}\tilde{Q}_t = -\frac{1}{1-F_t}\mathrm{d}Q_t, \quad t \in [0,T],$$

that follows from Lemma 5.1 of [5]. Since

$$\begin{aligned} \mathrm{d}[\Phi,\Psi]_t = -\hat{P}(t,T) &\left(\hat{S}^0_t - f(t)\frac{e^{-\frac{f(t)}{\hat{S}^0_t}}}{1-e^{-\frac{f(t)}{\hat{S}^0_t}}} \right) \sqrt{(\hat{S}^0_t)^{-1}\alpha_t} \\ &\cdot \left(\tilde{g}(t) - \frac{1-F_T}{1-F_t} \right) \mathrm{d}[W,Q]_t = 0, \quad t \in [0,T], \end{aligned}$$

by applying Itô's formula we obtain

$$\begin{aligned} &\mathrm{d}\hat{U}_H(t) \\ &= \Phi_t \mathrm{d}\Psi_t + \Psi_{t-}\mathrm{d}\Phi_t + \mathrm{d}[\Phi,\Psi]_t \\ &= \Phi_t \cdot \left(\tilde{g}(t) - \frac{1-F_T}{1-F_t} \right) \mathrm{d}Q_t \\ &\quad - \Psi_t \cdot \hat{P}(t,T) \left(\hat{S}^0_t - f(t) \cdot \frac{e^{-\frac{f(t)}{\hat{S}^0_t}}}{1-e^{-\frac{f(t)}{\hat{S}^0_t}}} \right) \sqrt{(\hat{S}^0_t)^{-1}\alpha_t}\mathrm{d}W_t. \end{aligned}$$

Hence the decomposition (3.4) for $\hat{H} = \hat{U}_H(T)$ is given by the expression

$$\hat{H} = \hat{P}(0,T) \cdot \mathbb{E}[g(\tau)] + \int_0^T \xi^{\hat{H},0}_s \mathrm{d}\hat{S}^0_s + L^{\hat{H}}_T,$$

where the benchmarked risk-minimizing strategy is of the form

$$\xi^{\hat{H},0}_t = P(t,T)\left(1 - \frac{f(t)}{\hat{S}^0_t} \cdot \frac{e^{-\frac{f(t)}{\hat{S}^0_t}}}{1-e^{-\frac{f(t)}{\hat{S}^0_t}}} \right) \Psi_t,$$

with $\xi^{\hat{H},j}_t = 0$ for $j \in \{1,2,\ldots,d\}$, and the benchmarked profit & loss appears as

$$\hat{C}^{\delta}_t = L^{\hat{H}}_t = \int_{]0,t]} \Phi_{s-} \left(\tilde{g}(s-) - \frac{1-F_T}{1-F_{s-}} \right) \mathrm{d}Q_s,$$

for every $t \in [0,T]$. Due to the boundness of $\bar{\delta}$, the function $F(z) = 1 - z\frac{e^{-z}}{1-e^{-z}}$ for $z \in [0,\infty]$ and the process Ψ, it is straightforward to verify that the resulting benchmarked hedgeable part of the claim forms a square-integrable $\mathbb{P}$-martingale, which makes the resulting strategy L^2-admissible. This gives an example for benchmarked risk-minimization of a defaultable zero coupon bond.

4.2 Defaultable Put on an Index

The numéraire portfolio S^{δ_*} can be realistically interpreted as a diversified equity index, see [22]. Index linked variable annuities or puts on the numéraire portfolio are products that are of particular interest to pension plans. The recent financial crisis made rather clear that the event of a potential default of the issuing bank has to be taken into account. As in the case of the defaultable zero coupon bond, we can now study similarly the problem of pricing and hedging a defaultable put on the numéraire portfolio with strike $K \in \mathbb{R}_+$ and maturity T using benchmarked local risk-minimization. The fair default free benchmarked price $\hat{p}_{T,K}(t)$ at time t is given by [22] in the form

$$\begin{aligned}\hat{p}_{T,K}(t) &= \mathbb{E}\left[\left.\frac{(K - S_T^{\delta_*})^+}{S_T^{\delta_*}}\right| \mathcal{F}_t\right] = \mathbb{E}\left[\left.\left(K\hat{S}_T^0 - 1\right)^+\right| \mathcal{F}_t\right] \\ &= -Z^2(d_1; 4, l_2) - K\hat{S}_t^1\left(Z^2(d_1; 0, l_2) - \exp\left\{-\frac{l_2}{2}\right\}\right),\end{aligned}$$

