Ambiguity Aversion in Standard and Extended Ellsberg Frameworks: $\alpha$-Maxmin versus Maxmin Preferences∗

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Abstract

We study optimal portfolio choice induced by $\alpha$-maxmin ($\alpha$-MEU) utility models. We find that in the standard Ellsberg framework ambiguity averse $\alpha$-MEU and maxmin preferences coincide. Only when there are three or more ambiguous states, the $\alpha$-MEU and maxmin models induce different attitudes toward ambiguity and portfolio choices. We derive novel theoretical implications for equilibrium asset prices, and revisit the laboratory experimental findings in Bossaerts, Ghirardato, Guarnaschelli, and Zame (2010).

JEL Classifications: G11, G12, C92, D53.

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1 Introduction

Over the last decades, the impact of Knightian ambiguity (Knight 1921) on financial decision making has received significant attention, as reflected by a large and growing literature. Theoretical models with ambiguity averse agents are consistent with a variety of documented phenomena such as nonparticipation, portfolio inertia and excess volatility of asset returns. These models are further motivated by the observation that ambiguity averse preferences are consistent with experimental evidence (Ellsberg 1961), and with recent portfolio choice experiments which suggest that investors’ preferences are heterogeneous and well approximated by subjective expected utility and ambiguity averse preferences; e.g. Bossaerts et al. (2010), and Ahn, Choi, Gale, and Kariv (2013).

The $\alpha$-maxmin expected utility ($\alpha$-MEU) model has become increasingly popular to describe the behavior of agents under ambiguity. This model generalizes the well-known maxmin expected utility model axiomatized by Gilboa and Schmeidler (1989). The utility of an $\alpha$-MEU agent from some state dependent wealth $w = (w_\sigma)_{\sigma \in S}$ is

$$U(w) = \alpha \min_{\pi \in C} \sum_{\sigma \in S} u(w_\sigma) \pi_\sigma + (1 - \alpha) \max_{\pi \in C} \sum_{\sigma \in S} u(w_\sigma) \pi_\sigma$$

(1.1)

where $u : \mathbb{R} \to \mathbb{R}$ is a utility function, $C$ is a set of priors on the state space $S$, and $\alpha \in [0, 1]$.\(^2\) The maxmin expected utility model (1-MEU) is obtained from (1.1) for $\alpha = 1$, whereas $\alpha = 0$ yields the maxmax expected utility model (0-MEU). With $\alpha$ ranging between 0 and 1, the $\alpha$-MEU seems to provide a large spectrum of ambiguity attitudes, beyond the maxmin and maxmax expected utility models. Thus, the $\alpha$-MEU has been adopted in theoretical and experimental work to study agents’ ambiguity attitudes; e.g.

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\(^1\)Theoretical papers that study how nonparticipation may arise in equilibrium with ambiguity averse agents are, e.g., Dow and Werlang (1992), Cao, Wang, and Zhang (2005), Easley and O’Hara (2009) and Illeditsch (2011). Other work showing that ambiguity aversion can capture other “puzzling” markets phenomena are: Chen and Epstein (2002), Uppal and Wang (2003), Trojan and Vanini (2004), Epstein and Schneider (2008), Cao, Han, Hirshleifer, and Zhang (2011) and Boyle, Garlappi, Uppal, and Wang (2012). Recent surveys on this topic include Epstein and Schneider (2010) and Gilboa and Marinacci (2013).

\(^2\)The $\alpha$-maxmin expected utility is a generalization of the Hurwicz’s model introduced by Hurwicz (1951a,b); see also Arrow and Hurwicz (1972) and Jaffray (1988). Properties of the $\alpha$-MEU model have been studied by Ghirardato, Klibanoff, and Marinacci (1998) and Marinacci (2002). For characterizations of subclasses of the $\alpha$-MEU preferences see Ghirardato, Maccheroni, and Marinacci (2004), Olszewski (2007), and Eichberger, Grant, Kelsey, and Koshevoy (2011).
Chen, Katuščák, and Ozdenoren (2007), Bossaerts et al. (2010), and Ahn et al. (2013).

The goal of this paper is to study portfolio choice induced by the \( \alpha \)-MEU model and to compare it with the portfolio choice under the maxmin model. The setting is a finite state space in which the future states of the economy correspond to draws from the Ellsberg (1961) urn.

First, we consider the standard Ellsberg framework that consists of three future states of the economy, one risky and two ambiguous. We find that, in this setting, \( \alpha \)-MEU preferences coincide with either maxmin, maxmax, or subjective expected utility (SEU) preferences. Specifically, we show that any \( \alpha \)-MEU utility with \( \alpha \in (1/2, 1) \) can be rewritten as a 1-MEU utility with a smaller set of priors, any \( \alpha \)-MEU utility with \( \alpha \in (0, 1/2) \) as a 0-MEU utility with a smaller set of priors, and any \( 1/2 \)-MEU utility as a SEU. This result holds true also for any number of risky states in the state space, as long as there are only two ambiguous states. The fact that \( \alpha \)-MEU preferences are indistinguishable from 1-MEU, 0-MEU and SEU preferences has implications for experimental studies. For example, this clarifies some recent experimental studies carried out in the standard Ellsberg framework in which the \( \alpha \)-MEU model is used as a generalization of the 1-MEU and 0-MEU models; e.g. Bossaerts et al. (2010), and Ahn et al. (2013).

We show that only when there are three or more ambiguous states, the \( \alpha \)-MEU model with \( \alpha \in (0, 1) \) does not reduce to the maxmin, maxmax, or SEU models. In this extended Ellsberg framework, the \( \alpha \)-MEU model induces different portfolio choice and attitude toward ambiguity. To study portfolio choice as function of the ambiguity averse parameter \( \alpha \), we consider \( \alpha \)-MEU models with the maximal set of priors consistent with the uncertainty structure of the Ellsberg framework. We denote this set by \( C_{\text{max}} \) and the corresponding model by \( \alpha \)-C\(_{\text{max}}\)-MEU.\(^3\) Interestingly, \( \alpha \)-C\(_{\text{max}}\)-MEU agents optimally choose only two types of portfolios: either with no exposure to ambiguity (that is, portfolios that provide equal wealth in all ambiguous states), or portfolios with a specific exposure to ambiguity, namely portfolios that allocate more wealth to the cheapest ambiguous state and equal wealth to all the other ambiguous states. When \( \alpha \) is below a certain threshold, which depends on the number of ambiguous states, the \( \alpha \)-C\(_{\text{max}}\)-MEU agent always takes

\(^3\)If \( C = C_{\text{max}} \) in (1.1), the parameter \( \alpha \) expresses the agent’s attitude toward ambiguity: an increase in \( \alpha \) corresponds to an increase in the agent’s ambiguity aversion. \( C = C_{\text{max}} \) is a common assumption in experimental-based papers studying the \( \alpha \)-MEU model; see, e.g., Chen et al. (2007) and Ahn et al. (2013).
a portfolio exposed to ambiguity; even if the state prices of the ambiguous states are all equal. The less the agent is ambiguity averse – namely, the smaller is $\alpha$ – the more the optimal portfolio will be exposed to ambiguity in the sense that the larger will be the difference between the wealth allocated to the cheapest ambiguous state compared to the wealth allocated to each of the other ambiguous states. Also, the less the agent is risk averse (namely, the less the utility function $u$ in (1.1) is concave), the more the optimal portfolio will be exposed to the ambiguity. As expected, the set of state prices for which an $\alpha$-$C_{\text{max}}$-MEU agent chooses an unambiguous portfolio increases with the ambiguity aversion $\alpha$. The limiting case is the 1-$C_{\text{max}}$-MEU agent who always takes an unambiguous portfolio, no matter what the state prices are.

We compare the portfolio choice of $\alpha$-$C_{\text{max}}$-MEU agents (with $\alpha \in (0, 1)$) and maxmin agents. We find that these two models induce significantly different portfolio choices. For instance, facing ambiguous states with equal prices, maxmin agents (no matter how large is the set of priors) typically choose an unambiguous portfolio. In contrast, all $\alpha$-$C_{\text{max}}$-MEU agents with an $\alpha$ below a certain threshold (that depends on the number of ambiguous states) choose the specific portfolios exposed to ambiguity described above. Thus, $\alpha$-$C_{\text{max}}$-MEU and maxmin models appear to induce very different attitudes toward ambiguity.

Finally, we derive theoretically equilibrium asset prices when the market is populated by agents with maxmin and SEU preferences, in the standard Ellsberg framework. We then revisit the laboratory experimental findings in Bossaerts et al. (2010). They run a series of experimental sessions in which a competitive financial market is embedded in the standard Ellsberg framework. Their findings show that a significant fraction of individuals is ambiguity averse, and provide evidence that ambiguity aversion matters for equilibrium prices. To interpret their experimental findings, Bossaerts et al. (2010) use a market model involving SEU and ambiguity averse $\alpha$-MEU agents. They do not derive equilibrium asset prices but only provide conjectures. We derive theoretically the equilibrium asset prices implied by their model. We show through which channels ambiguity aversion impacts equilibrium asset prices, and hence it does not wash out in aggregate. Also, we find a striking matching between our theoretical predictions and their experimental findings. Specifically, our theoretically derived rankings of the state-price/state-probability ratios fully explain the empirical rankings documented by Bossaerts et al. (2010), including “the
rankings [that] appear anomalous” from the perspective of their conjectures.

The structure of the paper is as follows. Section 2 introduces the setup. Section 3 presents our theoretical results on the $\alpha$-MEU model in the standard Ellsberg framework. Section 4 studies the portfolio choice of the $\alpha$-$C_{\text{max}}$-MEU model in the extended Ellsberg framework (i.e. with at least three ambiguous states), and compares it to the maxmin model of Gilboa and Schmeidler (1989). Section 5 derives the equilibrium state prices when the market is populated by ambiguity averse and SEU agents in the standard Ellsberg framework, and revisits the experimental findings in Bossaerts et al. (2010). Section 6 concludes. The Appendix collects proofs and technical results.

2 Setup

The market model considered in this paper is an Arrow–Debreu complete market for contingent claims with two dates, $t = 0$ and $t = 1$. $S$ is the finite state space containing all possible states of the economy at time $t = 1$, and $|S|$ is the number of states. At time $t = 0$ the agents face both uncertainty (risk) and ambiguity since they neither know which state in $S$ will realize at time $t = 1$ (uncertainty), nor what is the probability of the occurrence of some of the states in $S$ (ambiguity). For any state there is an Arrow security traded in the market which pays at time $t = 1$ one unit of currency in that state and nothing in the other states. Pricing rules $p = (p_\sigma)_{\sigma \in S} \in \mathbb{R}^{|S|}$ are normalized so that the price of the riskless and unambiguous portfolio $w = (1, \ldots, 1)$ is 1, that is $\sum_{i=1}^{|S|} p_i = 1$.

