Volatility targeting using delayed diffusions*†

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Abstract

A target volatility strategy (TVS) is a risky asset-riskless bond dynamic portfolio allocation which makes use of the risky asset historical volatility as an allocation rule with the aim of maintaining the instantaneous volatility of the investment constant at a target level. In a market with stochastic volatility, we consider a diffusion model for the value of a target volatility fund (TVF) which employs a system of stochastic delayed differential equations (SDDEs) involving the asset realized variance. Firstly we prove that, under some technical assumptions, contingent claim valuation on a TVF is approximately of Black-Scholes type, which is consistent with and supports the standing market practice. In second place, we develop a computational framework using recent results on Markovian approximations of SDDEs systems, which we then implement in the Heston variance model using an *ad hoc* Euler scheme. Our framework allows for efficient numerical valuation of derivatives on TVFs, whose typical purpose is the assessment of the guarantee costs of such funds for insurers.

Keywords: target volatility, portfolio strategy, stochastic delayed differential equations, finite-dimensional Markovian representation, stochastic volatility, guarantee costs, Euler scheme.
MSC 2010 classification: 91G10, 60H10, 91G20, 60H35.

1 Introduction

One of the categories of investors which is most suffering from the long-term consequences of the recent credit and sovereign debt crises is certainly that of the institutional investors. The generalized lowering of yields in the global economy resulting form the recent financial turmoil has brought these under the double pressure of having to generate long-term value in a low interest rate environment, while also having to comply with tighter regulatory constraints. Adding to the falling yields, outright equity investments are less and less attractive for pension funds and life insurers in the post-crisis market scenario, due either to regulatory disincentives, or the

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detrimental effect of the often-observed high implied volatility on hedge and guarantee costs. For example, during the 2007-2009 credit crisis, high volatilities and falling prices caused traditional products like constant proportion portfolio insurances to breach the minimum guarantee, which entailed increased market consistent insurance costs for the issuer as well as poor fund value for policyholders.

Some recent savings product innovations have bee specifically geared towards answering this dilemma. In particular, keeping under control guarantee costs while ensuring substantial access to the equity upside seems now to be a critical issue.

One attempt to achieve this was the introduction of structured products based on dynamically allocated target (or controlled) volatility mechanisms. Assume we construct a portfolio comprising of a risky asset (typically an equity, a basket or an index) and a reserve asset such as a zero coupon bond paying a risk-free rate. We define an allocation rule for this portfolio as follows: at certain fixed dates, the position in the risky asset is automatically adjusted proportionally to the ratio between a target constant and the measured asset volatility; the latter can be a forward-looking measure like VIX, or more commonly the historical volatility. In one such product, as the measured volatility rises the equity exposure decreases and the portfolio tends to be more invested in the bond; when volatility instead plunges, the portfolio increases its equity position. Assuming, as logical, a negatively-correlated asset price and volatility, we are effectively shielding the portfolio value in bearish markets by locking the earnings in the bond; conversely, we are able to secure consistent portions of equity returns as share prices rise back. This confers to the fund a downside protection feature which is a typical one for variable annuities funds and it is similar to those present in CPPIs. Also, this contrarian dynamic feeds back into the portfolio variability: in times of market turmoil the portfolio will be more invested in the bond, thereby reducing the total fund volatility; conversely, when low volatility regimes are detected, the targeting mechanism will increase the portfolio uncertainty as a result of equity investing. The net effect of a sequence of these adjustments should in principle stabilizing the volatility around the target level.

If this latter conclusion is correct, one could model the fair price of insuring against the fund not meeting a capital guarantee as a constant volatility hedge. That is, the price of capital insurance is reflected by the constant contractually-agreed target volatility value: therefore, unlike in the case of a CPPI, where insurers have to pay market consistent costs of the probability of the fund breaching the minimum insured amount, target volatility strategies allow to independently and transparently model the guarantee costs by calculating the fair value of a claim on a constant volatility asset, yielding to Black-Scholes valuation.

The final takeaway is that volatility targeting offers a built-in risk management system which not only, similarly to CPPI contracts, reduces the downside risk, but also controls the guarantee and hedging costs associated with underwriting one of these funds.

Even if a substantial number of volatility target funds are currently offered[1], volatility tar-

[1]Some examples are: State Street Target Volatility Triggers, J.P. Morgan Commodities Target Volatility

geting is objectively challenging to market. Besides the problem of effectively communicating its merits to non-specialist perspective clients, much of the evidence on which this concept relies is empirical and/or based on specific case studies. In this paper we address this issue by offering a realistic and mathematically rigorous model for target volatility investments.

Prior research on the topic has mainly focused on portfolio management and risk-return analysis. Chew [2], Goldsticker [11], and Xue [25] analyze empirical scenarios in which volatility targeting and related structured products offer an improved risk-return profile compared to equity or fails to do so. Assessing the Sharpe ratio, Romain et al. [19], suggest that constant volatility targeting offers a better risk-return tradeoff compared to a direct equity investment. Criticism is instead expressed by Vandenbroucke [21], who argues that in option-based capital protection it is only the issuer who ultimately benefits of volatility targeting in the form of lower costs, while the end investor should instead prefer standard option insurance if her realized volatility expectations are incompatible with the target level.

Jaschke [15] looks at the problem of hedging target volatility strategies in stochastic volatility markets with jumps, albeit without disclosing any particular model for the target volatility portfolio process. On similar lines of analysis, guarantee costs are more explicitly addressed by Morrison and Tadrowski [18], who study volatility smiles and skews of options on a target volatility fund and compare them to those from the naked equity, finding a substantial flattening of the smile/skew for models without jumps. However, they use a proprietary model for generating prices, and the dynamics assumed for the fund are also not shown. Coles [3] discusses the hedging problem and various limitations of volatility targeting connected with discrete time-sampling.

Finally, Stoyanov [20] is the main mathematical contribution. The author analyses the performance of a target volatility strategy in the Heston model using as a rebalancing factor the instantaneous volatility, and compares it to a fixed-mix allocation and an option-based capital insurance. The rebalancing factor used is thus a statistically estimated value and not a true market observable quantity, and as such is prone to both model and estimation error. In this paper we try to overcome this fundamental point by instead basing the portfolio rebalancing on the objective market historical volatility, which is the prevailing market practice.

A related financial product that has received some attention in the quantitative finance literature is the target volatility option (TVO). A TVO is an option where the total share exposure at maturity is given by the ratio of the target volatility and the realized volatility over the entire life of the option. Mathematical contributions on the pricing and hedging of this claim where offered by Di Graziano and Torricelli [5], Da Fonseca et al. [4], Grasselli and Marabel Romo [12], and Wang and Wang [22]. Although the TVO certainly shares common technical features with target volatility portfolio strategies, the two products remain structurally

Indices, S&P Risk Control Indexes, NASDAQ 100 Volatility target Index, SocGen SGI Vol Target Bric, BNP Risk Controlled Equity Indexes, Mitsubishi UFJ MAXIS TOPIX ETF, EuroStoxx 50 Risk Controlled Indexes and FTSE 100 Risk Target Excess.

different.

As far as we are aware, as of today, a continuous-time finance mathematical model for target volatility strategies based purely on observable market inputs is missing. The present paper aims at filling this gap. In a stochastic volatility framework, we build a portfolio process whose only drivers are an equity, a bond, and the equity realized volatility. In such a framework, for small estimation windows and large leverage caps, we are able to prove the proximity of derivative prices written on TVFs to Black-Scholes values with target volatility, which is one of the main purposes of these products.

In order to test the this main result we make use of a purpose-made Monte Carlo scheme of independent importance. A critical drawback in modeling financial products using SDDEs is that Monte Carlo simulations of the system are in general computationally costly, and converge slowly. This owes to the fact that delayed systems are not Markovian in the classic sense, but they are so only when considering the state space as also comprising of an infinitely-dimensional function space to which the past realized volatility function must belong. A way to circumvent this is to devise a finite-dimensional Markovian evolution that in some sense approximates the dynamics of the original delayed process, and using this approximating system for valuation (see e.g. Bernhart et al. [1]). We espouse this strategy by resorting to a multidimensional version of a theorem by Federico and Tankov [10] which allows us to reduce the infinitely-dimensional Markov problem as given at the outset by an SDDE, to an approximating finite-dimensional Markovian one. The additional dimension we need compared to Federico and Tankov treatment is due to the presence of the stochastic variance. The approximating Markov diffusion we obtain is numerically handled in the Heston model using an Euler scheme inspired by that of Higham and Mao [14]. A comparison with a plain Euler SDDE discretization shows a better tradeoff between precision and efficiency in the method we endorse.

This paper is organized as follows. In section 2 we lay down the model and derive the TVF wealth dynamics. In section 3 we discuss the martingale properties of TVF and prove the convergence of valuations on TVFs to the Black-Scholes formulae. In section 4 we enunciate the form of the theorem of Federico and Tankov [10] we need and derive the relevant approximating Markovian system as a corollary. In section 5 we formulate the Euler scheme allowing for Monte Carlo valuation of claims on the approximated value process. Section 6 shows some numerical experiments with the simulated portfolio process aiming at investigating its proximity to a Black-Scholes-Samuelson asset, namely: the pathwise behavior of price and volatility; the generation of volatility surfaces; the computation of Vega; some return/cost comparison between a TVF and the underlying equity; a comparison of the efficiency of our numerical method with a classic Euler SDDE approximation. Conclusions and views for further work are expressed in section 7. Proofs and technical points are in the appendices.

2 The portfolio process and the model

On a given filtered probability space $\mathbb{M} = (\Omega, \mathcal{F}, \mathbb{R}, \mathcal{F}_t, \mathbb{P})$ satisfying the usual assumptions for a market model, we define a riskless money market process B_t earning a constant fixed interest rate r as the solution of the ODE:

$$dB_t = rB_t dt, \quad B_0 = 1 \tag{2.1}$$

and a risky equity S_t following the geometric diffusion:

$$S_t = \mu_t S_t dt + \sqrt{v_t} S_t dW_t, \quad S_0 = s_0 > 0 \tag{2.2}$$

for an $\mathcal{F}_t$-adapted Brownian motion W_t, some deterministic continuously-compounded return rate $\mu_t > 0$, and an $\mathcal{F}_t$-adapted square-integrable stochastic variance process v_t. We also assume the minimal requirement that v_t is pathwise bounded. The aim is to build a self-financing portfolio in B_t and S_t which "targets", in some sense to be specified, a constant volatility level $\overline{\sigma} > 0$.

Let π_0 be the initial investors' endowment and (Θ_t^S, Θ_t^r), the self-financing trading strategy in S_t and B_t giving the fund position at time t. According to the general portfolio theory, the wealth process is given by:

$$\pi_t = \pi_0 + \int_0^t \Theta_u^B dB_u + \int_0^t \Theta_u^S dS_u. \tag{2.3}$$

Let $w_t = \Theta_t^S S_t / \pi_t$ be the time-t relative portfolio equity position. Using the self-financing condition we have:

$$\begin{aligned} d\pi_t &= \Theta_t^S dS_t + \Theta_t^r dB_t = w_t \pi_t \frac{dS_t}{S_t} + (1 - w_t)\pi_t \frac{dB_t}{B_t} \\ &= r\pi_t dt + w_t \pi_t (\mu_t - r) dt + w_t \sqrt{v_t} \pi_t dW_t. \end{aligned} \tag{2.4}$$

See also Stoyanov [20]. The portfolio process thus decomposes as a fixed riskless rate r paid on the portfolio, a risk premium $\mu_t - r$ weighted by the relative equity exposure w_t, and a noise term depending on both the market stochastic variance and the exogenous variable w_t.

To see how w_t should look like, we should first introduce the definition of *realized volatility* we shall use throughout the paper. Let $\mathcal{H}$ be the class of the L^2 positive functions with respect to the Lebesgue measure integrating to one:

$$\mathcal{H} = \left\{ h \in L^2(\mathbb{R}^+, \mathbb{R}^+; dt), \|h\|_1 = 1 \right\}. \tag{2.5}$$

An element $h \in \mathcal{H}$ acts an averaging rule for any square-integrable stochastic process by simply integrating against it. We can think of the elements of $\mathcal{H}$ as continuous-time variance estimators. In particular we can define a *realized variance* process RV_t^h as a functional of a h-average of v_t in the following way:

$$RV_t^h = \int_{\mathbb{R}^+} h(u) v_{t-u} du, \tag{2.6}$$

which is almost surely a finite quantity because the paths of v_t are bounded almost surely. The realized volatility will then be $\sqrt{RV_t^h}$. As an example, when $h(t) = \delta^{-1}\mathbb{I}_{\{0\leq t\leq \delta\}}$ for $\delta > 0$ the definition above produces the usual equal weight estimator for the quadratic variation of $\log S_t$ accrued in $[t-\delta, t]$.

The belief of the investment manager is that RV_t^h is a good proxy of the instantaneous equity variance. Her intuition comes from the mean integral theorem, stating that for $h \in \mathcal{H}$ and $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$, for all $\omega \in \Omega$ there exists a $\xi \in [a, b]$ such that:

$$RV_t^h(\omega) = v_{t-\xi}(\omega); \tag{2.7}$$

where ξ will be small so long as $h(t)$ has a sufficiently high mass around zero.