where $Z^2(x; \nu, l)$ denotes the non-central chi-square distribution function with ν degrees of freedom, non-centrality parameter l and which is taken at the level x. Here we have

$$d_1 = \frac{4\eta K}{\alpha_t\left(\exp\{\eta(T-t)\} - 1\right)}$$

and

$$l_2 = \frac{2\eta}{\alpha_t\left(\exp\{\eta(T-t)\} - 1\right)\hat{S}_t^1}.$$

By assuming the same default mechanism as in the previous example, also with the same recovery rate, the defaultable put has the benchmarked payoff

$$\hat{H} = \left(K\hat{S}_T^0 - 1\right)^+ \cdot \left(1 + (\bar{\delta} - 1)D_T\right).$$

By applying the real world pricing formula we obtain, as in (4.14), the relation

$$\hat{U}_H(t) = \mathbb{E}\left[\left.\left(K\hat{S}_T^1 - 1\right)^+\right| \mathcal{F}_t\right] \cdot \mathbb{E}\left[1 + (h(\tau \wedge T) - 1)D_T | \mathcal{F}_t\right] = \hat{p}_{T,K}(t) \cdot \Psi_t,$$

for all $t \in [0, T]$, with Ψ_t as in the previous example. Consequently,

$$\mathrm{d}\hat{U}_H(t) = \hat{p}_{T,K}(t)\mathrm{d}\Psi_t + \Psi_{t-}\mathrm{d}\hat{p}_{T,K}(t),$$

and, thus

$$\hat{H} = \hat{p}_{T,K}(0)\mathbb{E}\left[g(\tau)\right] + \int_0^T \xi_s^{\hat{H},0}\mathrm{d}\hat{S}_s^0 + L_T^{\hat{H}},$$

where the benchmarked risk-minimizing strategy is of the form

$$\xi_t^{\hat{H},0} = \Psi_{t-}\frac{\partial \hat{p}_{T,K}(t)}{\partial \hat{S}_t^0}$$

with $\xi_t^{\hat{H},j} = 0$, for $j \in \{1, 2, \ldots, d\}$. Here the benchmarked profit & loss appears as

$$\hat{C}_t^{\delta} = L_t^{\hat{H}} = \int_{]0,t]} \hat{p}_{T,K}(s-) \left(\tilde{g}(s-) - \frac{1 - F_T}{1 - F_{s-}} \right) \mathrm{d}Q_s,$$

for every $t \in [0, T]$, see (4.15). Due to the boundness of the hedge ratio $\frac{\partial \hat{p}_{T,K}(t)}{\partial \hat{S}_t^0}$, $\bar{\delta}$ and the process Ψ, the benchmarked hedgeable part of the contingent claim forms a square-integrable $\mathbb{P}$-martingale and the resulting strategy is L^2-admissible.

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APPENDIX

A Technical Proofs

Here we extend the result of Lemma 2.3 of [24] to the case of a general discounting factor. Our proof is similar to the one of Lemma 2.3 of [24], however it contains some differences due to the fact that all the strategy's components contribute to the cost.

Proof of Lemma 3.5. Suppose δ is a benchmarked risk-minimizing strategy. Fix $t_0 \in [0, T]$ and define a strategy $\tilde{\delta}$ by setting for each $t \in [0, T]$

$$\tilde{\delta}_t := \delta_t \mathbf{1}_{[0,t_0)}(t) + \eta_t \mathbf{1}_{[t_0,T]}(t),$$

where η is an $\mathbb{F}$-predictable process determined in a way such that the resulting strategy $\tilde{\delta}$ is L^2-admissible and

$$\hat{S}_t^{\tilde{\delta}} = \tilde{\delta}_t \cdot \hat{S}_t := \hat{S}_t^{\delta} \mathbf{1}_{[0,t_0)}(t) + \mathbb{E}\left[\hat{S}_T^{\delta} - \int_t^T \delta_s \cdot \mathrm{d}\hat{S}_s \middle| \mathcal{F}_t \right] \mathbf{1}_{[t_0,T]}(t).$$

Here we assume to work with an RCLL version. Then $\tilde{\delta}$ is an L^2-admissible strategy with $\hat{S}_T^{\delta} = \hat{S}_T^{\tilde{\delta}}$ and

$$\hat{C}_{t_0}^{\tilde{\delta}} = \mathbb{E}[\hat{C}_T^{\delta} | \mathcal{F}_{t_0}]. \tag{A.1}$$