Given $N$ agents in the market, each agent $n$ is characterized by an initial endowment $e^n \in \mathbb{R}^{|S|}$, where the $i$th coordinate of $e^n$ corresponds to the number of Arrow security that pays in the state $i$, and by a criterion $U^n$ representing her preferences, $n = 1, \ldots, N$. The total endowment in the market is $W := \sum_1^N e^n = (W_1, \ldots, W_{|S|})$, where := denotes definition. Let $\cdot$ denote the scalar product. Given the pricing rule $p$ on the Arrow securities, a portfolio $w^n = (w^n_\sigma)_{\sigma \in S} \in \mathbb{R}^{|S|}$ is said to be optimal for agent $n$ if $w^n$ satisfies the budget constraint $p \cdot w^n \leq p \cdot e^n$ and maximizes the utility $U^n$ over all portfolios $w \in \mathbb{R}^{|S|}$ subject to the budget constraint $p \cdot w \leq p \cdot e^n$, i.e.

$$U^n(w^n) = \max\{U^n(w) \mid w \in \mathbb{R}^{|S|}, p \cdot w \leq p \cdot e^n\}.$$ 

An equilibrium $(p; w^1, \ldots, w^N)$ consists of a pricing rule $p$ and individual portfolio choices $w^n$ such that
for each \( n = 1, \ldots, N \) the portfolio \( w^n \) is optimal for agent \( n \) given the pricing rule \( p \), and

- the market clears: \( \sum_i^N w^n = \sum_i^N e^n \).

The set of priors \( \mathcal{C} \) in (1.1) is closed and convex, and \( \alpha \) can take any value between \([0, 1]\). All utility functions \( u : \mathbb{R} \to \mathbb{R} \) are assumed to be differentiable, strictly concave and strictly increasing. To keep the analysis tractable we assume that \( u \) is defined on the whole real line. The majority of results in this paper (e.g. the \( \alpha \)-MEU portfolio characterization in Proposition 4.1) holds true also when \( u \) has a bounded domain, as long as the set of feasible portfolios remain convex and the utility differentiable.\(^4\)

### 3 Standard Ellsberg framework

Throughout this section we consider a standard Ellsberg framework, that is a three-dimensional state space \( S = \{R, G, B\} \) where the states correspond to draws from the Ellsberg (1961) urn. The probability of the state \( R \) (red) is known and equal to \( \pi_R \in (0, 1) \), while the probabilities of the two ambiguous states \( G \) (green) and \( B \) (blue) are unknown.

Any closed convex set of priors \( \mathcal{D} \), consistent with the above information on the Ellsberg framework, can be written as

\[
\mathcal{D} = \{(\pi_R, q, 1-q - \pi_R) : q \in [a, b]\}
\]

where \( \pi_R \), \( q \), and \( 1 - q - \pi_R \) are the probability weights on the states \( R \), \( G \), and \( B \), respectively, corresponding to a given prior in \( \mathcal{D} \), and \( 0 \leq a \leq b \leq 1 - \pi_R \). Thus, any \( \alpha \)-MEU utility \( U \) in (1.1) on the portfolio \( w = (w_R, w_G, w_B) \in \mathbb{R}^3 \) reads as

\[
U(w) = \alpha \min_{q \in [a, b]} [\pi_R u(w_R) + q u(w_G) + (1 - q - \pi_R) u(w_B)] + (1 - \alpha) \max_{q \in [a, b]} [\pi_R u(w_R) + q u(w_G) + (1 - q - \pi_R) u(w_B)]
\]

for some \( \alpha \in [0, 1] \). In the following Proposition 3.1, we show that any \( \alpha \)-MEU is equivalent to either 1-MEU or 0-MEU, over a smaller set of priors, or SEU. The proof is provided in Appendix A.

\(^4\)These properties can be insured, for instance, by requiring that the feasible portfolios are in the interior of the utility domain.
Proposition 3.1. Consider the utility $U$ in (3.2) and let $c := \alpha a + (1 - \alpha)b$ and $d := (1 - \alpha)a + \alpha b$.

(i) If $\alpha > 1/2$, then $U$ is a maxmin expected utility (1-MEU), i.e.

$$U(w) = \min_{q \in [c, d]} [\pi_R u(w_R) + q u(w_G) + (1 - q - \pi_R) u(w_B)]$$

with set of priors $C = \{(\pi_R, q, 1 - q - \pi_R) : q \in [c, d]\} \subset \mathcal{D}$.

(ii) If $\alpha = 1/2$, then $U$ is a subjective expected utility (SEU) with subjective prior

$$\left(\pi_R, \frac{(a + b)}{2}, 1 - \pi_R - \frac{(a + b)}{2}\right).$$

(iii) If $\alpha < 1/2$, then $U$ is a maxmax expected utility (0-MEU), i.e.

$$U(w) = \max_{q \in [d, c]} [\pi_R u(w_R) + q u(w_G) + (1 - q - \pi_R) u(w_B)]$$

with set of priors $C = \{(\pi_R, q, 1 - q - \pi_R) : q \in [d, c]\} \subset \mathcal{D}$.

Proposition 3.1 shows that any $\alpha$-MEU model with $\alpha > 1/2$ ($\alpha < 1/2$) and a generic set of prior $\mathcal{D}$ is equivalent to a unique maxmin model (respectively, maxmax model) over a set of priors $C$, which is smaller than $\mathcal{D}$, and univocally characterized by $\alpha$ and $\mathcal{D}$. Consequently, $\alpha$-MEU preferences in the standard Ellsberg framework cannot be distinguished from maxmin, maxmax or SEU preferences. The fact that $\alpha$-MEU preferences are indistinguishable from 0-MEU, 1-MEU or SEU preferences has implications for experimental studies. For instance, it clarifies some recent experimental studies carried out in the standard Ellsberg framework in which the $\alpha$-MEU model is used as a generalization of the 1-MEU and 0-MEU model; e.g. Bossaerts et al. (2010), and Ahn et al. (2013). We will discuss these aspects in Section 5.

The converse of Proposition 3.1 was already known from Siniscalchi (2006), namely that a maxmin model with a given set of priors $C$ can be rewritten as less parsimonious $\alpha$-MEU models, with set of priors $\mathcal{D}$ larger than $C$, for many different $(\alpha, \mathcal{D})$.

The $\alpha$-MEU preferences coincide with maxmin, maxmax, or SEU preferences also in a state space setting with more than one risky states, or with no risky states, as long as there are only two ambiguous states. When there are no risky states, Proposition 3.1

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The set of priors $C$ equals $\mathcal{D}$ when $\alpha = 1$ ($\alpha = 0$), and when $\alpha$ decreases (increases) to 1/2 shrinks up to only containing the prior (3.4).
holds true by setting \( \pi_R = 0 \). When there are \( m \geq 1 \) risky states, \( R_1, \ldots, R_m \), with known probabilities \( \pi_{R_i} \in (0, 1) \) which satisfy \( \sum_{i=1}^{m} \pi_{R_i} < 1 \), Proposition 3.1 holds true by replacing the prior in (3.4) by the prior \( (\pi_{R_1}, \ldots, \pi_{R_m}, (a+b)/2, 1-\sum_{i=1}^{m} \pi_{R_i}-(a+b)/2) \), and \( \pi_R u(w_R) \) in (3.3) and (3.5) by \( \sum_{i=1}^{m} \pi_{R_i} u(w_{R_i}) \).

In the following section we show that when there are three or more ambiguous states, the \( \alpha \)-MEU model with \( \alpha \in (0, 1) \) does not reduce to the maxmin, maxmax, or SEU models, and induces different portfolio choice and attitude toward ambiguity.

4 Extended Ellsberg framework

In this section we consider an extended Ellsberg framework with \( l \) ambiguous and \( m \) risky states. We consider \( l \geq 3 \) and denote by \( A \subset S \) the set of ambiguous states \( (|A| = l) \). Moreover, the known probabilities \( \pi_R \in (0, 1) \) of risky states \( R \in S \setminus A \) satisfy \( \sum_{R \in S \setminus A} \pi_R < 1 \).

4.1 The \( \alpha \)-MEU portfolio choice

To study the portfolio choice of the \( \alpha \)-MEU model as a function of the ambiguity aversion parameter \( \alpha \in [0, 1] \), we set \( C \) equal to \( C_{\text{max}} \), i.e. the maximal set of priors consistent with the information available in the extended Ellsberg framework.\(^6\) More precisely, \( C_{\text{max}} \) is the set of all possible priors such that probabilities on the risky states equal the known probabilities \( \pi_R, R \in S \setminus A \). We denote this class of models by \( \alpha-C_{\text{max}} \)-MEU. The \( \alpha \)-MEU utility in (1.1) with \( C = C_{\text{max}} \) can be rewritten as

\[
U(w) = \sum_{R \in S \setminus A} \pi_R u(w_R) + (1 - \sum_{R \in S \setminus A} \pi_R) (\alpha u(w_{\text{min}}^A) + (1 - \alpha) u(w_{\text{max}}^A))
\]

where \( w_{\text{min}}^A \) and \( w_{\text{max}}^A \) is the respectively smallest and largest wealth in the portfolio \( w \in \mathbb{R}^{m+l} \) allocated among the \( l \) ambiguous states, namely

\[
(4.2) \quad w_{\text{min}}^A := \min_{\sigma \in A} w_{\sigma}, \quad \text{and} \quad w_{\text{max}}^A := \max_{\sigma \in A} w_{\sigma}.
\]

Note that \( U \) is concave if and only if \( \alpha = 1 \).

\(^6\)\( C = C_{\text{max}} \) is an assumption made to interpret the \( \alpha \) in the \( \alpha \)-MEU model as the agent’s degree of aversion toward ambiguity and thereby allowing for comparative statics; e.g. Chen et al. (2007) and Alm et al. (2013).
The following Proposition 4.1 characterizes the optimal portfolios of the \( \alpha \)-\( \text{C}_{\text{max}} \)-MEU agents for \( \alpha \in [0,1] \). The proof is provided in Appendix B.

**Proposition 4.1.** Suppose that the pricing rule \( p \) satisfies \( p_\sigma > 0 \) for all \( \sigma \in S \). Let \( I \) denote the set of states with the lowest price among the ambiguous states, that is \( I := \{ \sigma \in A \mid p_\sigma = \min_{\eta \in A} p_\eta \} \). Consider an \( \alpha \)-\( \text{C}_{\text{max}} \)-MEU agent. Then the following holds true:

(i) If \( \alpha = 0 \), then there is no optimal portfolio.\(^7\)

(ii) Let \( \alpha \in (0, \frac{l-1}{l}) \) and suppose there is an optimal portfolio. Then there are \( |I| \) optimal portfolios. The optimal portfolios all coincide on the risky states, whereas on the ambiguous states they only take two different values \( \overline{w} \in \mathbb{R} \) and \( \underline{w} \in \mathbb{R} \) with \( \underline{w} < \overline{w} \), which are the same for all optimal portfolios. Every optimal portfolio is obtained by choosing a single ambiguous state \( \nu \in I \) and then setting

\[
\begin{align*}
  w_\nu &= \overline{w} \\
  w_\eta &= \underline{w} \text{ for the remaining } (l-1) \text{ ambiguous states } \eta \in A \setminus \{\nu\}.
\end{align*}
\]

Hence, for all optimal portfolios \( w_{\min}^A = \underline{w} \) and \( w_{\max}^A = \overline{w} \).