Therefore, by having in mind some desirable volatility level $\overline{\sigma}$ to be offered to the investor, she can exploit this belief in setting w_t, in such a way as having the log-returns of π_t to show approximately a constant standard deviation $\overline{\sigma}$. Let us consider the predictable process

$$w_t^h = \frac{\overline{\sigma}}{\sqrt{RV_t^h}}. \tag{2.8}$$

By taking into account (2.7), we have that at least pathwise, $w_t^h\sqrt{v_t}\pi_t dW_t \sim \overline{\sigma} dW_t$. That is, in a pathwise sense, the target volatility portfolio tracks a Black-Scholes-Samuelson asset with log-normally distributed returns of standard deviation $\overline{\sigma}$. Note also, that the TVS we are structuring is leveraged (excess of 100% equity position) if and only $\overline{\sigma} > \sqrt{RV_t^h}$. In any case, a cap $C \geq 1$ is generally put into place to avoid unreasonable leverage and thus excessive risk taking for the investor (typically $C = 1$ or $C = 1.5$). Therefore we modify equation (2.8) as:

$$w_t^h = \min\left\{C, \frac{\overline{\sigma}}{\sqrt{RV_t^h}}\right\} = \min\left\{C, \frac{\overline{\sigma}}{\left(\int_{\mathbb{R}^+} h(u)v_{t-u}du\right)^{1/2}}\right\} \tag{2.9}$$

yielding the full rebalancing process available to the asset manager to try and track the target volatility $\overline{\sigma}$. Setting $C = \infty$ in (2.9) recovers (2.8) and yields a TVS with no leverage cap. Normally, we will assume C to be finite, which also ensures the square-integrability of the process. In summary, we have the following definition:

Definition 2.1. In the market setting presented, a *target volatility portfolio strategy* (TVS) of underlying equity S_t, target volatility $\overline{\sigma}$, capping factor $C \in [1, \infty]$ and variance estimator $h \in \mathcal{H}$, is a self-financing, admissible trading strategy (Θ_t^S, Θ_t^r) such that $\Theta_t^S = w_t^h\pi_t/S_t$, where w_t^h is as in (2.9). The process $\pi_t = \pi_t(\pi_0, \overline{\sigma}, C, h)$ is the associated target volatility fund (TVF) value process, whose dynamics are given by (2.4) with $w_t \equiv w_t^h$.

We have introduced a stochastic portfolio in the form of a stochastic delayed process whose value at time t is given by a stochastic differential relation depending on a certain integral functional calculated up to time t. In particular, the stochastic system we are analyzing is not a Markovian one: the state of the system at any given time depends in principle on the entire system past evolution because of the presence of the lagged variance integral estimator. Alternatively, a Markov structure can be recovered by taking as a state variable a certain given variance realization belonging to some function space, yielding to an infinite-dimensional problem[2].

3 Martingale properties and contingent claim pricing

We now proceed to illustrate the problem of pricing derivatives and claims having a target volatility portfolio as an underlying. This is a crucial aspect for investors willing incorporate funds implementing TVSs in their portfolios, which allows them to correctly assess, hedge and risk manage the guarantee costs of these funds.

As customary, in order to proceed to the valuation of contingent claims on TVSs whose value process is given by (2.4) we assume the existence of a martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which all the financial assets show a return rate r. That is, we assume the existence of some variance market price of risk process γ_t such that $\mu_t - r = \gamma_t \sqrt{v_t}$. The associated Radon-Nikodym change of measure martingale to the risk-risk neutral measure is hence of the usual form:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = Z_t = \exp\left(-\int_0^t \gamma_u \sqrt{v_u} dW_u - \frac{1}{2}\int_0^t \gamma_u^2 v_u du\right). \tag{3.1}$$

Applying the Girsanov transform shows that the $\mathbb{Q}$-dynamics of S_t are:

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_t^* \tag{3.2}$$

for some $\mathbb{Q}$-Brownian motion W_t^*. The $\mathbb{Q}$-dynamics of π_t are then:

$$d\pi_t = r\pi_t dt + w_t^h \sqrt{v_t} \pi_t dW_t^*. \tag{3.3}$$

Now, we would like to find the value at time $t \leq T$ of a sufficiently regular European contingent claim F on π_T. By the general theory, this amounts to calculating the risk-neutral expectation:

$$V(t, \pi_t, v_t) = \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)} F(\pi_T)]. \tag{3.4}$$

We begin from forward contracts. Notably, if the risk-neutral dynamics of v_t are regular enough to make the discounted value of S_t into a $\mathbb{Q}$-martingale, then the same holds true for π_t. Indeed, we have the following proposition:

Proposition 3.1. *Under the measure $\mathbb{Q}$ the discounted wealth processes $\tilde{\pi}_t = e^{-rt}\pi_t$ is a local martingale. Furthermore, if $\tilde{S}_t$ is a true $\mathbb{Q}$-martingale, then so is $\tilde{\pi}_t$.*

[2]This contrasts with the special case of fixed-start averaging window where the problem can be effectively reduced to a finite-dimensional one: see e.g. Wilmott et al. [23].

This of course allows arbitrage-free valuation of forward contracts on TVFs. In order to proceed towards option pricing we prove a general theorem on the convergence of the log-returns of a TVF, which makes rigorous the ideas developed in section 2 and constitutes the main theoretical contribution of this paper.

Theorem 3.2. *Let $\tilde{S}_t$ be a martingale price process under $\mathbb{Q}$ on $[0, T]$, and $h^n(u)$ be a sequence in $\mathcal{H}$. Consider the target volatility strategies $\pi_t^{n,C} := \pi_t(\pi_0, \overline{\sigma}, C, h^n)$ and denote by w_t^n the process in (2.8) when $h \equiv h^n$, and with μ^n the measure whose Radon-Nikodym derivative with respect to the Lebesgue measure is h^n. Set finally*

$$X_t = \log(\pi_0) + (r - \overline{\sigma}^2/2)t + \overline{\sigma} W_t^*. \tag{3.5}$$

Assume that the following hold:

(A1) in the weak measure-theoretical sense

$$\lim_{n\to\infty} \mu^n = \delta_{\mathbf{0}} \tag{3.6}$$

where $\delta_{\mathbf{0}}$ is the Dirac delta function concentrated in 0;

(A2) there exist $p > 1$ and $q > p/(p-1)$ such that:

(A2.1) v_t is in $L^p(\Omega \times [0, T])$;

(A2.2) the sequence $(w_t^n)^2$ is bounded in $L^q(\Omega \times [0, T])$.

Then we have that

$$\lim_{n,C\to\infty} \log(\pi_t^{n,C}) = X_t \tag{3.7}$$

in $L^1(\Omega)$, for all $t \leq T$.

Assumption A1 has the natural interpretation of the sequence being given by averaging rules narrowing around the instantaneous value, and assumption A2.1 is clearly very mild. Assumption A2.2 seems to be the critical one: in general, L^q boundedness for $q > 1$ is sufficient to guarantee uniform integrability of the sequences considered, which lies at the heart of the convergence problem (see the appendix). Note also that the higher the maximum finite moment of v_t is, the less stringent A2.2 is.

As a consequence of theorem 3.2 we have convergence of Lipschitz-continuous claim values.

Corollary 3.3. *Let F be a Lipschitzian European claim on S_t maturing at time T. Under the assumptions of theorem 3.2, the $t < T$ value $V_t^{n,C}$ of $F(\log(\pi_T^{n,C}))$ satisfies*

$$\lim_{\substack{n\to\infty \\ C\to\infty}} V_t^{n,C} = V_t^{BS}(\overline{\sigma}) \tag{3.8}$$

where $V_t^{BS}(\overline{\sigma})$ is the $\overline{\sigma}$-Black-Scholes price of F.

Convergence of call values is obtained by call-put parity and combining corollary 3.3 with proposition 3.1.

As already mentioned, the case of put options is of particular importance to the financial operator. An insurer selling a unit-linked product typically offers a minimum return or capital guarantee on the contract. In order to meet his obligation the insurer has to hedge against the event of the fund not meeting such a guarantee, which can be attained for example by purchasing put options having the guarantee level as a strike price. Valuation of such a hedge is therefore of critical importance to a variable annuity provider for offering a transparent cost assessment to policyholders, or correctly valuing positions in balance sheets. By trading in a TVFs targeting some volatility level, the valuation of such a hedge can be, theoretically, attained using the Black-Scholes framework, at least whenever the estimation window is not too big and the capping level not too small.

We finish this section by establishing that the assumptions of theorem 3.2 are met by some popular volatility models and typical variance estimators, which will justify the analysis in the next sections.

Proposition 3.4. *The stochastic volatility model by Heston* [13] *and the 3/2 model with an appropriate non-explosion parametrization (e.g. Lewis* [16]*, Drimus* [7]*), with*

$$h^n(u) = n\,\mathbb{1}_{\{u\leq 1/n\}} \tag{3.9}$$

or

$$h^n(u) = \frac{\lambda e^{-\lambda u}}{1-e^{\lambda/n}}\mathbb{1}_{\{u\leq 1/n\}}, \quad \lambda > 0 \tag{3.10}$$

satisfy the assumptions of theorem 3.2.

The two sequences of estimators above correspond respectively to equal weighting an EWMA-type estimators. These classes will be employed further on in section 6.

Having laid out a theoretical basis justifying the market practice of devising TVSs based on the market realized volatility, in the next section we turn to illustrate our computational method for these structured products.

4 Markovian approximation of the TVF dynamics and approximating derivative values

Numerical simulation for derivative pricing on the process π_t^n can be of course performed by a full Euler discretization of the process π_t, which entails storing the variance path and calculating at each time step of the integrated variance over the previous period. This is computationally expensive, since sufficiently many integration points are needed at every step to keep the estimation accurate, which in turns requires a very fine time grid (see further on subsection 6.5).

Therefore we suggest here an alternative approach relying on a Markovian approximatng system of SDDEs depending on the delayed integrated system, whose theory is presented by Federico and Tankov [10]. The comparisons in subsection 6.5 seem to suggest that for European claim valuation this approach is superior to the simple Euler approximation of the SDDEs. Moreover, this approach can be straightforwardly extended for valuation of path-dependent options and portfolio optimization problems, which are computationally unmanageable by directly simulating an infinite-dimensional process.

In the following we shortly review the result of Federico and Tankov [10] we need. The authors main contribution is that controlled stochastic Ito diffusions depending on the lagged integral of the past of the process, admit a finite-dimensional Markovian approximation in the L^2-sup sense. Although the theorem is proved for one-dimensional autonomous diffusions only, it can be shown to also hold in the multi-dimensional non-autonomous setting ([10], remark 2.1). A short discussion on how to modify the proof for the multi-dimensional case is found in appendix.

We first need some preliminaries and notation. For $t > 0$, consider the n-dimensional delayed system:

$$dX_t = \alpha\left(X_t, \int_{\mathbb{R}^+} a(u)X_{t-u}du, t\right) + \sigma\left(X_t, \int_{\mathbb{R}^+} b(u)X_{t-u}du, t\right) dW_t \tag{4.1}$$

where W_t is a standard m-dimensional Brownian motion, $\alpha : \mathbb{R}^{2m} \times \mathbb{R}^+ \to \mathbb{R}^m$ is a measurable function, σ is a matrix of $n \times n$ measurable functions $\sigma_{i,j} : \mathbb{R}^{2n} \times \mathbb{R}^+ \to \mathbb{R}$, and $a = (a_1, \ldots, a_m)$, $b = (b_1, \ldots, b_m)$ are such that $a_i, b_i \in L^2(\mathbb{R}^+, \mathbb{R}; dt)$ for $i = 1, \ldots, m$. Clearly, the initial datum for (4.1) has to be a vector function $\underline{x}(t) = (\underline{x}_i(t))_{1\le i\le m}$ from $\mathbb{R}^-$ to $\mathbb{R}^m$. The product inside the integrals is taken componentwise.

We recall that the family of the *Laguerre polynomials* $\{L_k\}_{k\ge0}$ can be defined as

$$L_k(x) = \sum_{i=0}^{k} \binom{k}{i} \frac{(-x)^i}{i!}; \tag{4.2}$$

it is well-known that the Laguerre polynomials form an orthonormal basis of $L^2(\mathbb{R}^+, \mathbb{R}; e^{-t}dt)$. For $p > 0$ we then define the k-th *Laguerre function* as:

$$l_k^p(x) = e^{-px}\sqrt{2p}L_k(2px). \tag{4.3}$$

The sequence $\{l_k^p\}_{k\ge0}$ is easily shown to constitute an orthonormal basis for $L^2(\mathbb{R}^+, \mathbb{R}; dt)$. For $n \ge 1$ we look at the the subspaces V^n generated by $\{l_k^p\}_{k=0,\ldots,n-1}$. The projections of the functions a_i and b_i on V^n are given by:

$$a_{i,n} = \sum_{k=1}^{n} a_i^k l_{k-1}^p, \quad b_{i,n} = \sum_{k=1}^{n} b_i^k l_{k-1}^p \tag{4.4}$$

where we set $a_i^k = \langle a_i, l_{k-1}^p \rangle$, $b_i^k = \langle b_i, l_{k-1}^p \rangle$, with the brackets indicating the inner product on $L^2(\mathbb{R}^+, \mathbb{R}; dt)$. Finally, denote $a^n = (a_1^n, \dots, a_m^n)$ and $b^n = (b_1^n, \dots, b_m^n)$. For the initial datum $\underline{x}$ we use the analogous definitions for $\underline{x}_{i,n}$, $\underline{x}_i^n$ and $\underline{x}^n$. Throughout, by convention it is $\sum_{i=1}^0 \cdot := 0$.