Since $\hat{C}_T^{\delta} = \hat{C}_T^{\tilde{\delta}} + \int_{t_0}^T (\eta_u - \delta_u) \cdot \mathrm{d}\hat{S}_u$, we have

$$\hat{C}_T^{\delta} - \hat{C}_{t_0}^{\delta} = \hat{C}_T^{\tilde{\delta}} - \hat{C}_{t_0}^{\tilde{\delta}} + \mathbb{E}[\hat{C}_T^{\delta} | \mathcal{F}_{t_0}] - \hat{C}_{t_0}^{\delta} + \int_{t_0}^T (\eta_u - \delta_u) \cdot \mathrm{d}\hat{S}_u.$$

Taking the squares of both sides of the equation, we have

$$\begin{aligned}\left(\hat{C}^{\delta}_{T}-\hat{C}^{\delta}_{t_0}\right)^2 &= \left(\hat{C}^{\tilde{\delta}}_{T}-\hat{C}^{\tilde{\delta}}_{t_0}\right)^2 + \left(\mathbb{E}[\hat{C}^{\delta}_{T}|\mathcal{F}_{t_0}]-\hat{C}^{\delta}_{t_0}\right)^2 + \left(\int_{t_0}^{T}(\eta_u-\delta_u)\cdot \mathrm{d}\hat{S}_u\right)^2\\
&\quad + 2\left(\hat{C}^{\tilde{\delta}}_{T}-\hat{C}^{\tilde{\delta}}_{t_0}\right)\left(\mathbb{E}[\hat{C}^{\delta}_{T}|\mathcal{F}_{t_0}]-\hat{C}^{\delta}_{t_0}\right) + 2\left(\hat{C}^{\tilde{\delta}}_{T}-\hat{C}^{\tilde{\delta}}_{t_0}\right)\int_{t_0}^{T}(\eta_u-\delta_u)\cdot \mathrm{d}\hat{S}_u\\
&\quad + 2\int_{t_0}^{T}(\eta_u-\delta_u)\cdot \mathrm{d}\hat{S}_u\left(\mathbb{E}[\hat{C}^{\delta}_{T}|\mathcal{F}_{t_0}]-\hat{C}^{\delta}_{t_0}\right).\end{aligned}$$

Then conditioning with respect to $\mathcal{F}_{t_0}$, by (A.1) we obtain

$$\begin{aligned}\hat{R}^{\delta}_{t_0} &= \hat{R}^{\tilde{\delta}}_{t_0} + \left(\mathbb{E}[\hat{C}^{\delta}_{T}|\mathcal{F}_{t_0}]-\hat{C}^{\delta}_{t_0}\right)^2 + \mathbb{E}\left[\left(\int_{t_0}^{T}(\eta_u-\delta_u)\cdot \mathrm{d}\hat{S}_u\right)^2\Big|\mathcal{F}_{t_0}\right]\\
&\quad + 2\mathbb{E}\left[\left(\hat{C}^{\tilde{\delta}}_{T}-\hat{C}^{\tilde{\delta}}_{t_0}\right)\left(\mathbb{E}[\hat{C}^{\delta}_{T}|\mathcal{F}_{t_0}]-\hat{C}^{\delta}_{t_0}\right)\Big|\mathcal{F}_{t_0}\right]\\
&\quad + 2\mathbb{E}\left[\left(\hat{C}^{\tilde{\delta}}_{T}-\hat{C}^{\tilde{\delta}}_{t_0}\right)\int_{t_0}^{T}(\eta_u-\delta_u)\cdot \mathrm{d}\hat{S}_u\Big|\mathcal{F}_{t_0}\right]\\
&= \hat{R}^{\tilde{\delta}}_{t_0} + \left(\mathbb{E}[\hat{C}^{\delta}_{T}|\mathcal{F}_{t_0}]-\hat{C}^{\delta}_{t_0}\right)^2 + \mathbb{E}\left[\left(\int_{t_0}^{T}(\eta_u-\delta_u)\cdot \mathrm{d}\hat{S}_u\right)^2\Big|\mathcal{F}_{t_0}\right]\\
&\quad + 2\left(\mathbb{E}[\hat{C}^{\delta}_{T}|\mathcal{F}_{t_0}]-\hat{C}^{\delta}_{t_0}\right)\cdot\mathbb{E}\Big[\underbrace{\hat{C}^{\tilde{\delta}}_{T}+\int_{t_0}^{T}(\eta_u-\delta_u)\cdot \mathrm{d}\hat{S}_u}_{=\hat{C}^{\delta}_{T}}-\hat{C}^{\tilde{\delta}}_{t_0}\Big|\mathcal{F}_{t_0}\Big]\\
&\quad + 2\mathbb{E}\left[\left(\hat{C}^{\tilde{\delta}}_{T}-\hat{C}^{\tilde{\delta}}_{t_0}\right)\int_{t_0}^{T}(\eta_u-\delta_u)\cdot \mathrm{d}\hat{S}_u\Big|\mathcal{F}_{t_0}\right]\\
&= \hat{R}^{\tilde{\delta}}_{t_0} + \left(\mathbb{E}[\hat{C}^{\delta}_{T}|\mathcal{F}_{t_0}]-\hat{C}^{\delta}_{t_0}\right)^2 + \mathbb{E}\left[\left(\int_{t_0}^{T}(\eta_u-\delta_u)\cdot \mathrm{d}\hat{S}_u\right)^2\Big|\mathcal{F}_{t_0}\right]\\
&\quad + 2\mathbb{E}\left[\left(\hat{C}^{\tilde{\delta}}_{T}-\hat{C}^{\tilde{\delta}}_{t_0}\right)\int_{t_0}^{T}(\eta_u-\delta_u)\cdot \mathrm{d}\hat{S}_u\Big|\mathcal{F}_{t_0}\right].\end{aligned}$$