(iii) Let \( \alpha \in [\frac{l-1}{l},1) \) and suppose there is an optimal portfolio. Then, the optimal portfolio \( w \) is unique and unambiguous, i.e. \( w_{\max}^A = w_{\min}^A \), if and only if

\[
\forall \sigma \in A : \quad \frac{p_\sigma}{\sum_{\nu \in A} p_\nu} = \frac{p_\sigma}{\sum_{S \setminus A} p_R} \geq 1 - \alpha.
\]

If (4.4) does not hold, then the optimal portfolios are the same as in (ii).

(iv) If \( \alpha = 1 \), the optimal portfolio is unambiguous, i.e. \( w_{\max}^A = w_{\min}^A \).

**Corollary 4.2.** In the setting of Proposition 4.1, suppose that the prices of the ambiguous states are all equal, i.e. \( p_\nu = p_\eta \) for all \( \nu, \eta \in A \).

(i) If \( \alpha \in (0, \frac{l-1}{l}) \), then the optimal portfolios are the ambiguous portfolios described in (4.3). Since \( |I| = |A| = l \), the number of optimal portfolios equals the number of ambiguous states.

(ii) If \( \alpha \in [\frac{l-1}{l},1] \), then the optimal portfolio is unambiguous, i.e. \( w_{\max}^A = w_{\min}^A \).

\(^7\)A utility with bounded domain would imply the existence of an optimal portfolio for the 0-MEU agent. This is due to the fact that the bounded domain will prevent the agent from going arbitrarily short; see Lemma B.5.
The following example illustrates Proposition 4.1 when the state space $S$ contains $m = 1$ risky state and $l = 3$ ambiguous states.

**Example 4.3.** Let $S = \{R\} \cup A$ where $A = \{G,B,Y\}$. Consider an $\alpha$-C$_{max}$-MEU agent and let $w = (w_R, w_G, w_B, w_Y) \in \mathbb{R}^4$ be her optimal portfolio. Without loss of generality, let $0 < p_G \leq p_B \leq p_Y$. Let $\alpha \in (0,2/3)$. Then the optimal portfolio is always exposed to ambiguity. In particular there are $w_R, \overline{w}, w \in \mathbb{R}$ with $w > \overline{w}$ such that:

(i) if $p_G < p_B$ ($I = \{G\}$), the optimal portfolio is unique and reads $w = (w_R, \overline{w}, w, \overline{w})$

(ii) if $p_G = p_B < p_Y$ ($I = \{G,B\}$), then there are two optimal portfolios, namely $(w_R, \overline{w}, w, \overline{w})$ and $(w_R, w, \overline{w}, \overline{w})$

(iii) if $p_G = p_B = p_Y$ ($I = A$), then there are three optimal portfolios: $(w_R, \overline{w}, w, \overline{w})$, $(w_R, \overline{w}, \overline{w}, \overline{w})$, and $(w_R, w, \overline{w}, \overline{w})$.

Let $\alpha \in \left[\frac{2}{3},1\right]$. Then the optimal portfolio $w$ is unambiguous, i.e. $w_G = w_B = w_Y$, if and only if

$$\forall \sigma \in \{G,B,Y\} : \frac{p_\sigma}{p_G + p_B + p_Y} = \frac{p_\sigma}{1 - p_R} \geq 1 - \alpha.$$  

This will for instance always be the case if $p_G = p_B = p_Y$ or if $\alpha = 1$. If (4.5) does not hold, then either (i) or (ii) holds.

In the following sections we provide an economic interpretation of Proposition 4.1, Corollary 4.2 and Example 4.5.

### 4.1.1 Portfolio choice and the ambiguity aversion parameter $\alpha$

Proposition 4.1 shows that an $\alpha$-C$_{max}$-MEU agent with $\alpha \in \left[\frac{l-1}{l},1\right]$ may either choose a portfolio exposed to ambiguity or an unambiguous portfolio, while an $\alpha$-C$_{max}$-MEU agent with $\alpha \in (0,\frac{l-1}{l})$ always chooses a portfolio exposed to ambiguity, no matter what state prices are.

Interestingly, all $\alpha$-C$_{max}$-MEU agents only choose ambiguous portfolios with the specific type of exposure to ambiguity described in (4.3), namely equal wealth $\overline{w}$ on $(l - 1)$ ambiguous states, and larger wealth $\overline{w} > w$ in the cheapest (or one of the cheapest) ambiguous state. If there is just one state with lowest price among the ambiguous states, that is if $|I| = 1$, then the optimal portfolio will be unique. Otherwise, if the agent chooses...
a portfolio exposed to ambiguity and if \(|I| > 1\), then there are \(|I|\) optimal portfolios. Example 4.3 illustrates this situation. The extent to which the optimal ambiguous portfolio in (4.3) is exposed to ambiguity (i.e. how large is the difference \(\bar{w} - \underline{w}\)) depends on \(\alpha\) and on the utility function \(u\) in (4.1) of the \(\alpha\)-C\(_{\text{max}}\)-MEU agent. This follows from the first order conditions (B.2) derived in the proof of Proposition 4.1:

\[
\frac{u'(\underline{w})}{u'(\bar{w})} = \frac{\sum_{\nu \in A \setminus \{\sigma\}} p_{\nu} (1 - \alpha)}{p_{\sigma} \alpha}, \quad \text{and} \quad \frac{u'(w_R)}{u'(\bar{w})} = \frac{(1 - \alpha)(1 - \pi_R)p_R}{p_{\sigma} \pi_R}, \quad \text{and} \quad \frac{u'(w_R)}{u'(\underline{w})} = \frac{\alpha(1 - \pi_R)p_R}{\sum_{\nu \in A \setminus \{\sigma\}} p_{\nu} \pi_R},
\]

where \(\sigma \in I\) with \(w_\sigma = \bar{w}\) and \(w_\eta = \underline{w}\) for all \(\eta \in A \setminus \{\sigma\}\), and where we for simplicity assume that there is only one risky state \(R\). Given a state price vector \(p\) and a utility \(u\), the smaller is \(\alpha\), the larger will be \(\bar{w} - \underline{w}\) and thus the exposure of the optimal portfolio to ambiguity. The limit case is \(\alpha = 0\) in which \(\bar{w} - \underline{w} \to \infty\), and thus there is no optimum; Proposition 4.1 (i). Given a state price vector \(p\) and \(\alpha \in (0, 1)\), the less the utility function \(u\) is concave (i.e. the slower \(u'\) decreases), the larger \(\bar{w} - \underline{w}\).

Proposition 4.1 (iii) shows that when \(\alpha\) increases, not only the exposure to ambiguity of the ambiguous portfolio decreases, but when \(\alpha \geq \frac{l-1}{l}\), leads to a decrease in demand for the ambiguous portfolio. In other words, an increase of \(\alpha\) increases the set of prices for which an \(\alpha\)-C\(_{\text{max}}\)-MEU agent chooses an unambiguous portfolio (see inequalities (4.4)). The limit case is \(\alpha = 1\) in which the optimal portfolio is necessarily unambiguous.\(^8\)

Finally, from the equalities (4.6), one can also assess how \(\alpha\) impacts the allocation of wealth between the risky state \(R\) and the ambiguous states. An increase of \(\alpha\) decreases the difference \(\bar{w} - w_R\). In the limit case when \(\alpha \uparrow \frac{l-1}{l}\) the optimal portfolio tends to the unambiguous portfolio and the optimal allocation between risky and ambiguous security is the same as that of an SEU with the prior that assigns probability \(\pi_R\) to the risky state \(R\) and probability \(\frac{1 - \pi_R}{l - 1}\) to each ambiguous state.

### 4.1.2 Portfolio choice and the number of ambiguous states

The portfolio choice of an \(\alpha\)-C\(_{\text{max}}\)-MEU agent depends on the number \(l\) of ambiguous states. Specifically, from Proposition 4.1 (ii), the interval \((0, \frac{l-1}{l})\) of \(\alpha\)-values for which the associated \(\alpha\)-C\(_{\text{max}}\)-MEU agents always chooses an ambiguous portfolio is increasing

\(^8\)When \(\alpha \to 1\) then \(\frac{1 - \alpha}{\alpha} \to 0\).
in $l$. In other words, the larger is $l$, the larger the set of parameters $\alpha$ for which an $\alpha$-$C_{\text{max}}$-MEU agent always prefers ambiguous portfolios. For instance, fix $\alpha \in (0, 1)$, and suppose that the state prices of ambiguous states are equal. From Corollary 4.2, in a state space with $l \leq \frac{1}{1-\alpha}$, the $\alpha$-$C_{\text{max}}$-MEU agent chooses an unambiguous portfolio, while in a state space with $l > \frac{1}{1-\alpha}$ she always prefers an ambiguous portfolio of type (4.3).

To further investigate the impact of $l$ on the optimal portfolios, consider a state space $S$ with a single risky state $R$ and $l \geq 3$ ambiguous states, and suppose that all state prices are equal, that is $p_\eta = \frac{1}{l+1}$ for all $\eta \in S$. In this setting, an $\alpha$-$C_{\text{max}}$-MEU with $\alpha \in \left[\frac{l-1}{l}, 1\right]$ chooses an unambiguous portfolio $w = (w_R, w_a, \ldots, w_a)$ for some $w_R, w_a \in \mathbb{R}$, while an $\alpha$-$C_{\text{max}}$-MEU with $\alpha \in (0, \frac{l-1}{l})$ chooses an ambiguous portfolio as in (4.3).

The first order condition which characterizes the unambiguous portfolio

$$u'(w_a) = \frac{\pi_R l}{1 - \pi_R}$$

shows that the larger $l$, the larger $|w_R - w_a|$. For instance, if $\frac{1-\pi_R}{l} > \pi_R$, the larger $l$, the larger will be the wealth $w_a$ allocated in each of the $l$ ambiguous securities as compared to the wealth $w_R$ allocated in the risky security $R$. The SEU with prior $(\pi_R, \frac{1-\pi_R}{l}, \ldots, \frac{1-\pi_R}{l})$ would choose the same optimal portfolio.