Theorem 4.1 (Federico and Tankov 2015, multidimensional). *On the filtration $\sigma(W_t)_{t\geq 0}$ consider the stochastic delayed equation in (4.1). Assume further that α, σ are such that (4.1) possesses a unique strong solution. For $p > 0$ and $1 \leq j \leq n$, the $m \times (n+1)$-dimensional system of SDEs*

$$\begin{cases} dX_t^n &= \alpha\left(X_t^n, \sum_{j=1}^n a^j X_t^{n,j}, t\right) dt + \sigma\left(X_t^n, \sum_{j=1}^n b^j X_t^{n,j}, t\right) \cdot dW_t \\ dX_t^{n,j} &= \left(\sqrt{2p} X_t^n - 2p \sum_{i=1}^{j-1} X_t^{n,i} - p X_t^{n,j}\right) dt \end{cases} \tag{4.5}$$

with initial condition

$$((\underline{x}_i(0))_{i=1,\dots,m}, (\underline{x}_i^j)_{i=1,\dots,m, j=1,\dots,n}), \tag{4.6}$$

satisfies

$$\mathbb{E}\left[\max_{j=1,\dots,n}\left(\sup_{t\in[0,T]} (X_t^j - X_t)^2\right)\right] \leq C_{p,T,a,b}(\|a^n - a\| + \|b^n - b\|) \tag{4.7}$$

for $C_{p,T,a,b} > 0$, where $\|\cdot\|$ is an equivalent norm in the product Hilbert space $\prod_{i=1}^m L^2(\mathbb{R}^+, \mathbb{R}; dt)$.

The theorem holds for any L^2 function basis, but we formulate it using directly the Laguerre polynomials as this is the practical choice; see also and Bernhart et al. [1].

Now if we specify for the variance process v_t in the previous section an Ito diffusion, as a consequence of theorem (4.5) we obtain a finite-dimensional representation for a TVF.

Corollary 4.2. *In the market setting of section 2, let Z_t be another Brownian motion supported in $\mathbb{M}$ and independent of W_t, and define the SDE*

$$\begin{aligned} dv_t &= \eta(t, v_t)dt + \epsilon(t, v_t)\left(\rho dW_t + \sqrt{1-\rho^2} dZ_t\right), \quad && t > 0 \\ v_t &= \underline{v}(t) && t \leq 0 \end{aligned} \tag{4.8}$$

for some function $\underline{v}(t) : \mathbb{R}^- \to \mathbb{R}^+$, $\rho \in [-1, 1]$ and $\eta, \epsilon : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ being such that there exists exactly one strong solution to the one-dimensional equation (4.8) restricted to $\mathbb{R}^+$ with initial condition $v_0 = \underline{v}(0)$. Then for $n \geq 1$ the stochastic delayed diffusion $X_t = (\pi_t, v_t)$ has the following finite-dimensional Markovian approximation $X_t^n = (\pi_t^n, v_t, v_t^1, \dots, v_t^n)$ in the sense of theorem 4.1:

$$\begin{cases} d\pi_t^n = r\pi_t^n dt + w_t^{h,p,n} \pi_t^n (\mu_t - r) + w_t^{h,p,n} \pi_t^n \sqrt{v_t} dW_t \\ dv_t = \eta(t, v_t)dt + \epsilon(t, v_t)(\rho dW_t + \sqrt{1-\rho^2} dZ_t) \\ dv_t^k = \left(\sqrt{2p} v_t - 2p \sum_{i=1}^{k-1} v_t^i - p v_t^k\right) dt, \qquad k = 1, \dots, n, \end{cases} \tag{4.9}$$

with initial condition $(\pi_0, v_0, v_0^1, \dots, v_0^n)$, $v_0^k = \langle \underline{v}, l_{k-1}^p \rangle$, *for* $k = 1, \dots, n$, *and*

$$RV_t^{h,p,n} = \sum_{k=1}^{n} h^{p,k} v_t^k, \tag{4.10}$$

$$w_t^{h,p,n} = \min \left\{ C, \frac{\overline{\sigma}}{\sqrt{RV_t^{h,p,n}}} \right\}, \tag{4.11}$$

where $h^{p,k} = \langle h, l_{k-1}^p \rangle$.

The processes v_t^k are denominated *Laguerre processes* (compare Bernhart et al. [1]). Conditionally on v_T, the equations for v_t^k can be readily solved using the variation of constants method[3]; however, at a first analysis this does not seem to provide any particular insight on the system analytical properties.

In order to apply the corollary for derivative pricing, the hope is that by replacing the delayed diffusion X_t with the approximating diffusion X_t^n, we can get an approximated value for V by computing instead:

$$V^n(t, \pi_t^n, v_t) = \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)} F(\pi_T^n)]. \tag{4.12}$$

Under the measure $\mathbb{Q}$ induced by (3.1), equation (4.8) rewrites as:

$$dv_t = \eta^*(t, v_t)dt + \epsilon(t, v_t)(\rho dW_t^* + \sqrt{1-\rho^2} dZ_t), \tag{4.13}$$

where η^* is the risk-neutral drift coefficient. Now it is not too hard to see that the proof proposition 3.1 works also for the process π_t^n, so that for forward contracts we have $V(t, \pi_t, v_t) = V^n(t, \pi_t^n, v_t)$, for all t, n. That we can use the process π_t^n to approximately valuate general Lipschitz-continuous payoffs F of π_t is instead guaranteed by the following result:

Proposition 4.3. *Let* $F(x)$ *be a square-integrable Lipschitzian contingent European claim with Lipschitz constant* c_L. *Let* $X_t^n = (\pi_t^n, v_t, v_t^1, \dots, v_t^n)$ *be as in corollary 4.2 and*

$$V^n(t, \pi_t^n, v_t, v_t^1, \dots, v_t^n) = \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)} F(\pi_T^n)]. \tag{4.14}$$

For some constant K *depending on* $t, T, p, h, \overline{\sigma}, C$ *and* c_L *we have*

$$|V^n - V| \leq K \|h^n - h\|_2. \tag{4.15}$$

[3] This makes use of the fact that the system $\overline{v}_t = (v_t^k)_{k=1,\dots,n}$, writes as $\overline{v}_t = A(p)\overline{v}_0 + v_t \cdot \mathbb{1}$ where $A(p)$ is a triangular matrix. The explicit solution reads, for $t \leq T$:

$$v_t^k = e^{-pt} \left(\sum_{j=0}^{k-1} v_0^{k-j} \sum_{i=0}^{j} \frac{(-2pt)^i}{i!} + I_t^k(p) \right)$$

where

$$I_t^k(p) = \sqrt{2p} \sum_{j=0}^{k} \int_0^t e^{ps} v_s \frac{(-2p(t-s))^j}{j!} ds.$$

Again, the proposition above covers put options, and the value of the non-Lipschitzian call option claim is recovered as customary by using the call-put parity, provided we are under the assumptions of proposition 3.1.

5 A numerical scheme for the Heston model specification

Having derived from the infinitely-dimensional delayed system X_t the approximating Markovian one X_t^n, we now devise a numerical simulation scheme for X_t^n which can be practically implemented. We place ourselves in the situation where the stochastic variance follows a mean-reverting square root process, which is the popular model suggested by Heston [13]. We thus assume in (4.13) the coefficients

$$\eta^*(x,y) = \kappa(\theta - y), \qquad \epsilon(x,y) = \nu\sqrt{y} \tag{5.1}$$

where $\kappa, \theta, \nu > 0$ are respectively the risk-free mean reversion rate, speed and volatility of volatility parameters. The Yamada conditions being satisfied, implies that this equation has a unique strong solution for all parameter sets. Furthermore, when the Feller condition $2\kappa\theta \geq \nu^2$ is in effect, v_t is almost surely positive. Also, the parametrization above implies a market price of risk of the form $\lambda\sqrt{v_t}$ for some constant λ. Under these assumptions it can also be verified, e.g. by using Feller explosion test that S_t is a true martingale, i.e., we are under the hypotheses of proposition 3.1 (for further details see e.g. Lewis [16]).

A well-known practical difficulty in the Heston model discretization is that while the Feller condition implies that $v_t > 0$ almost surely, its raw Euler scheme can become negative with positive probability. Various modifications of the natural discretization have been proposed to circumvent this problem, involving reflecting and/or flooring the variance process at time instants when it becomes negative. Our scheme makes use of the reflection fix by Higham and Mao [14] which takes the absolute value of the variance in the next step diffusion every time it goes negative. This is known to guarantee positivity of the discretized variance, as well as keeping the discretized process very close to the actual variance process[4]. The Laguerre processes are discretized using a simple forward scheme; such a choice is computationally efficient and it does not appear to generate stability issues for the time step sizes used in our numerical tests.

Let $\Delta = \{0 = t_0, \ldots, t_m = T\}$ be a partition of $[0,T]$ with $\Delta_t = t_{i+1} - t_i$, and i.i.d. random variables $\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i} \sim \mathcal{N}(0, \Delta_t)$, $\Delta Z_{t_i} = Z_{t_{i+1}} - Z_{t_i} \sim \mathcal{N}(0, \Delta_t)$, for all $i = 1, \ldots, m$.

[4]Other choices are possible, for example the full truncation scheme by Lord et al. [17].

We define our log-Euler scheme $X_t^{n,\Delta}$ for X_t^n as:

$$\begin{cases} \log \pi_{t_{i+1}}^{n,\Delta} = \log \pi_{t_i}^{n,\Delta} + \left(r - (w_{t_i}^{h,p,n,\Delta})^2 |v_{t_i}^{\Delta}|/2\right) \Delta_t + w_{t_i}^{h,p,n,\Delta} \sqrt{|v_{t_i}^{\Delta}|} \Delta W_{t_i} \\ v_{t_{i+1}}^{\Delta} = v_{t_i}^{\Delta} + \kappa \left(\theta - v_{t_i}^{\Delta}\right) \Delta_t + \nu \sqrt{|v_{t_i}^{\Delta}|} \left(\rho \Delta W_{t_i} + \sqrt{1-\rho^2} \Delta Z_{t_i}\right) \\ v_{t_{i+1}}^{k,\Delta} = v_{t_i}^{k,\Delta} + \left(\sqrt{2p} v_{t_i}^{\Delta} - 2p \sum_{j=1}^{k-1} v_{t_i}^{j,\Delta} - p v_{t_i}^{k,\Delta}\right) \Delta_t, \ k = 1, \dots n, \end{cases} \tag{5.2}$$

where:

$$RV_{t_i}^{h,p,n,\Delta} = \sum_{k=1}^{n} h^{p,k} v_{t_i}^{k,\Delta} \tag{5.3}$$

and

$$w_{t_i}^{h,p,n,\Delta} = \min \left\{ C, \frac{\overline{\sigma}}{\sqrt{RV_t^{h,p,n,\Delta}}} \right\}. \tag{5.4}$$

This scheme allows the pricing of call and put options on the approximate dynamics π_t^n of a TVF. Indeed, we have the following weak convergence result:

Theorem 5.1. *For $\log \pi_T^{n,\Delta}$ as defined by the Euler scheme (5.2), let $K > 0$, $F_1(x) = (K - e^x)^+$, $F_2(x) = (e^x - K)^+$. Then*

$$\lim_{\Delta_t \to 0} |\mathbb{E}^{\mathbb{Q}}[F_i(\log \pi_T^{n,\Delta})] - \mathbb{E}^{\mathbb{Q}}[F_i(\log \pi_T^n)]| = 0 \tag{5.5}$$

for $i = 1, 2$.

The proof of this theorem can be found in appendix B.

6 Numerical analysis

In this section we conduct several numerical experiments aiming at establishing if our model supports the financial practice of assuming a TVF to be a constant volatility investment. Also we compare its efficiency against a standard Euler discretization of equation (2.4).

The scheme (5.2) has been implemented in JAVA 8 using the `finmath` library[5] and ran on a machine with 4 GB RAM and a 2 GhZ processor. The parameters for the Heston model are taken from the calibration of the Hang Seng volatility index (VHSI) of Stoyanov [20], table 18, and are:

$$\kappa = 0.3765, \quad v_0 = \theta = 0.0426, \quad \nu = 0.1714, \quad \rho = -0.8235. \tag{6.1}$$

The risk-free interest rate is $r = 0.02$, the spot price $S_0 = 100$, and the past realized variance $\underline{v}(t)$ is assumed to be constant at v_0. The target volatility has been set at $\overline{\sigma} = 0.1$, and the

[5] `http://finmath.net/finmath-lib`.

fund is assumed to be unleveraged, i.e. $C = 1$. We use the families in $\mathcal{H}$ already introduced in proposition 3.4:

$$h_1(x;\delta) = \delta^{-1}\mathbb{1}_{\{0\le x\le\delta\}}, \quad h_2(x;\delta,\lambda) = \frac{\lambda e^{-\lambda x}}{1-e^{-\lambda\delta}}\mathbb{1}_{\{0\le x\le\delta\}}. \tag{6.2}$$

These correspond respectively to an equally-weighted average of window length δ and an EWMA-type estimator with weighting exponential factor λ and sampling window length δ. Interestingly, the Fourier coefficients $h_i^{p,k}$ for $i = 1, 2$, can be analytically derived, which saves much computational time. The formulae can be found in appendix C.

We use $\delta_1 = 1/12$ and $\delta_2 = 1/50$ corresponding to a monthly and weekly set of return data. The parameter λ can be determined by the criterium that at the final point δ for the variance recording, the contribution of the the last return data point is weighted only a fraction $q < 1$ of the most recent return considered. We can achieve this by setting $\lambda_i = -\log(q)/\delta_i$; choosing further $q = 0.5$ gives $\lambda_1 = 8.317$ and $\lambda_2 = 34.657$.