Because δ is benchmarked risk-minimizing, it has minimal risk. If $\tilde{\delta}$ is also risk-minimizing, we must have

$$\hat{R}^{\delta}_{t_0} = \hat{R}^{\tilde{\delta}}_{t_0}, \quad t\in[0,T]$$

and

$$\mathbb{E}\left[\left(\hat{C}^{\tilde{\delta}}_{T}-\hat{C}^{\tilde{\delta}}_{t_0}\right)\int_{t_0}^{T}(\eta_u-\delta_u)\cdot \mathrm{d}\hat{S}_u\Big|\mathcal{F}_{t_0}\right]=0,$$

since the residual optimal cost $\hat{C}^{\tilde{\delta}}_{T}-\hat{C}^{\tilde{\delta}}_{t_0}$ must be orthogonal to all integrals of the form $\int_{t_0}^{T}\xi_u \mathrm{d}\hat{S}_u$, with ξ L^2-admissible. Consequently, we obtain

$$\left(\mathbb{E}[\hat{C}^{\delta}_{T}|\mathcal{F}_{t_0}]-\hat{C}^{\delta}_{t_0}\right)^2 + \mathbb{E}\left[\left(\int_{t_0}^{T}(\eta_u-\delta_u)\cdot \mathrm{d}\hat{S}_u\right)^2\Big|\mathcal{F}_{t_0}\right]=0$$

and we can easily conclude that

$$\hat{C}^{\delta}_{t_0} = \mathbb{E}[\hat{C}^{\delta}_T | \mathcal{F}_{t_0}] \quad \mathbb{P}-\text{a.s.}.$$

Since t_0 is arbitrary, the assertion follows.

□

B Some Useful Definitions

We recall briefly the definition of $\mathbb{F}$-predictable projection of a measurable process endowed with some suitable integrability properties and the definition of $\mathbb{F}$-predictable dual projection of a raw integrable increasing process.

Theorem B.1 (predictable projection)**.** *Let X be a measurable process either positive or bounded. There exists an $\mathbb{F}$-predictable process Y such that*

$$\mathbb{E}\left[X_\tau \mathbf{1}_{\{\tau<\infty\}} \middle| \mathcal{F}_{\tau-} \right] = Y_\tau \mathbf{1}_{\{\tau<\infty\}} \quad \text{a.s.}$$

for every predictable $\mathbb{F}$-stopping time τ.

Proof. See [6] or [23] for the proof. □

Definition B.2. *Let A be a raw integrable increasing process. The $\mathbb{F}$-predictable dual projection of A is the $\mathbb{F}$-predictable increasing process B defined by*

$$\mathbb{E}\left[\int_{[0,\infty[} X_s \mathrm{d}B_s \right] = \mathbb{E}\left[\int_{[0,\infty[} {}^p X_s \mathrm{d}A_s \right].$$

For a further discussion on this issue, see e.g. [6].

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