The first order conditions (see (B.2) in Lemma B.4) that characterize the ambiguous portfolios, i.e. $w_R, \bar{w}$ and $\bar{w}$, here read

$$\frac{u'(w_R)}{u'(\bar{w})} = \frac{(1-\alpha)(1-\pi_R)}{\pi_R}, \quad \text{and} \quad \frac{u'(w)}{u'(\bar{w})} = \frac{(l-1)(1-\alpha)(1-\pi_R)}{\alpha(1-\pi_R)}.$$

These first order conditions show that, for fixed $\alpha$, the larger $l$, the larger the difference $\bar{w} - \bar{w}$, and thus the more ambiguous is the portfolio.\(^9\)

### 4.1.3 Portfolio inertia and risk premium

We now consider a state space $S$ with no risky states, i.e. $m = 0$, and $l \geq 3$ ambiguous states. In this setting an unambiguous portfolio $(w_1, \ldots, w_l) = (w, \ldots, w)$, $w \in \mathbb{R}$, is also a riskfree portfolio. Any $\alpha$-$C_{\text{max}}$-MEU agent with $\alpha \in \left[\frac{l-1}{l}, 1\right]$ exhibits portfolio inertia at certainty, i.e. when the initial portfolio is unambiguous riskfree. Indeed, for any $\alpha \in \left[\frac{l-1}{l}, 1\right]$, there is a set of prices $\{p \in \mathbb{R}_+^l \mid \forall \sigma \in S: p_\sigma \geq 1 - \alpha\}$ (see (4.4)) such that it is optimal for the $\alpha$-$C_{\text{max}}$-MEU agent to hold, i.e. not rebalance, the initial unambiguous

---

\(^9\)Since $\alpha \in (0, \frac{l-1}{l})$ implies that $\frac{u'(w)}{u'(\bar{w})} \geq 1$, $\bar{w} - \bar{w} > 0$. 

riskfree portfolio. An increase in $\alpha$ leads to more portfolio inertia at certainty. The larger is $\alpha$, the larger becomes the set of prices for which it is optimal to hold an unambiguous riskfree portfolio.

In contrast, $\alpha$-C$_{\text{max}}$-MEU agents with $\alpha \in (0, \frac{l-1}{l})$ do not exhibit portfolio inertia at certainty. For such agents, no matter what state prices are, it always optimal to hold ambiguous portfolios of type (4.3). An increase in the number of ambiguous states $l$ shrinks the interval $[\frac{l-1}{l}, 1]$ of $\alpha$-values for which $\alpha$-C$_{\text{max}}$-MEU agents exhibit portfolio inertia.

According to (4.4), an $\alpha$-C$_{\text{max}}$-MEU agent takes an ambiguous portfolio if and only if amongst the ambiguous states there is at least one state $\sigma$ for which $p_\sigma < 1 - \alpha$. Hence, the larger is $\alpha$, the smaller the state price $p_\sigma$ must be in order to make the agent choose an ambiguous portfolio. An increase in $\alpha$ always leads to an increase in the risk premium for the ambiguous security.

### 4.2 Comparison of $\alpha$-C$_{\text{max}}$-MEU and maxmin models

We now compare the portfolio choice of the $\alpha$-C$_{\text{max}}$-MEU, $\alpha < 1$, with the portfolio choice of the maxmin model. The maxmin (1-MEU) agent is represented by a convex and closed set of priors $C \subseteq C_{\text{max}}$ and the corresponding criterion

\[
U(w) = \sum_{R \in S \setminus A} \pi_R u(w_R) + \min_{\pi \in C} \sum_{\sigma \in A} \pi_\sigma u(w_\sigma).
\]

Thanks to the concavity of the utility $U$ in (4.7), the optimal portfolio of such an agent can be derived by studying the supergradient of $U$. As an illustration, Appendix C provides the portfolio choice of the maxmin agent in the standard Ellsberg framework. For instance, for any given portfolio, there exists a set of priors $C$ for which the associated maxmin agent will choose that portfolio as the optimal one. Already in this respect, the maxmin model exhibits a clear difference with the $\alpha$-C$_{\text{max}}$-MEU model. We recall that for $\alpha$-C$_{\text{max}}$-MEU agents only unambiguous portfolio and ambiguous portfolios with the specific exposure to ambiguity given in (4.3) can be optimal.

Another difference between $\alpha$-C$_{\text{max}}$-MEU and maxmin models is the following. If there are three or more ambiguous states, the $\alpha$-MEU utility with $\alpha \in (0, 1)$ cannot in general be rewritten as a 1-MEU, SEU, or 0-MEU utility. One way to see this is to observe that the $\alpha$-C$_{\text{max}}$-MEU utility from a portfolio $w \in \mathbb{R}^{m+l}$ on the ambiguous states only depends
on \( w_{\text{min}}^A \) and \( w_{\text{max}}^A \) (see equation (4.1)), which are respectively the smallest and largest wealth in the portfolio \( w \in \mathbb{R}^{m+l} \) allocated among the \( l \) ambiguous states. This is not the case for a maxmin (maxmax) utility model, as soon as in the state space there are more than two ambiguous states. Indeed, the utility of the maxmin model from a portfolio \( w \in \mathbb{R}^{m+l} \) will be a function of the portfolio’s smallest wealth \( w_{\text{min}}^A \) (respectively, the portfolio’s largest wealth \( w_{\text{max}}^A \)) and then, depending on the set of priors, of the second smallest wealth (respectively the second largest wealth) and so on, until the sum of the probabilities of the states in which these wealths are allocated reaches \( 1 - \sum_{R \in S \setminus A} \pi_R \).\(^{10}\)

To investigate the different attitudes toward ambiguity of maxmin and \( \alpha \)-MEU agents, we contrast their portfolio choices when the prices of the ambiguous states are equal.\(^{11}\) In this setting, the ambiguous states are all “equally ambiguous” and thus interchangeable. It appears natural that the set of priors \( C \subseteq C_{\text{max}} \) of the maxmin agent includes the prior \( \tilde{\pi} \) which assigns equal probability, i.e. \((1 - \sum_{R \in S \setminus A})/l\), to all ambiguous states.\(^{12}\) The following lemma shows that maxmin agents, no matter how large is the set of priors, always choose an unambiguous portfolio.

**Lemma 4.4.** Suppose that \( p_\sigma = p_\eta \) for all \( \sigma, \eta \in A \). Then any maxmin agent with a set of priors \( C \) such that \( \tilde{\pi} \in C \) takes an unambiguous portfolio.

**Proof.** Let \( w \) be an optimal portfolio of the maxmin agent and assume that \( w_\sigma \neq w_\eta \) for \( \sigma, \eta \in A \). Consider the portfolio \( \hat{w} \) given by \( \hat{w}_R = w_R \) for any risky state \( R \in S \setminus A \) and \( \hat{w}_\sigma = z \) for any ambiguous state \( \sigma \in A \) where
\[
 z := \frac{\sum_{\sigma \in A} p_\sigma w_\sigma}{\sum_{\sigma \in A} p_\sigma} = \frac{1}{l} \sum_{\sigma \in A} w_\sigma.
\]

\(^{10}\)Another way to see that \( \alpha \)-MEU utility with \( \alpha \in (0, 1) \) cannot be rewritten as a 1-MEU or SEU utility is to observe that the \( \alpha \)-\( C_{\text{max}} \)-MEU utility (as soon as there are more than two ambiguous states) is concave if and only if \( \alpha = 1 \) while the SEU and the maxmin utility are always concave if \( u \) is concave.

\(^{11}\)A comparatively high price in one of the ambiguous state could make the agents believe that this state has a higher probability of occurrence than the other ambiguous states, even though an exact knowledge of the probabilities is not available.

\(^{12}\)This is always the case for example when the set of priors \( C \) is permutation invariant (symmetric) in the ambiguous coordinates. Note that \( C_{\text{max}} \) contains \( \tilde{\pi} \) and is permutation invariant in the ambiguous states.
The portfolio \( \hat{w} \) satisfies the budget constraint and

\[
U(\hat{w}) = \sum_{R \in S \setminus A} \pi_R u(w_R) + (1 - \sum_{R \in S \setminus A} \pi_R) u(z) > \sum_{R \in S \setminus A} \pi_R u(w_R) + \frac{1}{l}(1 - \sum_{R \in S \setminus A} \pi_R) \left( \sum_{\sigma \in A} u(w_\sigma) \right) \geq U(w)
\]

where the strict inequality follows from the strict concavity of \( u \) and the last inequality is due to \( \tilde{\pi} \in \mathcal{C} \). This contradicts the optimality of \( w \).

Lemma 4.4 shows that when the unambiguous states are interchangeable, all maxmin agents (who considers the prior \( \tilde{\pi} \)) optimally choose an unambiguous portfolio. In contrast, any \( \alpha\)-\( \text{C}_{\text{max}} \)-MEU agent with \( \alpha \in (0, (l - 1)/l) \) always prefers to be exposed to ambiguity, despite the fact that she also sees the ambiguous states as interchangeable.\(^{13}\) Moreover, as pointed out in Section 4.1.2, the interval \((0, \frac{l-1}{l})\) of \( \alpha \)-values for which \( \alpha\)-\( \text{C}_{\text{max}} \)-MEU agents always choose an ambiguous portfolio increases with the number of ambiguous states \( l \) in the state space, whereas the optimal portfolio of maxmin agents do not depend on \( l \).

5 Market equilibrium in the standard Ellsberg framework

In this section we derive theoretically equilibrium asset prices in the standard Ellsberg framework. We then revisit the laboratory experimental evidence in Bossaerts et al. (2010). They run a series of experimental sessions in which a competitive financial market is embedded in the standard Ellsberg framework. Their findings show that investors’ preferences are well approximated by ambiguity averse and subjective expected utility preferences. They also find that ambiguity aversion matters for portfolio choices and equilibrium prices, and it does not wash out in aggregate.\(^{14}\)

\(^{13}\)Such an agent is indifferent on which state amongst the ambiguous ones the larger wealth \( \bar{w} \) is allocated.

\(^{14}\)Ahn et al. (2013) also run portfolio choice experiments in the standard Ellsberg framework and provide evidence of considerable heterogeneity in subject’s preferences. They find that one half of the subjects in the pull are well approximated by SEU preferences, the remaining half have a significant degree of ambiguity aversion while there is no evidence of ambiguity seeking behaviors.
Bossaerts et al. (2010) do not derive equilibrium asset prices, and provide conjectures to interpret their experimental findings. In the following, we derive theoretically the equilibrium asset prices implied by their model. Notably, we find a striking matching between our theoretical predictions and their experimental findings. Specifically, our theoretically derived rankings of the state-price/state-probability ratios fully explain the empirical rankings documented by Bossaerts et al. (2010).

5.1 Market equilibrium with ambiguity averse agents

We now derive equilibrium state-price/state-probability ratios for the market model considered by Bossaerts et al. (2010). We exploit the fact that their market is indeed a market populated by agents with maxmin and SEU preferences, and as such it is tractable because all utilities are concave.