Finally, we have to pick an order n for the Markovian system (4.2) as well as the Laguerre function parameter p. By looking at figures 1 and 2, we see that the two estimators respond differently to variations of p and n; the high error in the approximation of h_1 is due to the oscillatory behavior of the Laguerre approximating function around the discontinuity point, which is to a great extent relaxed by the introduction of the smoothing exponential factor in h_2. In view of these results we choose throughout $p = 20$ and $n = 15$.

The simulations of the Heston asset S_t have been performed using the scheme by Higham and Mao [14] which is the choice consistent with (5.2). In the following, we shall often abuse of notation by identifying the stochastic processes with their discretized counterparts, implicitly using theorem 5.1 where necessary.

6.1 Path properties

In order to illustrate the pathwise properties of a TVS wealth process, for some given realizations of the processes W_t and Z_t we graphically represent the diffusive volatility and value evolution of two TVFs using the estimator h_1 at both lags δ_1 and δ_2, and compare it to those of the the asset S_t. The time horizon is 3 years, covered by 500 simulated steps; the dynamics shown are the physical ones, with $\mu_t = 0.05$.

In both the figures 3 and 6 we plot the process $w_t^{h,p,n}\sqrt{v_t}$ and compare them to the diffusion $\sqrt{v_t}$. The corresponding TVF price processes are shown in figure 4 and 7, and compared to a Black-Scholes-Samuelson asset of constant volatility $\overline{\sigma}$ and drift μ_t. Finally, in figures 5 and 8 we plot the process $w_t^{h,p,n}$ in terms of the total percentage portfolio equity exposure.

We have selected two typical but opposing scenarios. In figure 3 the asset volatility realizes at a very high level. In this case the rebalancing mechanism of both the TVSs manages to track the target volatility level fairly well, and the TVF volatility varies in a very limited range around $\overline{\sigma}$. In addition, as it can be seen in figure 4, while the price of S_t experiences substantial drops due to the negative return/volatility correlation, the TVFs manage to stay roughly level. The TVS rebalancing mechanism effectively cancels out the equity/volatility leverage effect: this is the downside reduction feature embedded in a TVS.

On the other hand, in figure 6 we observe a prolonged volatility drop, yielding to an increase in the value of S_t. As volatility lowers the rebalancing process eventually exceeds the capping constant and the fund becomes fully invested in equity. From the moment the cap is struck, the TVF evolves according to exactly the same dynamics as S_t, and the leverage effect is reintroduced. However, the fund cannot regain the ground lost to the equity when volatility was dropping and the price rising, since in that time frame a nonzero bond position was present in the portfolio. The final outcome is that the TVF is dominated by S_t in value, as visible in figure 7. Still, we observe that in one such circumstance the fund value being lower than equity is contingent upon assuming $C = 1$. Indeed, an inspection of equation (2.4) reveals that under this assumption the drift of π_t^n is lower than μ_t. But if we let $C > 1$ then this inequality no longer necessarily holds true; in such a case the TVF may have shown a higher drift than the equity. Further to this, the constant C act as a multiplier of the leverage effect. In other words, allowing for leverage when volatility is low generates a higher potential portfolio growth, as expected.

6.2 Volatility surfaces

Effectively, the approximation of option prices on TVFs by the Black-Scholes formula, can be understood as the superposition of four different approximations: two structural, depending on the contractual parameters δ and C, one functional, from the Markovian representation, and one analytical from our Euler scheme. By looking at the volatility surfaces, in this section we test how well this procedure performs.

Using our Euler scheme, we compare here the implied volatility surfaces extracted from call options on the asset S_t evolving according the Heston model, and that of various TVFs based on such an asset. We use 100.000 sample paths and consider a strike range between -20% to $+20\%$ moneynesses, with maturities ranging from 6 months up to 3 years. The time step is $\Delta_t = 0.01$, representing a bi-weekly portfolio rebalancing. For the rather computationally demanding chosen values of p and n, the runtime for generating a volatility surface with a maturity and moneyness step of respectively 3 months and 5% (99 points in total) is of about 200 seconds on average on the machine we used. We consider all the four possible combinations of δ_i and h_i.

In figures 10 to 13 we plot the volatility surfaces for all the possible considered estimation window and weighting function combinations. The level of the surfaces is found to be consistently close to $\overline{\sigma}$; with the exception of the 6 months maturity section which shows some surviving skew, a nearly flat surface is attained in all the plots. In contrast, we observe in figure 9 that with the given parameters the skew of the Heston asset volatility surface is persistent across the whole term structure.

Looking at table 1 we see that for monthly averaging, the estimator h_2 seems to be performing slightly better than h_1, in the sense of keeping the volatility surface level closer to the target value; for weekly averaging we observe the opposite trend, although the evidence is not very clear cut. More importantly, for each given h_i, we find implied volatilities to be nearer to the target level when using the lag δ_2, which is consistent with the main results of section 3. Finally, we observe that results improve as maturity increases.

6.3 Vega

Another feature emerging as an effect of the convergence to a constant volatility assets, consists in the reduction of the volatility sensitivity of option prices.

Let us in the present set-up define the Vega $\mathcal{V}_t$ of an option written on a TVF as the derivative $\mathcal{V}_t = \partial V_t / \partial \sigma$ of the option value V_t with respect to the instantaneous volatility $\sigma_t = \sqrt{v_t}$ of the equity component. As a consequence of corollary 3.3, and assuming that the technical conditions for interchanging the limit on δ and C and differentiation hold, we see that the Vega of the option tends to 0 as $\delta \to 0$ and C grows bigger.

To verify this, we numerically compute these values using the central differences method, i.e.:

$$\mathcal{V}_t = \frac{\partial V_t}{\partial \sigma} \sim \frac{V_t(\sigma_t + h) - V_t(\sigma_t - h)}{2h} \tag{6.3}$$

for $h \sim 0$. In figure 14 we represent the values of Vega of call options written on S_t and π_t^n with $\sigma_0 = \sqrt{v_0}$, for a given fixed strike, as a function of time to maturity. Since typically Vega (for example, in the Black-Scholes model) peaks around at the ATM forward point, there is where it has maximum importance; therefore we choose to plot the direction $K = S_0 e^{rt}$. We observe that the level is greatly reduced with respect to the Heston asset. Also, while the Vega of S_t increases with time to maturity, that of π_t^n tends to zero, in accordance to the increased similarity of a TVF to a constant volatility asset for longer time horizons. In figure 15 we represent the call option Vegas of S_t and π_t^n at time $T = 1$ as a function of the forward strike value. The familiar pattern peaking around the ATM forward point and tending to zero for extreme moneynesses is observed for both. Yet again, the difference in levels is striking.

The reduction in the instantaneous volatility sensitivity has the strong implication that options written on a TVF require little volatility hedging. This means that for a re-insurer the cost of hedging a guarantee on a TVF is potentially sensibly inferior to other typical variable annuities products, whose hedge generally requires to be closely monitored.

6.4 Portfolio insurance and returns

We turn now to see how the result on the skew and level reduction reflects on the cost of insuring and managing a TVF. In this subsection we use δ_1 and h_1. In order to see how the volatility surface level reduction affects insurance costs we consider for instance an option-based portfolio insurance (OBPI), that is, a structured product consisting in a portfolio of an asset and a put option written on it. We consider a first OBPI directly based on the equity S_t, and a second one on a TVF π_t derived from S_t. Let P_t and be the put option value as function of the strike, maturity and underlying. The values of these two products can be written as:

$$V^{OBPI}(S_t) = S_t + P_t(S_t, K, T) \tag{6.4}$$

$$V^{OBPI}(\pi_t) = \pi_t + P_t(\pi_t, K, T) \sim \pi_t + P_t(\pi_t^n, K, T) \tag{6.5}$$

remembering in the second equation proposition 4.3. As a concrete example, by setting $K = S_0$ and $T = 1$ we have $P_t(S_t, K, T) = 7.0316$ and $P_t(\pi_t^n, K, T) = 3.0115$[6]. By trading in a TVF instead of the equity, the total reduction of insuring the capital π_0 thus amounts to 57.17%.

On the other hand, lowering volatility also impacts the portfolio growth rate. As already observed, when the TVF is unleveraged, i.e. $C = 1$, the drift of π_t is less than μ_t. This amounts to say that the arithmetic mean of the returns of π_t is always less than that of S_t. However, when observing the expected log-return of π_t at T, given by the formula:

$$\mathbb{E}[\log(\pi_T/\pi_0)] = rT + (\mu_T - r)\int_0^T w_u^h du - \frac{1}{2}\int_0^T (w_u^h)^2 v_u du, \tag{6.6}$$

this does not have to be the case, because the term $(w_t^h)^2$ also depresses the downward return adjustment due to compounding. For example when $\mu_t = 0.05$ we calculate for S_t an average one year arithmetic return of 5.15%, which for π_t^n reduces to 3.71%; instead, when geometrically averaging the returns (i.e., first-order arithmetically averaging the log-returns) the values are respectively 2.95% and 3.14%. In other words, the risk-return tradeoff between the two investment options he depends on the averaging rules applied.

6.5 Comparison with a standard Euler discretization

We tested the efficiency of our Markovian approximation method against a direct implementation of the Euler scheme for th SDDE of π_t given by (2.4)-(2.9). Each variance path is stored, and at each time step the variance estimator is adjourned by removing the oldest value in the window and adding the new instantaneous variance realization. The integrated variance is then computed numerically using the trapezoid quadrature.

Employing 50.000 paths simulations and the estimator h_1 we considered a 1-year ATM option on a TVF and performed the following exercises. Firstly, we fixed a number of 100 time steps and compared the option prices and runtimes for the two methods using both δ_1 and δ_2 and benchmarking against the Black-Scholes target value: the results are in table 2. One observes that the better performance of the full Euler scheme is illusory; valuations using this method, for the given number of time steps, are simply too distant from the Black-Scholes price. This is evidently because too few variance values are considered in the integral quadrature, which

[6]For comparison, the Black-Scholes $\overline{\sigma}$ put price is 3.0367.

is especially problematic for δ_2. Instead, the slower (for the time steps number considered) Markovian approximation does not suffer from this drawback and achieves better precision.

Therefore, we increased the number of steps in the pure Euler discretization to try to reduce the discretization error in the realized variance integral. As observed in table 3, the quality of the simulation then improves, but to a great expense of computational time. Indeed, to get a precision similar to the Markovian approximation method we need 400 time steps at the lag δ_1 and 800 time steps at the lag δ_2, implicating an increase of computational time of respectively a factor 6 and 10, well exceeding that of our suggested method. Especially for the narrower window δ_2 the Markovian approximation seems then to be preferable to the simple Euler scheme for SDDEs.

7 Conclusions and future possible improvements

Target volatility strategies are risk-controlled dynamic portfolio allocations operated by some funds with the aim of maintaining a target global volatility level of the investment. The demand for such products is mainly driven by long term investor seeking to gain access to the equity market while being able to control/reduce the overall volatility risk.

In this paper we presented the first continuous-time stochastic model of such investment strategies which uses as a driver for the allocation rule an actual market observable quantity, i.e. the equity realized variance. In order to do so, we found natural to model the target volatility strategy wealth dynamics using a system of differential delayed stochastic equations. The model also allows the flexibility of choosing between different variance estimators.

Motivated by the general willingness of insurers of offering minimum return guarantees, which are typically met by taking a long position in a put option on the TVF, we tackled the valuation problem of contingent claims written on these funds.

On a theoretical level, under some technical assumption compatible with actual stochastic volatility models, we proved a mean-convergence theorem of the log-price of a TVF to the normal distribution, from which the convergence of option prices to Black-Scholes values follows. This reconciles the market practice with the natural interpretation of a TVF as a lagged diffusion.

On a computational side, we used an approximate finite-dimensional Markovian representation for the TVF dynamics to formulate an Euler scheme which allows for improved Monte Carlo valuations compared to the simple SDDE Euler discretization, and proved its convergence.

Using this scheme, we have finally produced abundant numerical evidence confirming that in a market with stochastic variance following the square root process, the TVS works as expected, in the sense of determining a generalized lowering of the implied volatilities in the market, and attaining implied price distributions much more similar to lognormal compared to the skewed/leptokurtic market consistent ones.

In our view further work on the topic should in first place concentrate on if, and how, the SDDE modeling strategy for a TVS can be extended to the case in which the underlying is modeled through a jump-diffusion. It has been suggested (see e.g. Morrison and Tadrowski [18], Xue [25]) that the presence of jumps breaks down the quality of the approximation of the instantaneous variance process with the realized volatility estimation, because after a jump a TVF is systematically underweight equity, as the jumps stick in the variance estimator even when the equity process has resumed its normal diffusive behavior. Further mathematical investigation of the correctness of these claims can be helpful to establish the viability and effectiveness of TVSs in markets exhibiting jumps in returns and/or volatility.

A second possible improvement would be introducing, in this or other models, the presence of triggers. In order to reduce transaction costs, it is common practice for funds not to rebalance at every given discretization date, but only when the volatility raises or lowers more than by a threshold amount. How and if this feature impacts the results obtained here could be subject to further investigation.

8 Appendices

Throughout the appendices all the expectations and dynamics are unless otherwise stated considered in the risk-neutral measure.

Appendix A: proofs and discussion of the main results

Proof of proposition 3.1. That the discounted values are local martingales can be directly read off from the asset dynamics. It has been observed that the martingale property for an exponential asset model is verified if and only if the variance process does not explode before T under the probability measure whose Radon-Nykodym derivative is induced by the discounted asset value (see Lewis [16], Wong and Heyde [24]). Once this is assumed for S_t, boundedness of w_t^h extends such a property to π_t. To prove this, following [24] define ,for a square-integrable process β_t, the stopping time:

$$\tau_n^{\beta} = \inf\left(t : \int_0^t \beta_u du \geq n\right) \tag{8.1}$$

and let τ^{β} be the pointwise limit of the sequence above. Define $\mathbb{Q}^*$ as the measure induced by the stochastic exponential given by the normalized discounted asset price, i.e.

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \frac{\tilde{S}_t}{S_0}. \tag{8.2}$$

Since by assumption $\tilde{S}_t$ is a martingale obtained as the stochastic exponential of $\sqrt{v_t}$, from [24], theorem 1, follows $\mathbb{Q}^*(\tau^v > T) = 1$. Being $\tilde{\pi}_t$ the stochastic exponential of $w_t^h\sqrt{v_t}$, since $w_t^h\sqrt{v_t} \leq C\sqrt{v_t}$ almost surely; we have that

$$\tau_n^{w^2 v} \geq \tau_n^{C^2 v}. \tag{8.3}$$

But now $\tau^{C^2 v} = \tau^v$ because $n_k = \lceil k/C^2 \rceil$ is such that $\tau^{C^2 v}_{n_k}(\omega) > \tau^v_k(\omega)$ for all ω, so that by taking limits in (8.3) one has $\tau^{w^2 v} \geq \tau^v$. Consequently, for all $T > 0$:

$$\mathbb{Q}^*(\tau^{w^2 v} > T) \geq \mathbb{Q}^*(\tau^v > T) = 1, \tag{8.4}$$

and the claim follows again by [24], theorem 1. □

For the concept of uniform integrability used in the proofs below, one reference is Diestel [6].

Proof of theorem 3.2. Define the processes $\pi^{n,\infty}_t = \pi_t(\pi_0, \overline{\sigma}, \infty, h^n)$ and $\Pi^{\cdot,\cdot}_t = \log(\pi^{\cdot,\cdot}_t)$. Then, for all $t \leq T$:

$$\mathbb{E}[||\Pi^{n,C}_t - X_t|] < \mathbb{E}[|\Pi^{n,\infty}_t - X_t|] + \mathbb{E}[|\Pi^{n,C}_t - \Pi^{n,\infty}_t|]. \tag{8.5}$$

Let us consider the first summand in (8.5). By assumption A1:

$$\int_{\mathbb{R}+} h^n(u) v_{t-u}(\omega) du \to v_t(\omega) \tag{8.6}$$

almost surely in Ω. Thus $(w^n_t)^2 v_t \to \overline{\sigma}^2$ almost surely for all t. Consider then the system

$$\begin{cases} q_0 > 1 \\ q_1 \geq 1 \\ q_0 q_1 < q \\ \dfrac{q_0 q_1}{q_0 - 1} < p \end{cases} \tag{8.7}$$

whose set of solutions is non empty because $q > p/(p-1)$. For a pair q_0, q_1 solving the above, let p_0 and p_1 be the respective Hölder conjugates. Using the Hölder inequality and combining assumptions A2.1 and A2.2 one sees that

$$\int_0^T \mathbb{E}[(w^n_t)^{2q_1} v^{q_1}_t] dt < \left(\int_0^T \mathbb{E}\left[(w^n_t)^{2q_1 q_0}\right] dt \right)^{1/q_0} \left(\int_0^T \mathbb{E}[v^{q_1 p_0}_t] dt \right)^{1/p_0} < c_q c_p \tag{8.8}$$

for some constants $c_q, c_p > 0$. In particular taking $q_1 = 1$ shows the square integrability of $w^n_t \sqrt{v_t}$. Instead if $q_1 > 1$, by e.g. [6], example 1, equation (8.8) implies that the sequence $(w^n_t)^2 v_t$ is uniformly integrable in $(\Omega \times [0,T], \mathbb{Q} \otimes \mu_T)$, where μ_T is the normalized Lebesgue measure on $[0,T]$. Therefore, we can invoke Vitali's theorem[7] to conclude $(w^n_t)^2 v_t \to \overline{\sigma}^2$ in $L^1(\Omega \times [0,T])$.

Next, solving the log-equation for (2.4) in the risk-neutral measure yields, for $t \leq T$:

$$\Pi^{n,\infty}_t = \log(\pi_0) + rt - \frac{1}{2} \int_0^t (w^n_u)^2 v_u du + \int_0^t w^n_t \sqrt{v_t} dW^*_u. \tag{8.9}$$

[7] Clearly, $(w^n_t)^2 v_t$ is not a dominated sequence in L^1.

Then, using Fubini's theorem, Ito isometry and simple estimates:

$$
\begin{aligned}
\mathbb{E}[|\Pi_t^{n,\infty} - X_t|] <& \frac{1}{2}\mathbb{E}\left[\int_0^t |(w_u^n)^2 v_u - \overline{\sigma}^2| du\right] + \mathbb{E}\left[\left(\int_0^t (w_u^n \sqrt{v_u} - \overline{\sigma}) dW_u^*\right)^2\right]^{1/2} \\
&< \frac{1}{2}\int_0^t \mathbb{E}\left[|(w_u^n)^2 v_u - \overline{\sigma}^2|\right] du + \sqrt{2}\left(\int_0^t \mathbb{E}\left[|(w_u^n)^2 v_u - \overline{\sigma}^2|\right] du\right)^{1/2} \qquad (8.10)
\end{aligned}
$$

and the second line tends to zero by the previous part.

Regarding the second summand in (8.5), first observe that $q_p = p/(p-1)$ is the Hölder conjugate of p. Therefore we can choose $p' < p$ such that its conjugate q' satisfies $q_p < q' < q$. By A2.1 we then have the following estimate:

$$
\begin{aligned}
\sup_n \mathbb{E}[|\Pi_t^{n,C} - \Pi_t^{n,\infty}|] <& \frac{1}{2}\sup_n \int_0^t \mathbb{E}\left[(w_u^n)^2 v_u \mathbb{I}_{\{(w_u^n)^2 > C\}}\right] du \\
&+ \sqrt{2}\sup_n \left(\int_0^t \mathbb{E}\left[(w_u^n)^2 v_u \mathbb{I}_{\{(w_u^n)^2 > C\}}\right] du\right)^{1/2} \\
<& \frac{c_{p'}}{2}\left(\sup_n \int_0^t \mathbb{E}\left[(w_u^n)^{2q'} \mathbb{I}_{\{(w_u^n)^{2q'} > C^{q'}\}}\right] du\right)^{1/q'} \\
&+ \sqrt{2c_{p'}}\sup_n \left(\int_0^t \mathbb{E}\left[(w_u^n)^{2q'} \mathbb{I}_{\{(w_u^n)^{2q'} > C^{q'}\}}\right] du\right)^{1/2q'} \qquad (8.11)
\end{aligned}
$$

with $c_{p'} > 0$. Since assumption A2.2 implicates that also the sequence $(w_u^n)^{2q'}$ is uniformly integrable in $(\Omega \times [0,T], \mathbb{Q} \otimes \mu_T)$, we see that the last line tends to 0 as $C \to \infty$.

In conclusion we then have that for some sequences K_n and $K_C^{p'}$ tending to zero:

$$
\mathbb{E}[|\Pi_t^{n,C} - X_t|] < K_n + K_C^{p'}. \qquad (8.12)
$$

Since K_n does not depend on C and $K_C^{p'}$ does not depend on n, the two iterated limits on C and n exist, and are both X_t. But by (8.11) the limit on C is uniform on n: therefore, by the Moore-Osgood theorem the double limit of $\Pi_t^{n,C}$ in L^1 exists, and is X_t. □

Proof of corollary 3.3. Let L_F be the Lipschitz constant of F; in the notation of the proof of theorem 3.2 we have

$$
\mathbb{E}_t[e^{-r(T-t)}|F(\Pi_T^{n,C}) - F(X_T)|] \leq L_F e^{-r(T-t)}\mathbb{E}_t[|\Pi_T^{n,C} - X_T|] \qquad (8.13)
$$

and the claim follows by an application of the Markov property and the theorem. □

Proof of proposition 3.4. Assumption A1 is clear. Jointly, the two estimator sequences can be represented in the form

$$
h^n(u) = c_n^{-1} e^{-\lambda u} \mathbb{I}_{\{u \leq 1/n\}} \qquad (8.14)
$$

for $\lambda \geq 0$ and some monotonous decreasing sequence c_n. Let S_t follow the Heston model, i.e. the variance process follows the SDE:

$$dv_t = \kappa(\theta - v_t)dt + \eta\sqrt{v_t}dZ_t \tag{8.15}$$

for some Brownian motion Z_t possibly correlated with W_t^*, and positive risk-neutral parameters κ, θ, η. Then assumption A2.1 is verified because of e.g. Dufresne [8], theorem 3.1

Consider now the process $\tilde{v}_t = e^{\lambda t} v_t$: Ito's lemma yields that $\tilde{v}_t$ also follows a square root process of the above form, with same initial condition, but with time-dependent parameters:

$$\begin{aligned} \tilde{\kappa}_t &= e^{\lambda t}(\kappa - \lambda) \\ \tilde{\theta} &= \frac{\theta\kappa}{\kappa - \lambda} \\ \tilde{\eta}_t &= e^{\lambda t}\eta. \end{aligned} \tag{8.16}$$

Further, denote the integrated process $Y_t = \int_0^t \tilde{v}_u du$ and define the homogeneous Markov process $Y_t^x = \int_{t-x}^t e^{-\lambda t}\tilde{v}_u du$; note that $Y_t = e^{\lambda t} Y_t^t$. Therefore, using the integral substitution $s = t - u$ and the Markov homogeneity, we have for all $q > 0$:

$$\begin{aligned} \mathbb{E}[(w_t^n)^q] &= \left(c_n \overline{\sigma}^2\right)^{q/2} \mathbb{E}\left[\frac{1}{\left(Y_t^{1/n}\right)^{q/2}}\right] \\ &= \left(c_n \overline{\sigma}^2\right)^{q/2} \mathbb{E}\left[\mathbb{E}\left[\frac{1}{\left(Y_t^{1/n}\right)^{q/2}} \middle| \sigma(\tilde{v}_s, s < t - 1/n)\right]\right] \\ &= \left(c_n e^{\lambda/n} \overline{\sigma}^2\right)^{q/2} \mathbb{E}[Y_{1/n}^{-q/2}] < \left(c_1 e^{\lambda} \overline{\sigma}^2\right)^{q/2} \mathbb{E}[Y_{1/n}^{-q/2}] \end{aligned} \tag{8.17}$$

up to changing v_0. Now, by Dufresne [8], theorem 4.1, all the moments and inverse moments of Y_t exist. Thus using the Laplace transform representation of the function $1/x^q$, $q > 0$, we have:

$$\mathbb{E}[Y_{1/n}^{-q}] = \frac{1}{\Gamma(q)} \int_0^\infty u^{q-1} \mathcal{L}(Y_{1/n}; u) du \tag{8.18}$$

where $\mathcal{L}(Y_t; u)$ is the Laplace transform of the law of Y_t evaluated at u, given by (formula (4.1) of [8], combined with (8.16)):

$$\begin{aligned} \mathcal{L}(Y_t; u) = &\left(\frac{e^{-\tilde{k}_t t/2}}{\cosh(Pe^{\lambda t} t/2) - \frac{\kappa - \lambda}{P}\sinh(Pe^{\lambda t} t/2)}\right)^{\frac{2\kappa\theta}{\eta^2 e^{\lambda t}}} \\ &\exp\left(-\frac{u v_0}{Pe^{\lambda t}} \frac{2\sinh(Pe^{\lambda t} t/2)}{\cosh(Pe^{\lambda t} t/2) - \frac{\kappa - \lambda}{P}\sinh(Pe^{\lambda t} t/2)}\right) \\ P = &\sqrt{\kappa^2 + 2\eta^2 u}. \end{aligned} \tag{8.19}$$

The function $\mathcal{L}(Y_t; u)$ is continuous in u and t; it is not hard to see that $\lim_{t\to 0} \mathcal{L}(Y_t; u) = 1$ and $\lim_{t\to\infty} \mathcal{L}(Y_t; u) = 0$ for all u. Moreover, $\mathcal{L}(Y_t; u)$ is bounded around $u = 0$ and

$\lim_{u\to\infty} \mathcal{L}(Y_t; u) = 0$ for all $t > 0$. Therefore $\mathcal{L}(Y_t; u)$ attains a global maximum M on $[0, T] \times \mathbb{R}^+$. and thus

$$\mathbb{E}[Y_{1/n}^{-q}] < \frac{M}{\Gamma(q)} \int_0^\infty u^{q-1} du = C(q) < \infty. \tag{8.20}$$

Substituting in (8.17) proves in particular assumption A2.2.

For a non-exploding 3/2 model, we can write the instantaneous variance process v_t as $v_t = 1/\nu_t$ where ν_t follows the square root process with some parameters (see e.g. Drimus [7]). Then, Jensen inequality applied to the Lebesgue measure on intervals yields:

$$\mathbb{E}[Y_{1/n}^{-q}] < c\,\mathbb{E}\left[\left(\int_0^{1/n} \nu_u du\right)^q\right] \tag{8.21}$$

for some constant $c > 0$. The verification of A2.2 can then be repeated using estimates similar to those above. □

Discussion of theorem 4.1. This result is a multidimensional version of a special case of proposition 4.7 of [10]. The authors discuss general optimal stopping problems connected to controlled delayed diffusions whose payout function depend on both the diffusion and the past integrated process. In extreme synthesis, the Markovian approximation for the system is derived by expanding h^p in its Fourier series on an L^2 basis, projecting the expansion on the n-dimensional subspace generated by the first n basis functions, and observing that on such finite-dimensional subspace the differential operator generating the system acts in the way described.