Proposition 5.1 characterizes equilibrium state-price/state-probability ratios induced by maxmin (1-MEU) and SEU agents. Appendix C provides the proof, and a concise treatment of maxmin and SEU portfolio choice. Recall that the probability of the state $R$ is known and equal to $\pi_R \in (0,1)$, while the probabilities of the two ambiguous states $G$ and $B$ are unknown. Denote by $w = (w_R, w_G, w_B)$ the optimal portfolio of a maxmin agent, and by $y = (y_R, y_G, y_B)$ the the optimal portfolio of a SEU agent.

**Proposition 5.1.** Suppose the market is in equilibrium and populated by maxmin agents who take unambiguous portfolios and SEU agents with prior $\pi = (\pi_R, \pi_G, \pi_B)$, with $\pi_R, \pi_G, \pi_B > 0$. Denote by $W = (W_R, W_G, W_B) \in \mathbb{R}^3$ the total endowment of the market.

1) If $W_R > W_G > W_B$, then two rankings of the state-price/state-probability ratios are possible:

\[ \frac{p_B}{\pi_B} > \frac{p_R}{\pi_R} > \frac{p_G}{\pi_G} \]

and the optimal portfolios $y$ of any SEU agent and $w$ of any maximin agent satisfy

\[ y_G > y_R > y_B \quad \text{and} \quad w_R > w_G = w_B. \]

The other possible ranking is:

\[ \frac{p_B}{\pi_B} > \frac{p_G}{\pi_G} > \frac{p_R}{\pi_R} \quad \text{or} \quad \frac{p_B}{\pi_B} > \frac{p_G}{\pi_G} = \frac{p_R}{\pi_R} \]
and the optimal portfolios \( y \) of any SEU agent and \( w \) of any maximin agent satisfy

\[ y_R > y_G > y_B \quad (\text{or} \quad y_R = y_G > y_B) \quad \text{and} \quad w_R > w_G = w_B. \]

2) If \( W_G > W_R > W_B \), then the only possible ranking of the state-price/state-probability ratios is:

\[
\frac{p_B}{\pi_B} > \frac{p_R}{\pi_R} > \frac{p_G}{\pi_G}
\]

and the optimal portfolios \( y \) of any SEU agent and \( w \) of any maximin agent satisfy

\[ y_G > y_R > y_B \quad \text{and} \quad w_R > w_G = w_B \quad (\text{or} \quad w_R < w_G = w_B). \]

3) If \( W_G > W_B > W_R \), then two rankings of the state-price/state-probability ratios are possible:

\[
\frac{p_B}{\pi_B} > \frac{p_R}{\pi_R} > \frac{p_G}{\pi_G}
\]

and the optimal portfolios \( y \) of any SEU agent and \( w \) of any maximin agent satisfy

\[ y_G > y_R > y_B \quad \text{and} \quad w_R < w_G = w_B. \]

The other possible ranking is:

\[
\frac{p_R}{\pi_R} > \frac{p_B}{\pi_B} > \frac{p_G}{\pi_G} \quad \left( \text{or} \quad \frac{p_R}{\pi_R} = \frac{p_B}{\pi_B} \right) > \frac{p_G}{\pi_G}
\]

and the optimal portfolios \( y \) of any SEU agent and \( w \) of any maximin agent satisfy

\[ y_G > y_B > y_R \quad (\text{or} \quad y_G > y_B = y_R) \quad \text{and} \quad w_R < w_G = w_B. \]

Depending on the distribution of the total endowment \( W = (W_R, W_G, W_B) \) in the market, in equilibrium only particular rankings of state-price/state-probability ratios can occur. The interesting case is when maximin agents take unambiguous portfolio, i.e. \( w_G = w_B \), and \( W_G \neq W_B \); without loss of generality we assume that \( W_G > W_B \). Market clearing implies that SEU agents have to hold portfolios which in aggregate pay strictly more on state \( G \) than on state \( B \). This occurs if and only if \( \frac{p_B}{\pi_B} > \frac{p_G}{\pi_G} \); see (C.3) in Appendix C. This condition excludes all state-price/state-probability rankings in which \( \frac{p_B}{\pi_B} < \frac{p_G}{\pi_G} \).
Proposition 5.1 confirms the conjecture in Bossaerts et al. (2010, Page 1339), that the ranking (5.3) is more likely to occur when \( W_G > W_R > W_B \). Indeed, this is the only theoretically possible ranking. However, Proposition 5.1 contradicts the conjecture that no matter how aggregate wealth \( W_R \) is ranked with respect to aggregate wealth in the ambiguous states \( W_G, W_B \), any ranking of the state-price/state-probability ratio for the risky state \( p_R/\pi_R \) with respect to \( p_B/\pi_B > p_G/\pi_G \) is theoretically possible; Bossaerts et al. (2010, Page 1339). In fact, the theoretically possible rankings do depend on the ranking of \( W_R \) with respect to \( W_G \) and \( W_B \).

Proposition 5.1 nicely explains the experimental findings in Bossaerts et al. (2010). Consider the empirical distribution functions of the state-price/state-probability ratios from their experimental session reproduced in Figure 1.

![Figure 1: Empirical distribution functions of state-price/state-probability ratios from the experimental session of eight trading periods in Bossaerts et al. (2010) with \( W_G = 272, W_B = 162, \) and \( W_R = 81 \). The distribution function with circles is for \( p_R/\pi_R \); the one with arrows pointing to the right is for \( p_B/\pi_B \); the one with arrows pointing to the left is for \( p_G/\pi_G \). This figure is a copy of Figure 8, right panel, in Bossaerts et al. (2010).](image)

The experimental findings summarized in Figure 1 provide evidence of two rankings: \( p_R/\pi_R > p_B/\pi_B > p_G/\pi_G \), and \( p_B/\pi_B > p_R/\pi_R > p_G/\pi_G \). Bossaerts et al. (2010,
Page 1351) report that “the rankings appear anomalous,” because they only expect to see the second ranking (Bossaerts et al. 2010, Page 1339). When $W_G > W_B > W_R$, Proposition 5.1 shows that two rankings are possible, namely (5.4) and (5.5). Remarkably, these are exactly the two rankings appearing in Figure 1.

Proposition 5.1 further shows that the ranking (5.5) prevails when $W_B - W_R$ is large enough to imply an optimal portfolio of the SEU agents with more Arrow securities that pay in the ambiguous state $B$ than in the risky state $R$. This provides a potential explanation why in Figure 1 the prices do not settle in favor of one of the two rankings: the values $W_R$, $W_G$, and $W_B$ in the experimental section in Figure 1 are close to the point at which the change from (5.4) to (5.5) takes place. Example 5.2.2 illustrates this point.

Bossaerts et al. (2010) perform other experimental sessions in which $W_G > W_B > W_R$. Although the most common ranking of state-price/state-probability ratios is (5.4), the empirical distribution functions of $p_R/\pi_R$ and $p_B/\pi_B$ are very close; see Figure 7 in Bossaerts et al. (2010). Proposition 5.1 predicts that to observe a clean separation of the rankings in (5.4) and (5.5), the aggregate wealth $W_B$ should be chosen closer to $W_R$ or $W_G$, respectively.

Finally, Proposition 5.1 shows why ambiguity aversion does not wash out in aggregate. When rankings in (5.1) or (5.4) occur, state-price/state-probability ratios are not ranked opposite to aggregate wealth. In these cases, an expected utility representative agent would rank state-price/state probability ratios differently than SEU agents in the market. The following section further studies how ambiguity aversion impacts equilibrium asset prices.

5.2 Illustrating Proposition 5.1

We now illustrate Proposition 5.1 in the case of exponential and quadratic utilities. Recall that the market is populated by SEU agents with prior $\pi = (\pi_R, \pi_G, \pi_B)$, with $\pi_R, \pi_G, \pi_B > 0$, and by maxmin agents who are sufficiently ambiguity averse to hold an unambiguous portfolio, i.e. $w_G = w_B$. The total endowment $W = (W_R, W_G, W_B)$ is such that $W_G > W_B$. 
5.2.1 CARA utility

There are $L$ SEU agents and $M$ maxmin agents, all having exponential utilities $u(z) = 1 - \frac{e^{-\delta z}}{\delta}$. Let $\delta = a$ and $\delta = b$ be the risk aversion parameter of the SEU agents and maxmin agents, respectively. Then, the equilibrium state prices are:

$$
P_R = \frac{\pi_R}{\pi_R + e^{-\frac{a}{M+L}} \pi_G \frac{bL}{aM+L} q \frac{aM}{aM+L} + e^{-\frac{a}{M+L}} \frac{bL}{aM+L} \pi_B \frac{bL}{aM+L} (1 - \pi_R - q) \frac{aM}{aM+L}}$$

$$
P_G = \frac{bL}{aM+L} \pi_G \frac{aM}{aM+L} + \pi_R e^{-\frac{a}{M+L}} \frac{bL}{aM+L} \pi_B \frac{bL}{aM+L} (1 - \pi_R - q) \frac{aM}{aM+L}$$

$$
P_B = \frac{bL}{aM+L} \frac{aM}{aM+L} + \pi_R e^{-\frac{a}{M+L}} \frac{bL}{aM+L} \pi_B \frac{bL}{aM+L} (1 - \pi_R - q) \frac{aM}{aM+L} q \frac{aM}{aM+L}$$

where

$$q := \frac{\pi_G + \pi_B}{\pi_G + \pi_B e^{\frac{a}{L}(W_G - W_B)}}$$

The above formulae show that the equilibrium prices are the same prices that would be obtained in a market populated by only SEU agents, $L$ of which with prior $(\pi_R, \pi_G, \pi_B)$ and risk aversion $a$, and $M$ of which with prior $(\pi_R, q, 1 - \pi_R - q)$ and risk aversion $b$. However, this equilibrium is significantly different than an equilibrium resulting from the interaction of SEU agents with different priors (or heterogeneous beliefs). The reason is that $q$ in (5.6) is a function of $W_G - W_B$, of the number $L$ of SEU agents in the market, and of their risk aversion $a$. In particular, the dependence of $q$ on $W_G - W_B$ illustrates one channel through which ambiguity aversion impacts asset prices. In aggregate maxmin agents do not hold the imbalance $W_G - W_B$, which is left to the SEU agents to be absorbed.

The impact of an increase of $\frac{a}{L}(W_G - W_B)$ on the securities prices that pay in the ambiguous states is clear: An increase of $\frac{a}{L}(W_G - W_B)$ will decrease $q$ and increase $(1 - \pi_R - q)$, and consequently will decrease $p_G$, and increase $p_B$. When $M$ ambiguity averse

---

15 The fact that $q$ does not depend neither on the number of $M$ of maxmin agents nor on their risk aversion $b$ is a peculiarity of the exponential utility.

16 The prior $(\pi_R, q, 1 - \pi_R - q)$ is the only prior in the set $C$ of the maxmin agents which allows the $L$ SEU agents to clear the imbalance $W_G - W_B$ and reach equilibrium.