As the authors themselves state in remark 2.1, while this proposition originally deals with a one-dimensional state space, it can be straightforwardly generalized to multidimensional processes, which is the version we provide here. The proof involves enlarging all the spaces defined in [10] to the natural topological product space endowed with e.g. the *max* norm. One can then repeat formally the same steps as in the one-dimensional case using instead the product norm, and then reduce each estimate to its one-dimensional counterpart for which the results are established. The extension to non-autonomous diffusions can be obtained in our setup by choosing, in the notation of [10], $u_t = t$. For more details we refer the reader directly to the paper in question.

Proof of corollary 4.2. The application of the theorem is justified, since the strong uniqueness of the solution to X_t follows from that of v_t and w_t^h. We set $m = 2$, $a_1 = b_1 = 0, a_2 = b_2 = h(t)$, $\alpha(x_1, x_2, y_1, y_2, t) = (rx_1 - x_1(\mu_t - r)\min\{C, \overline{\sigma}/\sqrt{y_2}\}, \eta(t, x_2))$, and

$$\sigma(x_1, x_2, y_1, y_2, t) = \begin{pmatrix} x_1\sqrt{x_2}\min\{C, \overline{\sigma}/\sqrt{y_2}\} & 0 \\ \rho\epsilon(t, x_2) & \sqrt{1-\rho^2}\epsilon(t, x_2) \end{pmatrix} \tag{8.22}$$

which yields the result. Note that in the notation of theorem 4.1, $v_t^n = v_t$, because a_1 and b_1 are zero. □

Proof of proposition 4.3. Using the Lipschitz property, from the familiar L^p inequalities and corollary 4.2 we have:

$$|V_t^n - V_t| \le e^{-r(T-t)}\mathbb{E}_t[|F(\pi_T^n) - F(\pi_T)|^2] \le c_L^2 e^{-r(T-t)}\mathbb{E}_t\left[\sup_{s\in[t,T]}(\pi_s^n - \pi_s)^2\right]$$
$$\le e^{-r(T-t)}c_L^2 C_T||h^n - h||_2 \quad (8.23)$$

for some constant C_T, and h as in corollary 4.2. □

Appendix B: proof of theorem 5.1

For notational convenience, in this section we drop the exponents h and p wherever relevant. Throughout we refer to as v_t^Δ (resp. $v_t^{k,\Delta}$, $\pi_t^{n,\Delta}$) as the lower piece-wise continuous time extension of the processes in (5.2), that is, the stochastic processes defined by $v_t^\Delta = v_{t_i}^\Delta$ (resp. $v_t^{k,\Delta} = v_{t_i}^{k,\Delta}$, $\pi_t^{n,\Delta} = \pi_{t_i}^{n,\Delta}$) for all $t \in [t_i, t_{i+1}]$, $i \le m$. $RV_t^{n,\Delta}$ and $w_t^{n,\Delta}$ are defined accordingly.

Lemma 8.1. *For all $k > 0$, we have that* $\sup_{t\le T}\mathbb{E}\left[|v_t^{k,\Delta} - v_t^k|\right] \le K_k(\Delta_t)e^T$ *where* $K_k(\Delta_t) \to 0$ *decreasingly.*

Proof. When $k > 0$, by strong induction on k, Fubini's theorem and standard integral estimates we have:

$$\mathbb{E}\left[|v_t^{k,\Delta} - v_t^k|\right] < \mathbb{E}\left[\int_0^t |\sqrt{2p}(v_s^\Delta - v_s) + 2p\sum_{j=0}^{k-1}(v_s^j - v_s^{j,\Delta}) + p(v_s^k - v_s^{k,\Delta})|ds\right]$$
$$< 2\sqrt{2p}\int_0^t \mathbb{E}\left[|v_s^\Delta - v_s|\right]ds + 4p\int_0^t\sum_{j=0}^{k-1}\mathbb{E}\left[|v_s^{j,\Delta} - v_s^j|\right]ds + 2p\int_0^t\mathbb{E}\left[|v_s^{k,\Delta} - v_s^k||\right]ds$$
$$<2\sqrt{2p}\,h(\Delta_t) + 4pe^{kT}\sum_{j=0}^{k-1}K_j(\Delta_t) + 2p\int_0^t\mathbb{E}\left[|v_s^{k,\Delta} - v_s^k|\right]ds. \quad (8.24)$$

The calculation for $k = 1$ is similar but omits the central sum. By theorem 3.2 of Higham and Mao [14] $h(\Delta_t) \to 0$ decreasingly; using the induction assumption and an application of the Gronwall lemma then yields the result. □

Proposition 8.2. *For all n we have that*

$$sup_{t\le T}\mathbb{E}\left[(w_t^{n,\Delta} - w_t^n)^2\right] \to 0 \quad (8.25)$$

decreasingly, as $\Delta_t \to 0$.

Proof. Define:

$$B^{\Delta,t} = \{RV_t^{n,\Delta} < \overline{\sigma}^2/C^2, RV_t^n < \overline{\sigma}^2/C^2\} \quad (8.26)$$
$$I_1^{\Delta,t} = \{RV_t^{n,\Delta} > \overline{\sigma}^2/C^2, RV_t^n < \overline{\sigma}^2/C^2\} \quad (8.27)$$
$$I_2^{\Delta,t} = \{RV_t^{n,\Delta} < \overline{\sigma}^2/C^2, RV_t^n > \overline{\sigma}^2/C^2\}. \quad (8.28)$$

For all $t < T$ we have:

$$\mathbb{E}[(w_t^{n,\Delta} - w_t^n)^2] < \mathbb{E}\left[\overline{\sigma}^2 \left(\frac{1}{\sqrt{RV_t^{n,\Delta}}} - \frac{1}{\sqrt{RV_t^n}}\right)^2 1\!\!1_B\right] + C^2 \left(\mathbb{P}(I_1^{\Delta,t}) + \mathbb{P}(I_2^{\Delta,t})\right)$$
$$< C^2 \left(\overline{\sigma}^2 \mathbb{E}\left[\left(\sqrt{RV_t^{n,\Delta}} - \sqrt{RV_t^n}\right)^2\right] + \mathbb{P}(I_1^{\Delta,t}) + \mathbb{P}(I_2^{\Delta,t})\right)$$
$$< C^2 \left(\overline{\sigma}^2 \mathbb{E}\left[|RV_t^{n,\Delta} - RV_t^n|\right] + \mathbb{P}(I_1^{\Delta,t}) + \mathbb{P}(I_2^{\Delta,t})\right)$$
$$< C^2 \left(\overline{\sigma}^2 \sum_{k=1}^n h^k \mathbb{E}\left[|v_t^{k,\Delta} - v_t^k|\right] + \mathbb{P}(I_1^{\Delta,t}) + \mathbb{P}(I_2^{\Delta,t})\right). \quad (8.29)$$

The supremum on t of the first term tends to zero decreasingly by lemma 8.1. Note also that the sets $I_1^{\Delta,t}$ and $I_2^{\Delta,t}$ are respectively contained in $M_1^{\Delta,t} = \{RV_t^{n,\Delta} > RV_t^n\}$ and $M_2^{\Delta,t} = \{RV_t^{n,\Delta} < RV_t^n\}$. But lemma 8.1 also implies that RV_t^n converges decreasingly almost surely to RV_t^n uniformly in t, so that both $\sup_{t<T} \mathbb{P}(M_1^{\Delta,t})$ and $\sup_{t<T} \mathbb{P}(M_2^{\Delta,t})$ tend to zero with Δ_t. □

Having established the sup-L^2 convergence of the discretized weighting coefficient, convergence for the option prices can be proved along the lines of Higham and Mao [14].

We introduce the linearly interpolating processes:

$$\bar{v}_t^{\Delta} = v_0 + \int_0^t \kappa(\theta - v_s^{\Delta})ds + \nu \int_0^t \sqrt{|v_s^{\Delta}|}dW_s^1 \quad (8.30)$$

for the Brownian motion W_t^1 whose Cholesky decomposition is given in X_t, and

$$\bar{\pi}_t^{n,\Delta} = \pi_0 + r\int_0^t \pi_s^{n,\Delta}ds + \int_0^t w_s^{n,\Delta}\pi_s^{n,\Delta}\sqrt{|v_s^{\Delta}|}dW_s \quad (8.31)$$

we have the following result:

Lemma 8.3. *For any $i, j > 0$ define the stopping time*

$$\tau_{i,j} = \inf\{t : \pi_t^n > i \ \text{or} \ |\bar{v}_t^{\Delta}| > j\} \quad (8.32)$$

Then

$$\mathbb{E}\left[\sup_{t\le T}(\bar{\pi}_{t\wedge\tau_{i,j}}^{n,\Delta} - \pi_{t\wedge\tau_{i,j}}^n)^2\right] \to 0 \quad (8.33)$$

as $\Delta_t \to 0$ for all i, j.

Proof. By definition we see that for $t \le T$

$$\bar{\pi}_{t\wedge\tau_{i,j}}^{n,\Delta} - \pi_{t\wedge\tau_{i,j}}^n = r\int_0^{t\wedge\tau_{i,j}} (\pi_s^{n,\Delta} - \pi_s^n)ds + \int_0^{t\wedge\tau_{i,j}} (\pi_s^{n,\Delta}w_s^{n,\Delta}\sqrt{|v_s^{\Delta}|} - \pi_s^n w_s^n\sqrt{v_s})dW_s$$
$$= r\int_0^{t\wedge\tau_{i,j}} (\pi_s^{n,\Delta} - \pi_s^n)ds + \int_0^{t\wedge\tau_{i,j}} (\pi_s^{n,\Delta}w_s^{n,\Delta} - \pi_s^n w_s^n)\sqrt{|v_s^{\Delta}|}dW_s +$$
$$\int_0^{t\wedge\tau_{i,j}} \pi_s^n w_s^n(\sqrt{v_s} - \sqrt{|v_s^{\Delta}|})dW_s. \quad (8.34)$$

Remember that $v_t > 0$ a.s. For $t \le t_0 \le T$ we can apply Doob's martingale inequality to $\sup_{t_0 \le t}(\bar{\pi}^{n,\Delta}_{t_0\wedge\tau_{i,j}} - \pi^n_{t_0\wedge\tau_{i,j}})$: combining this with the bound of a sum of squares, Jensen's inequality and Ito isometry we have:

$$\mathbb{E}\left[\sup_{t_0<t}(\bar{\pi}^{n,\Delta}_{t_0\wedge\tau_{i,j}} - \pi^n_{t_0\wedge\tau_{i,j}})^2\right] < 3r^2 t\mathbb{E}\left[\int_0^{t\wedge\tau_{i,j}} (\pi^{n,\Delta}_s - \pi^n_s)^2 ds\right] +$$
$$12\mathbb{E}\left[\int_0^{t\wedge\tau_{i,j}} |v^\Delta_s|(\pi^{n,\Delta}_s w^{n,\Delta}_s - \pi^n_s w^n_s)^2 ds\right] + 12\mathbb{E}\left[\int_0^{t\wedge\tau_{i,j}} (\pi^n_s w^n_s)^2(\sqrt{v_s} - \sqrt{|v^\Delta_s|})^2 ds\right]$$
$$\le 3r^2 t\mathbb{E}\left[\int_0^{t\wedge\tau_{i,j}} (\pi^{n,\Delta}_s - \pi^n_s)^2 ds\right] +$$
$$12j\int_0^{t\wedge\tau_{i,j}} \mathbb{E}[(\pi^{n,\Delta}_s w^{n,\Delta}_s - \pi^n_s w^n_s)^2]ds + 12i^2C^2\mathbb{E}\left[\int_0^{t\wedge\tau_{i,j}} |v_s - v^\Delta_s| ds\right]$$
$$\le (3r^2T + 24C^2 j)\mathbb{E}\left[\int_0^{t\wedge\tau_{i,j}} (\pi^{n,\Delta}_s - \pi^n_s)^2 ds\right] + 12i^2C^2\mathbb{E}\left[\int_0^{t\wedge\tau_{i,j}} |v_s - v^\Delta_s| ds\right] +$$
$$24ji^2\mathbb{E}\left[\int_0^{t\wedge\tau_{i,j}} (w^{n,\Delta}_s - w^n_s)^2 ds\right] = k(\Delta_t) + h(\Delta_t) + (3r^2T + 24C^2 j)\mathbb{E}\left[\int_0^{t\wedge\tau_{i,j}} (\pi^{n,\Delta}_s - \pi^n_s)^2 ds\right]. \tag{8.35}$$

Now, $k(\Delta_t)$ and $h(\Delta_t) \to 0$ decreasingly, by an application respectively of proposition 8.2 and combining lemma 3.2 and the corollary to theorem 3.1 of Higham and Mao [14]. But:

$$\mathbb{E}\left[\int_0^{t\wedge\tau_{i,j}} (\pi^{n,\Delta}_s - \pi^n_s)^2 ds\right] \le 2\mathbb{E}\left[\int_0^{t\wedge\tau_{i,j}} (\bar{\pi}^{n,\Delta}_s - \pi^n_s)^2 ds\right] + 2\mathbb{E}\left[\int_0^{t\wedge\tau_{i,j}} (\bar{\pi}^{n,\Delta}_s - \pi^{n,\Delta}_s)^2 ds\right] \tag{8.36}$$

and

$$\mathbb{E}\left[\int_0^{t\wedge\tau_{i,j}} (\bar{\pi}^{n,\Delta}_s - \pi^n_s)^2 ds\right] \le \int_0^t \mathbb{E}[(\bar{\pi}^{n,\Delta}_{s\wedge\tau_{i,j}} - \pi^n_{s\wedge\tau_{i,j}})^2]ds \le \int_0^t \mathbb{E}\left[\sup_{t_0\le s}(\bar{\pi}^{n,\Delta}_{s\wedge\tau_{i,j}} - \pi^n_{s\wedge\tau_{i,j}})^2\right] ds. \tag{8.37}$$

Next, a direct computation shows

$$(\bar{\pi}^{n,\Delta}_s - \pi^{n,\Delta}_s)^2 < 2r^2 i^2 \Delta_t^2 + 2Cji^2(W_s - W_{[s/\Delta_t]\Delta_t})^2 \tag{8.38}$$

from which after directly integrating one obtains:

$$\mathbb{E}\left[\int_0^{t\wedge\tau_{i,j}} (\pi^{n,\Delta}_s - \bar{\pi}^{n,\Delta}_s)^2 ds\right] \le 2r^2 i^2 \Delta_t^2 T + 2Cji^2\Delta_t := q(\Delta_t). \tag{8.39}$$

In conclusion:

$$\mathbb{E}\left[\sup_{t<T}(\bar{\pi}^{n,\Delta}_{t\wedge\tau_{i,j}} - \pi^n_{t\wedge\tau_{i,j}})^2\right] \le k(\Delta_t) + h(\Delta_t) +$$
$$(3r^2T + 24C^2 j)\left(q(\Delta_t) + \int_0^T \mathbb{E}\left[\sup_{t_0\le s}(\bar{\pi}^{n,\Delta}_{s\wedge\tau_{i,j}} - \pi^n_{s\wedge\tau_{i,j}})^2\right] ds\right), \tag{8.40}$$

and the claim follows by applying Gronwall lemma and letting Δ_t tend to 0. □

From this one easily gets convergence of put option prices.

Proof of theorem 5.1. Let $\epsilon > 0$ and for positive integers i, j define:

$$A_i = \{\omega : \pi_t^n(\omega) > i, \forall t \le T\}, \tag{8.41}$$

$$B_j = \{\omega : |\bar{v}_t^\Delta(\omega)| > j, \forall t \le T\}. \tag{8.42}$$

Let $\epsilon > 0$. Because of lemma 6.2 in Higham and Mao [14] combined to Markov's inequality there exist j big enough such that $\mathbb{Q}(B_j) < \epsilon/(6K)$. Also in the Heston model, S_t is a true $\mathbb{Q}$-martingale for every choice of parameters (see e.g. Lewis [16]) so we can apply proposition 3.1 which in particular ensures $\mathbb{E}[\pi_T^n] < \infty$. Hence, by Doob's martingale inequality we can choose i such that $\mathbb{Q}(A_i) < \epsilon/(6K)$. Let then $E_{ij} = A_i \cup B_j$ and $\tau_{i,j}$ the stopping time defined in lemma 8.3. We have:

$$\begin{aligned}
&\left|\mathbb{E}\left[(K - \pi_T^{n,\Delta})^+\right] - \mathbb{E}[(K - \pi_T^n)^+]\right| \le \mathbb{E}\left[|(K - \pi_T^{n,\Delta})^+ - (K - \pi_T^n)^+|\mathbb{1}_{E_{ij}^c}\right] \\
&+\mathbb{E}\left[|(K - \pi_T^{n,\Delta})^+ - (K - \pi_T^n)^+|\mathbb{1}_{E_{ij}}\right] < \mathbb{E}\left[|\pi_T^{n,\Delta} - \pi_T^n|\mathbb{1}_{\{\tau_{i,j} > T\}}\right] + K\mathbb{Q}(E_{ij}) \\
&< \mathbb{E}\left[|\bar{\pi}_{T\wedge\tau_{i,j}}^{n,\Delta} - \pi_{T\wedge\tau_{i,j}}^n|\right] + \mathbb{E}\left[|\bar{\pi}_T^{n,\Delta} - \pi_T^{n,\Delta}|\mathbb{1}_{\{\tau_{i,j} > T\}}\right] + \epsilon/3.
\end{aligned} \tag{8.43}$$

Applying lemma 8.3 we can find Δ_t small enough such that the first term is smaller than $\epsilon/3$. Regarding the second summand we see that:

$$\begin{aligned}
\mathbb{E}\left[|\bar{\pi}_T^{n,\Delta} - \pi_T^{n,\Delta}|\mathbb{1}_{\{\tau_{i,j} > T\}}\right] &\le \mathbb{E}\left[|ri(T - [T/\Delta_t]\Delta_t) + iC\sqrt{j}(W_T - W_{[T/\Delta_t]\Delta_t})|\mathbb{1}_{\{\tau_{i,j} > T\}}\right] \\
&< ri\Delta_t + iC\sqrt{j\Delta_t},
\end{aligned} \tag{8.44}$$

which can also be made smaller than $\epsilon/3$ with Δ_t. By call-put parity the convergence result above also holds for call options. ☐

Note that from the set of results above strong convergence of $\pi_t^{n,\Delta}$ to π_t^n cannot be deduced.

Appendix C: calculation of the L^2 Fourier coefficients of the Laguerre function basis for various weighting functions

Preliminarily we compute the Laplace transform of the functions $L_k(x)I_{\{0 \le x \le \delta\}}$, $k \ge 0$. We have, for $s > 0$:

$$\int_0^\delta e^{-sx} L_k(x)dx = \sum_{i=0}^{k} \binom{k}{i}(-1)^i \frac{x^i}{i!}\int_0^\delta x^i e^{-sx}dx = \sum_{i=0}^{k} \binom{k}{i}(-1)^i \frac{x^i}{s^{i+1}i!}\gamma(i+1, s\delta) \tag{8.45}$$

where $\gamma(m, s)$ is the lower incomplete Gamma function. Using the recursive properties of γ it is easy to prove by backward induction that

$$\gamma(m+1, t) = m!\left(1 - e^{-t}\sum_{j=0}^{m} \frac{t^j}{j!}\right). \tag{8.46}$$

Therefore:

$$\int_0^\delta e^{-sx}L_k(x)dx = \lambda^{-1}\sum_{i=0}^k \binom{k}{i}(-1)^i(-s^{-1})^i - e^{-s\delta}\sum_{i=0}^k\binom{k}{i}\frac{(-1)^i}{s^{i+1}}\sum_{j=0}^i\frac{(s\delta)^j}{j!}$$

$$\frac{(s-1)^k}{s^{k+1}} + e^{-s\delta}\sum_{i=0}^k\binom{k}{i}(-s^{-1})^{i+1}\sum_{j=0}^i\frac{(s\delta)^j}{j!}. \tag{8.47}$$

We notice that as $\delta \to \infty$ the above produces the usual Laplace transform $(s-1)^k/s^{k+1}$ of the k-th Laguerre polynomial, which gives us the initial values $v_0^k = (-1)^k v_0\sqrt{2p}/p$ of the Laguerre processes. Now we have, for $\lambda, \delta, p > 0$:

$$h_1^{p,k}(\delta) = c_1(\delta)\sqrt{2p}\int_0^\delta e^{-px}L_k(2px)dx, \tag{8.48}$$

$$h_2^{p,k}(\delta,\lambda) = c_2(\delta,\lambda)\sqrt{2p}\int_0^\delta e^{-(\lambda+p)x}L_k(2px)dx \tag{8.49}$$

with $c_1(\delta) = \delta^{-1}$ and $c_2(\delta,\lambda) = \lambda/(1-e^{-\lambda\delta})$. We observe that the integral in the right hand side of the equations (8.48) is the particular case $\lambda = 0$ of the integral in the right hand side of (8.49), which using (8.47) we compute as:

$$\sqrt{2p}\int_0^\delta e^{-(\lambda+p)x}L_k(2px)dx = \frac{1}{\sqrt{2p}}\int_0^{2p\delta} e^{-\left(\frac{\lambda+p}{2p}\right)y}L_k(y)dy =$$

$$\sqrt{2p}\left[\frac{(\lambda-p)^k}{(\lambda+p)^{k+1}} + e^{-(\lambda+p)\delta}\sum_{i=0}^k\binom{k}{i}\frac{(2p)^i}{(-\lambda-p)^{i+1}}\sum_{j=0}^i\frac{(\lambda+p)^j\delta^j}{j!}\right] \tag{8.50}$$

so that[8]

$$h_1^{p,k}(\delta) = \frac{\sqrt{2p}}{p\delta}\left[(-1)^k - e^{-p\delta}\sum_{i=0}^k\binom{k}{i}(-2)^i\sum_{j=0}^i\frac{p^j\delta^j}{j!}\right] \tag{8.51}$$

$$h_2^{p,k}(\delta,\lambda) = \frac{\lambda\sqrt{2p}}{1-e^{-\lambda\delta}}\left[\frac{(\lambda-p)^k}{(\lambda+p)^{k+1}} + e^{-(\lambda+p)\delta}\sum_{i=0}^k\binom{k}{i}\frac{(2p)^i}{(-\lambda-p)^{i+1}}\sum_{j=0}^i\frac{(\lambda+p)^j\delta^j}{j!}\right]. \tag{8.52}$$

References

[1] M. Bernhart, P. Tankov, and X. Warin. A finite dimensional approximation for pricing moving average options. *SIAM Journal in Financial Mathematics*, 2, 2011.

[2] L. Chew. Target volatility asset allocation strategy. Society of Actuaries International News, 2011.

[8] Bernhart et al. [1] find an elegant closed formula for $h_1^{p,k}$ (equation 21) in terms of sums of the first n Laguerre polynomials. However, for computational purposes (8.48) seems to be better since it involves the calculation of only $k+1$ binomial coefficients instead of $k(k+1)/2$ as required by their formula.

[3] R. Coles. Vol target options. CitiGroup presentation, 2014.

[4] J. Da Fonseca, A. Gnoatto, and M. Grasselli. Analytic pricing of volatility-equity options within affine models: an efficient conditioning technique. *Operations Research Letters*, 43: 601–607, 2015.

[5] G. Di Graziano and L. Torricelli. Target volatility option pricing. *International Journal of Theoretical and Applied Finance*, 15, 2012.

[6] J. Diestel. *Uniform integrability: an introduction.* Rendiconti dell'Istituto di Matematica dell'Università di Trieste. An international Journal of Mathematics, 1991.

[7] G. D. Drimus. Options on realized variance by transform methods: a non-affine stochastic volatility model. *Quantitative Finance*, 12:1679–1694, 2012.

[8] D. Dufresne. *The integrated square-root process.* University of Montreal research paper number 90, 2001.

[9] S. Eason, P. Barker, G. Foroughi, J. Harsant, D. Hunter, S. Jarvis, G. Jones, et al. Is there a place in the uk defined contribution pensions market for a guaranteed savings product? Presentation to The Institute and Faculty of Actuaries London, 2012.

[10] S. Federico and P. Tankov. Finite-dimensional representations for controlled diffusions with delay. *Applied Mathematics and Optimization*, 71:165–194, 2015.

[11] R. P. Goldsticker. Target volatility may miss the mark. BNY Mellon asset management, 2012.

[12] M. Grasselli and J. Marabel Romo. Stochastic skew and target volatility options. *Journal of Futures Markets*, 36:174–193, 2015.

[13] S. L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6:327–343, 1993.

[14] D. J. Higham and X. Mao. Convergence of monte carlo simulations involving the mean-reverting square root process. *Journal of Computational Finance*, 8:35–61, 2005.

[15] S. Jaschke. Pricing and hedging derivatives on actively managed funds. Investment-based insurance guarantees conference, 2013.

[16] A. Lewis. *Option Valuation under Stochastic Volatility.* Finance Press, 2000.

[17] R. Lord, R. Koekkoek, and D. Van Dijk. A comparison of biased simulation schemes for stochastic volatility models. *Quantitative Finance*, 10:177–194, 2010.

[18] S. Morrison and L. Tadrowski. Guarantees and target volatility funds. Moody's Analytics B&H Research series, 2013.

[19] P. Romain, R. L. de Carvalho, and P. Moulin. Inter-temporal risk parity: a constant volatility framework for factor investing. *Journal of Investment Strategies*, 4:1–23, 2014.

[20] S. Stoyanov. Structured equity investment strategies for long-term Asian investors. *EDHEC Risk Publications*, 2011.

[21] J. Vandenbroucke. Fallacies of risk control. *Applied Economics and Finance*, 2, 2015.

[22] X. Wang and Y. Wang. Variance-optimal hedging for target volatility options. *Journal of Industrial and Management Optimization*, 10:207–218, 2013.

[23] P. Wilmott, S. Howison, and J. Dewynne. *The mathematics of financial derivatives*. Cambridge University Press, 1995.

[24] B. Wong and C. C. Heyde. On the martingale property of stochastic exponentials. *Journal of Applied Probability*, 41:654–664, 2004.

[25] Y. Xue. Target volatility: an effective risk management tool for VA? Society of Actuaries, 2012.

Figures and tables

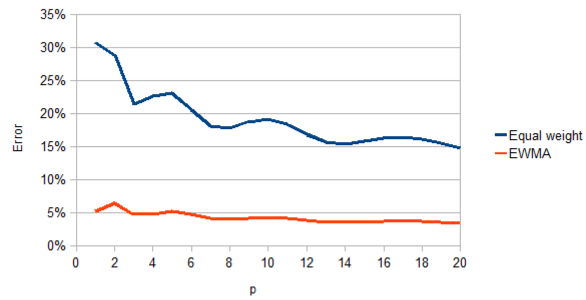

Figure 1: L^2 percentage error as a function of the parameter p with $n = 15$.