17 Depending on the particular rank of the total endowment $W = (W_R, W_B, W_G)$, the price of the Arrow security that pays in the state $R$ will increase or decrease with $\frac{a}{L}(W_G - W_B)$. 

20
agents are in the market, the equilibrium price \(p_G\) (\(p_B\)) will always be lower (higher) than the equilibrium price \(p_G\) (\(p_B\)) that would result when \(L + M\) SEU agents with the same prior \((\pi_R, \pi_G, \pi_B)\) are in the market.

The equilibrium price ratios are:

\[
\begin{align*}
\frac{p_G}{p_B} &= \frac{\pi_G^{\frac{aM}{aM+bL}}}{\pi_B^{\frac{aM}{aM+bL}}} e^{-\frac{ab}{aM+bL}(W_G-W_B)} \\
\frac{p_G}{p_R} &= \frac{\pi_G^{\frac{aM}{aM+bL}}}{\pi_R^{\frac{aM}{aM+bL}}} e^{-\frac{ab}{aM+bL}(W_R-W_B)} \\
\frac{p_B}{p_R} &= \frac{\pi_B^{\frac{aM}{aM+bL}}}{\pi_R^{\frac{aM}{aM+bL}}} e^{-\frac{ab}{aM+bL}(W_B-W_R)}.
\end{align*}
\]

Thus, all rankings of the state-price/state-probability ratios, that are possible accord to Proposition 5.1, can indeed occur. For instance, consider the case \(W_G > W_B > W_R\). The first two equations in (5.7) show that always \(\frac{p_B}{p_B} > \frac{p_G}{p_G}\) and \(\frac{p_R}{p_R} > \frac{p_G}{p_G}\). The third equation in (5.7) shows that both \(\frac{p_B}{p_B} > \frac{p_R}{p_R}\) and \(\frac{p_B}{p_B} < \frac{p_R}{p_R}\) can occur and, consequently, the corresponding rankings (5.4) and (5.5).

### 5.2.2 Quadratic utility

The quadratic utility is:

\[
u_a(x) = \begin{cases} 
  x - ax^2/2 & , x \leq 1/a \\
  1/(2a) & , x > 1/a 
\end{cases}
\]

for some \(a > 0\). To apply our results we assume that the feasible portfolios live on the strictly increasing part of the utility function. Let \(a, b > 0\) and suppose SEU agents have quadratic utility \(u_a\), whereas maxmin agents have \(u_b\). The fixed point equations for the equilibrium prices are

\[
\begin{align*}
p_R &= \frac{\pi_R(c-W_R)}{c-W} \\
p_G &= \frac{\pi_G(c-W_G)}{c-W} \left( \frac{D - \left( \frac{a}{b} - X^{MEU} \right)(c - \pi \cdot W)(1 + \frac{W_G-W_B}{c-W_G} - \pi_B)}{D - \left( \frac{a}{b} - X^{MEU} \right)(c - \pi \cdot W)} \right) \\
p_B &= \frac{\pi_B(c-W_B)}{c-W} \left( \frac{D - \left( \frac{a}{b} - X^{MEU} \right)(c - \pi \cdot W)(1 - \frac{W_G-W_B}{c-W_B} - \pi_G)}{D - \left( \frac{a}{b} - X^{MEU} \right)(c - \pi \cdot W)} \right)
\end{align*}
\]

where \(c := \frac{1}{a} + \frac{1}{b}, D := \pi_R(c-W_R)^2 + (1 - \pi_R)(c - \frac{\pi_GW_G+\pi_BW_B}{1-\pi_R})^2\) and \(X^{MEU}\) is the initial wealth of the maxmin agents. Note that \(p_G\) is lower (\(p_B\) is higher) than the price.
\[
\frac{\pi_G(c-W_G)}{c-\pi_G} \quad \text{(respectively } \frac{\pi_B(c-W_B)}{c-\pi_B}) \text{ that would result in a market equilibrium with SEU agents sharing the same prior } (\pi_R, \pi_G, \pi_B).
\]

Let \( W_G > W_B > W_R \). Figure 2 shows the state-price/state-probability ratios of the equilibrium prices as a function of the difference \( W_B - W_R \), computed for fixed \( W_G = 272 \) and \( W_R = 81 \), as in Figure 1. The parameters \( a \) and \( b \) in (5.8) are set to 0.001 in the left graph, and to \( a = 0.0015 \) and \( b = 0.001 \) in the right graph.

![Figure 2](image_url)

(a) \( a = b = 0.001 \).  
(b) \( a = 0.0015 \) and \( b = 0.001 \).

Figure 2: State-price/state-probability ratios (y-axis) of the equilibrium prices as a function of the difference \( W_B - W_R \) (x-axis), computed for fixed \( W_G = 272 \) and \( W_R = 81 \), as in Figure 1. The SEU prior is \( \pi_R = \pi_G = \pi_B = 1/3 \). The parameters \( a \) and \( b \) in (5.8) are set to 0.001 in the left graph, and to \( a = 0.0015 \) and \( b = 0.001 \) in the right graph. The line marked with circles represents \( p_R/\pi_R \), the one marked with arrows pointing to the right represents \( p_B/\pi_B \), and the one marked with arrows pointing to the left represents \( p_G/\pi_G \).

In both cases there is a clear change of rankings of state-price/state-probability ratios: as \( W_B - W_R \) increases the ranking switches from (5.4) to (5.5). When \( W_B - W_R \) is approximately 81 as in Figure 1, the switch of the rankings occurs. This confirms that to observe a clean ranking of state-price/state-probability ratios in laboratory experiment, the difference in aggregate wealth \( W_B - W_R \) should be chosen either relatively large or small.
6 Conclusion

The $\alpha$-MEU model has been adopted in recent work to study the impact of ambiguity aversion on optimal portfolio choice in a standard Ellsberg framework. We find, however, that in this framework $\alpha$-MEU preferences coincide with either maxmin, maxmax or subjective expected utility preferences. We then derive the theoretical equilibrium asset prices when the market is populated by maxmin and subjective expected utility agents. We show theoretically that ambiguity aversion does not wash out in equilibrium. Moreover, our theoretical predictions on equilibrium asset prices are strikingly consistent with the laboratory experimental evidence in Bossaerts et al. (2010).

We show that in an extended Ellsberg framework, with more than three ambiguous states, the $\alpha$-MEU model does not reduce to maxmin, maxmax or subjective expected utility models. We characterize the optimal portfolio choice of an $\alpha$-C$_{\text{max}}$-MEU agent. This agent chooses only two types of optimal portfolios: either unambiguous portfolios, or ambiguous portfolios that allocate more wealth to the cheapest ambiguous state and equal wealth to all other ambiguous states. Comparing $\alpha$-C$_{\text{max}}$-MEU and maxmin models, we find that these models induce significantly different portfolio choices and attitudes toward ambiguity. For instance, facing ambiguous states with equal prices, maxmin agents (no matter how large is the set of priors) typically choose an unambiguous portfolio. In contrast, all $\alpha$-C$_{\text{max}}$-MEU agents with an $\alpha$ lower than a certain threshold (that depends on the number of ambiguous states) choose ambiguous portfolios.
A Proof of Proposition 3.1

Let $U$ be as in (3.2). Then it follows that indeed

$$U(w) = \begin{cases} 
\pi_R u(w_R) + \alpha [au(w_G) + (1 - \pi_R - a)u(w_B)] + \\
(1 - \alpha) [bu(w_G) + (1 - \pi_R - b)u(w_B)] & \text{if } w_G \geq w_B \\
\pi_R u(w_R) + \alpha [bu(w_G) + (1 - \pi_R - b)u(w_B)] + \\
(1 - \alpha) [au(w_G) + (1 - \pi_R - a)u(w_B)] & \text{if } w_G < w_B 
\end{cases}$$

1st case: Suppose that $\alpha > 1/2$. Then

$$cu(w_G) + (1 - \pi_R - c)u(w_B) \leq du(w_G) + (1 - \pi_R - d)u(w_B)$$

whenever $w_G \geq w_B$ and

$$du(w_G) + (1 - \pi_R - d)u(w_B) \leq cu(w_G) + (1 - \pi_R - c)u(w_B)$$

if $w_G < w_B$. Thus the assertion of Proposition 3.1 (i) follows.

2nd case: If $\alpha = 1/2$, then $c = d$ and $U(\omega) = u(w_R) + cu(w_G) + (1 - \pi_R - c)u(w_B)$.

3rd case: If $\alpha < 1/2$, similar arguments as in the first case yield the assertion of Proposition 3.1 (iii).

B Proof of Proposition 4.1

Proposition 4.1 follows from Lemmas B.1–B.5 in the following.

Lemma B.1. Suppose that the pricing rule $p = (p_\sigma)_{\sigma \in S}$ satisfies $p_\sigma > 0$ for all $\sigma \in S$. Consider an $\alpha$-$C_{\text{max}}$-MEU agent with $\alpha \in (0, 1)$. Let $w = (w_\sigma)_{\sigma \in S} \in \mathbb{R}^n$ be an optimal portfolio for the $\alpha$-$C_{\text{max}}$-MEU agent. Then, either $w$ takes the same value on all ambiguous states, or there exist two disjoint subsets $\overline{A}$ and $\underline{A}$ of the set of ambiguous states $A$ such that $\overline{A} \cup \underline{A} = A$ and two values $\overline{w}, \underline{w} \in \mathbb{R}$ such that $w_\sigma = \overline{w} > \underline{w} = w_\eta$ for all $\sigma \in \overline{A}$ and all $\eta \in \underline{A}$.

Proof. Note that the only portfolio values on the ambiguous states on which the utility $U$ in (4.1) depends are $w^{\overline{A}}_{\text{max}}$ and $w^{\underline{A}}_{\text{min}}$. We order the set of ambiguous states $A = \{\sigma_1, \ldots, \sigma_l\}$
such that

\[(B.1) \quad w_{\sigma_1} \leq w_{\sigma_2} \leq \ldots \leq w_{\sigma_l}.\]

Let \(s\) be the number of strict inequalities in \((B.1)\). Consider states \(\nu_1, \ldots, \nu_{s+1} \in A\) such that \(w_{\nu_1} < w_{\nu_2} < \ldots < w_{\nu_{s+1}}\). Suppose there is a state \(\eta \in A\) such that \(w_\eta \neq w_\eta^A\) and \(w_\eta \neq w_{\min}^A\), namely suppose that \(s \geq 2\). We now consider the function \(U\) in \((4.1)\) as defined on \(\mathbb{R}^{m+s+1}\), where we merge those ambiguous states in which \(w\) takes the same value. Let \(\tilde{w} \in \mathbb{R}^{m+s+1}\) such that \(\tilde{w}_R = w_R\) for all risky states \(R \in S \setminus A\) and otherwise \(\tilde{w}_\sigma = w_\sigma\) for \(i = 1, \ldots, s + 1\). Then, \(\tilde{w}\) is a maximizer for the function \(U\) restricted to the open set \(C := \{x \in \mathbb{R}^{m+s+1} \mid x_{\sigma_1} < x_{\sigma_2} < \ldots < x_{\sigma_{s+1}}\}\), which we call \(U_C\), given the budget constraint \(\tilde{p} \cdot \tilde{w} \leq p \cdot e\). Here \(e\) is the initial portfolio and \(\tilde{p} \in \mathbb{R}^{m+s+1}\) is obtained from \(p\) by summing up the prices of those states which are merged when forming \(\tilde{w}\). As \(U_C\) is concave, according to \((D.3)\), a multiple of \(\tilde{p}\) is in the supergradient of \(U_C\) at \(\tilde{w}\). However, this supergradient is equal to zero in any \(x_{\sigma_i}\)-direction, \(i \in \{2, \ldots, s\}\), because only the largest value and the smallest value on the ambiguous states matter for \(U\). This contradicts the assumption \(p_{\sigma_i} > 0\) for \(i \in \{2, \ldots, s\}\). \(\square\)

**Lemma B.2.** In the setting of Lemma B.1, if \(p_\sigma < p_\eta\) for \(\sigma, \eta \in A\), then the optimal portfolio \(w\) satisfies \(w_\eta \leq w_\sigma\).