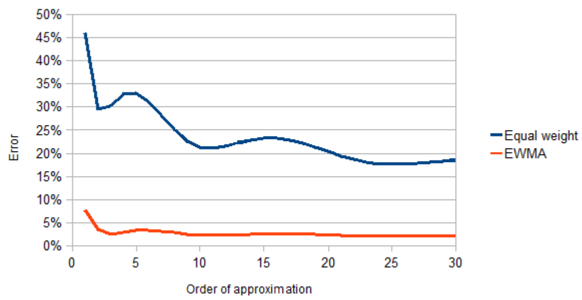

Figure 2: L^2 percentage error as a function of the approximation order n when $p = 20$.

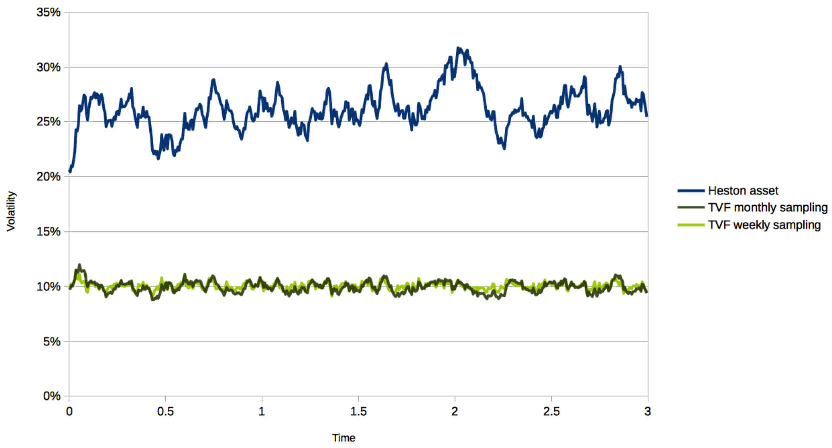

Figure 3: Diffusive volatility of TVFs using both sampling lengths compared to Heston, first path.

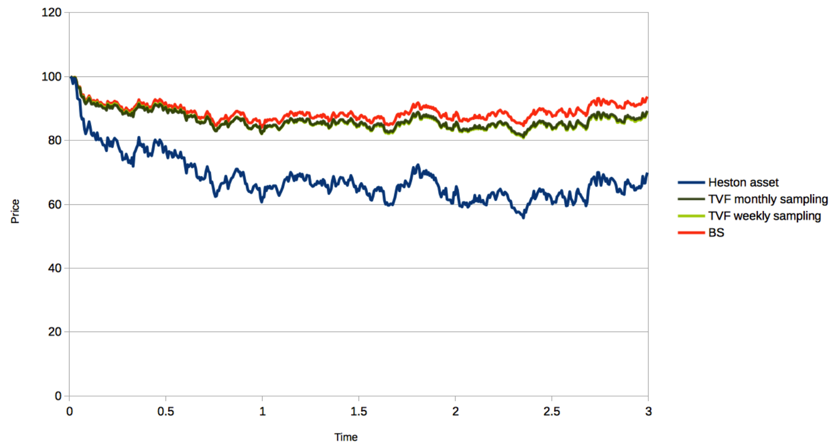

Figure 4: Price of TVFs using for both sampling lengths, compared to Heston and Black-Scholes, first path.

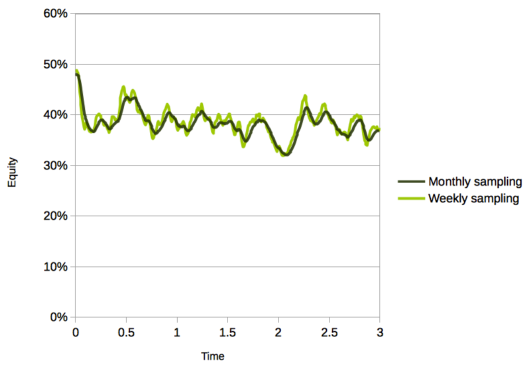

Figure 5: Equity percentage exposure of TVFs using both sampling lengths, first path.

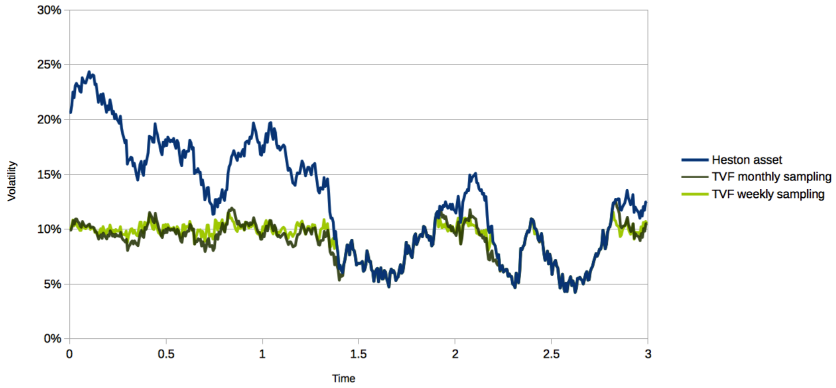

Figure 6: Diffusive volatility of TVFs using both sampling lengths compared to Heston, second path.

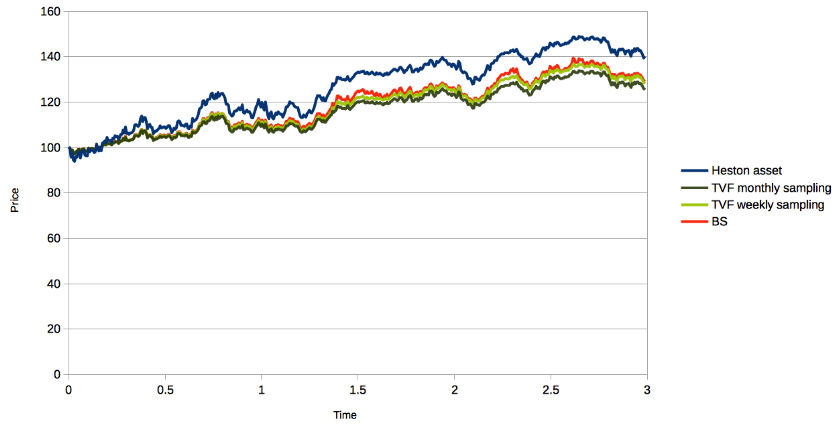

Figure 7: Price of TVFs using both sampling lengths compared to Heston and Black-Scholes assets, second path.

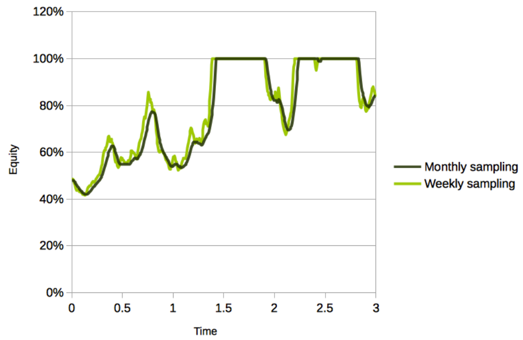

Figure 8: Equity percentage exposure of TVFs using both sampling lengths, second path.

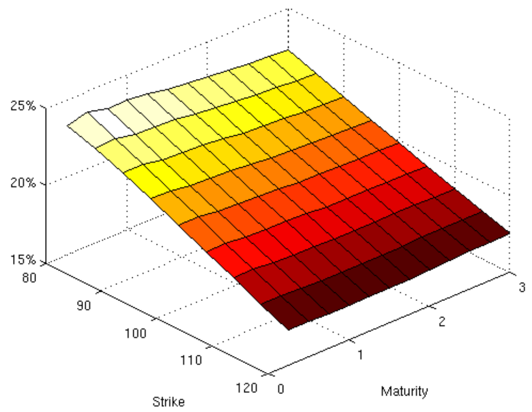

Figure 9: Implied volatility surface of the Heston asset S_t.

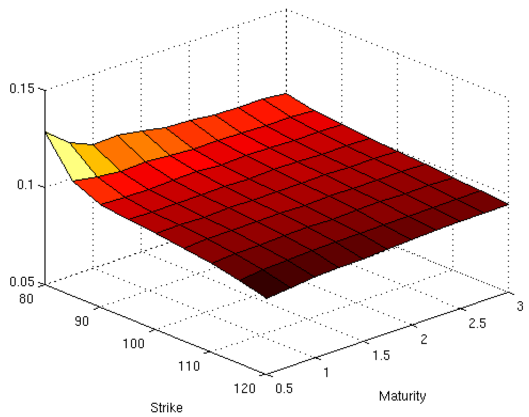

Figure 10: Implied volatlity surface of a TVF using h_1 and δ_1.

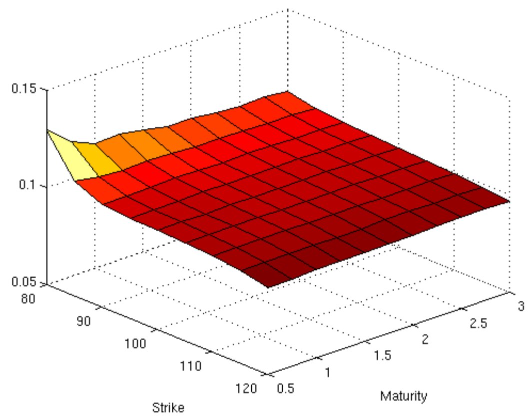

Figure 11: Implied volatlity surface of a TVF using h_1 and δ_2.

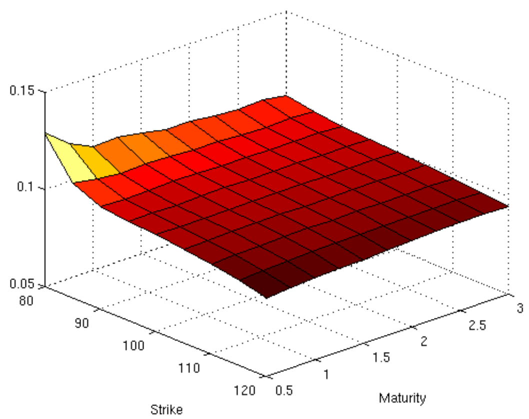

Figure 12: Implied volatlity surface of TVF using h_2 and δ_1.

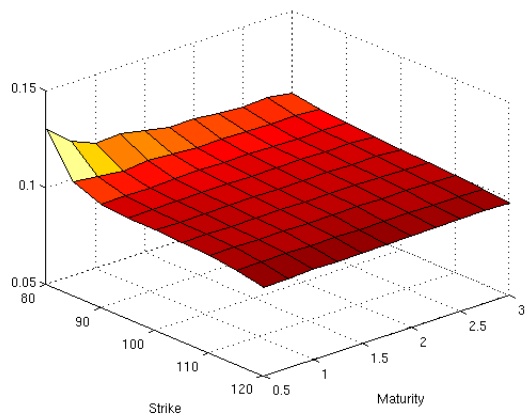

Figure 13: Implied volatlity surface of TVF using h_2 and δ_2.

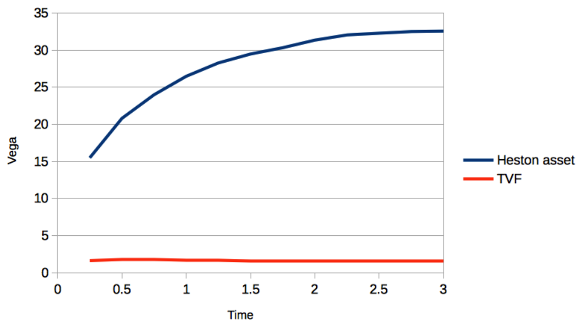

Figure 14: Vega as a function of time. The strike direction plotted is S_0e^{rt}.

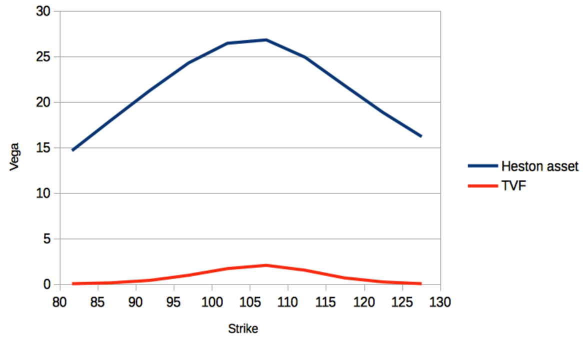

Figure 15: Vega as a function of strike at $t = 1$, centered at S_0e^r.

T	δ_1		δ_2	
	h_1	h_2	h_1	h_2
6 months	7.56%	7.23%	5.92%	5.87%
18 months	4.30%	4.11%	3.82%	3.96%
3 years	2.78%	2.68%	2.67%	2.81%
Overall surface average	4.19%	4.04%	3.69%	3.83%

Table 1: Average relative error across strikes of the TVF time sections compared to $\overline{\sigma}$, for given lags and estimators.

Method	δ_1	δ_2
Markovian Approximation	4.9269(2579)	5.0283(2577)
Euler SDDE discretization	5.3519(1189)	6.5513(1109)

Table 2: Call option prices and runtimes in milliseconds (in parentheses) for the Markovian method and the Euler SDDE scheme. $K = S_0 = 100$, $T = 1$, 50.000 simulations, 100 time steps. The analytical target Black-Scholes value is 5.0167.

Lag	100	200	400	800
δ_1	5.3519(1189)	5.1943(2676)	5.0870(7315)	5.0630(20089)
δ_2	6.5513(1109)	5.5741(2158)	5.2710(5221)	5.1403(12147)

Table 3: Call option prices for the Euler SDDE scheme for increasing time steps. Parameters as in table 2.