**Proof.** Suppose that the optimal portfolio \(w\) is such that \(w_\eta > w_\sigma\). Let \(\tilde{w}\) given by \(\tilde{w}_\nu = w_\nu\) for all \(\nu \in S \setminus \{\sigma, \eta\}\) and \(\tilde{w}_\sigma = w_\eta\) and \(\tilde{w}_\eta = w_\sigma\). Then \(U(\tilde{w}) = U(w)\), but \(p \cdot \tilde{w} < p \cdot w\) because \(p \cdot (w - \tilde{w}) = (p_\eta - p_\sigma)(w_\eta - w_\sigma) > 0\). This contradicts the optimality of \(w\), because increasing the wealth \(\tilde{w}_\sigma\) one could achieve a strictly higher utility while still respecting the budget constraint. \(\square\)

**Lemma B.3.** In the setting of Lemma B.1, if the sets \(\overline{A}\) and \(\underline{A}\) associated to the optimal portfolio \(w\) are not empty, then \(\overline{A} = \{\overline{\sigma}\}\) for a state \(\overline{\sigma} \in I := \{\sigma \in A \mid p_\sigma = \min_{\eta \in A} p_\eta\}\). Moreover, any portfolio which equals \(w\) on the risky states and assigns the weight \(w^A_{\max}\) to a single state in \(I\) and \(w^A_{\min}\) to all the other ambiguous states is optimal. Hence, there are \(|I|\) optimal portfolios.

**Proof.** By contradiction suppose that there are two different states \(\sigma_1\) and \(\sigma_2\) in \(\overline{A}\), i.e. that the optimal portfolio \(w\) is such that \(w_{\sigma_1} = w_{\sigma_2} = w^A_{\max}\), and without loss of generality we assume that \(p_{\sigma_1} \leq p_{\sigma_2}\). Consider \(\tilde{w}\) given by \(\tilde{w}_\eta = w_\eta\) for all \(\eta \in S \setminus \{\sigma_1, \sigma_2\}\) and
\[ \tilde{w}_{\sigma_1} = 2w_{\text{max}}^A - w_{\text{min}}^A \quad \text{and} \quad \tilde{w}_{\sigma_2} = w_{\text{min}}^A. \]

Then \( p \cdot \tilde{w} \leq p \cdot w \), so \( \tilde{w} \) satisfies the budget constraint, and \( U(\tilde{w}) > U(w) \) since \( \tilde{w}_{\text{max}}^A = \tilde{w}_{\sigma_1} > w_{\text{max}}^A \) and \( \tilde{w}_{\text{min}}^A = w_{\text{min}}^A \). This is a contradiction to optimality of \( w \). Lemma B.2 implies that \( \bar{\sigma} \in I \). The last statement of the lemma follows by observing that all these portfolios share the same price and utility.

\[ \tilde{w}_{\sigma_1} = 2w_{\text{max}}^A - w_{\text{min}}^A \quad \text{and} \quad \tilde{w}_{\sigma_2} = w_{\text{min}}^A. \]

Then \( p \cdot \tilde{w} \leq p \cdot w \), so \( \tilde{w} \) satisfies the budget constraint, and \( U(\tilde{w}) > U(w) \) since \( \tilde{w}_{\text{max}}^A = \tilde{w}_{\sigma_1} > w_{\text{max}}^A \) and \( \tilde{w}_{\text{min}}^A = w_{\text{min}}^A \). This is a contradiction to optimality of \( w \). Lemma B.2 implies that \( \bar{\sigma} \in I \). The last statement of the lemma follows by observing that all these portfolios share the same price and utility.

**Lemma B.4.** In the setting of Lemma B.1, let \( \alpha < 1 \). Then \( w \) is unambiguous, i.e. \( w_\sigma = w_\nu \) for all \( \sigma, \nu \in A \), if and only if (4.4) holds. In this case \( w \) is the only optimal portfolio. Condition (4.4) can only be satisfied if \( \alpha \geq \frac{l-1}{l} \).

**Proof.** Suppose \( \bar{A} = \{\sigma\} \) and thus \( A = A \setminus \{\sigma\} \). Then, the first order conditions imply

\[
\frac{p_R}{\pi_R u'(w_R)} = \frac{p_\sigma}{(1 - \alpha)(1 - \sum_{R \in S \setminus A} \pi_R) u'(w_{\text{max}}^A)} = \frac{\sum_{\nu \in A \setminus \{\sigma\}} p_\nu}{\alpha(1 - \sum_{R \in S \setminus A} \pi_R) u'(w_{\text{min}}^A)}
\]

where \( R \) denotes any risky state among the \( m \) ones. Thus,

\[
\frac{p_\sigma}{\sum_{\nu \in A \setminus \{\sigma\}} p_\nu} = \frac{(1 - \alpha)u'(w_{\text{max}}^A)}{\alpha u'(w_{\text{min}}^A)} < \frac{1 - \alpha}{\alpha}
\]

as \( w_{\text{max}}^A > w_{\text{min}}^A \). Consequently, if there are no \( \sigma \in A \) for which (B.3) is satisfied, i.e. if the condition (4.4) holds true, then \( w \) must be unambiguous. In order to prove necessity of (4.4), assume that (B.3) holds for some \( \sigma \in A \). In the following we show that in this case the unambiguous portfolio cannot be optimal. To this end, suppose by contradiction that the unambiguous portfolio \( w \) is optimal and let \( z := w_{\text{max}}^A = w_{\text{min}}^A \). Then \( \epsilon = 0 \) needs to maximize the function

\[ F : \mathbb{R} \ni \epsilon \mapsto \alpha u(z - \epsilon) + (1 - \alpha) u(z + \delta(\epsilon)) \]

over all \( \epsilon \geq 0 \), where \( \delta(\epsilon) := \epsilon \frac{\sum_{\sigma \in A \setminus \{\epsilon\}} p_\nu}{p_\sigma} \) is chosen such that the portfolio which invests \( z - \epsilon \) in the states \( \nu \in A \), and \( z + \delta(\epsilon) \) in the state \( \sigma \) satisfies the budget constraint (while the investment in the risky states is unaltered). \( F \) is a concave function and the first order condition reads

\[ \frac{u'(z + \delta(\epsilon))}{u'(z - \epsilon)} = \frac{\alpha}{(1 - \alpha) \sum_{\sigma \in A \setminus \{\epsilon\}} p_\nu} p_\sigma. \]

By assumption, the right hand side of the above equation is strictly smaller than 1. Hence, \( F \) attains its optimum for \( \epsilon > 0 \), which contradicts the optimality at 0 over all \( \epsilon \geq 0 \).

Finally, note that summing up (4.4) over all \( \sigma \in A \) yields:

\[ \alpha \sum_{\sigma \in A} p_\sigma \geq (1 - \alpha)(l - 1) \sum_{\nu \in A} p_\nu \quad \Leftrightarrow \quad \alpha \geq \frac{l - 1}{l}. \]
Lemma B.5. In the setting of Lemma B.1, if \(\alpha = 1\), then \(w\) is unambiguous. If \(\alpha = 0\), then there is no optimal portfolio.

Proof. If \(\alpha = 1\), then (4.1) is a maxmin agent and also \(\tilde{\pi} \in C_{\text{max}}\). Hence, Lemma 4.4 proves the claim.

The optimization problem of a 0-MEU agent with the maximal set of priors \(C_{\text{max}}\) is

\[
\begin{align*}
(B.4) & \quad \sum_{R \in S \setminus A} \pi_R u(w_R) + (1 - \sum_{R \in S \setminus A} \pi_R) u(w_{A_{\text{max}}}) \to \max \\
& \text{subject to } p \cdot w \leq p \cdot e
\end{align*}
\]

where \(e\) denotes her initial endowment. Since the agent may go arbitrarily long in the ambiguous state \(\sigma\) with \(w_\sigma = w_{A_{\text{max}}}\) and satisfy the budget constraint by going arbitrarily short in an other ambiguous state, the optimal value in (B.4) cannot be attained. \(\square\)

C Proof of Proposition 5.1

In the following, we briefly summarize how the interaction among SEU and maxmin agents impacts the equilibrium asset prices. This will provide us the tools to prove Proposition 5.1. Assume that \((p; w^1, \ldots, w^n)\) is an equilibrium with \(p_\sigma > 0\) for all \(\sigma \in \{R, G, B\}\). Then, the equilibrium price \(p\) satisfies

\[
(C.1) \quad \lambda_n p \in \partial U^n(w^n)
\]

for some \(\lambda_n > 0\); see (D.3). Here \(\partial U^n(w)\) denotes the supergradient of the criterion \(U^n\) of agent \(n\) at \(w \in \mathbb{R}^3\). The supergradient of a SEU-agent with prior \(\pi = (\pi_R, \pi_G, \pi_B)\) is simply the gradient

\[
(C.2) \quad \partial U^n(w) = \{(\pi_R u'_R(w_R), \pi_G u'_G(w_G), \pi_B u'_B(w_B))\}.
\]

From (C.2) and the strict concavity of the utility function follows the well known fact that the optimal portfolio \(w = (w_R, w_G, w_B)\) of a SEU agent is always such that the optimal choices of state dependent wealth are ranked opposite to the state-price/state-probability ratios, i.e.

\[
(C.3) \quad w_\sigma > w_\nu \Leftrightarrow \frac{p_\sigma}{\pi_\sigma} < \frac{p_\nu}{\pi_\nu}, \quad \sigma, \nu \in \{R, B, G\}.
\]
The supergradient of an agent with maxmin (1-MEU) preferences represented as in (3.4) is

\[
\partial U^m(w) = \begin{cases} 
\{ (\pi_R u'(w_R), cu'(w_G), (1 - \pi_R - c)u'(w_B)) \} & \text{if } w_G > w_B \\
\{ (\pi_R u'(w_R), du'(w_G), (1 - \pi_R - d)u'(w_B)) \} & \text{if } w_G < w_B \\
\{ (\pi_R u'(w_R), (\lambda c + (1 - \lambda)d)u'(w_G), \\
(1 - \pi_R - (\lambda c + (1 - \lambda)d))u'(w_B)) | \lambda \in [0, 1] \} & \text{if } w_G = w_B.
\end{cases}
\]

Using (C.1) and the shape of the supergradients we easily obtain the optimal portfolio choices that were already derived in Bossaerts et al. (2010). In particular, from (C.4) and the strict concavity of \( u \) it follows that

\[
w_G > w_B \text{ if and only if } \frac{p_G}{p_B} < \frac{c}{1 - \pi_R - c} \\
w_G < w_B \text{ if and only if } \frac{p_G}{p_B} > \frac{d}{1 - \pi_R - d} \\
w_G = w_B \text{ if and only if } \frac{p_G}{p_B} \in \left[ \frac{c}{1 - \pi_R - c}, \frac{d}{1 - \pi_R - d} \right]
\]

where \( x/0 := \infty \). The larger the set of priors \( C \) in (3.4), the more likely a maxmin agent will take an unambiguous portfolio \( (w_B = w_G) \). In particular this will be always the case if \( C = C_{\text{max}} := \{ (\pi_R, q, 1 - q - \pi_R) : q \in [0, 1 - \pi_R] \} \), because then the second respectively third coordinate of the supergradient in (C.4) will be 0 if either \( w_G > w_B \) or \( w_G < w_B \).

Hence, \( p_\sigma > 0 \) for all \( \sigma \in \{ R, G, B \} \) and (C.1) imply that in equilibrium this agent will only take an unambiguous portfolios \( w \). If \( c > 0 \) and/or \( d < 1 - \pi_R \) in (3.4), then the multiple prior agent may also take an ambiguous portfolio in equilibrium. We observe that a maxmin agent holding an unambiguous optimal portfolio behaves as a SEU-agent who is not differentiating between the ambiguous states \( G \) and \( B \), but merges them to an unambiguous state \( \{ G, B \} \) with probability \((1 - \pi_R)\). Indeed, from (C.4) and (C.1) it follows that

\[
P_{\{G,B\}} = \frac{(1 - \pi_R)u'(w_{\{G,B\}})}{\pi_R u'(w_R)} \begin{cases} 
< \frac{(1 - \pi_R)}{\pi_R} & \text{iff } w_{\{G,B\}} > w_R \\
> \frac{(1 - \pi_R)}{\pi_R} & \text{iff } w_{\{G,B\}} < w_R
\end{cases}
\]

and thus

\[
P_{\{G,B\}} \frac{(1 - \pi_R)}{\pi_R} < \frac{p_R}{\pi_R} \iff w_{\{G,B\}} > w_R
\]

\[
P_{\{G,B\}} \frac{(1 - \pi_R)}{\pi_R} > \frac{p_R}{\pi_R} \iff w_{\{G,B\}} < w_R \quad \text{(compare this to (C.3))},
\]
where  \( p_{(G,B)} := p_G + p_B \) and  \( w_{(G,B)} := w_G = w_B \).

**Proof of Proposition 5.1**

**Case 1:** Let  \( W_R > W_G > W_B \). Since the 1-MEU agents take an unambiguous portfolio, the optimal portfolio of some SEU agent must satisfy  \( y_G > y_B \) which according to (C.3) is equivalent to

\[
\frac{p_B}{\pi_B} > \frac{p_G}{\pi_G},
\]

which only leaves the ranking of  \( p_R/\pi_R \) within (C.9) an open question. Suppose that the ranking of the ratios state-price/state-probability is as follows:

\[
\frac{p_R}{\pi_R} \geq \frac{p_B}{\pi_B} > \frac{p_G}{\pi_G}.
\]

Then (C.3) implies that  \( y_G > y_B \geq y_R \) for any SEU agent, and rearranging (C.10) yields

\[
\frac{p_G + p_B}{1 - \pi_R} = \frac{p_G + p_B}{\pi_G + \pi_B} < \frac{p_R}{\pi_R}.
\]

Consequently, according to (C.6), we must have for each 1-MEU agent that  \( w_R < w_G = w_B \). But this contradicts the clearing of the market and  \( W_R > W_G > W_B \). If the ranking is (5.1), then we have  \( y_G > y_R > y_B \) for each SEU agent according to (C.3). Denote by  \( y^\Sigma = (y_R^\Sigma, y_G^\Sigma, y_B^\Sigma) \) the sum over all optimal portfolios of the SEU agents and similarly by  \( w^\Sigma = (w_R^\Sigma, w_G^\Sigma, w_B^\Sigma) \) the sum over all optimal portfolios of the 1-MEU agents. The market clearing condition says  \( W_\sigma = y_\sigma^\Sigma + w_\sigma^\Sigma \) for every  \( \sigma \in \{R, G, B\} \). Since  \( y_G^\Sigma > y_R^\Sigma \) we conclude that

\[
w_R^\Sigma = W_R - y_R^\Sigma > W_G - y_G^\Sigma = w_G^\Sigma.
\]

Thus there must be at least one 1-MEU agent who’s portfolio  \( w = (w_R, w_G, w_B) \) satisfies  \( w_R > w_G = w_B \) which implies that  \( (p_G + p_B)/p_R > (1 - \pi_R)/\pi_R \) due to (C.6). But then, again by (C.6), we must have  \( w_R > w_G = w_B \) for all 1-MEU agents. In case of (5.2) (C.3) and (C.6) imply the claimed ranking of payoffs in the portfolios  \( y \) and  \( w \).

**Case 2:** Let  \( W_G > W_R > W_B \). As in case one we conclude that  \( y_G > y_B \). Assume that the ranking of the ratio state-price/state-probability is as follows:

\[
\frac{p_R}{\pi_R} \geq \frac{p_B}{\pi_B} > \frac{p_G}{\pi_G}.
\]

Then as in case 1 it follows that  \( y_G > y_B \geq y_R \) and  \( w_R < w_G = w_B \) which together with the clearing of the market contradicts  \( W_R > W_B \). Similarly it follows that the ranking

\[
\frac{p_B}{\pi_B} > \frac{p_G}{\pi_G} > \frac{p_R}{\pi_R}
\]

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is not possible, since it would imply that $y_R \geq y_G > y_B$ and $w_R > w_G = w_B$ due to (C.6), again contradicting the assumed ranking of the aggregate wealth.

**Case 3:** Let $W_G > W_B > W_R$: Suppose that

$$\frac{p_B}{\pi_B} > \frac{p_G}{\pi_G} \geq \frac{p_R}{\pi_R}$$

then, $y_R \geq y_G > y_B$, and in view of (C.6) we obtain $w_R > w_G = w_B$ for every 1-MEU agent which again contradicts the market clearing and the assumed ranking $W_G > W_B > W_R$. Again (C.3), (C.6), and the clearing of the market imply the claimed ranking of payoffs in the portfolios $y, w$ for the remaining possible rankings.

## D Optimization in the partially concave case

Consider the optimization problem

$$\text{(D.1)} \quad \max_{x \in C} U(x) \quad \text{subject to} \quad px \leq pe$$

where $C \neq \emptyset$ is a convex subset of $\mathbb{R}^n$, $p, e \in \mathbb{R}^n$, and $U : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ is a concave function with dom $U = C$.

**Lemma D.1.** If the optimal value in (D.1) is not $+\infty$ and if there exists at least one $\bar{x} \in \text{ri} C$ with $p\bar{x} \leq pe$, then there is a multiplier $\lambda \geq 0$ such that the supremum of $h_\lambda(x) = U(x) - \lambda p(x-e)$, $x \in \mathbb{R}^n$, is finite and equal to the optimal value in (D.1). Moreover, suppose that $\lambda > 0$ and that $D$ is the set of points $x \in \mathbb{R}^n$ where $h$ attains its maximum intersected with the set of points satisfying $px = pe$, then $D$ is the set of all optimal solutions to (D.1).

**Proof.** see Theorem 28.1 and Corollary 28.2.2 in Rockafellar (1997). \hfill \Box

Now suppose that agent $n$ with choice criterium $U^n : \mathbb{R}^{|S|} \to \mathbb{R}$ maximizes her utility over all portfolios $w \in \mathbb{R}^{|S|}$ satisfying the budget constraint $pw \leq pe^n$ for some $p \in \mathbb{R}^{|S|}$ with $p_i > 0$ for all $i = 1, \ldots, |S|$. Furthermore, assume that an optimal portfolio $\hat{w}$ exists and that $\hat{w} \in C$ for a convex set $C \subset \mathbb{R}^{|S|}$ such that the restriction $U^n_C$ of $U^n$ to $C$ is concave. Then, we may view $U^n_C$ as defined on all $\mathbb{R}^{|S|}$ by defining $U^n_C(x) := -\infty$ for $x \notin C$, and we are thus in the setting of Lemma D.1 where $\hat{w}$ is a solution to problem
(D.1) with $U = U^n_C$. Hence, if there exists $x \in \text{ri} C$ with $px \leq pe^n$, which is satisfied if for instance $\hat{w} \in \text{ri} C$, then there exists a multiplier $\lambda \geq 0$ such that

\begin{equation}
U^n_C(\hat{w}) = \sup_{x \in \mathbb{R}^n} h_\lambda(x)
\end{equation}

with $h_\lambda$ as in Lemma D.1. If $C = C + \mathbb{R}_+ \cdot (1, 0, \ldots, 0)$ and given that the utility function $u$ is strictly increasing we deduce that $\lambda > 0$, since otherwise

$$h_\lambda(\hat{w} + (1, 0, \ldots, 0)) = U^n_C(\hat{w} + (1, 0, \ldots, 0)) > U^n_C(\hat{w}).$$

Moreover, any solution $\hat{x}$ to the right hand side of (D.2) with $p\hat{x} = pe^n$ is a solution to the portfolio optimization problem, and in particular $\hat{w}$ is such a solution. Additionally, for any solution $\hat{x}$ to the right hand side of (D.2) we have for all $y \in \mathbb{R}^{|S|}$ that

$$U^n_C(y) - \lambda p(y - e^n) \leq U^n_C(\hat{x}) - \lambda p(\hat{x} - e^n)$$

which shows that

\begin{equation}
\lambda p \in \partial U^n_C(\hat{x})
\end{equation}

where $\partial U^n_C(w)$ denotes the supergradient of $U^n_C$ at $w$, i.e.

$$\partial U^n_C(w) := \{\nu \in \mathbb{R}^{|S|} \mid \forall y \in \mathbb{R}^{|S|}, U^n_C(y) \leq U^n_C(w) + \nu \cdot (y - w)\}.$$
References


